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EXPOSÉ X

GABBER'S MODIFICATION THEOREM (LOG SMOOTH CASE)

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In this exposé we state and prove a variant of the main theorem of VIII (see VIII-1.1) for schemes X which are log smooth over a base S with trivial G-action. See 1.1 for a precise statement. The proof is given in § 1 and in the remaining part of the exposé we deduce refinements of classical theorems of de Jong, for schemes of finite type over a field or a trait, where the degree of the alteration is made prime to a prime ℓ invertible on the base. Sections 2 and 3 are independent and contain two different proofs of such a refinement, so let us outline the methods briefly.

For concreteness, assume that k is a field, $S = \mathrm{Spec}(k)$, and X is a separated S-scheme of finite type. Two methods to construct regular ℓ'-alterations of X are: (1) use a pluri-nodal fibration to construct a regular G-alteration $X' \to X$ and then factor X' by an ℓ-Sylow subgroup of G, and (2) construct a regular ℓ'-alteration by induction on $\dim(S)$ so that one factors by an ℓ-Sylow subgroup at each step of the induction. The first approach is presented in § 2. It is close in spirit to the approach of [de Jong, 1997] and its strengthening by Gabber-Vidal, see [Vidal, 2004, § 4]. The weak point of this method is that one uses inseparable Galois alterations. In particular, even when k is perfect, one cannot obtain a separable alteration of X.

The second approach is realized in § 3, using [Temkin, 2010]; it outperforms the method of § 2 when k is perfect. Moreover, developing this method the second author discovered Theorem 3.5 that generalizes Gabber's theorems 2.1 and 2.4 to the case of a general base S satisfying a certain resolvability assumption (see § 3.3). In addition, if S is of characteristic zero then the same method allows to use modifications instead of ℓ'-alterations, see Theorem 3.9. As an application, in Theorem 3.10 we

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generalize Abramovich-Karu's weak semistable reduction theorem. Finally, we minimize separatedness assumptions in § 3, and for this we show in § 3.1 how to weaken the separatedness assumptions in Theorems 1.1 and VIII-1.1.

1. The main theorem

1.1. Theorem. — *Let $f : X \to S$ be an equivariant log smooth map between fs log schemes endowed with an action of a finite group G. Assume that:*

(i) *G acts trivially on S;*

(ii) *X and S are noetherian, qe, separated, log regular, and f defines a map of log regular pairs $(X, Z) \to (S, W)$ (see VI-1.4: (X, Z) and (S, W) are log regular pairs and $f(X - Z) \subset S - W$);*

(iii) *G acts tamely and generically freely on X.*

Let T be the complement of the largest open subset of X over which G acts freely. Then there exists an equivariant projective modification $h : X' \to X$ such that, if $Z' = h^{-1}(Z \cup T)$, the pair (X', Z') is log regular, the action of G on X' is very tame, and (X', Z') is log smooth over (S, W) as well as the quotient $(X'/G, Z'/G)$ when G acts admissibly on X ([SGA 1 v 1.7]*).*

1.1.1. Remark. — (a) In the absence of the hypothesis (i) it may not be possible to find a modification h satisfying the properties of 1.1, as the example at the end of VIII-1.2 shows.

(b) By [Kato, 1994, 8.2] the log smoothness of f and the log regularity of S imply the log regularity of X. Conversely, according to Gabber (private communication), if X is log regular and f is log smooth and surjective, then S is log regular.

(c) We will deduce Theorem 1.1 from Theorem VIII-1.1. Recall that in the latter theorem we assumed that X is qe, though Gabber has a subtler argument that works for a general X. This forces us to assume that S (and hence X) is qe in Theorem 1.1. However, our argument also shows that once one removes the quasi-excellence assumption from VIII-1.1, one also obtains the analogous strengthening of Theorem 1.1.

For the proof of 1.1 we will use the following result on the local structure of equivariant log smooth maps.

***1.2. Proposition* (Gabber's preparation lemma).** — *Let $f : X \to Y$ be an equivariant log smooth map between fine log schemes endowed with an action of a finite group G. Let x be a geometric point of X, with image y in Y. Assume that G is the inertia group at x and is of order invertible on Y. Assume furthermore that G acts trivially on $\overline{M}_x$ and*

$\overline{M}_y$ [(ii)] *and we are given an equivariant chart* $a : Y \to \operatorname{Spec} \Lambda[Q]$ *at* y, *modeled on some pairing* $\chi : G^{\mathrm{ab}} \otimes Q^{\mathrm{gp}} \to \mu = \mu_N(\mathbf{C})$ *(in the sense of (VI-3.3)), where* Q *is fine,* $\Lambda = \mathbf{Z}[1/N, \mu]$, *with* N *the exponent of* G. *Then, up to replacing* X *by an inert equivariant étale neighborhood of* x, *there is an equivariant chart* $b : X \to \operatorname{Spec} \Lambda[P]$ *extending* a, *such that* $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ *is injective, the torsion of its cokernel is annihilated by an integer invertible on* X, *and the resulting map* $b' : X \to X' = Y \times_{\operatorname{Spec} \Lambda[Q]} \operatorname{Spec} \Lambda[P]$ *is smooth. Moreover, up to further shrinking* X *around* x, b' *lifts to an inert equivariant étale map* $c : X \to X' \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \operatorname{Sym}_\Lambda(V)$, *where* V *is a finitely generated projective* Λ*-module equipped with a* G*-action. If* X, Y, *and* Q *are fs, with* Q *sharp, then* P *can be chosen to be fs with its subgroup of units* P^* *torsionfree.*

Proof of 1.2. — This is an adaptation of the proof of [Kato, 1988, 3.5] to the equivariant case. Consider the canonical homomorphism of *loc. cit.*

$$k(x) \otimes_{\mathscr{O}_{X,x}} \Omega^1_{X/Y,x} \to k(x) \otimes_{\mathbf{Z}} \overline{M}^{\mathrm{gp}}_{X/Y,x} \tag{1.2.1}$$

sending $1 \otimes d\log t$ to the class of $1 \otimes t$, where

$$\overline{M}^{\mathrm{gp}}_{X/Y,x} = M^{\mathrm{gp}}_{X,x} / (\mathscr{O}^*_{X,x} + \operatorname{Im} f^{-1}(M^{\mathrm{gp}}_{Y,y})).$$

It is surjective, and as G fixes x, it is G-equivariant. As G is of order invertible in $k(x)$ and acts trivially on the right hand side, (1.2.1) admits a G-equivariant decomposition

$$k(x) \otimes_{\mathscr{O}_{X,x}} \Omega^1_{X/Y,x} = V_0 \oplus (k(x) \otimes_{\mathbf{Z}} \overline{M}^{\mathrm{gp}}_{X/Y,x}), \tag{1.2.2}$$

where V_0 is a finite dimensional $k(x)$-vector space, endowed with an action of G. Let $(t_i)_{1 \le i \le r}$ be elements of M^{gp}_x such that the classes of $1 \otimes t_i$ form a basis of $k(x) \otimes_{\mathbf{Z}} \overline{M}^{\mathrm{gp}}_{X/Y,x}$. By the method of (VI-3.5) we can modify the t_i's to make them eigenfunctions of G. More precisely, for $g \in G$, we have

$$g t_i = z_i(g) t_i,$$

with $z_i(g) \in \mathscr{O}^*_x$, and $g \mapsto z_i(g)$ is a 1-cocycle of G with values in $\mathscr{O}^*_x$. By reduction mod $\mathfrak{m}_x$, it gives a 1-cocycle $\psi_i \in Z^1(G, \mu) = \operatorname{Hom}(G, \mu)$, as μ is naturally embedded in $k(x)^*$ since X is over Λ. Lifting μ in $\mathscr{O}^*_x$, $g \mapsto z_i(g)/\psi_i(g)$ is a 1-cocycle of G with values in $1 + \mathfrak{m}_x$, hence a coboundary $\delta_i \in B^1(G, 1 + \mathfrak{m}_x)$, $g \mapsto \delta_i(g) = g u_i / u_i$, for $u_i \in 1 + \mathfrak{m}_x$. Replacing t_i by $t_i u_i^{-1}$, we may assume that $z_i = \psi_i$, *i.e.*

$$g t_i = \psi_i(g) t_i,$$

for characters

$$\psi_i : G \to \mu.$$

(ii) If M is the sheaf of monoids of a log scheme, $\overline{M}$ denotes, as usual, the quotient $M/\mathscr{O}^*$.

Let Z be the free abelian group with basis $(e_i)_{1\leq i\leq r}$, and $h : Z \to M_x^{\mathrm{gp}}$ the homomorphism sending e_i to t_i. As in the proof of [Kato, 1988, 3.5], consider the homomorphism

$$u : Z \oplus Q^{\mathrm{gp}} \to M_x^{\mathrm{gp}}$$

defined by h on Z and the composition $Q^{\mathrm{gp}} \to M_y^{\mathrm{gp}} \to M_x^{\mathrm{gp}}$ on the second factor. We have

$$gu(a) = \psi(g \otimes a)u(a)$$

for some homomorphism

$$\psi : G^{\mathrm{ab}} \otimes (Z \oplus Q^{\mathrm{gp}}) \to \mu$$

extending χ and such that $\psi(g \otimes e_i)u(e_i) = \psi_i(g)h(e_i)$. As in *loc. cit.*, if $\overline{u}$ denotes the composition

$$\overline{u} : Z \oplus Q^{\mathrm{gp}} \to M_x^{\mathrm{gp}} \to \overline{M}_x^{\mathrm{gp}} (= M_x^{\mathrm{gp}}/\mathscr{O}_x^*)$$

we see that $k(x) \otimes \overline{u}$ is surjective, hence the cokernel C of $\overline{u}$ is killed by an integer m invertible in $k(x)$. Using that $\mathscr{O}_{X,x}^*$ is m-divisible, one can choose elements $a_i \in M_x^{\mathrm{gp}}$ and $b_i \in Z \oplus Q^{\mathrm{gp}}$ $(1 \leq i \leq n)$ such that the images of the a_i's generate $\overline{M}_x^{\mathrm{gp}}$ and $a_i^m = u(b_i)$. Let E be the free abelian group with basis e_i $(1 \leq i \leq n)$, and let F be the abelian group defined by the push-out diagram

(1.2.3)
$$\begin{array}{ccc} E & \xrightarrow{m} & E \\ \downarrow & & \downarrow \\ Z \oplus Q^{\mathrm{gp}} & \xrightarrow{w} & F \end{array},$$

where the left vertical arrow sends e_i to b_i. The lower horizontal map is injective and its cokernel is isomorphic to E/mE, in particular, killed by m. The relation $a_i^m = u(b_i)$ implies that u extends to a homorphism

$$v : F \to M_x^{\mathrm{gp}}$$

whose composition $\overline{v} : F \to M_x^{\mathrm{gp}} \to \overline{M}_x^{\mathrm{gp}}$ is surjective. Associated with v is a morphism

$$\varphi : G^{\mathrm{ab}} \otimes F \to \mu$$

extending ψ, such that $gv(a) = \varphi(g \otimes a)v(a)$ for $a \in F$. Let $P := v^{-1}(M_x) \subset F$. Then P is a fine monoid containing Q, $P^{\mathrm{gp}} = F$, and v sends P to M_x. As in VI-3.5, VI-3.10 we get a G-equivariant chart of $X_{(x)}$ associated with φ, which, up to replacing X by an inert equivariant étale neighborhood at x, extends to an equivariant chart

$$b : X \to \operatorname{Spec} \Lambda[P]$$

extending the chart $a : Y \to \operatorname{Spec} \Lambda[Q]$. The homomorphism $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, and the torsion part of its cokernel injects into the cokernel of $w : Z \oplus Q^{\mathrm{gp}} \to F$ in

(1.2.3), which is killed by m. Consider the resulting map

$$b' : X \to X' = Y \times_{\mathrm{Spec}\ \Lambda[Q]} \mathrm{Spec}\ \Lambda[P].$$

This map is strict. Showing that the underlying schematic map is smooth at x is equivalent to showing that b' is log smooth at x. To do this, as X and X' are log smooth over Y, by the jacobian criterion [Kato, 1988, 3.12] it suffices to show that the map

$$k(x) \otimes \Omega^1_{X'/Y} \to k(x) \otimes \Omega^1_{X/Y}$$

induced by b' is injective. We have

$$k(x) \otimes \Omega^1_{X'/Y} = k(x) \otimes P^{\mathrm{gp}}/Q^{\mathrm{gp}} = k(x) \otimes Z$$

(the last equality by the fact that $F/(Z \oplus Q^{\mathrm{gp}})$ is killed by m), and by construction (cf. (1.2.2)), we have

$$k(x) \otimes Z = k(x) \otimes \overline{M}^{\mathrm{gp}}_{X/Y,x},$$

which by the map induced by b' injects into $k(x) \otimes \Omega^1_{X/Y}$.

Let us now prove the second assertion. For this, as b' is strict, we may forget the log structures of X and X', and by changing notations, we may assume that $X' = Y$ and the log structures of X and Y are trivial. In particular, we have

$$k(x) \otimes \Omega^1_{X/Y} = V_0,$$

with the notation of (1.2.2). As the question is étale local on X, and closed points are very dense in the fiber X_y, in particular, any point has a specialization at a closed point of X_y, we may assume that x sits over a closed point of X_y, and even, up to base changing Y by a finite radicial extension, that x is a rational point of X_y. We then have

$$k(x) \otimes \Omega^1_{X/Y} = \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \mathscr{O}_x),$$

where $\mathfrak{m}$ denotes a maximal ideal. By a classical result in representation theory (see 1.3 below) there is a finitely generated projective $\Lambda[G]$-module V such that $V_0 = k(x) \otimes V$. The homomorphism $V \to \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \mathscr{O}_x)$ therefore lifts to a homomorphism of $\Lambda[G]$-modules

$$V \to \mathfrak{m}_x,$$

inducing an isomorphism $k(x) \otimes V \to k(x) \otimes \Omega^1_{X/Y}$. By the jacobian criterion, it follows that the (G-equivariant) map

$$X \to Y \times_{\mathrm{Spec}\ \Lambda} \mathrm{Spec}\ \mathrm{Sym}_\Lambda(V)$$

is étale at x, and as in VI-3.10 can be made inert by shrinking X.

Let us prove the last assertion. Now X and Y are fs, and Q is fs and sharp. First of all, as M_x is saturated, $P = \nu^{-1}(M_x)$ is fs. Then (cf. [Gabber & Ramero, 2013, 3.2.10]) we have a split exact sequence

$$0 \to H \to P \to P_0 \to 0$$

with P_0^* torsionfree and H a finite group. As Q is fs and sharp, Q^{gp} is torsionfree, so the composition $Q^{\mathrm{gp}} \to P^{\mathrm{gp}} \to P_0^{\mathrm{gp}}$ is still injective, as well as the composition $H \to P^{\mathrm{gp}} \to P^{\mathrm{gp}}/Q^{\mathrm{gp}}$, hence H is contained in the torsion part of $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$, and we have an exact sequence

$$0 \to H \to (P^{\mathrm{gp}}/Q^{\mathrm{gp}})_{\mathrm{tors}} \to (P_0^{\mathrm{gp}}/Q^{\mathrm{gp}})_{\mathrm{tors}} \to 0,$$

where the subscript *tors* denotes the torsion part. Thus $(P_0^{\mathrm{gp}}/Q^{\mathrm{gp}})_{\mathrm{tors}}$ is killed by an integer invertible on X. As $\overline{M}_x$ is torsionfree, the composition $P \to M_x \to \overline{M}_x$ factors through P_0, into a map $v_0 : P_0 \to \overline{M}_x$. Consider the diagram

$$\begin{array}{ccccc} & & M_x & \longrightarrow & M_x^{\mathrm{gp}} \\ & & \downarrow & & \downarrow \\ P_0 & \xrightarrow{v_0} & \overline{M}_x & \longrightarrow & \overline{M}_x^{\mathrm{gp}}, \end{array}$$

where the square is cartesian. As P_0^{gp} is torsionfree, the map $P_0^{\mathrm{gp}} \to \overline{M}_x^{\mathrm{gp}}$ defined by the lower row admits a lifting $s : P_0^{\mathrm{gp}} \to M_x^{\mathrm{gp}}$, sending P_0 to M_x. One can adjust s to make it compatible with the morphism $\tilde{a} : Q^{\mathrm{gp}} \to M_y^{\mathrm{gp}} \to M_x^{\mathrm{gp}}$ given by the chart $a : Q \to M_Y$. Indeed, if $j : Q^{\mathrm{gp}} \hookrightarrow P_0^{\mathrm{gp}}$ is the inclusion, the homomorphisms $sj/\tilde{a} : Q^{\mathrm{gp}} \to \mathscr{O}_{X,x}^*$ can be extended to P_0^{gp} as the torsion part of $P_0^{\mathrm{gp}}/Q^{\mathrm{gp}}$ is killed by an integer invertible on X. Assume that this adjustment is done. As v is a chart, $P/v^{-1}(\mathscr{O}_x^*) \to \overline{M}_x$ is an isomorphism, and since H is contained in $v^{-1}(\mathscr{O}_x^*)$, $P_0/s^{-1}(\mathscr{O}_x^*) \to \overline{M}_x$ is an isomorphism as well, hence $s : P_0 \to M_x$ is a chart at x compatible with a. A second adjustment is needed to make it G-equivariant. To do so, one can proceed as above, by considering the 1-cocycle z of G with values in $\mathrm{Hom}(P_0^{\mathrm{gp}}, \mathscr{O}_x^*)$ given by

$$gs(p) = z(g,p)s(p).$$

The image of z in $Z^1(G, \mathrm{Hom}(P_0^{\mathrm{gp}}, k(x)^*))$ is a homomorphism

$$\varphi_0 : G^{\mathrm{ab}} \otimes P_0^{\mathrm{gp}} \to \mu.$$

The quotient $g \mapsto (p \mapsto z(g,p)/\varphi_0(g,p))$ belongs to $B^1(G, \mathrm{Hom}(P_0^{\mathrm{gp}}/Q^{\mathrm{gp}}, 1 + \mathfrak{m}_x))$, hence can be written $g \mapsto (p \mapsto g\rho(p)/\rho(p))$ for $\rho : P_0^{\mathrm{gp}}/Q^{\mathrm{gp}} \to 1 + \mathfrak{m}_x$. So, replacing z by $g \mapsto z(g,p)\rho(p)^{-1}$, we may assume that $z = \varphi_0$, in other words, the map

$$b_0 : X \to \mathrm{Spec}\ \Lambda[P_0]$$

defined by the pair (s, φ_0) is an equivariant chart of X at x (extending a).

One can give an alternate, shorter proof of the last assertion which does not use the above decomposition of P into $H \oplus P_0$. Consider again the cokernel C of the map $\overline{u}$ introduced a few lines above diagram (1.2.3). Write C as a direct sum of cyclic groups of orders $m_i | m$. Choose $a_i \in M_x^{\mathrm{gp}}$ and $b_i \in Z \oplus Q^{\mathrm{gp}}$ $(1 \leq i \leq n)$ such that $a_i^{m_i} = u(b_i)$, and the a_i's induce an isomorphism

$$\bigoplus_i \mathbf{Z}/m_i\mathbf{Z} \xrightarrow{\sim} C.$$

Replace diagram (1.2.3) by the following push-out diagram

$$\begin{array}{ccc} \bigoplus_i \mathbf{Z}e_i & \longrightarrow & \bigoplus_i \mathbf{Z}(\frac{1}{m_i}e_i) \\ \downarrow & & \downarrow \\ Z \oplus Q^{\mathrm{gp}} & \xrightarrow{\quad w \quad} & F, \end{array} \tag{1.2.4}$$

where the upper horizontal map is the natural inclusion and the left vertical one sends e_i to b_i. In this way, we have $F = P^{\mathrm{gp}} \supset Z \oplus Q^{\mathrm{gp}}$ and $P^{\mathrm{gp}}/(Z \oplus Q^{\mathrm{gp}}) \xrightarrow{\sim} C$. As X is fs, $\overline{M}_x^{\mathrm{gp}}$ is torsionfree, so the map $\overline{v} : F \to \overline{M}_x^{\mathrm{gp}}$, defined similarly as above (using (1.2.4) instead of (1.2.3)), sends $(P^{\mathrm{gp}})_{\mathrm{tors}}$ to 0, hence

$$(P^{\mathrm{gp}})_{\mathrm{tors}} = (Z \oplus Q^{\mathrm{gp}})_{\mathrm{tors}} = (Q^{\mathrm{gp}})_{\mathrm{tors}},$$

which finishes the proof. □

1.3. Lemma. — *Let G be a finite group of exponent n, let $\Lambda = \mathbf{Z}[\mu_n][1/n]$, let k be a field over Λ, and let L be a finitely generated $k[G]$-module. There exists a finitely generated projective $\Lambda[G]$-module V such that $L = k \otimes_\Lambda V$.*

Proof. — First, observe that since n is invertible in Λ, any $\Lambda[G]$-module which is finitely generated and projective over Λ is projective over $\Lambda[G]$ [Serre, 1978, § 14.4, Lemme 20].

Suppose first that $\mathrm{char}(k) = 0$, and let $\overline{k}$ be an algebraic closure of k. Then, L descends to a $\mathbf{Q}[\mu_n][G]$-module W, as $\overline{k} \otimes_k L$ descends [Serre, 1978, § 12.3] and the homomorphism $R_k(G) \to R_{\overline{k}}(G)$ given by extension of scalars is injective [Serre, 1978, § 14.6]. One can then take for V a G-stable Λ-lattice in W (projective over Λ), which is necessarily projective over $\Lambda[G]$ by the above remark.

Suppose now that $\mathrm{char}(k) = p > 0$. Let $I \twoheadrightarrow k$ be a Cohen ring for k. As Λ is étale over $\mathbf{Z}$, $\Lambda \to k$ lifts (uniquely) to $\Lambda \to I$. On the other hand, as L is projective of finite type over $k[G]$, by [Serre, 1978, § 14.4, Prop. 42, Cor. 3] L lifts to a finitely generated projective $I[G]$-module E, free over I. Let K be the fraction field of I. Then $E \otimes K$ descends to a $\mathbf{Q}[\mu_n][G]$-module E'. Choose a G-stable Λ-lattice V in E' (projective over Λ, hence, projective of finite type over $\Lambda[G]$). By [Serre, 1978, § 15.2, Th. 32], $k \otimes_\Lambda V$ has the same class in $R_k(G)$ as L. But, as $k[G]$ is semisimple by Maschke's theorem, L and $k \otimes_\Lambda V$ are isomorphic as $k[G]$-modules. □

Proof of 1.1 (beginning).

The strategy is to check that, at each step of the proof of the absolute modification theorem (VIII-1.1), the log smoothness of X/S is preserved, and, at the end, that of the quotient $(X/G)/S$ as well. For some of them, this is trivial, as the modifications performed are log blow ups. Others require a closer inspection.

1.4. — *Preliminary reductions.* We may assume that conditions (1) and (2) at the beginning of (VIII-4) are satisfied, namely:

(1) *X is regular,*

(2) *Z is a G-strict snc divisor in X.*

Indeed, these conditions are achieved by G-equivariant saturated log blow up towers (VIII-4.1.1, VIII-4.1.6).

We will now exploit Gabber's preparation lemma 1.2 to give a local picture of f displaying both the log stratification and the inertia stratification of X. We work étale locally at a geometric point x in X with image s in S. Up to replacing X by the G_x-invariant neighborhood X' constructed at the beginning of the proof of VIII-5.3.8, and G by G_x, where G_x is the inertia group at x, we may assume that $G = G_x$. Indeed, the morphism $(X', G_x) \to (X, G)$ is strict and inert, and by VIII-5.4.4 the tower $f_{(G,X,Z)}$ is functorial with respect to such morphisms.

We now apply 1.2. Let N be the exponent of G. Assume S strictly local at s. We may replace $\Lambda = \mathbf{Z}[1/N, \mu]$ by its localization at the (Zariski) image of s, so that Λ is either the cyclotomic field $\mathbf{Q}(\mu)$ or its localization at a finite place of its ring of integers, of residue characteristic $p = \mathrm{char}(k(s))$ not dividing n. Choose a chart

$$a : S \to \mathrm{Spec}\ \Lambda[Q]$$

with Q fs and the inverse image of $\mathscr{O}^*_{S,s}$ in Q equal to $\{1\}$, so that Q is sharp and $Q \xrightarrow{\sim} \overline{M}_s$. Let C denote $k(s)$ if $\mathscr{O}_{S,s}$ contains a field, and a Cohen ring of $k(s)$ otherwise. Let $(y_i)_{1 \le i \le m}$ be a family of elements of $\mathfrak{m}_s$ such that the images of the y_i's in $\mathscr{O}_{S,s}/I_s$ form a regular system of parameters, where $I_s = I(s, M_s)$ is the ideal generated by the image of $M_s - \mathscr{O}^*_{S,s}$ by the canonical map $\alpha : M_s \to \mathscr{O}_{S,s}$. By [Kato, 1994, 3.2], the chart a extends to an isomorphism

$$C[[y_1, \dots, y_m]][[Q]]/(g) \xrightarrow{\sim} \widehat{\mathscr{O}}_{S,s}, \tag{1.4.1}$$

where $g \in C[[y_1, \dots, y_m]][[Q]]$ is 0 if $C = k(s)$, and congruent to $p = \mathrm{char}(k(s)) > 0$ modulo the ideal generated by $Q - \{1\}$ and $(y_1, \dots, y_m)$ otherwise. By 1.2, up to shrinking X around x, we can find a G-equivariant commutative diagram (with trivial action of G on the bottom row)

$$\begin{array}{ccccc} X & \xrightarrow{c} & X' & \xrightarrow{b} & \operatorname{Spec}(\Lambda[P]\otimes_\Lambda \operatorname{Sym}_\Lambda(V)) \\ & \searrow & \downarrow & & \downarrow \\ & & S & \xrightarrow{a} & \operatorname{Spec}\Lambda[Q], \end{array} \tag{1.4.2}$$

where:

(i) the square is cartesian;

(ii) a, b, and c are strict, where the log structure on Spec $\Lambda[Q]$ (resp. Spec $(\Lambda[P]\otimes_\Lambda \operatorname{Sym}_\Lambda(V))$) is the canonical one, given by Q (resp. P); P is an fs monoid, with P^* torsionfree; G acts on $\Lambda[P]$ by $g(\lambda p) = \lambda\chi(g,p)p$, for some homomorphism

$$\chi : G^{\mathrm{ab}} \otimes P^{\mathrm{gp}} \to \mu$$

(iii) V is a free, finitely generated Λ-module, equipped with a G-action;

(iv) the right vertical arrow is the composition of the projection onto the factor Spec $\Lambda[P]$ and Spec $\Lambda[h]$, for a homomorphisme $h : Q \to P$ such that h^{gp} is injective and the torsion part of Cokerh^{gp} is annihilated by an integer invertible on X;

(v) c is étale and inert.

(vi) Consider the map

$$v : P \to M_x$$

defined by the chart $X \to$ Spec $\Lambda[P]$ induced by bc. Up to localizing on X' around x, we may assume that v factors through the localization $P_{(\mathfrak{p})}$ of P at the prime ideal $\mathfrak{p}$ complementary of the face $v^{-1}(\mathscr{O}^*_{X,x})$. Replacing P by $P_{(\mathfrak{p})}$, P decomposes into

$$P = P^* \oplus P_1, \tag{1.4.3}$$

with $P^* = v^{-1}(\mathscr{O}^*_{X,x})$ free finitely generated over $\mathbf{Z}$, and P_1 sharp, and the image of x by bc in the factor Spec $\Lambda[P_1]$ is the rational point at the origin. Then v induces an isomorphism $P_1 \xrightarrow{\sim} \overline{M}_x$. By the assumptions (1), (2), we have $\overline{M}_x \xrightarrow{\sim} \mathbf{N}^r$. One can therefore choose $(e_i \in P_1)$ $(1 \le i \le r)$ forming a basis of P_1. Then $v(e_i) = t_i \in M_x \subset \mathscr{O}_{X,x}$ is a local equation for a branch Z_i of Z at x, $(Z_i)_{1\le i\le r}$ is the set of branches of Z at x, and G acts on t_i through the character $\chi_i = \chi(-,e_i) : G \to \mu$.

Furthermore:

(vii) The square in (1.4.2) is tor-independent.

Indeed, by the log regularity of S and the choice of the chart a, we have, by [Kato, 1994, 6.1], $\operatorname{Tor}_i^{\mathbf{Z}[\mathbf{Q}]}(\mathscr{O}_{S,s}, \mathbf{Z}[P]) = 0$ for $i > 0$.

Though this will not be needed, one can describe the local structure of (1.4.2) more precisely as follows. Let

$$(1.4.4) \qquad Y := \operatorname{Spec}(\Lambda[P] \otimes_\Lambda \operatorname{Sym}_\Lambda(V)) = \operatorname{Spec}(\Lambda[P^*] \otimes_\Lambda \Lambda[P_1] \otimes_\Lambda \operatorname{Sym}_\Lambda(V))$$

and let $Y' := \operatorname{Spec} C[[y_1, \dots, y_m]][[Q]] \times_{\operatorname{Spec}\Lambda} Y$, with the notation of 1.4.1. We may assume that $X = X'$. Then the completion of X at x is either isomorphic to the completion of Y' at x, or a regular divisor in it, defined by the equation $g' = 0$, where g' is the image of g in $\widehat{\mathscr{O}}_{Y',x}$, with the notation of 1.4.1.

1.5. — *Step 3 and log smoothness (beginning).* We will now analyze the modifications performed in the proof of Step 3 in VIII-4.1.9, VIII-4.2.13. The permissible towers used in *loc. cit.* are iterations of operations of the form: for a subgroup H of G, blow up the fixed point (regular) subscheme X^H, and replace Z by the union of its strict transform Z^{st} and the exceptional divisor E. Though such a blow up is not a log blow up in general, we will see that it still preserves the log smoothness of X over S.

We work étale locally around x, so we can assume $X = X'$ in 1.4.2. We then have a cartesian square

$$(1.5.1) \qquad \begin{array}{ccc} X^H & \xrightarrow{b^H} & Y^H \\ \downarrow f & & \downarrow \\ X & \xrightarrow{b} & Y, \end{array}$$

with Y as in (1.4.4). We also have cartesian squares

$$(1.5.2) \qquad \begin{array}{ccc} Z & \xrightarrow{b} & T \\ \downarrow f & & \downarrow \\ X & \xrightarrow{b} & Y, \end{array}$$

where $T \subset Y$ is the snc divisor $\sum T_i$, T_i defined by the equation $e_i \in P_1$ (1.4.3), and

$$(1.5.3) \qquad \begin{array}{ccc} Z \times_X X^H & \longrightarrow & T \times_Y Y^H \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

1.6. Lemma. — *The squares* (1.5.1), (1.5.2), *and* (1.5.3) *are tor-independent.*

Proof. — For (1.5.2), this is because Z (resp. T) is a divisor in X (resp. Y) (cf. [SGA 6 VII 1.2]). For (1.5.1), as the square (1.4.2) is tor-independent (by 1.4 (vii)), it is enough to show that the composite (cartesian) square

$$\begin{array}{ccc} X^H & \longrightarrow & Y^H \\ \downarrow & & \downarrow \\ S & \longrightarrow & \operatorname{Spec} \Lambda[Q] \end{array} \tag{1.6.1}$$

is tor-independent. We have a decomposition

$$Y^H = (\operatorname{Spec} \Lambda[P^*])^H \times (\operatorname{Spec} \Lambda[P_1])^H \times (\operatorname{Spec} \operatorname{Sym}_\Lambda(V))^H, \tag{1.6.2}$$

(products taken over Spec Λ), and the map to Spec $\Lambda[Q]$ is the composition of the projection onto $(\operatorname{Spec} \Lambda[P^*])^H \times (\operatorname{Spec} \Lambda[P_1])^H$ and the canonical map induced by $\operatorname{Spec} \Lambda[Q] \to \operatorname{Spec} \Lambda[P]$, which factors through the fixed points of H, G acting trivially on the base. Let us examine the three factors.

(a) We have

$$(\operatorname{Spec} \operatorname{Sym}_\Lambda(V))^H = \operatorname{Spec} \operatorname{Sym}_\Lambda(V_H),$$

where V_H is the module of coinvariants, a free module of finite type over Λ, as H is of order invertible in Λ. Therefore $\operatorname{Spec} \Lambda[Q] \times_{\operatorname{Spec} \Lambda} (\operatorname{Spec} \operatorname{Sym}_\Lambda(V))^H$ is flat over Spec $\Lambda[Q]$, and its enough to check that $(\operatorname{Spec} \Lambda[P^*])^H \times (\operatorname{Spec} \Lambda[P_1])^H$ is tor-independent of S over Spec $\Lambda[Q]$.

(b) The restriction to $P^* = v^{-1}(\mathscr{O}^*_{X,x})$ of the 1-cocycle $z(v) \in Z^1(H, \operatorname{Hom}(P, k(x)^*))$ associated with $v : P \to M_x$ ($hv(a) = z(v)(h,a)v(a)$ for $h \in H$, $a \in P$, see the proof of 1.2 and (VI-3.5), is a 1-coboundary, hence trivial, as $B^1(H, \operatorname{Hom}(P, k(x)^*)) = 0$. Therefore

$$(\operatorname{Spec} \Lambda[P^*])^H = \operatorname{Spec} \Lambda[P^*].$$

(c) Recall that

$$P_1 = \bigoplus_{1 \le i \le r} \mathbf{N} e_i,$$

with e_i sent by v to a local equation of the branch Z_i of Z, and that G acts on $\Lambda[\mathbf{N}e_i]$ through the character $\chi_i : G \to \mu$. Let $A \subset \{1, \dots, r\}$ be the set of indices i such that $\chi_i | H$ is trivial. Then

$$(\operatorname{Spec} \Lambda[P_1])^H = \operatorname{Spec} \Lambda[\bigoplus_{i \in A} \mathbf{N} e_i].$$

Let I be the ideal of P generated by $\{e_i\}_{i \notin A}$. It follows from (b) and (c) that

$$(\operatorname{Spec} \Lambda[P])^H = \operatorname{Spec} \Lambda[P]/(I),$$

where (I) is the ideal of $\Lambda[P]$ generated by I. By [Kato, 1994, 6.1], $\operatorname{Tor}_i^{\Lambda[Q]}(\mathscr{O}_S, \Lambda[P]/(I)) = 0$ for $i > 0$, and therefore (1.6.1), hence (1.5.1) is tor-independent. It remains to show the tor-independence of (1.5.3). For this, again it is enough to show the tor-independence of

(1.6.3)
$$\begin{array}{ccc} Z \times_X X^H & \longrightarrow & T \times_Y Y^H \\ \downarrow & & \downarrow \\ S & \longrightarrow & \operatorname{Spec} \Lambda[Q]. \end{array}$$

By (a), (b), (c), we have

$$T \times_Y Y^H = \sum_{i \in A} \operatorname{Spec} \Lambda[P]/(J_i) \times \operatorname{Spec} \operatorname{Sym}_\Lambda(V_H),$$

where $J_i \subset P$ is the ideal generated by $e_i \in P_1$, and (J_i) the ideal generated by J_i in $\Lambda[P]$. The desired tor-independence follows from the vanishing of $\operatorname{Tor}_i^{\Lambda[Q]}(\mathscr{O}_S, \Lambda[P]/(J_B))$, where for a subset B of A, J_B denotes the ideal generated by the e_i's for $i \in B$. □

1.7. Lemma. — *Consider a cartesian square*

(1.7.1)
$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g} & X, \end{array}$$

where the right vertical arrow is a regular immersion. If (1.7.1) *is tor-independent, then the left vertical arrow is a regular immersion, and*

$$\mathrm{Bl}_{V'}(X') = X' \times_X \mathrm{Bl}_V(X).$$

Let $W \to X$ be a second regular immersion, such that $V \times_X W \to W$ is a regular immersion, and let $W' = X' \times_X W$. If moreover the squares

(1.7.2)
$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g} & X, \end{array}$$

and

(1.7.3)
$$\begin{array}{ccc} V' \times_{X'} W' & \longrightarrow & V \times_X W \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g} & X, \end{array}$$

are tor-independent, then the left vertical arrows are regular immersions, and

$$W'^{\mathrm{st}} = X' \times_X W^{\mathrm{st}},$$

where W^{st} (resp. W'^{st}) is the strict transform of W (resp. W') in $\mathrm{Bl}_V(X)$ *(resp.* $\mathrm{Bl}_{V'}(X')$*).*

Proof. — Let I (resp. I') be the ideal of V (resp. V') in X (resp. X'). By the tor-independence of (1.7.1), if $u : E \to I$ is a local surjective regular homomorphism [SGA 6 VII 1.4], the Koszul complex $g^\star K(u)$ is a resolution of $\mathcal{O}_{V'}$, hence $V' \to X'$ is a regular immersion. Moreover, by [SGA 6 VII 1.2], for any $n \geq 0$, the natural map $g^\star I^n \to I'^n$ is an isomorphism, and therefore $\mathrm{Bl}_{V'}(X') = X' \times_X \mathrm{Bl}_V(X)$. The tor-independence of (1.7.2) and (1.7.3) imply that of

$$\begin{array}{ccc} V' \times_{X'} W' & \longrightarrow & V \times_X W \\ \downarrow & & \downarrow \\ W' & \longrightarrow & W. \end{array}$$

The second assertion then follows from the first one and the formulas (VIII-2.1.3 (ii))

$$W^{\mathrm{st}} = \mathrm{Bl}_{V \times_X W} W,$$
$$W'^{\mathrm{st}} = \mathrm{Bl}_{V' \times_{X'} W'} W'.$$

□

1.8. — *Step 3 and log smoothness (end).* As recalled at the beginning of 1.5, we have to show that, if H is a subgroup of G, then the log regular pair (X_1, Z_1) is log smooth over S, where $X_1 := \mathrm{Bl}_{X^H}(X)$ and Z_1 is the snc divisor $Z^{\mathrm{st}} \cup E$, Z^{st} (resp. E) denoting the strict transform of Z (resp. the exceptional divisor) in the blow-up $h : X_1 \to X$.

The question is again étale local above X around x, so we may assume that $X = X'$ and we look at the cartesian square (1.4.2)

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Spec}\ \Lambda[Q], \end{array}$$

with Y as in (1.4.4), and the associated cartesian squares (1.5.1), (1.5.2), and (1.5.3).

Claim. We have

$$\mathrm{Bl}_{X^H}(X) = X \times_Y \mathrm{Bl}_{Y^H}(Y), \tag{1.8.1}$$

$$Z^{\mathrm{st}} = X \times_Y T^{\mathrm{st}}. \tag{1.8.2}$$

Proof. — In view of 1.6 and 1.7, (1.8.1) follows from the fact that the immersion $Y^H \to Y$ is regular. For (1.8.2) recall that

$$T = T_0 \times_{\mathrm{Spec}\ \Lambda} \mathrm{Spec}\ \mathrm{Sym}_\Lambda(V),$$

where $T_0 \subset \mathrm{Spec}\ \Lambda[P]$ is the snc divisor

$$T_0 = \sum_{1 \leq i \leq r} \mathrm{div}(z_i)$$

with $z_i \in \Lambda[P]$ the image of $e_i \in P_1$ as in 1.4.3. Hence

$$T = \sum_{1 \leq i \leq r} T_i, \tag{1.8.3}$$

where $T_i = \mathrm{div}(z_i) \times_{\mathrm{Spec}\,\Lambda} \mathrm{Sym}_\Lambda(V)$, and $T^{\mathrm{st}} = \sum_{1 \leq i \leq r} T_i^{\mathrm{st}}$. We have (1.6.2)

$$Y^H = (\mathrm{Spec}\,\Lambda[P])^H \times \mathrm{Spec}\,\mathrm{Sym}_\Lambda(V_H),$$

with $(\mathrm{Spec}\,\Lambda[P])^H$ defined by the equations $(z_i = 0)_{i \notin A}$, with the notations of 1.6 (c). In particular, the immersion $Y^H \times_Y Z_i \to Z_i$ is regular, hence, by 1.7, we have $Z_i^{\mathrm{st}} = X \times_Y T_i^{\mathrm{st}}$, hence (1.8.2), which finishes the proof of the claim. □

Since the map $S \to \mathrm{Spec}\,\Lambda[Q]$ is strict, in order to prove the desired log smoothness, we may, by this claim, replace the triple (X, X^H, Z) over S by (Y, Y^H, T) over $\mathrm{Spec}\,\Lambda[Q]$. We choose coordinates on P^*, $P_1 = \mathbf{N}^r$, V:

$$P^* = \bigoplus_{1 \leq i \leq t} \mathbf{Z} f_i,\ P_1 = \bigoplus_{1 \leq i \leq r} \mathbf{N} e_i,\ V = \bigoplus_{1 \leq i \leq s} \Lambda y_i$$

$$\Lambda[P] = \Lambda[u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1, \ldots, z_r],\ \mathrm{Sym}_\Lambda(V) = \Lambda[y_1, \ldots, y_s],$$

with u_i (resp. z_i) the image of f_i (resp. e_i) in $\Lambda[P]$, in such a way that

$$\Lambda[P]^H = \Lambda[u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_{m+1}, \ldots, z_r],$$

i.e. is defined in $\Lambda[P]$ by the equations $(z_1 = \cdots = z_m = 0)$, for some m, $1 \leq m \leq r$, and

$$\Lambda[V_H] = \Lambda[y_{n+1}, \ldots, y_s],$$

i.e. is defined in $\Lambda[V]$ by the equations $y_1 = \cdots = y_n = 0$ for some n, $1 \leq n \leq s$. Then

$$Y^H \subset Y = \mathrm{Spec}\,\Lambda[u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1, \ldots, z_r, y_1, \ldots, y_s]$$

is defined by the equations

$$z_1 = \cdots = z_m = y_1 = \cdots = y_n = 0.$$

Then

$$Y' := \mathrm{Bl}_{Y^H}(Y)$$

is covered by affine open pieces:

$$U_i = \mathrm{Spec}\,\Lambda[(u_j^{\pm 1})_{1 \leq j \leq t}, z'_1, \ldots, z'_{i-1}, z_i, z'_{i+1}, \ldots, z'_m, z_{m+1}, \ldots, z_r, y'_1, \ldots, y'_n, y_{n+1}, \ldots, y_s]$$

$(1 \leq i \leq m)$, with $U_i \to Y$ given by $z_j \to z_i z'_j$ for $1 \leq j \leq m$, $j \neq i$, $y_j \to z_i y'_j$, $1 \leq j \leq n$, and the other coordinates unchanged, and

$$V_i = \mathrm{Spec}\,\Lambda[(u_j^{\pm 1})_{1 \leq j \leq t}, z'_1, \ldots, z'_m, z_{m+1}, \ldots, z_r, y'_1, \ldots, y'_{i-1}, y_i, y'_{i+1}, \ldots, y'_n, y_{n+1}, \ldots, y_s]$$

$(1 \leq i \leq n)$, with $V_i \to Y$ given by $z_j \mapsto y_i z'_j$ for $1 \leq j \leq m$, $y_j \mapsto y_i y'_j$, $1 \leq j \leq n$, $j \neq i$, and the other coordinates unchanged. Recall that Y has the log structure defined by the log regular pair (Y, T), where T is the snc divisor

$$T = (z_1 \cdots z_r = 0),$$

and Y' is given the log structure defined by the log regular pair (Y', T'), where T' is the snc divisor

$$T' = F \cup T^{\mathrm{st}},$$

where F is the exceptional divisor of the blow up of Y^H and T^{st} the strict transform of T. Consider the canonical morphisms

$$Y' \xrightarrow{\ b\ } Y \xrightarrow{\ g\ } \Sigma := \mathrm{Spec}\ \Lambda[Q]\ .$$

They are both morphisms of log schemes. The morphism g is given by the homomorphism of monoids $\gamma : Q \to P$, i.e.

$$q \in Q \mapsto (\gamma_1(q), \ldots, \gamma_t(q), \gamma_{t+1}(q), \ldots, \gamma_{t+r}(q), 0, \ldots, 0)$$
$$\in \Lambda[u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1, \ldots, z_r, y_1, \ldots, y_s].$$

The blow up b has been described above in the various charts. Note that b is not log étale, or even log smooth, in general. However, the composition $gb : Y' \to \Sigma$ is log smooth. We will check this on the charts (U_i), (V_i).

(a) *Chart of type U_i*. We have $F = (z_i = 0)$, $T^{\mathrm{st}} = (\prod_{1 \leq j \leq r, j \neq i} z'_i = 0)$. Hence the log strucure of U_i is given by the canonical log structure of $\Lambda[\mathbf{N}^r]$ in the decomposition

$$U_i = \mathrm{Spec}\ \Lambda[\mathbf{Z}^t] \times \mathrm{Spec}\ \Lambda[\mathbf{N}^r] \times \mathrm{Spec}\ \Lambda[y'_1, \ldots, y'_n, y_{n+1}, \ldots, y_s]$$

with the basis element e_k of $\mathbf{N}^r$ sent to the k-th place in $(z'_1, \ldots, z'_{i-1}, z_i, z'_{i+1}, \ldots, z'_m, z_{m+1}, \ldots, z_r)$ (and the basis element f_k of $\mathbf{Z}^t$ sent to u_k), the third factor having the trivial log structure. Checking the log smoothness of $gb : U_i \to \Sigma$ amounts to checking the log smoothness of its factor Spec $\Lambda[P] \to \Sigma =$ Spec $\Lambda(Q)$, which is defined by the composition of homomorphisms of monoids

$$Q \xrightarrow{\ \gamma\ } \mathbf{Z}^t \oplus \mathbf{N}^r \xrightarrow{Id \oplus \beta} \mathbf{Z}^t \oplus \mathbf{N}^r\ ,$$

where β is the homomorphism $\mathbf{N}^r \to \mathbf{N}^r$ sending e_j to $e_j + e_i$ for $1 \leq j \leq m, j \neq i$, e_i to e_i, and e_j to e_j for $m+1 \leq j \leq r$. Recall ((1.4.2), (iv)) that γ^{gp} is injective and the torsion part of its cokernel is invertible in Λ. As β^{gp} is an isomorphism, the same holds for the composition $(Id \oplus \beta)\gamma$, hence $gb : U_i \to \Sigma$ is log smooth.

(b) *Chart of type V_i*. We have $F = (y_i = 0)$, $T^{\mathrm{st}} = \prod_{1 \leq j \leq m} z'_j \prod_{j \geq m+1} z_i$. Hence the log structure of V_i is given by the canonical log structure of $\Lambda[\mathbf{N}^{r+1}]$ in the decomposition

$$V_i = \mathrm{Spec}\ \Lambda[\mathbf{Z}^t] \times \mathrm{Spec}\ \Lambda[\mathbf{N}^{r+1}] \times \mathrm{Spec}\ \Lambda[(y'_j)_{1 \leq j \leq n, j \neq i}, y_{n+1}, \ldots, y_s]$$

with the basis element e_k of $\mathbf{N}^{r+1}$ sent to the k-th place in $(z'_1, \ldots, z'_m, z_{m+1}, \ldots, z_r)$ if $k \leq r$, and e_{r+1} sent to y_i (and the basis element f_k of $\mathbf{Z}^t$ sent to u_i), the third factor having the trivial log structure. Again, checking the log smoothness of $gb : V_i \to \Sigma$ amounts to checking the log smoothness of its factor Spec $\Lambda[\mathbf{Z}^t] \times$ Spec $\Lambda[\mathbf{N}^{r+1}] \to$ Spec $\Lambda(Q)$. This factor is defined by the composition of homomorphisms of monoids

$$Q \xrightarrow{\gamma} \mathbf{Z}^t \oplus \mathbf{N}^r \xrightarrow{Id \oplus \beta} \mathbf{Z}^t \oplus \mathbf{N}^{r+1}$$

where $\beta : \mathbf{N}^r \to \mathbf{N}^{r+1}$ sends e_j to $e_j + e_{r+1}$ for $1 \leq j \leq m$, and to e_j for $m+1 \leq j \leq r$. Then β^{gp} is injective, and its cokernel is isomorphic to $\mathbf{Z}$, hence $(\beta\gamma)^{\mathrm{gp}}$ is injective, and we have an exact sequence

$$0 \to \mathrm{Coker}\, \gamma^{\mathrm{gp}} \to \mathrm{Coker}\, (\beta\gamma)^{\mathrm{gp}} \to \mathbf{Z} \to 0.$$

In particular, the torsion part of $\mathrm{Coker}(\beta\gamma)^{\mathrm{gp}}$ is isomorphic to that of $\mathrm{Coker}\, \gamma^{\mathrm{gp}}$, hence of order invertible in Λ, which implies that $gb : V_i \to \Sigma$ is log smooth.

This finishes the proof that Step 3 preserves log smoothness.

1.9. — *End of proof of 1.1.* We may now assume that in addition to conditions (1) and (2) of 1.4, condition (3) is satisfied as well, namely

(3) *G acts freely on $X - Z$ (i.e. $Z = Z \cup T$ in the notation of 1.1 or (VIII-1.1)), and, for any geometric point $x \to X$, the inertia group G_x is abelian.*

We have to check:

Claim. If $f_{(G,X,Z)} : (X', Z') \to (X, Z)$ is the modification of (VIII-5.4.4), then (X', Z') and $(X'/G, Z'/G)$ are log smooth over S.

Working étale locally around a geometric point x of X, we will first choose a strict rigidification $(X, \overline{Z})$ of (X, Z) such that $(X, \overline{Z})$ is log smooth over S. We will define $(X, \overline{Z})$ as the pull-back by $S \to \Sigma =$ Spec $\Lambda[Q]$ of a rigidification $(Y, \overline{T})$ of (Y, T) which is log smooth over Σ, with the notation of (1.4.4). Using that G $(= G_x)$ is abelian, one can decompose V into a sum of G-stable lines, according to the characters of G:

$$V = \bigoplus_{1 \leq i \leq s} \Lambda y_i$$

with G acting on Λy_i through a character $\chi_i : G \to \mu_N$, i.e. $g y_i = \chi_i(g) y_i$. We define $\overline{T}$ to be the divisor $z_1 \cdots z_r y_1 \cdots y_s = 0$ in $Y =$ Spec $\Lambda[u_1^{\pm 1}, \ldots, u_t^{\pm 1}, z_1, \ldots, z_r, y_1, \ldots, y_s]$. The action of G on $(Y, \overline{T})$ is very tame at x because the log stratum at x is Spec $\Lambda[u_1^{\pm 1}, \ldots, u_t^{\pm 1}]$, hence very tame in a neighborhood of x by (VIII-5.3.2) (actually on the whole of Y, cf. (VIII-4.6, VIII-4.7(a)). On the other hand, (Spec $\Lambda[y_1, \ldots, y_s], y_1 \cdots y_s = 0$) is log smooth over Spec Λ, and as Spec $\Lambda[P]$ is log smooth over Σ, $(Y, \overline{T})$ is log smooth over Σ. Since $f_{(G,X,Z)}$ is compatible with base change by strict inert morphisms, it is enough to check that if

$f_{(G,Y,T)} = f_{(G,Y,T,\overline{T})} : (Y',T') \to (Y,T)$ is the modification of (VIII-5.4.4) then (Y',T') is log smooth over Σ. Recall (VIII-5.3.9) that we have a cartesian G-equivariant diagram

$$\begin{array}{ccc} (Y',\overline{T'}) & \xrightarrow{\overline{h'}} & (Y,\overline{T}) \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ (Y'_1,\overline{T'_1}) & \xrightarrow{\overline{h_1}} & (Y_1,\overline{T_1}), \end{array} \tag{1.9.1}$$

where the horizontal maps are the compositions of saturated log blow up towers, and the vertical ones Kummer étale G-covers. From (1.9.1) is extracted the relevant diagram involving $h := f_{(G,Y,T,\overline{T})}$,

$$\begin{array}{ccc} (Y',T') & \xrightarrow{h} & (Y,T) \\ \downarrow{\scriptstyle \beta} & & \\ (Y'_1,T'_1), & & \end{array}$$

where $T'_1 = \overline{h_1}^{-1}(T_1)$, with $T_1 = T/G$, $T' = \alpha'^{-1}(T'_1)$, and h (resp. β) is the restriction of $\overline{h'}$ (resp. α') over (Y,T) (resp. (Y'_1,T'_1)). In particular, β is a Kummer étale G-cover (as Kummer étale G-covers are stable under any fs base change). As G acts trivially on S, this diagram can be uniquely completed into a commutative diagram

$$\begin{array}{ccc} (Y',T') & \xrightarrow{h} & (Y,T) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle f} \\ (Y'_1,T'_1) & \xrightarrow{g} & \Sigma. \end{array} \tag{1.9.2}$$

Here f is log smooth and β is a Kummer étale G-cover. Though $\overline{h'}$ and $\overline{h_1}$ are log smooth, h and h_1 are not, in general. However, it turns out that:

(∗) *$g : (Y'_1,T'_1) \to \Sigma$, hence $g\beta = fh : (Y',T') \to \Sigma$, are log smooth,*

which will finish the proof of the claim, hence of 1.1. We first prove

(∗∗) *With the notation of (1.9.2), $(Y_1,\overline{T_1})$ is log smooth over Σ.*

Let us write $Y = \operatorname{Spec} \Lambda[\overline{P}]$, with

$$\overline{P} = P \times \mathbf{N}^s = \mathbf{Z}^t \times \mathbf{N}^r \times \mathbf{N}^s. \tag{1.9.3}$$

As G acts very tamely on $(Y,\overline{T})$, the quotient pair $(Y_1 = Y/G, \overline{T_1} = \overline{T}/G)$ is log regular. More precisely, by the calculation in (VI-3.4(b)), this pair consists of the log scheme $Y_1 = \operatorname{Spec} \Lambda[\overline{R}]$ with its canonical log structure, where

$$\overline{R} = \operatorname{Ker}(\overline{P}^{\mathrm{gp}} \to \operatorname{Hom}(G,\mu_N)) \cap \overline{P},$$

$\overline{P}^{\mathrm{gp}} \to \mathrm{Hom}(G, \mu_N)$ being the homomorphism defined by the pairing $\chi : G \otimes \overline{P}^{\mathrm{gp}} \to \mu_N$. The inclusion $\overline{R} \subset \overline{P}$ is a Kummer morphism, and $\overline{P}^{\mathrm{gp}}/\overline{R}^{\mathrm{gp}}$ is annihilated by an integer invertible in Λ. As $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, with the torsion part of its cokernel annihilated by an integer invertible in Λ, the same is true for $Q^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}}$, hence also for $Q^{\mathrm{gp}} \to \overline{R}^{\mathrm{gp}}$. Thus $(Y_1, \overline{T_1}) = \mathrm{Spec}\ \Lambda[\overline{R}]$ is log smooth over Σ.

Finally, let us prove $(*)$. It is enough to work locally on Y_1' so we can replace the log blow up sequence $(Y_1', \overline{T}_1') \to (Y_1, \overline{T}_1)$ with an affine chart (i.e. we replace the first log blow up with a chart, then do the same for the second one, etc.). Then $Y_1' = \mathrm{Spec}\,\Lambda[\overline{R}']$, and $\overline{R}^{\mathrm{gp}} \xrightarrow{\sim} \overline{R}'^{\mathrm{gp}}$ by VIII-3.1.19. Note that $\overline{R}' \xrightarrow{\sim} \mathbf{Z}^a \times \mathbf{N}^b$ where $D_1, \ldots, D_b$ are the components of $\overline{T}_1'$. We can assume that $D_1, \ldots, D_c \subset T_1'$ and $D_{c+1}, \ldots, D_b$ are not contained in T_1'. Let $R' \xrightarrow{\sim} \mathbf{Z}^a \times \mathbf{N}^c$ denote the submonoid $\overline{R}'$ that defines the log structure of (Y_1', T_1'). Note that R' consists of all elements $g' \in \overline{R}'$ such that $(g' = 0) \subset T_1'$ (as a set). Also, by $\nu : \overline{R} \to \overline{R}'$ we will denote the homomorphism that defines $(Y_1', \overline{T}_1') \to (Y_1, \overline{T}_1)$.

We showed in VIII-5.3.9 that $T_1 = T/G$ is a $\mathbf{Q}$-Cartier divisor in Y_1 and observed that therefore T_1' is a Cartier divisor in Y_1'. Note that the inclusion $R \subset \overline{R}$, where

$$R = \mathrm{Ker}(P^{\mathrm{gp}} \to \mathrm{Hom}(G, \mu_N)) \cap \overline{P}$$

defines a log structure on Y_1. Denote the corresponding log scheme (Y_1, T_1). We obtain the following diagram of log schemes (on the left). The corresponding diagram of groups is placed on the right; we will use it to establish log smoothness of g. Existence of dashed arrows requires an argument; we will construct them later.

(1.9.4)
$$\begin{array}{ccccc} (Y_1', \overline{T}_1') & \xrightarrow{\overline{h}_1} & (Y_1, \overline{T}_1) & & \\ \downarrow & & \downarrow & \searrow & \\ (Y_1', T_1') & \dashrightarrow & (Y_1, T_1) & \longrightarrow & \Sigma \end{array} \qquad \begin{array}{ccccc} \overline{R}'^{\mathrm{gp}} & \xleftarrow[\nu^{\mathrm{gp}}]{\sim} & \overline{R}^{\mathrm{gp}} & & \\ \uparrow & & \uparrow & \nwarrow & \\ R'^{\mathrm{gp}} & \dashleftarrow & R^{\mathrm{gp}} & \longleftarrow & Q^{\mathrm{gp}}. \end{array}$$

Part (ii) of the following remark clarifies the notation (Y_1, T_1). It will not be used so we only sketch the argument.

1.10. Remark. — (i) Note that (Y_1, T_1) may be not log smooth over Σ. For example, even when Σ is log regular, e.g. $\mathrm{Spec}\,k$ with trivial log structure, (Y_1, T_1) does not have to be log regular, as T_1 may even be non-Cartier. Nevertheless, as $\overline{h_1}$ is log smooth (even log étale), $(Y_1', \overline{T_1'})$ is log smooth over Σ. Moreover, Y_1' is regular, and $\overline{T_1'}$ an snc divisor in it.

(ii) Although T_1 may be bad, one does have that $R\mathscr{O}_{Y_1}^* = \mathscr{O}_{Y_1} \cap i_*\mathscr{O}_{Y_1 \setminus T_1}^*$ for the embedding $i : Y_1 \setminus T_1 \hookrightarrow Y_1$. This can be deduced from the formulas for R and $\overline{R}$ and the fact that $\overline{R}\mathscr{O}_{Y_1}^* = \mathscr{O}_{Y_1} \cap j_*\mathscr{O}_{Y_1 \setminus \overline{T}_1}^*$ by log regularity of $(Y_1, \overline{T}_1)$.

Note that $Q \to \overline{P}$ factors through P, hence $Q \to \overline{R}$ factors through $R = \overline{R} \cap P$. It follows from (1.9.3) that P consists of all elements $f \in \overline{P}$ whose divisor $(f = 0)$ is contained in T (as a set). Therefore $g \in \overline{R}$ lies in R if and only if $(g = 0) \subset T_1$ (as a set). This fact and the analogous description of R' observed earlier imply that $\nu : \overline{R} \to \overline{R}'$ takes R to R'. Thus, we have established the dashed arrows in (1.9.4).

Let $\varphi : Q \to \overline{R}'$ be the homomorphism defining the composition $(Y_1', \overline{T_1'}) \to (Y_1, \overline{T_1}) \to \Sigma$. Since the latter is log smooth, φ is injective, and the torsion part of $\mathrm{Coker}(\varphi^{\mathrm{gp}})$ is annihilated by an integer m invertible in Λ. Note that $R^{\mathrm{gp}} \hookrightarrow R'^{\mathrm{gp}} \hookrightarrow \overline{R}'^{\mathrm{gp}}$, and therefore we also have that $Q^{\mathrm{gp}} \hookrightarrow R'^{\mathrm{gp}}$ and the torsion of its cokernel is annihilated by m. Therefore, (Y_1', T_1') is log smooth over Σ, which finishes the proof of $(*)$, hence of 1.1.

1.11. Remark. — In the proof of $(*)$ above, we first proved that g is log smooth, and deduced that $g\beta$ is, too. In fact, as β is a Kummer étale G-cover, the log smoothness of $g\beta$ implies that of g. More generally, we have the following descent result, due to Kato-Nakayama ([Nakayama, 2009, 3.4]):

1.12. Theorem. — *Let $X' \xrightarrow{g} X \xrightarrow{f} Y$ be morphisms of fs log schemes. If g is surjective, log étale and exact, and fg is log smooth, then f is log smooth.*

The assumption on g is equivalent to saying that g is a Kummer étale cover (cf. [Illusie, 2002, 1.6]).

2. Prime to ℓ variants of de Jong's alteration theorems

Let X be a noetherian scheme, and ℓ be a prime number. Recall that a morphism $h : X' \to X$ is called an **ℓ'-alteration** if h is proper, surjective, generically finite, maximally dominating (i.e., (II-1.1.2) sends each maximal point to a maximal point) and the degrees of the residual extensions $k(x')/k(x)$ over each maximal point x of X are prime to ℓ. The next theorem was stated in Intro.-3 (1):

2.1. Theorem. — *Let k be a field, ℓ a prime number different from the characteristic of k, X a separated and finite type k-scheme, $Z \subset X$ a nowhere dense closed subset. Then there exists a finite extension k' of k, of degree prime to ℓ, and a projective ℓ'-alteration $h : \tilde{X} \to X$ above* Spec $k' \to$ Spec k*, with $\tilde{X}$ smooth and quasi-projective over k', and $h^{-1}(Z)$ is the support of a relative strict normal crossings divisor.*

Recall that a relative strict normal crossings divisor in a smooth scheme T/S is a divisor $D = \sum_{i \in I} D_i$, where I is finite, $D_i \subset T$ is an S-smooth closed subscheme of codimension 1, and for every subset J of I the scheme-theoretic intersection $\bigcap_{i \in J} D_i$ is smooth over S of codimension $|J|$ in T.

We will need the following variant, due to Gabber-Vidal (proof of [Vidal, 2004, 4.4.1]), of de Jong's alteration theorems [de Jong, 1997, 5.7, 5.9, 5.11], cf. [Zheng, 2009, 3.8]:

2.2. Lemma. — *Let X be a proper scheme over $S = \operatorname{Spec} k$, normal and geometrically reduced and irreducible, $Z \subset X$ a nowhere dense closed subset. We assume that a finite group H acts on $X \to S$, faithfully on X, and that Z is H-stable. Then there exists a finite extension k_1 of k, a finite group H_1, a surjective homomorphism $H_1 \to H$, and an H_1-equivariant diagram with a cartesian square (where $S = \operatorname{Spec} k$, $S_1 = \operatorname{Spec} k_1$)*

(2.2.1)
$$\begin{array}{ccccc} X & \xleftarrow{\ b\ } & X_1 & \xleftarrow{\ a\ } & X_2 \\ \downarrow & & \downarrow & \swarrow & \\ S & \longleftarrow & S_1 & & \end{array}$$

satisfying the following properties:

(i) *$S_1/\operatorname{Ker}(H_1 \to H) \to S$ is a radicial extension;*

(ii) *X_2 is projective and smooth over S_1;*

(iii) *$a : X_2 \to X_1$ is projective and surjective, maximally dominating and generically finite and flat, and there exists an H_1-admissible dense open subset $W \subset X_2$ over a dense open subset U of X, such that if $U_1 = S_1 \times_S U$ and $K = \operatorname{Ker}(H_1 \to \operatorname{Aut}(U_1))$, $W \to W/K$ is a Galois étale cover of group K and the morphism $W/K \to U_1$ induced by a is a universal homeomorphism;*

(iv) *$(ba)^{-1}(Z)$ is the support of a strict normal crossings divisor in X_2.*

Proof. — We may assume X of dimension $d \geq 1$. We apply [Vidal, 2004, 4.4.3] to X/S, Z, and $G = H$. We get the data of *loc. cit.*, namely an equivariant finite extension of fields $(S_1, H_1) \to (S, H)$ such that $S_1/\operatorname{Ker}(H_1 \to H) \to S$ is radicial, an H_1-equivariant pluri-nodal fibration $(Y_d \to \cdots \to Y_1 \to S_1, \{\sigma_{ij}\}, Z_0 = \varnothing)$, and an H_1-equivariant alteration $a_1 : Y_d \to X$ over S, satisfying the conditions (i), (ii), (iii) of *loc. cit.* (in particular $a_1^{-1}(Z) \subset Z_d$). Then, as in the proof of [Vidal, 2004, 4.4.1], successively applying [Vidal, 2004, 4.4.4] to each nodal curve $f_i : Y_i \to Y_{i-1}$, one can replace Y_i by an H_1-equivariant projective modification Y_i' of it such that Y_i' is regular, and the inverse image Z_i' of $Z_i := \bigcup_j \sigma_{ij}(Y_{i-1}) \cup f_i^{-1}(Z_{i-1})$ in Y_i' is an H_1-equivariant strict snc divisor. Then, $X_2 := Y_d'$ is smooth over S_1 and Z_d' is a relative snc divisor over S_1. This follows from the analog of the remark following [Vidal, 2004, 4.4.4] with "semistable pair over a trait" replaced by "pair consisting of a smooth scheme and a relative snc divisor over a field". In particular, $(ba)^{-1}(Z)_{\mathrm{red}}$ is a subdivisor of Z_d', hence an snc divisor. After replacing H_1 by $H_1/\operatorname{Ker}(H_1 \to \operatorname{Aut}(X_2))$ the open subsets U and V of (iii) are obtained as at the end of the proof of [Vidal, 2004, 4.4.1]. □

2.3. — *Proof of 2.1.* There are three steps.

Step 1. Preliminary reductions. By Nagata's compactification theorem [Conrad, 2007], there exists a dense open immersion $X \subset \overline{X}$ with $\overline{X}$ proper over S. Up to replacing X by $\overline{X}$ and Z by its closure $\overline{Z}$, we may assume X proper over S. By replacing X by the disjoint sum of its irreducible components, we may further assume X irreducible, and geometrically reduced (up to base changing by a finite radicial extension of k). Up to blowing up Z in X me may further assume that Z is a (Cartier) divisor in X. Finally, replacing X by its normalization X', which is finite over X, and Z by its inverse image in X', we may assume X normal.

Step 2. Use of 2.2. Choose a finite Galois extension k_0 of k such that the irreducible components of $X_0 = X \times_S S_0$ ($S_0 = \operatorname{Spec} k_0$) are geometrically irreducible. Let $G = \operatorname{Gal}(k_0/k)$ and $H \subset G$ the decomposition subgroup of a component Y_0 of X_0. We apply 2.2 to $(Y_0/S_0,\ Z_0 \cap Y_0)$, where $Z_0 = S_0 \times_S Z$. We find a surjection $H_1 \to H$ and an H_1-equivariant diagram of type 2.2.1:

$$\begin{array}{ccccc} Y_0 & \xleftarrow{b'} & Y_1 & \xleftarrow{a'} & Y_2 \\ \downarrow & & \downarrow & \swarrow & \\ S_0 & \longleftarrow & S_1, & & \end{array} \tag{2.3.1}$$

satisfying conditions (i), (ii), (iii), (iv) with S replaced by S_0, and $X_2 \to X_1 \to X$ by $Y_2 \to Y_1 \to Y_0$. As G transitively permutes the components of X_0, X_0 is, as a G-scheme over S_0, the contracted product

$$X_0 = Y_0 \times^H G,$$

i.e. the quotient of $Y_0 \times G$ by H acting on Y_0 on the right and on G on the left (cf. proof of VIII-5.3.8), and similarly $Z = Z_0 \times^H G$. Choose an extension of the diagram $H_1 \xrightarrow{u} H \xrightarrow{i} G$ into a commutative diagram of finite groups

$$\begin{array}{ccc} H_1 & \xrightarrow{i_1} & G_1 \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ H & \xrightarrow{i} & G \end{array}$$

with i_1 injective and v surjective (for example, take i_1 to be the graph of iu and v the projection). Define

$$X_1 := Y_1 \times^{H_1} G_1, \;\; X_2 := Y_2 \times^{H_1} G_1.$$

Then (2.3.1) extends to a G_1-equivariant diagram of type 2.2.1

(2.3.2)
$$\begin{array}{ccccc} X_0 & \xleftarrow{b} & X_1 & \xleftarrow{a} & X_2 \\ \downarrow & & \downarrow & \swarrow & \\ S_0 & \longleftarrow & S_1, & & \end{array}$$

satisfying again (i), (ii), (iii), (iv). In particular, the composition $h : X_2 \to X_1 \to X_0 \to X$ is an alteration above the composition $S_1 \to S_0 \to S$, X_2 is projective and smooth over S_1, and $h^{-1}(Z)$ is the support of an snc divisor. However, as regard to 2.1, the diagram

$$\begin{array}{ccc} X & \xleftarrow{h} & X_2 \\ \downarrow & & \downarrow \\ S & \longleftarrow & S_1 \end{array}$$

deduced from (2.3.2) has two defects:

(a) the extension $S_1 \to S$ is not necessarily of degree prime to ℓ,

(b) the alteration h is not necessarily an ℓ'-alteration.

We will first repair (a) and (b) at the cost of temporarily losing the smoothness of X_2/S_1 and the snc property of $h^{-1}(Z)$. By (i), $S_1/\mathrm{Ker}(G_1 \to G) \to S_0$ is a radicial extension, hence $S_1/G_1 \to S = S_0/G$ is a radicial extension, too. Similarly, by (iii), $X_2/G_1 \to X$ is an alteration over $S_1/G_1 \to S$, which is a universal homeomorphism over a dense open subset. Now let L be an ℓ-Sylow subgroup of G_1. Then $S_1/L \to S_1/G_1$ is of degree prime to ℓ, and $X_2/L \to X_2/G_1$ is a finite surjective morphism of generic degree prime to ℓ. Let $S' := \mathrm{Spec}\, k' = S_1/L$, $X' = S' \times_S X$. We get a commutative diagram with cartesian square

(2.3.3)
$$\begin{array}{ccccccc} X & \longleftarrow & X' & \longleftarrow & X_2/L & \longleftarrow & X_2 \\ \downarrow & & \downarrow & \swarrow & & \swarrow & \\ S & \longleftarrow & S' & \longleftarrow & S_1, & & \end{array}$$

where $S' \to S$ is an extension of degree prime to ℓ, $S_1 \to S'$ a Galois extension of group L,

$$h_2 : X_2/L \to X$$

an ℓ'-alteration, X_2/S_1 is projective and smooth, and if now h denotes the composition $X_2 \to X$, $Z_1 := h^{-1}(Z)$ is an snc divisor in X_2. If X_2/L was smooth over S' and Z_1/L an snc divisor in X_2/L, we would be finished. However, this is not the case in general. We will use Gabber's theorem 1.1 to fix this.

Step 3. Use of 1.1. Let Y be a connected component of X_2, $(Z_1)_Y = h^{-1}(Z) \cap Y$, D the stabilizer of Y in L, $I \subset D$ the inertia group at the generic point of Y. Then D

acts on Y through $K := D/I$, and this action is generically free. As Y is smooth over S_1 and $(Z_1)_Y$ is an snc divisor in Y, $(Y, (Z_1)_Y)$ makes a log regular pair, log smooth over S_1, hence over $S' = S_1/L$ (equipped with the trivial log structure). We have a K-equivariant commutative diagram

(2.3.4)
$$\begin{array}{ccc} (Y/K, (Z_1)_Y/K) & \longleftarrow & (Y, (Z_1)_Y) \\ \downarrow & \overset{f}{\swarrow} & \\ S', & & \end{array}$$

where K acts trivially on S', and f is projective, smooth, and log smooth (S' having the trivial log structure). We now apply 1.1 to $(f : (Y, (Z_1)_Y) \to S' = (S', \varnothing), K)$, which satisfies conditions (i)–(iii) of *loc. cit.* We get a D-equivariant projective modification $g : Y_1 \to Y$ (D acting through K) and a D-strict snc divisor E_1 on Y_1, containing $Z_1 := g^{-1}((Z_1)_Y)$ as a subdivisor, such that the action of D on (Y_1, E_1) is very tame, and (Y_1, E_1) and $(Y_1/D, E_1/D)$ are log smooth over S'. Pulling back g to the orbit $Y \times^D L$ of Y under L, i.e. replacing g by $g \times^D L$, and working separately over each orbit, we eventually get an L-equivariant commutative square

(2.3.5)
$$\begin{array}{ccc} (Y_2/L, E_2/L) & \longleftarrow & (Y_2, E_2) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ (X_2/L, Z_1/L) & \longleftarrow & (X_2, Z_1), \end{array}$$

where u, v are projective modifications (and $Z_1 = h^{-1}(Z)$, $Z_1/L = h_2^{-1}(Z)$ as above), with the property that the pair $(Y_2/L, E_2/L)$ is an fs log scheme log smooth over S' $(= S_1/L)$, and $v^{-1}(h_2^{-1}(Z)) \subset E_2/L)$. Let $w : (\tilde{X}, \tilde{E}) \to (Y_2/L, E_2/L)$ be a projective, log étale modification such that $\tilde{X}$ is regular, and $\tilde{E} = w^{-1}(E_2/L)$ is an snc divisor in $\tilde{X}$. For example, one can take for w the saturated monoidal desingularization $\tilde{\mathscr{F}}^{\log}$ of (VIII-3.4.9). We then apply 1.2 to the log smooth morphism $\tilde{X} \to S'$. By a special case of the (1.4.2), with $Q = \{1\}$, $G = \{1\}$, as P^* is torsion free, $\tilde{X}$ is not only regular, but smooth over S', and $\tilde{E}$ a relative snc on $\tilde{X}$. Let

$$\tilde{h} : \tilde{X} \to X$$

be the composition

$$\tilde{X} \xrightarrow{w} Y_2/L \xrightarrow{v} X_2/L \xrightarrow{h_2} X\ .$$

This is a projective ℓ'-alteration, and it fits in the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\tilde{h}} & \tilde{X} \\ \downarrow & & \downarrow \\ S & \longleftarrow & S', \end{array}$$

where S' is an extension of S of degree prime to ℓ, $\tilde{X}$ is projective and smooth over S', and $\tilde{h}^{-1}(Z)_{\mathrm{red}}$ is a sub-divisor of the S'-relative snc divisor $\tilde{E}$, hence a relative snc divisor as well. This finishes the proof of 2.1.

Recall now the theorem stated in Intro.-3 (2):

2.4. Theorem. — *Let S be a separated, integral, noetherian, excellent, regular scheme of dimension 1, with generic point η, X a scheme separated, flat and of finite type over S, ℓ a prime number invertible on S, $Z \subset X$ a nowhere dense closed subset. Then there exists a finite extension η' of η of degree prime to ℓ and a projective ℓ'-alteration $h : \tilde{X} \to X$ above $S' \to S$, where S' is the normalization of S in η', with $\tilde{X}$ regular and quasi-projective over S', a strict normal crossings divisor $\tilde{T}$ on $\tilde{X}$, and a finite closed subset Σ of S' such that:*

(i) *outside Σ, $\tilde{X} \to S'$ is smooth and $\tilde{T} \to S'$ is a relative divisor with normal crossings;*

(ii) *étale locally around each geometric point x of $\tilde{X}_{s'}$, where $s' = \operatorname{Spec} k'$ belongs to Σ, the pair $(\tilde{X}, \tilde{T})$ is isomorphic to the pair consisting of*

$$\begin{aligned} X' &= S'[u_1^{\pm 1}, \dots, u_s^{\pm 1}, t_1, \dots, t_n]/(u_1^{b_1} \cdots u_s^{b_s} t_1^{a_1} \cdots t_r^{a_r} - \pi), \\ T' &= V(t_{r+1} \cdots t_m) \subset X' \end{aligned}$$

around the point $(u_i = 1), (t_j = 0)$, with $1 \leq r \leq m \leq n$, for positive integers $a_1, \dots, a_r, b_1, \dots, b_s$ satisfying $\gcd(p, a_1, \dots, a_r, b_1, \dots, b_s) = 1$, p the characteristic exponent of k', π a local uniformizing parameter at s';

(iii) *$\tilde{h}^{-1}(Z)_{\mathrm{red}}$ is a sub-divisor of $\bigcup_{s' \in \Sigma} (\tilde{X}_{s'})_{\mathrm{red}} \cup \tilde{T}$.*

The proof follows the same lines as that of 2.1. We need again a Gabber-Vidal variant of de Jong's alteration theorems (cf. [Zheng, 2009, 3.8]). This is essentially [Vidal, 2004, 4.4.1]), except for the additional data of $Z \subset X$ and the removal of the hypothesis that S is a strictly local trait:

2.5. Lemma. — *Let X be a normal, proper scheme over S, whose generic fiber is geometrically reduced and irreducible, $Z \subset X$ a nowhere dense closed subset. We assume that a finite group H acts on $X \to S$, faithfully on X, and that Z is H-stable. Then there exists a finite group H_1, a surjective homomorphism $H_1 \to H$, and an*

H_1-equivariant diagram with a cartesian square

$$(2.5.1) \qquad \begin{array}{ccccc} X & \xleftarrow{\;b\;} & X_1 & \xleftarrow{\;a\;} & X_2 \\ \downarrow & & \downarrow & \swarrow & \\ S & \longleftarrow & S_1, & & \end{array}$$

satisfying the following properties:

(i) *$S_1 \to S$ is the normalization of S in a finite extension η_1 of η such that $\eta_1/\mathrm{Ker}(H_1 \to H) \to \eta$ is a radicial extension (where η is the generic point of S);*

(ii) *X_2 is projective and strictly semistable over S_1 (i.e. is strictly semistable over the localizations of S_1 at closed points* [de Jong, 1996, 2.16]*);*

(iii) *$a : X_2 \to X_1$ is projective and surjective, maximally dominating and generically finite and flat, and there exists an H_1-admissible dense open subset $W \subset X_{2\eta_1}$ over a dense open subset U of X_η, such that if $U_1 = \eta_1 \times_\eta U$ and $K = \mathrm{Ker}(H_1 \to \mathrm{Aut}(U_1))$, $W \to W/K$ is a Galois étale cover of group K and the morphism $W/K \to U_1$ induced by a is a universal homeomorphism;*

(iv) *$(ba)^{-1}(Z)$ is the support of a strict normal crossings divisor in X_2, and $(X_2, (ba)^{-1}(Z))$ is a strict semistable pair over S_1 (i.e. over the localizations of S_1 at closed points* [de Jong, 1996, 6.3]*).*

Note that (ii) and (iv) imply that there exists a finite closed subset Σ of S_1 such that, outside Σ, the pair $(X_2, (ba)^{-1}(Z))$ consists of a smooth morphism and a relative strict normal crossings divisor.

Proof. — Up to minute modifications the proof is the same as that of 2.2. We may assume the generic fiber X_η is of dimension $d \geq 1$. We apply [Vidal, 2004, 4.4.3] to X/S, Z, and $G = H$. We get the data of *loc. cit.*, namely an equivariant finite surjective morphism $(S_1, H_1) \to (S, H)$, with S_1 regular (hence equal to the normalization of S in the extension η_1 of the generic point η of S) such that $\eta_1/\mathrm{Ker}(H_1 \to H) \to \eta$ is radicial, an H_1-equivariant pluri-nodal fibration $(Y_d \to \cdots \to Y_1 \to S_1, \{\sigma_{ij}\}, Z_0)$, and an H_1-equivariant alteration $a_1 : Y_d \to X$ over S, satisfying the conditions (i), (ii), (iii) of *loc. cit.* (in particular $a_1^{-1}(Z) \subset Z_d$). Then, as in the proof of [Vidal, 2004, 4.4.1], successively applying [Vidal, 2004, 4.4.4] to each nodal curve $f_i : Y_i \to Y_{i-1}$, one can replace Y_i by an H_1-equivariant projective modification Y_i' of it such that Y_i' is regular, and the inverse image of $Z_i := \bigcup_j \sigma_{ij}(Y_{i-1}) \cup f_i^{-1}(Z_{i-1})$ in Y_i' is an H_1-equivariant strict snc divisor. Then, by the remark following [Vidal, 2004, 4.4.4] $X_2 := Y_d'$ is strict semistable over S_1 and (X_2, Z_d) is a strict semistable pair over S_1. In particular, $(ba)^{-1}(Z)_{\mathrm{red}}$ is a subdivisor of Z_d, hence an snc divisor, and $(X_2, (ba)^{-1}(Z))_{\mathrm{red}}$ is a strict semistable pair over S_1. The open subsets U and W as in (iii) are constructed as at the end of the proof of [Vidal, 2004, 4.4.1]. □

2.6. — *Proof of 2.4.* It is similar to that of 2.1. There are again three steps. We will indicate which modifications should be made.

Step 1. Preliminary reductions. Up to replacing X by the disjoint union of the schematic closures of the reduced components of its generic fiber, and working separately with each of them, we may assume X integral (and $X_\eta \neq \varnothing$). Applying Nagata's compactification theorem, we may further assume X proper and integral. Base changing by the normalization of S in a finite radicial extension of η and taking the reduced scheme, we reduce to the case where, in addition, X_η is irreducible and geometrically reduced. Then we blow up Z in X and normalize as in the previous step 1. Here we used the excellency of S to guarantee the finiteness of the normalizations.

Step 2. Use of 2.5. Let S_0 be the normalization of S in a finite Galois extension η_0 of η such that the irreducible components of the generic fiber of $X_0 = X \times_S S_0$ ($S_0 = \operatorname{Spec} k_0$) are geometrically irreducible. Let $G = \operatorname{Gal}(\eta_0/\eta)$ and $H \subset G$ the decomposition subgroup of a component Y_0 of X_0. We apply 2.5 to $(Y_0/S_0, Z_0 \cap Y_0)$, where $Z_0 = S_0 \times_S Z$. We find a surjection $H_1 \to H$ and an H_1-equivariant diagram of type (2.5.1) satisfying conditions (i), (ii), (iii), (iv) with S replaced by S_0, and $X_2 \to X_1 \to X$ by $Y_2 \to Y_1 \to Y_0$. We then, as above, extend $H_1 \to H$ to a surjection $G_1 \to G$ and obtain a G_1-equivariant diagram of type (2.5.1)

(2.6.1)
$$\begin{array}{ccccc} X_0 & \xleftarrow{\ b\ } & X_1 & \xleftarrow{\ a\ } & X_2 \\ \downarrow & & \downarrow & \swarrow & \\ S_0 & \longleftarrow & S_1, & & \end{array}$$

satisfying again (i), (ii), (iii), (iv). In particular, the composition $h : X_2 \to X_1 \to X_0 \to X$ is an alteration above the composition $S_1 \to S_0 \to S$, X_2 is projective and strictly semistable over S_1 and $h^{-1}(Z)$ is the support of an snc divisor forming a strict semistable pair with X_2/S_1. It follows from (i) that $S_1/G_1 \to S = S_0/G$ is generically radicial, and by (iii) that $X_2/G_1 \to X$ is an alteration over $S_1/G_1 \to S$, which is a universal homeomorphism over a dense open subset. As above, choose an ℓ-Sylow subroup L of G_1. Then S_1/L is regular, $S_1/L \to S_1/G_1$ is finite surjective of generic degree prime to ℓ, and $X_2/L \to X_2/G_1$ is a finite surjective morphism of generic degree prime to ℓ. Putting $S' = S_1/L$, $X' = S' \times_S X$, we get a commutative diagram with cartesian square

(2.6.2)
$$\begin{array}{ccccccc} X & \longleftarrow & X' & \longleftarrow & X_2/L & \longleftarrow & X_2 \\ \downarrow & & \downarrow & \swarrow & & \swarrow & \\ S & \longleftarrow & S', & \longleftarrow & S_1 & & \end{array}$$

where S' is regular, $S' \to S$ is finite surjective of generic degree prime to ℓ, $S_1 \to S'$ generically étale of degree the order of L,

$$h_2 : X_2/L \to X$$

an ℓ'-alteration, X_2/S_1 is projective and strictly semistable, and if h denotes the composition $X_2 \to X$, $Z_1 := h^{-1}(Z)_{\mathrm{red}}$ is an snc divisor in X_2, forming a strictly semistable pair with X_2/S_1.

Step 3. Use of 1.1. Defining Y, $(Z_1)_Y$, $I \subset D$, $K = D/I$ as in the former step 3, K acts generically freely on Y. As the pair $(Y, (Z_1)_Y)$ is strictly semistable over S_1, there exists a finite closed subset Σ_1 of S_1 such that $(Y, Y_{\Sigma_1} \cup (Z_1)_Y)$ forms a log regular pair, log smooth over S_1 equipped with the log structure defined by Σ_1. As $S_1 \to S' = S_1/L$ is Kummer étale, $(Y, (Y_{\Sigma'})_{\mathrm{red}} \cup (Z_1)_Y)$ (where Σ' is the image of Σ_1) is also log smooth over S' (equipped with the log structure given by Σ'), and we get a K-equivariant commutative diagram (2.3.4), with trivial action of K on S' and f projective and log smooth over S'. We then apply 1.1 to $f : (Y, (Y_{\Sigma'})_{\mathrm{red}} \cup (Z_1)_Y) \to S'$, and the proof runs as above. We get a D-equivariant projective modification $g : Y_1 \to Y$ (D acting through K) and a D-strict snc divisor E_1 on Y_1, containing $(g^{-1}((Z_1)_Y) \cup (Y_1)_{\Sigma'})_{\mathrm{red}}$ as a subdivisor, such that the action of D on (Y_1, E_1) is very tame, and (Y_1, E_1) and $(Y_1/D, E_1/D)$ are log smooth over S'. After extending from D to L we get an L-equivariant commutative square of type (2.3.5), with $(Y_2/L, E_2/L)$ log smooth over S' ($= S_1/L$), and $(v^{-1}(h_2^{-1}(Z)) \cup (Y_2/L)_{\Sigma'})_{\mathrm{red}} \subset E_2/L$. As above, we take a projective, log étale modification such that $\tilde{X}$ is regular, and $\tilde{E} = w^{-1}(E_2/L)$ is an snc divisor in $\tilde{X}$.

We now apply 1.2 to the log smooth morphism $(\tilde{X}, \tilde{E}) \to S'$. It's enough to work étale locally on $\tilde{X}$ around some geometric point x of $\tilde{X}_{s'}$, with $s' \in \Sigma'$. We replace S' by its strict localization at the image of x, and consider (1.4.2), with $Q = \mathbf{N}$, $G = \{1\}$, $\Lambda = \mathbf{Z}_{(p)}$ if $p > 1$ and $\mathbf{Q}$ otherwise, and the chart $a : \mathbf{N} \to M_{S'}, 1 \mapsto \pi$, π a uniformizing parameter of S'. In (1.4.3) we have $P^* = \mathbf{Z}^\nu$, $P_1 = \mathbf{N}^\mu$, for nonnegative integers μ, ν, hence

$$P = \mathbf{Z}^\nu \oplus \mathbf{N}^\mu.$$

Let $((b_i), (a_i))$ be the image of $1 \in \mathbf{N}$ in P in the above decomposition, and let $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_s)$ be the sets of those a_i's and b_i's which are $\neq 0$. We may assume $b_i > 0$ if $b_i \neq 0$. As the torsion part of $\mathrm{Coker}(\mathbf{Z} \to P^{\mathrm{gp}})$ is annihilated by an integer invertible on $\tilde{X}$, we have $\gcd(p, a_1, \ldots, a_r, b_1, \ldots, b_s) = 1$, where p is the characteristic exponent of k. We have $P = \mathbf{Z}^s \oplus \mathbf{N}^r \oplus \mathbf{Z}^{\nu - s} \oplus \mathbf{N}^{\mu - r}$. Choosing a basis

$t_{\mu+1}, \dots, t_n$ of V, we get that étale locally around x, $\tilde{X}$ is given by a cartesian square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \operatorname{Spec} \Lambda[u_1^{\pm 1}, \dots, u_s^{\pm 1}, t_1, \dots, t_n] \\ \downarrow & & \downarrow \\ S' & \longrightarrow & \operatorname{Spec} \Lambda[z] \end{array}$$

with x going to the point $(u_i = 1), (t_j = 0)$, and $z \mapsto \pi$, $z \mapsto u_1^{b_1} \cdots u_s^{b_s} t_1^{a_1} \cdots t_r^{a_r}$, in other words,

$$\tilde{X} = S'[u_1^{\pm 1}, \dots, u_s^{\pm 1}, t_1, \dots, t_n]/(u_1^{b_1} \cdots u_s^{b_s} t_1^{a_1} \cdots t_r^{a_r} - \pi),$$

Finally, $\tilde{E}$ is the union of the special fiber $\tilde{X}_{s'}$ and a horizontal divisor $\tilde{T}$, étale locally given by the equation $t_{r+1} \cdots t_m = 0$, where $m = \mu$ and $1 \leq r \leq m \leq n$. As $\tilde{h}^{-1}(Z)_{\text{red}}$ is a sub-divisor of $(\tilde{X}_{s'})_{\text{red}} \cup \tilde{T}$, this finishes the proof of 2.4.

3. Resolvability, log smoothness, and weak semistable reduction

3.1. Elimination of separatedness assumptions. — The main aim of § 3.1 is to weaken the separatedness assumptions in Theorems 1.1 and VIII-1.1.

3.1.1. — Recall, see VI-4.1, that if a finite group G acts on a scheme X then the fixed point subscheme X^G is the intersection of graphs of the translations $g \colon X \to X$. In particular, X^G is closed whenever X is separated. The definition obviously makes sense for non-separated schemes, and the only novelty is that X^G is a subscheme that does not have to be closed.

3.1.2. Inertia specializing actions. — An action of a finite group G on a scheme X is **inertia specializing** if for any point $x \in X$ with a specialization $y \in X$ one has that $G_x \subset G_y$.

3.1.3. Lemma. — *An action of G on X is inertia specializing if and only if for each subgroup $H \subset G$ the subscheme X^H is closed.*

Proof. — Note that a subscheme is closed if and only if it is closed under specializations. If X^H is not closed then there exists a point $x \in X^H$ with a specialization $y \notin X^H$. Thus, $H \subset G_x$ and $H \not\subset G_y$, and the action is not inertia specializing. The opposite direction is proved similarly. □

3.1.4. Remark. — (i) A large class of examples of inertia specializing actions can be described as follows. The following conditions are equivalent and imply that the action is inertia specializing: (a) any G orbit is contained in an open separated subscheme of X, (b) X admits a covering by G-equivariant separated open subschemes X_i. In particular, any admissible action is inertia specializing.

(ii) If (G,X,Z) is as in Theorem VIII-1.1, but instead of separatedness of X one only assumes that it possesses a covering by G-equivariant separated open subschemes X_i, then the assertion of the theorem still holds true. Indeed, the theorem applies to the G-equivariant log schemes $(X_i, Z_i = Z|_{X_i})$, and by Theorem VIII-5.6.1 the modifications $f_{(G,X_i,Z_i)}$ agree on the intersections and hence glue to a required modification $f_{(G,X,Z)}$ of X.

A quick analysis of the proof of VIII-1.1 is required to obtain the following stronger result.

3.1.5. Theorem. — (i) *Theorem VIII-1.1 and its complement VIII-5.6.1 hold true if the assumption that X is separated is weakened to the assumption that the action of G on X is inertia specializing.*

(ii) *Theorem 1.1 holds true if the assumptions that X and S are separated are replaced with the single assumption that the action of G on X is inertia specializing.*

Proof. — The construction of modification $f_{(G,X,Z)}$ in the proof of VIII-1.1 runs in four steps. The first two steps are determined by X and Z, see VIII-4. These steps do not use any separatedness assumption. In Step 3, one blows up the inertia strata, see VIII-4.1.9. Here one only needs to know that the inertia strata are closed, and by Lemma 3.1.3 this happens if and only if the action is inertia specializing. Finally, let us discuss the main part of the construction, see VIII-5 and VIII-5.5.5. Using Lemma VIII-5.3.8, one finds an appropriate equivariant covering $(X',Z',G) \to (X,Z,G)$ with an affine X' and reduces the problem to studying the source. Thus, the separatedness assumption is only used in Lemma VIII-5.3.8. In fact, the only property of the G-action used in the proof of the latter is that for any $x \in X$ the set $X \setminus \bigcup_{H \not\subseteq G_{\overline{x}}} X^H$ is open. Thus, in this case too, one only uses that the action is inertia specializing.

The proof of Theorem 1.1 runs as follows. One considers the modification $f_{(G,X,Z)}$ from VIII-1.1 and checks that it satisfies the additional properties asserted by 1.1. This check is local on X and hence applies to non-separated schemes as well. Since by part (i) of 3.1.5, $f_{(G,X,Z)}$ is well defined whenever G acts inertia specializing on X, we obtain (ii). □

3.1.6. Pseudo-projective morphisms and non-separated Chow's lemma. — We conclude § 3.1 with recalling a non-separated version of Chow's lemma due to Artin-Raynaud-Gruson, see [Raynaud & Gruson, 1971, I 5.7.13]. It will be needed to avoid unnecessary separatedness assumptions in the future. We prefer to use the following non-standard terminology: a finite type morphism $f\colon X \to S$ is **pseudo-projective** if it can be factored into a composition of a **local isomorphism** $X \to \overline{X}$ (i.e. X admits an open covering $X = \bigcup_i X_i$ such that the morphisms $X_i \to \overline{X}$ are open immersions) and a projective morphism $\overline{X} \to S$.

3.1.7. Remark. — (i) We introduce pseudo-projective morphisms mainly for terminological convenience. Although pseudo-projectivity is preserved by base changes, it can be lost under compositions. Moreover, even if X is pseudo-projective over a field k, its blow up X' does not have to be pseudo-projective over k (thus giving an example of a projective morphism $f\colon X' \to X$ and a pseudo-projective one $X \to \operatorname{Spec}(k)$ so that the composition is not pseudo-projective). Indeed, let X be an affine plane with a doubled origin $\{o_1, o_2\}$, and let X' be obtained by blowing up o_1. By η we denote the generic point of $C_1 = f^{-1}(o_1)$. The ring $\operatorname{Spec}(\mathcal{O}_{X',\eta})$ is a DVR and its spectrum has two different k-morphisms to X': one takes the closed point to η and another one takes it to o_2. It then follows from the valuative criterion of separatedness that any k-morphism $g\colon X' \to Y$ with a separated target takes o_2 and η to the same point of Y. In particular, such g cannot be a local isomorphism, and hence X' is not pseudo-projective over k.

(ii) Note that a morphism f is separated (resp. proper) and pseudo-projective if and only if it is quasi-projective (resp. projective). So, the following result extends the classical Chow's lemma to non-separated morphisms.

3.1.8. Proposition. — *Let $f\colon X \to S$ be a finite type morphism of quasi-compact and quasi-separated schemes, and assume that X has finitely many maximal points. Then there exists a projective modification $g\colon X' \to X$ (even a blow up along a finitely generated ideal with a nowhere dense support) such that the morphism $X' \to S$ is pseudo-projective.*

Proof. — As a simple corollary of the flattening theorem, it is proved in [Raynaud & Gruson, 1971, I 5.7.13] that there exists a modification $X' \to X$ such that $X' \to S$ factors as a composition of an étale morphism $X' \to \overline{X}$ that induces an isomorphism of dense open subschemes and a projective morphism $\overline{X} \to S$. (In loc.cit. one works with algebraic spaces and assumes that f is locally separated, but the latter is automatic for any morphism of schemes.) Our claim now follows from the following lemma (which fails for locally separated morphisms between algebraic spaces). □

3.1.9. Lemma. — *Assume that $\phi\colon Y \to Z$ is an étale morphism of schemes that restricts to an open embedding on a dense open subscheme $Y_0 \hookrightarrow Y$. Then ϕ is a local isomorphism.*

Proof. — Let us prove that if, in addition, ϕ is separated then ϕ is an open immersion. Since Y possesses an open covering by separated subschemes, this will imply the lemma. The diagonal $\Delta_\phi\colon Y \to Y \times_Z Y$ is an open immersion, and by the separatedness of ϕ, it is also a closed immersion. Thus, Y is open and closed in $Y \times_Z Y$, and since both Y and $Y \times_Z Y$ have dense open subschemes that map isomorphically onto Y_0, Δ_ϕ is an isomorphism. This implies that ϕ is a monomorphism, but any étale monomorphism is an open immersion by [ÉGA IV_4 17.9.1]. □

3.2. Semistable curves and log smoothness

3.2.1. Log structure associated to a closed subset. — Let S be a reduced scheme. Any closed nowhere dense subset $W \subset S$ induces a log structure $j_\star \mathscr{O}_U^* \cap \mathscr{O}_S \hookrightarrow \mathscr{O}_S$, where $j\colon U \hookrightarrow S$ is the embedding of the complement of W. The associated log scheme will be denoted (S, W). By VI-1.4, any log regular log scheme is of the form (S, W), where W is the non-triviality locus of the log structure.

3.2.2. Semistable relative curves. — Following the terminology of [Temkin, 2010], by a **semistable multipointed relative curve** over a scheme S we mean a pair (C, D), where C is a flat finitely presented S-scheme of pure relative dimension one and with geometric fibers having only ordinary nodes as singularities, and $D \hookrightarrow C$ is a closed subscheme which is étale over S and disjoint from the singular locus of $C \to S$. We do not assume C to be neither proper nor even separated over S.

3.2.3. Proposition. — *Assume that (S, W) is a log regular log scheme and (C, D) is a semistable multipointed relative S-curve such that the morphism $f\colon C \to S$ is smooth over $S \setminus W$. Then the morphism of log schemes $(C, D \cup f^{-1}(W)) \to (S, W)$ is log smooth.*

Proof. — See VI-1.9. □

3.3. ℓ'-resolvability

3.3.1. Alterations. — Assume that S' and S are reduced schemes with finitely many maximal points and let $\eta' \subset S'$ and $\eta \subset S$ denote the subschemes of maximal points. Let $f\colon S' \to S$ be an **alteration**, i.e. a proper, surjective, generically finite, and maximally dominating morphism. Recall that f is an **ℓ'-alteration** if one has that $([k(x) : k(f(x))], l) = 1$ for any $x \in \eta'$. We say that f is **separable** if $k(\eta')$ is a separable $k(\eta)$-algebra (i.e. $\eta' \to \eta$ has geometrically reduced fibers). If, in addition, S' and S are provided with an action of a finite group G such that f is G-equivariant, the action on S is trivial, the action on η' is free, and $\eta'/G \xrightarrow{\sim} \eta$, then we say that f is a **separable Galois alteration** of group G, or just **separable G-alteration**.

3.3.2. Remark. — We add the word "separable" to distinguish our definition from Galois alterations in the sense of de Jong (see [de Jong, 1997]) or Gabber-Vidal (see [Vidal, 2004, p. 370]). In the latter cases, one allows alterations that factor as $S' \to S'' \to S$, where $S' \to S''$ is a separable Galois alteration and $S'' \to S$ is purely inseparable.

3.3.3. Universal ℓ'-resolvability. — Let X be a locally noetherian scheme and let ℓ be a prime invertible on X. Assume that for any alteration $Y \to X_{\mathrm{red}}$ and nowhere dense closed subset $Z \subset Y$ there exists a surjective projective morphism $f\colon Y' \to Y$ such that Y' is regular and $Z' = f^{-1}(Z)$ is an snc divisor. (By a slight abuse of language, by saying that a closed subset is an snc divisor we mean that it is the support of an snc divisor.) If, furthermore, one can always choose such f to be a modification, separable ℓ'-alteration, ℓ'-alteration, or alteration, then we say that X is **universally resolvable**, **universally separably ℓ'-resolvable**, **universally ℓ'-resolvable**, or **universally $\mathbf{Q}$-resolvable**, respectively.

3.3.4. Remark. — (i) Due to resolution of singularities in characteristic zero, any qe scheme over $\mathrm{Spec}(\mathbf{Q})$ is universally resolvable. This is essentially due to Hironaka, [Hironaka, 1964], though an additional work was required to treat qe schemes that are not algebraic in Hironaka's sense, see [Temkin, 2008] for the noetherian case and [Temkin, 2012] for the general case.

(ii) It is hoped that all qe schemes admit resolution of singularities (in particular, are universally resolvable). However, we are, probably, very far from proving this. Currently, it is known that any qe scheme of dimension at most two admits resolution of singularities (see [Cossart et al., 2009] for a modern treatment). In particular, any qe scheme of dimension at most two is universally resolvable.

(iii) One can show that any universally $\mathbf{Q}$-resolvable scheme is qe, but we prefer not to include this proof here, and will simply add quasi-excellence assumption to the theorems below.

(iv) On the negative side, we note that there exist regular (hence resolvable) but not universally $\mathbf{Q}$-resolvable schemes X. They can be constructed analogously to examples from I-11.5. For instance, there exists a discretely valued field K whose completion $\widehat{K}$ contains a non-trivial finite purely inseparable extension K'/K (e.g. take an element $y \in k((x))$ which is transcendental over $k(x)$ and set $K = k(x, y^p) \subset K' = k(x, y) \subset k((x))$ with the induced valuation). The valued extension K'/K has a defect in the sense that $e_{K'/K} = f_{K'/K} = 1$. In other words, the DVR's A' and A of K' and K, have the same residue field and satisfy $m_{A'} = m_A A'$. Since A' is A-flat, it cannot be A-finite. On the other hand, A' is the integral closure of A in K', and we obtain that A is not qe. In addition, although $X = \mathrm{Spec}(A)$ is regular, any X-finite integral scheme X' with $K' \subset k(X')$ possesses a non-finite normalization and hence does not admit a desingularization. Thus, X is not universally $\mathbf{Q}$-resolvable.

Our main goal will be to show that universal ℓ'-resolvability of a qe base scheme S is inherited by finite type S-schemes whose structure morphism $X \to S$ is maximally dominating (see Theorem 3.5 below, where a more precise result is formulated). The proof will be by induction on the relative dimension, and the main work is done when dealing with the case of generically smooth relative curves.

3.4. Theorem. — *Let S be an integral, noetherian, qe scheme with generic point $\eta = \mathrm{Spec}(K)$, let $f\colon X \to S$ be a maximally dominating (II-1.1.2) morphism of finite type, and let $Z \subset X$ be a nowhere dense closed subset. Assume that S is universally ℓ'-resolvable (resp. universally separably ℓ'-resolvable), $X_\eta = X \times_S \eta$ is a smooth curve over K, and $Z_\eta = Z \times_S \eta$ is étale over K. Then there exist a projective ℓ'-alteration (resp. a separable projective ℓ'-alteration) $a\colon S' \to S$, a projective modification $b\colon X' \to (X \times_S S')^{\mathrm{pr}}$, where $(X \times_S S')^{\mathrm{pr}}$ is the proper transform of X, i.e. the schematic closure of $X_\eta \times_S S'$ in $X \times_S S'$,*

$$\begin{array}{ccccccc} X' & \xrightarrow{b} & (X \times_S S')^{\mathrm{pr}} & \hookrightarrow & X \times_S S' & \longrightarrow & X \\ & \searrow^{f'} & & & \downarrow & & \downarrow f \\ & & & & S' & \xrightarrow{a} & S \end{array}$$

and divisors $W' \subset S'$ and $Z' \subset X'$ such that S' and X' are regular, W' and Z' are snc, the morphism $f'\colon X' \to S'$ is pseudo-projective (§ 3.1.6), $(X', Z') \to (S', W')$ is log smooth, and $Z' = c^{-1}(Z) \cup f'^{-1}(W')$, where c denotes the alteration $X' \to X$.

We also note if f is separated (resp. proper) then f' is even quasi-projective (resp. projective) by Remark 3.1.7 (ii).

Proof. — It will be convenient to represent Z as $Z_h \cup Z_v$, where the horizontal component Z_h is the closure of Z_η and the vertical component Z_v is the closure of $Z \setminus Z_h$. The following observation will be used freely: if $a_1\colon S_1 \to S$ is a (resp. separable) projective ℓ'-alteration with an integral S_1 and $b_1\colon X_1 \to (X \times_S S_1)^{\mathrm{pr}}$ is a projective modification, then it suffices to prove the theorem for $f_1\colon X_1 \to S_1$ and the preimage $Z_1 \subset X_1$ of Z (note that the generic fiber of f_1 is smooth because it is a base change of that of f). So, in such a situation we can freely replace f by f_1, and Z will be updated automatically without mentioning, as a rule. We will change S and X a few times during the proof. We start with some preliminary steps.

Step 1. *We can assume that f is quasi-projective.* By Proposition 3.1.8, replacing X with its projective modification we can achieve that f factors through a local isomorphism $X \to \overline{X}$, where $\overline{X}$ is S-projective. Let $X_1 \subset \overline{X}$ be the image of X. Then the induced morphism $X_1 \to S$ is quasi-projective and with smooth generic fiber. If the theorem holds for f_1 and the image $Z_1 \subset X_1$ of Z, i.e., there exist $a\colon S' \to S$ and $b_1\colon X_1' \to (X_1 \times_S S')^{\mathrm{pr}}$ that satisfy all assertions of the theorem, then the theorem also holds for f and Z because we can keep the same a and take

$b = b_1 \times_{X_1} X$. This completes the step, and in the sequel we assume that f is quasi-projective. As we will only use projective modifications $b_1\colon X_1 \to (X \times_S S_1)^{\mathrm{pr}}$, the quasi-projectivity of f will be preserved automatically.

Step 2. *We can assume that f and $Z_h \to S$ are flat.* Indeed, due to the flattening theorem of Raynaud-Gruson, see [Raynaud & Gruson, 1971, I 5.2.2], this can be achieved by replacing S with an appropriate projective modification S', replacing X with the proper transform, and replacing Z with its preimage. From now on, the proper transforms of X will coincide with the base changes.

Step 3. *Use of the stable modification theorem.* By the stable modification theorem [Temkin, 2010, 1.5 and 1.1] there exist a separable alteration $\overline{a}\colon \overline{S} \to S$ with an integral $\overline{S}$ and a projective modification $\overline{X} \to X \times_S \overline{S}$ such that $(\overline{X}, \overline{Z}_h)$ is a *semistable multipointed* $\overline{S}$-curve (see § 3.2.2), where $\overline{Z}_h \subset \overline{X}$ is the horizontal part of the preimage $\overline{Z}$ of Z. Enlarging $\overline{S}$ we can assume that it is integral and normal.

$$\begin{array}{ccccc} \overline{X} & \longrightarrow & X \times_S \overline{S} & \longrightarrow & X \\ & \overset{\overline{f}}{\searrow} & \downarrow & & \downarrow f \\ & & \overline{S} & \xrightarrow{\ \overline{a}\ } & S. \end{array}$$

Step 4. *We can assume that $\overline{a}$ is a separable projective G-alteration, where G is an ℓ-group.* Since semistable multipointed relative curves are preserved by base changes, we can just enlarge $\overline{S}$ by replacing it with any separable projective Galois alteration that factors through $\overline{S}$. Once $\overline{S} \to S$ is Galois, let $\overline{G}$ denote its Galois group and let $G \subset \overline{G}$ be any Sylow ℓ-subgroup. Since $\overline{S} \to \overline{S}/G$ is a separable G-alteration and $\overline{S}/G \to S$ is a separable projective ℓ'-alteration, we can replace S with $\overline{S}/G$, replace X with $X \times_S (\overline{S}/G)$, and update Z accordingly, accomplishing the step.

Step 5. *The action of G on $X \times_S \overline{S}$ via $\overline{S}$ lifts equivariantly to $\overline{X}$. In particular, $\overline{f}$ becomes G-equivariant and $\overline{X} \to X$ becomes a separable projective G-alteration.* This follows from [Temkin, 2010, 1.6].

Step 6. *The action of G on $\overline{X}$ is inertia specializing.* Indeed, any covering of X by separated open subschemes induces a covering of $\overline{X}$ by G-equivariant separated open subschemes. So, it remains to use Remark 3.1.4 (i).

Step 7. *We can assume that $\overline{S} \to S$ is finite.* By Raynaud-Gruson there exists a projective modification $S' \to S$ such that the proper transform $\overline{S}'$ of $\overline{S}$ is flat over S'. Let η denote the generic point of S and S' and let η' denote the generic point of $\overline{S}$ and $\overline{S}'$. Since the morphisms $\overline{S} \times_S S' \to S'$ and $\eta' \to \eta$ are G-equivariant and $\overline{S}'$ is the schematic closure of η' in $\overline{S} \times_S S'$, we obtain that the morphism $\overline{S}' \to S'$ is a separable projective G-alteration. Replacing $\overline{S} \to S$ with $\overline{S}' \to S'$, and updating X and $\overline{X}$ accordingly, we achieve that $\overline{S} \to S$ becomes flat, and hence finite. All conditions of steps 1–6 are preserved with the only exception that $\overline{S}$ may be non-normal. So, we replace $\overline{S}$ with its normalization and update $\overline{X}$. This operation preserves the finiteness of $\overline{S} \to S$, so we complete the step.

Step 8. *Choice of* W. Fix a closed subset $W \subsetneq S$ such that $\overline{S} \to S$ is étale over $S \setminus W$, $\overline{f}(\overline{Z}_v) \subset \overline{W}$, where $\overline{Z}_v$ is the vertical part of $\overline{Z}$ and $\overline{W} = \overline{a}^{-1}(W)$, and $\overline{f}$ is smooth over $\overline{S} \setminus \overline{W}$.

Step 9. *We can assume that* S *is regular and* W *is snc.* Indeed, by our assumptions on S there exists a projective ℓ'-alteration (resp. a separable projective ℓ'-alteration) $a\colon S' \to S$ such that S' is regular and $a^{-1}(W)$ is snc. Choose any preimage of η in $S' \times_S \overline{S}$ and let $\overline{S}'$ be the normalization of its closure. Then $\overline{S}' \to S'$ is a separable projective Galois alteration with Galois group $G' \subset G$, so we can replace $S, \overline{S}, G$ and X with $S', \overline{S}', G'$ and $X \times_S S'$, respectively, and update W, $\overline{W}$ and Z accordingly (i.e. replace them with their preimages). Note that step 9 is the only step where a non-separable alteration of S may occur.

Step 10. *The morphism* $(\overline{S}, \overline{W}) \to (S, W)$ *is Kummer étale.* Indeed, $\overline{S} \to S$ is an étale G-covering outside of W, and $\overline{S}$ is the normalization of S in this covering, so the assertion follows from IX-2.1.

Consider the G-equivariant subscheme $\overline{T} = \overline{Z} \cup \overline{f}^{-1}(\overline{W})$ of $\overline{X}$. The morphism $(\overline{X}, \overline{T}) \to (\overline{S}, \overline{W})$ is log smooth by Proposition 3.2.3, hence so is the composition $(\overline{X}, \overline{T}) \to (S, W)$ and we obtain that $(\overline{X}, \overline{T})$ is log regular. The group G acts freely on $\overline{S} \setminus \overline{W}$ and hence also on $\overline{X} \setminus \overline{T}$. Also, its action on $\overline{X}$ is tame and inertia specializing (step 6), hence we can apply Theorem 1.1 to $(\overline{X}, \overline{T}) \to (S, W)$. As a result, we obtain a projective G-equivariant modification $(\overline{X}', \overline{T}') \to (\overline{X}, \overline{T})$ such that $\overline{T}'$ is the preimage of $\overline{T}$, G acts very tamely on $(\overline{X}', \overline{T}')$, and $(X', Z') = (\overline{X}'/G, \overline{T}'/G)$ is log smooth over (S, W) (the quotient exists as a scheme as f is quasi-projective by Step 1 and the morphisms $\overline{S} \to S$ and $\overline{X}' \to \overline{X} \to X \times_S \overline{S}$ are projective). Clearly, X' is a projective modification of X and Z' is the union of the preimages of W and Z, hence it only remains to achieve that X' is regular and Z' is snc. For this it suffices to replace (X', Z') with its projective modification $\widetilde{\mathscr{F}}^{\log}(X', Z')$ introduced in VIII-3.4.9. □

3.4.1. Remark. — It is natural to compare Theorem 3.4 and the classical de Jong's result recalled in IX-1.2. The main differences are as follows.

(i) One considers non-proper relative curves in 3.4, and this is the only point that requires to use the stable modification theorem instead of de Jong's result. The reason is that although the problem easily reduces to the case of a quasi-projective f (see Step 2), one cannot make f projective, as the compactified generic fiber $\overline{X}_\eta$ does not have to be smooth (i.e. *geometrically* regular) at the added points.

(ii) One uses ℓ'-alterations in 3.4. This is more restrictive than in IX-1.2, but the price one has to pay is that the obtained log smooth morphism $(X', Z') \to (S', W')$

does not have to be a nodal curve (e.g. $X' \to S'$ may have non-reduced fibers). The construction of such $b\colon X' \to X$ involves a quotient by a Sylow subgroup, and is based on Theorem 1.1. (Note also that it seems probable that instead of 1.1 one could use the torification argument of Abramovich-de Jong, see [Abramovich & de Jong, 1997, § 1.4.2].)

Now, we are going to use Theorem 3.4 to prove the main result of § 3.

3.5. Theorem. — *Let $f\colon X \to S$ be a maximally dominating (II-1.1.2) morphism of finite type between noetherian qe schemes, let $Z \subset X$ be a nowhere dense closed subset, and assume that S is universally ℓ'-resolvable, then:*

(i) *X is universally ℓ'-resolvable.*

(ii) *There exist projective ℓ'-alterations $a\colon S' \to S$ and $b\colon X' \to X$ with regular sources, a pseudo-projective (§ 3.1.6) morphism $f'\colon X' \to S'$ compatible with f*

$$\begin{array}{ccc} X' & \xrightarrow{b} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{a} & S \end{array}$$

and snc divisors $W' \subset S'$ and $Z' \subset X'$ such that $Z' = b^{-1}(Z) \cup f'^{-1}(W')$ and the morphism $(X', Z') \to (S', W')$ is log smooth.

(iii) *If $S = \mathrm{Spec}(k)$, where k is a perfect field, then one can achieve in addition to* (ii) *that a is an isomorphism and the alteration b is separable. In particular, X is universally separably ℓ'-resolvable in this case.*

Proof. — Note that (i) follows from (ii) because any alteration X_1 of X is also of finite type over S, so we can apply (ii) to X_1 as well. Thus, our aim is to prove (ii) and its complement (iii). We will view Z both as a closed subset and a reduced closed subscheme. We start with a few preliminary steps, that reduce the theorem to a special case. We will tacitly use that if $S_1 \to S$ and $X_1 \to X$ are projective ℓ'-alterations, separable in case (iii), and $f_1\colon X_1 \to S_1$ is compatible with f, then it suffices to prove the theorem for f_1 and the preimage $Z_1 \subset X_1$ of Z. So, in such situation we can freely replace f with f_1, and Z will be updated automatically without mentioning, as a rule.

Step 1. *We can assume that X and S are integral and normal.* For a noetherian scheme Y, let Y^{nor} denote the normalization of its reduction. Since f is maximally dominating, it induces a morphism $f^{\mathrm{nor}}\colon X^{\mathrm{nor}} \to S^{\mathrm{nor}}$, and replacing f with f^{nor} we can assume that S and X are normal. Since we can work separately with the connected components, we can now assume that S and X are integral.

Step 2. *We can assume that f is projective.* By Proposition 3.1.8 there exists a projective modification $X_1 \to X$ such that the morphism $X_1 \to S$ factors into a composition of a local isomorphism $X_1 \to \overline{X}$ and a projective morphism $\overline{f}\colon \overline{X} \to S$. Replacing X with X_1 we can assume that X itself admits a local isomorphism $g\colon X \to \overline{X}$ with an S-projective target. Let $\overline{Z}$ be the closure of $g(Z)$. Then it suffices to solve our problem for $\overline{f}$ and $\overline{Z}$, as the corresponding alteration of $\overline{X}$ will induce an alteration of X as required. Thus, replacing X and Z with $\overline{X}$ and $\overline{Z}$, we can assume that f is projective.

Step 3. *It suffices to find f' which satisfies all assertions of the theorem except the formula for Z', while the latter is weakened as $b^{-1}(Z) \cup f'^{-1}(W') \subset Z'$.* Given such an f' note that $Z'' = b^{-1}(Z) \cup f'^{-1}(W')$ is a subdivisor of Z', hence it is an snc divisor too. We claim that X', Z'', S', W' satisfy all assertions of the theorem, and the only thing one has to check is that the morphism $(X', Z'') \to (S', W')$ is log smooth. The latter follows from Lemma 3.5.3 whose proof will be given below.

Step 4. *In the situation of* (iii) *we can assume that the field k is infinite.* Assume that $S = \mathrm{Spec}(k)$ where k is a finite field and fix an infinite algebraic ℓ'-prime extension $\overline{k}/k$ (i.e. it does not contain the extension of k of degree ℓ). We claim that it suffices to prove (ii) and (iii) for $\overline{S} = \mathrm{Spec}(\overline{k})$ and the base changes $\overline{X} = X \times_S \overline{S}$ and $\overline{Z} = Z \times_S \overline{S}$. Indeed, assume that $a\colon \overline{X}' \to \overline{X}$ is a separable ℓ'-alteration with a regular source and such that $\overline{Z}' = a^{-1}(\overline{Z})$ is an snc divisor (obviously, we can take $\overline{S}' = \overline{S}$ and $\overline{W}' = \varnothing$). Since $\overline{S} = \lim_i S_i$ where $S_i = \mathrm{Spec}(k_i)$ and k_i/k run over finite subextensions of $\overline{k}/k$, [ÉGA IV$_3$ 8.8.2(ii)] implies that there exists i and a finite type morphism $X'_i \to X_i = X \times_S S_i$ such that $\overline{X}' \xrightarrow{\sim} X'_i \times_{S_i} \overline{S}$. For any finite subextension $k_i \subset k_j \subset \overline{k}$ set $X'_j = X'_i \times_{S_i} S_j$ and $X_j = X_i \times_{S_i} S_j$, and let $Z'_j \subset X'_j$ be the preimage of Z. Then it follows easily from [ÉGA IV$_3$ 8.10.5] and [ÉGA IV$_4$ 17.7.8] that $X'_j \to X_j$ is an ℓ'-alteration and $X'_j \to S_j$ is smooth for large enough k_j. In the same manner one achieves that Z'_j is an snc divisor. Now, it is obvious that $(X'_j, Z'_j) \to (S, \varnothing)$ is log smooth and $X'_j \to X_j \to X$ is an ℓ'-alteration.

Now we are in a position to prove the theorem. We will use induction on $d = \mathrm{tr.deg.}(k(X)/k(S))$, with the case of $d = 0$ being obvious. Assume that $d \geq 1$ and the theorem holds for smaller values of d.

Step 5. *Factorizing f through a relative curve.* After replacing X by a projective modification, we can factor f as $h \circ g$, where $h\colon Y \to S$ is projective, Y is integral, $g\colon X \to Y$ is maximally dominating and $\mathrm{tr.deg.}(k(X)/k(Y)) = 1$. Indeed, one can obviously construct such a rational map $g'\colon X \dashrightarrow Y$ even without modifying X (i.e. g' is well defined on a non-empty open subscheme $U \subset X$). Let X' be the schematic image of the morphism $U \hookrightarrow X \times Y$. Then $X' \to X$ is a projective modification (an isomorphism over U), and the morphism $X' \to S$ factors through $g\colon X \to Y$.

Let $\eta = \mathrm{Spec}(k(Y))$ denote the generic point of Y, $X_\eta = X \times_Y \eta$ and $Z_\eta = Z \times_Y \eta$. We claim that in addition to factoring f through Y one can achieve that the following condition is satisfied:

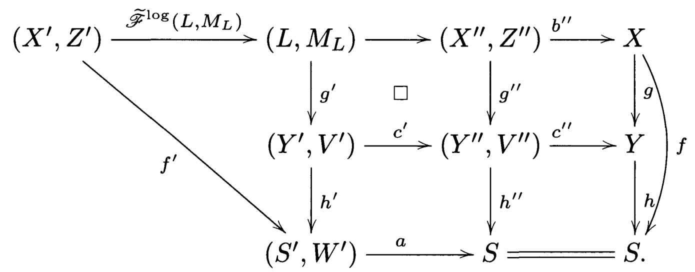
$(*)$ X_η and Z_η are smooth over η.

In general, this is achieved by replacing X and Y by inseparable alterations. Pick up any finite purely inseparable extension $K/k(Y)$ such that $Z_K = (Z \times_\eta \operatorname{Spec}(K))^{\mathrm{nor}}$ (i.e. just the reduction) and $X_K = (X \times_\eta \operatorname{Spec}(K))^{\mathrm{nor}}$ are smooth, extend $K/k(Y)$ to a projective alteration $Y' \to Y$, and replace Y and X with Y' and the schematic closure of X_K in $X \times_Y Y'$, respectively. Clearly, $(*)$ holds after this replacement.

It remains to deal with the case (iii). This time we should avoid inseparable alterations, so g and Y should be chosen more carefully. If $k = \overline{k}$ is algebraically closed and $S = \operatorname{Spec}(k)$ then such g and Y exist by [de Jong, 1996, 4.11], and the general assertion of (iii) will be proven similarly. Let us recall the main line of the proof of [de Jong, 1996, 4.11]. Fix a closed immersion $X \hookrightarrow \mathbf{P}^N_k$ and for each linear subspace L of dimension $N-d$ consider the classical projection $\mathrm{Bl}_L(\mathbf{P}^N_k) \to Y = \mathbf{P}^{d-1}_k$. If L is general then it does not contain X and hence the strict transform $X'_L \hookrightarrow \mathrm{Bl}_L(\mathbf{P}^N_k)$ is a modification of X. de Jong shows that if $k = \overline{k}$ then for a general choice of L the projection $X'_L \to Y$ satisfies $(*)$.

In the general case, the schemes $\overline{X} = X \otimes_k \overline{k}$ and $\overline{Z} = Z \otimes_k \overline{k}$ are reduced since k is perfect. Hence a general $\overline{L} \hookrightarrow \mathbf{P}^N_{\overline{k}}$ induces a modification $\overline{X}'_{\overline{L}} \to \overline{X}$ and a curve fibration $g_{\overline{L}} \colon \overline{X}'_{\overline{L}} \to \mathbf{P}^{d-1}_{\overline{k}}$ that satisfies $(*)$. Since k is infinite we can choose $\overline{L}$ to be defined over k, i.e. $\overline{L} = L \otimes_k \overline{k}$ for $L \hookrightarrow \mathbf{P}^N_k$. We obtain thereby a modification $X'_L \to X$ and a curve fibration $g_L \colon X'_L \to \mathbf{P}^{d-1}_k$. Since $g_{\overline{L}}$ is the flat base change of g_L, the latter satisfies $(*)$ by descent.

Step 6. *Use of Theorem 3.4.* So far, we have constructed the right column of the following diagram



By Theorem 3.4 there exists a projective ℓ'-alteration $c'' \colon Y'' \to Y$ with regular source, a projective modification $X'' \to (X \times_Y Y'')^{\mathrm{pr}}$ with regular source, a projective morphism $g'' \colon X'' \to Y''$ compatible with g, and snc divisors $V'' \subset Y''$ and $Z'' \subset X''$ such that $(X'', Z'') \to (Y'', V'')$ is log smooth and $b''^{-1}(Z) \subset Z''$. In case (iii), Y is universally separably ℓ'-resolvable by the induction assumption, hence we can take c'' to be separable, and then $b'' \colon X'' \to X$ is also separable. In addition, by the induction assumption applied to $h'' \colon Y'' \to S$ and $V'' \subset Y''$ there exist projective

ℓ'-alterations $a\colon S' \to S$ and $c'\colon Y' \to Y''$ with regular sources and snc divisors $W' \subset S'$ and $V' \subset Y'$ and a projective morphism $h'\colon Y' \to S'$ compatible with h'' such that $(Y', V') \to (S', W')$ is log smooth, $c'^{-1}(V'') \subset V'$, and c' is separable if the assumption of (iii) is satisfied.

Set $(L, M_L) = (X'', Z'') \times^{\mathrm{fs}}_{(Y'',V'')} (Y', V')$, where the product is taking place in the category of fs log schemes. To simplify notation we will write P^{sat} instead of $(P^{\mathrm{int}})^{\mathrm{sat}}$ for monoids and log schemes. Recall that $(L, M_L) = (F, M_F)^{\mathrm{sat}}$, where (F, M_F) is the usual log fibered product, and $F = X'' \times_{Y''} Y'$ by [Kato, 1988, 1.6]. Furthermore, we have local Zariski charts for c' and g'' modeled, say, on $P_i \to P'_i$ and $P_i \to Q_i$. Hence (F, M_F) is a Zariski log scheme with charts modeled on $R_i = P'_i \oplus_{P_i} Q_i$, and (L, M_L) is a Zariski log scheme with charts modeled on R_i^{sat}. Furthermore, the saturation morphism $L \to F$ is finite hence the composition $L \to F \to X''$ is projective. The morphism $g'\colon (L, M_L) \to (Y', V')$ is a saturated base change of the log smooth morphism $g''\colon (X'', Z'') \to (Y'', V'')$, hence it is log smooth. As (Y', V') is log regular, (L, M_L) is also log regular. Applying to (L, M_L) the saturated monoidal desingularization functor $\widetilde{\mathscr{F}}^{\log}$ from VIII-3.4.9 we obtain a log regular Zariski log scheme (X', Z') with a regular X'. Then Z' is a normal crossings divisor, which is even an snc divisor since the log structure is Zariski.

We claim that (X', Z') and (S', W') are as asserted by the theorem except of the weakening dealt with in Step 3. Indeed, the morphism $(X', Z') \to (S', W')$ is log smooth because it is the composition $(X', Z') \to (L, M_L) \to (Y', V') \to (S', W')$ of log smooth morphisms. The preimage of Z in X'' is contained in Z'', which is the non-triviality locus of the log structure of (X'', Z''), hence its preimage in X' is also contained in the non-triviality locus of the log structure of (X', Z'), which is Z'. Clearly, Z' also contains the preimage of W'. By the construction, $S' \to S$ is a projective ℓ'-alteration, and it remains to check that $X' \to X$ is also a projective ℓ'-alteration. Since $\widetilde{\mathscr{F}}^{\log}(L, M_L)$ is a saturated log blow up tower and (L, M_L) is log regular, the underlying morphism of schemes $X' \to L$ is a projective modification by VIII-3.4.6 (i). The projective morphism $L \to X''$ is an ℓ'-alteration because generically (where the log structures are trivial) it is a base change of the projective ℓ'-alteration $Y' \to Y''$. And $X'' \to X$ is a projective ℓ'-alteration by the construction. It remains to recall that in the situation of (iii) the alterations $c'\colon Y' \to Y''$ and $b''\colon X'' \to X$ are separable, hence so are $(L, M_L) \to X''$ and the total composition $X' \to X$. □

3.5.1. Remark. — The only place where inseparable alterations are used is the argument at step 5, where we had to choose an inseparable extension $K/k(Y)$ when X_η or Z_η is not geometrically regular.

3.5.2. Remark. — Analogs of Theorems 3.4 and 3.5 hold also for the class of universally **Q**-resolvable schemes. In a sense, this is the "$\ell = 1$" case of these theorems. One can prove this by the same argument but with ℓ replaced by 1. In fact, few arguments become vacuous (though formally true); for example, in steps 4-6 in the proof of Theorem 3.4, an ℓ-group G should be replaced by the trivial group, so the steps 5 and 6 collapse.

3.5.3. Lemma. — *Assume that S and X are regular schemes, $W \subset S$ and $Z \subset X$ are snc divisors, and $f : X \to S$ is a morphism such that $f^{-1}(W) \subset Z$ and the induced morphism of log schemes $h\colon (X, Z) \to (S, W)$ is log smooth. Then for any intermediate divisor $f^{-1}(W) \subset Z' \subset Z$ the morphism $h'\colon (X, Z') \to (S, W)$ is log smooth.*

Proof. — We can work locally at a geometric point $\overline{x} \to X$. Let $x \in X$ and $s \in S$ be the images of $\overline{x}$, and let $q_1, \dots, q_r \in \mathscr{O}_{S,s}$ define the irreducible components of W through s. Set $Q = \bigoplus_{i=1}^r q_i^{\mathbf{N}}$. Shrinking S we obtain a chart $c\colon S \to \operatorname{Spec}(\mathbf{Z}[Q])$ of (S, W). By Proposition 1.2 applied to c, h, and $G = 1$, after localizing X along $\overline{x}$ one can find an fs chart of h consisting of c, $X \to \operatorname{Spec}(\mathbf{Z}[P])$, and $\phi\colon Q \to P$ such that the morphism $X \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[P])$ is smooth, P^* is torsion free, ϕ is injective, and the torsion of $\operatorname{Coker}(\phi^{\mathrm{gp}})$ is annihilated by an integer n invertible on S.

Let $p_1, \dots, p_t \in \mathscr{O}_{X,x}$ define the irreducible components of Z through x. Our next aim is to adjust the chart similarly to 1.4.2 (vi) to achieve that $\overline{P} \xrightarrow{\sim} N = \bigoplus_{j=1}^t p_j^{\mathbf{N}}$. Note that $\overline{M}_{X,x} \xrightarrow{\sim} N$, where $M_X \hookrightarrow \mathscr{O}_X$ is the log structure of (X, Z). The homomorphism $\psi\colon P \to M_{X,x}$ factors through the fs monoid $R = P[T^{-1}]$ where $T = \psi^{-1}(M^*_{X,x})$. Clearly, R^* is torsion free, $\overline{R} \xrightarrow{\sim} N$, and shrinking X around x we obtain a chart $X \to \operatorname{Spec}(\mathbf{Z}[R])$. Since $\operatorname{Spec}(\mathbf{Z}[R])$ is open in $\operatorname{Spec}(\mathbf{Z}[P])$ the morphism $X \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[R])$ is smooth. So, we can replace P with R achieving that $\overline{P} \xrightarrow{\sim} N$, and hence $P \xrightarrow{\sim} \mathbf{N}^t \oplus \mathbf{Z}^u$.

Without restriction of generality Z' is defined by the vanishing of $\prod_{j=1}^{t'} p_j$ for $t' \leq t$. Since $f^{-1}(W) \subset Z'$, the image of Q in $\overline{P}$ is contained in $\overline{P}' = \bigoplus_{j=1}^{t'} p_j^{\mathbf{N}}$. Hence ϕ factors through a homomorphism $\phi'\colon Q \to P' = \overline{P}' \oplus P^*$, and we obtain a chart of h' consisting of c, $X \to \operatorname{Spec}(\mathbf{Z}[P'])$, and ϕ'. By [Kato, 1994, 3.5, 3.6], to prove that h' is log smooth it remains to observe that ϕ' is injective, the torsion of $\operatorname{Coker}(\phi'^{\mathrm{gp}})$ is annihilated by n, and the morphism

$$X \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[P]) \to S \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}(\mathbf{Z}[P'])$$

is smooth because $\operatorname{Spec}(\mathbf{Z}[P]) \to \operatorname{Spec}(\mathbf{Z}[P'])$ is so. □

3.5.4. Comparison with Theorems 2.1 and 2.4. — Theorems 2.1 and 2.4 follow by applying Theorem 3.5 (ii) to $X \to S$ and Z (where one takes $S = \operatorname{Spec}(k)$ in 2.1).

Indeed, the main part of the proofs of 2.1 and 2.4 was to construct ℓ'-alterations $X' \to X$ and $S' \to S$ with regular sources, snc divisors $Z' \subset X'$ and $W' \subset S'$, and a log smooth morphism $f'\colon (X', Z') \to (S', W')$ compatible with f. Then, in the last paragraphs of both proofs, Proposition 1.2 was used to obtain a more detailed description of X', Z', and f'. In particular, for a zero-dimensional base this amounted to saying that X' is S-smooth and Z' is relatively snc over S, and for a one-dimensional base this amounted to the conditions (i) and (ii) of 2.4.

Conversely, Theorem 2.1 (resp. 2.4) implies assertion (ii) of Theorem 3.5 under the assumptions of 2.1 (resp. 2.4) on X and S. Moreover, the non-separated Chow's lemma could be used in their proofs as well, so the separatedness assumption there could be easily removed. In such case, Theorems 2.1 and 2.4 would simply become the low dimensional (with respect to S) cases of Theorem 3.5 (ii) plus an explicit local description of the log smooth morphism f'. The strengthening 3.5 (iii), however, was not achieved in 2.1, and required a different proof of the whole theorem.

3.6. Saturation. — In Theorems 3.4 and 3.5 we resolve certain morphisms $f\colon X \to S$ with divisors $Z \subset X$ by log smooth morphisms $f'\colon (X', Z') \to (S', W')$. However, as we insisted to use only ℓ'-alterations and to obtain regular X' and snc Z', we had to compromise a little on the "quality" of f'. For example, our f' may have non-reduced fibers. Due to de Jong's theorem, if the relative dimension is one, then one can make f' a nodal curve. We will see that a similar improvement of f' is possible in general if one uses arbitrary alterations and allows non-regular X'. The procedure reduces to saturating f' and is essentially due to Tsuji and Illusie-Kato-Nakayama ([Illusie et al., 2005, A.4.4 and A.4.3]).

3.6.1. Saturated morphisms. — Recall that a homomorphism $P \to Q$ of fs (resp. fine) monoids is **saturated** (resp. **integral**) if for any homomorphism $P \to P'$ with fs (resp. fine) target the pushout $Q \oplus_P P'$ is fs (resp. fine). A morphism of fs (resp. fine) log schemes $f\colon (Y, M_Y) \to (X, M_X)$ is **saturated** (resp. **integral**) if so are the homomorphisms $\overline{M}_{X,f(y)} \to \overline{M}_{Y,y}$.

3.6.2. Remark. — (i) Integral morphism were introduced already by Kato in [Kato, 1988, § 4]. Kato also introduced the notion of saturated morphisms, which was first seriously studied by Tsuji in [Tsuji, 1997]. Actually, one can define saturated morphisms for arbitrary fine log schemes, but the definition is more involved than we use. For fs log schemes our definition coincides with the usual one due to [Tsuji, 1997, II 2.13(2)].

(ii) The following two basic properties of saturated morphisms follow from the definition: (a) a composition of saturated morphisms between fs log schemes is saturated, (b) if $f\colon Y \to X$ is a saturated morphism between fs log schemes, then, for any morphism of fs log schemes $X' \to X$, the base change $f'\colon Y' \to X'$ of f in the category of log schemes is a saturated morphism of fs log schemes. (Also, it is proved in [Tsuji, 1997, II 2.11] that analogous properties hold for saturated morphisms between arbitrary integral log schemes.)

(iii) Let $f\colon Y \to X$ be a morphism of fs log schemes. It is shown in [Tsuji, 1997, II 3.5] that if f can be modeled on charts corresponding to saturated homomorphisms of fs monoids $P_i \to Q_i$ then f is saturated. Let us remark that the converse is also true: if f is saturated then it can be modeled on charts corresponding to $P_i \to Q_i$ as above.

3.6.3. Integrality and saturatedness for log smooth morphisms. — We recall the following result that relates the notions of integral and saturated morphisms to certain properties of the underlying morphisms of schemes.

3.6.4. Proposition. — *Let $f\colon (Y, M_Y) \to (X, M_X)$ be a log smooth morphism between fs log schemes and assume that f is integral. Then,*

(i) *the morphism $Y \to X$ is flat,*

(ii) *f saturated if and only if $Y \to X$ has reduced fibers.*

Proof. — The first claim is proved in [Kato, 1988, 4.5] and the second one is proved in [Tsuji, 1997, II 4.2]. □

One can also go in the opposite direction: from flatness to integrality.

3.6.5. Proposition. — *Let $f\colon (Y, M_Y) \to (X, M_X)$ be a log smooth morphism between fs log schemes and assume that the morphism $Y \to X$ is flat and (X, M_X) is log smooth over a field k with the trivial log structure. Then f is integral.*

Proof. — It suffices to show that if $\overline{y} \to Y$ is a geometric point and $\overline{x} = f(\overline{y})$ then the homomorphism $\overline{\phi}\colon \overline{M}_{X,\overline{x}} \to \overline{M}_{Y,\overline{y}}$ is integral. By Proposition 1.2 and the argument in 1.4 (vi), localizing X and Y along these points we can assume that X possesses a chart $a\colon X \to X_0 = \mathrm{Spec}(k[Q])$ with smooth a and f is modeled on a chart $Y_0 = \mathrm{Spec}(k[P]) \to X_0$ corresponding to a homomorphism $\phi\colon Q \to P$ so that the morphism $g\colon Y \to Z = X \times_{X_0} Y_0$ is smooth (in particular, flat), and ϕ has the following properties: $Q = \overline{M}_{X,\overline{x}}$, P is fs, P^* is torsion free, the composition $Q \to P \to \overline{P} = P/P^*$ coincides with $\overline{\phi}$, the kernel of ϕ^{gp} is finite, killed by an integer invertible at x, as well as the torsion part of its cokernel (but we will not need these last two properties). Since Q is sharp and saturated, Q^{gp} is torsion free, so ϕ is injective. We claim that $\overline{\phi}$ is integral if and only if ϕ is integral. To see this note that if $Q \to R$ is a homomorphism of monoids, then $R \oplus_Q \overline{P}$ is isomorphic to the quotient of $R \oplus_Q P$ by the image of P^*, and hence either both pushouts are integral or neither of them is integral. Thus, we only need to prove that ϕ is integral.

Note that the morphism $h\colon Z \to X$ is flat at the (Zariski) image $z \in Z$ of $\overline{y}$ because f and g are flat. Note that a takes $x = h(z)$ to the origin of X_0 and $Y_0 \to X_0$ is flat at the image $y_0 \in Y_0$ of z by flat descent with respect to a. In other words, if $I \subset k[P]$ is the ideal corresponding to y_0 then the homomorphism $k[Q] \to k[P]_I$ is flat. The preimage of m_y under $k[P] \to \mathcal{O}_{Y,y}$ contains the set $m_P = P \setminus P^*$, hence $m_P \subset I$ and we obtain that I contains $J = k[m_P]$. Note that the ideal J is prime as $k[P]/J \xrightarrow{\sim} k[P^*]$ is a domain due to P^* being torsion free. Thus, the localization $k[P]_J$ makes sense, and we obtain a flat homomorphism $\psi\colon k[Q] \to k[P]_J$.

It is proved in [Kato, 1988, 4.1], that if the homomorphisms $K[\phi]\colon K[Q] \to K[P]$ are flat for any field K then ϕ is integral. The proof consists of two parts. First one checks that ϕ is injective, which is automatic in our case. This is the only argument in loc.cit. where a play with different fields is needed. We claim that the second part of the proof of the implication (iii) $\implies$ (v) in [Kato, 1988, 4.1] works fine with a single field k, and, moreover, it suffices to only use that $k[Q] \to k[P]_J$ is flat. Let us indicate how the argument in loc.cit. should be adjusted.

Assume that, as in the proof of [Kato, 1988, 4.1], we are given $a_1, a_2 \in Q$ and $b_1, b_2 \in P$ such that $\phi(a_1)b_1 = \phi(a_2)b_2$. Let S be the kernel of the homomorphism of $k[Q]$-modules $k[Q] \oplus k[Q] \to k[Q]$ given by $(x, y) \mapsto a_1x - a_2y$. By the flatness, the kernel of $k[P]_J \oplus k[P]_J \to k[P]_J$, $(x,y) \mapsto \phi(a_1)x - \phi(a_2)y$ is generated by the image of S. Hence there exist representations $b_1 = \sum_{i=1}^r \phi(c_i)\frac{f_i}{s}$ and $b_2 = \sum_{i=1}^r \phi(d_i)\frac{f_i}{s}$ with $c_i, d_i \in k[Q]$, $f_i \in k[P]$, $s \in k[P] \setminus J$, and $a_1c_i = a_2d_i$. Moreover, multiplying s and f_i's by an appropriate unit $u \in P^*$ we can assume that $s = 1 + s'$ for $s' \in \mathrm{Span}_k(P \setminus \{1\})$. Then $b_1 + \sum_{1 \le \alpha \le m} \lambda_\alpha t_\alpha = \sum_{1 \le i \le r} \phi(c_i) f_i$, with $\lambda_\alpha \in k^*$, and the $t_\alpha \in P$ pairwise distinct and distinct from b_1, so we see that there exist $a_3 \in Q$, $b \in P$, and $1 \le i \le r$, such that a_3 appears in c_i, b appears in f_i, and $b_1 = \phi(a_3)b$. The remaining argument copies that of the loc.cit. verbatim, and one obtains in the end that ϕ satisfies the condition (v) from [Kato, 1988, 4.1]. Thus, ϕ is integral and we are done. □

Before going further, let us discuss an incarnation of saturated morphisms in (more classical) toroidal geometry.

3.6.6. Remark. — In toroidal geometry an analog of saturated morphisms was introduced by Abramovich and Karu in [Abramovich & Karu, 2000]. In the language of log schemes toroidal morphisms can be interpreted as log smooth morphisms $f\colon (X, Z) \to (S, W)$ between log regular schemes (with the toroidal structure given by the triviality loci of the log structures). If f is a toroidal morphism as above then Abramovich-Karu called it weakly semistable when the following conditions hold: S is regular, f is locally equidimensional, and the fibers of f are reduced. Furthermore, they remarked that the equidimensionality condition is equivalent to flatness of f

whenever S is regular, see [Abramovich & Karu, 2000, 4.6]. Thus, the weak semistability condition is nothing else but saturatedness of f and regularity of the target. In particular, saturated log smooth morphisms between log regular log schemes may be viewed as the generalization of weakly semistable morphisms to the case of an arbitrary log regular (or toroidal) base.

Now, we are going to prove our main result about saturation.

3.7. Theorem. — *Assume that $f\colon (X,Z) \to (S,W)$ is a log smooth morphism such that (S,W) is log regular and S is universally $\mathbf{Q}$-resolvable (§ 3.3.3). Then there exists an alteration $h\colon S' \to S$ such that S' is regular, $W' = g^{-1}(W)$ is an snc divisor, and the fs base change $f'\colon (X',Z') \to (S',W')$ is a saturated morphism.*

Recall that $(X',Z') = (X,Z) \times^{\mathrm{fs}}_{(S,W)} (S',W')$ and f' is log smooth because the saturation morphism is log smooth.

Proof. — By VIII-3.4.9, applying to (S,W) an appropriate saturated log blow up tower and replacing (X,Z) with the fs base change we can achieve that S is regular and W is normal crossings. By an additional sequence of log blow ups we can also make W snc (see VIII-4.1.6), so (S,W) becomes a Zariski log scheme. Now, we can étale-locally cover f by charts $f_i\colon (X_i,Z_i) \to (S_i,W_i)$ modeled on $P_i \to Q_i$ such that S_i are open subschemes in S. By [Illusie et al., 2005, A.4.4, A.4.3], for each i there exists a morphism $h_i\colon (S'_i,W'_i) \to (S_i,W_i)$ such that h_i is a composition of a Kummer morphism and a log blow up, and the fs base change of f_i is saturated. (Although the proof in loc.cit. is written in the context of log analytic spaces, it translates to our situation almost verbatim. The only changes are that we have to distinguish étale and Zariski topology on the base (in order to construct log blow ups), and h_i does not have to be log étale as there might be inseparable Kummer morphisms.)

Note that $W'_i = h_i^{-1}(W_i)$. In addition, $S'_i \to S_i$ is a projective alteration by VIII-3.4.6. Extend each h_i to a projective alteration $g_i\colon T_i \to S$, and let $h\colon S' \to S$ be a projective alteration that factors through each T_i. By the universal $\mathbf{Q}$-resolvability assumption we can enlarge S' so that it becomes regular and $Z' = h^{-1}(Z)$ becomes snc. We claim that h is as claimed. It suffices to check that the fs base change of each morphism $f'_i\colon (X_i,Z_i) \to (S,W)$ is saturated. However, already the fs base change of f'_i to $(T_i, g_i^{-1}(W))$ is saturated by the construction, hence so is its further base change to (S',W'). □

3.7.1. Remark. — Our proof is an easy consequence of [Illusie et al., 2005, A.4.4 and A.4.3]. The first cited result shows that (locally) any log smooth morphism can be made exact by an appropriate log blow up of the base. This result is somewhat analogous to the flattening theorem of Raynaud-Gruson. The second cited result shows that by a Kummer extension of the base one can (locally) saturate an exact log smooth morphism. It is somewhat analogous to the reduced fiber theorem of Bosch-Lütkebohmert-Raynaud ([Bosch et al., 1995]) which implies that if $f\colon Y \to X$ is a finite type morphism between reduced noetherian schemes then there exists an alteration $X' \to X$ such that the normalized base change $f'\colon (Y \times_X X')^{\mathrm{nor}} \to X'$ has reduced fibers. Although the proof of the latter is far more difficult.

3.8. Characteristic zero case. — Theorem 3.5 can be substantially strengthened when S is of characteristic zero, i.e., the morphism $S \to \mathrm{Spec}(\mathbf{Z})$ factors through $\mathrm{Spec}(\mathbf{Q})$.

3.9. Theorem. — *Assume that S is a reduced, noetherian, qe scheme of characteristic zero, $f\colon X \to S$ is a maximally dominating morphism of finite type with reduced source, and $Z \subset X$ is a nowhere dense closed subset. Then there exist projective modifications $a\colon S' \to S$ and $b\colon X' \to X$ with regular sources, a pseudo-projective morphism $f'\colon X' \to S'$ compatible with f, and snc divisors $W' \subset S'$ and $Z' \subset X'$ such that $Z' = b^{-1}(Z) \cup f'^{-1}(W')$ and the morphism $(X', Z') \to (S', W')$ is log smooth.*

Proof. — The proof is very close to the proof of Theorem 3.5, so we will just say which changes in that proof should be made. First, we note that any S-scheme Y of finite type is noetherian and qe. Thus, if Y is reduced and $T \subset Y$ is a nowhere dense closed subset then the pair (Y, T) can be desingularized by [Temkin, 2008] in the following sense: there exists a projective modification $h\colon Y' \to Y$ with regular source and such that $h^{-1}(T)$ is an snc divisor. This result replaces the ℓ'-resolvability assumption in Theorem 3.5, and it allows to apply the proof of that theorem to our situation with the only changes that one always uses projective modifications instead of projective ℓ'-alterations, and Theorem 3.4 is replaced with Lemma 3.9.1 below. (Note that Lemma 3.9.1 is weaker than Theorem 3.9, while Theorem 3.4 does not follow from Theorem 3.5.) □

3.9.1. Lemma. — *Let S be an integral, noetherian, qe scheme with generic point $\eta = \mathrm{Spec}(K)$, let $f\colon X \to S$ be a maximally dominating morphism of finite type, and let $Z \subset X$ be a nowhere dense closed subset. Assume that $X_\eta = X \times_S \eta$ is a smooth curve over K, and $Z_\eta = Z \times_S \eta$ is étale over K. Then there exist projective modifications $a\colon S' \to S$ and $b\colon X' \to X$ with regular sources, a pseudo-projective morphism $f'\colon X' \to S'$ compatible with f and snc divisors $W' \subset S'$ and $Z' \subset X'$ such that $Z' = b^{-1}(Z) \cup f'^{-1}(W')$ and the morphism $(X', Z') \to (S', W')$ is log smooth.*

Proof. — The proof copies the proof of Theorem 3.4 with the only difference that instead of an ℓ-Sylow subgroup $G \subset \overline{G}$ one simply takes $G = \overline{G}$. The latter is possible because the schemes are of characteristic zero and hence any action of $\overline{G}$ is tame. □

Combining Theorem 3.9 and 3.7 we obtain the following weak semistable reduction theorem.

3.10. Theorem. — *Assume that S is a reduced, noetherian, qe scheme of characteristic zero, $f\colon X \to S$ is a maximally dominating morphism of finite type with reduced source, and $Z \subset X$ is a nowhere dense closed subset. Then there exists an alteration $S' \to S$, a modification $X' \to (X \times_S S')^{\mathrm{pr}}$ of the proper transform of X, a pseudo-projective morphism $f'\colon X' \to S'$ compatible with f, and divisors $W' \subset S'$ and $Z' \subset X'$ such that S' is regular, W' is snc, $Z' = b^{-1}(Z) \cup f'^{-1}(W')$, and the morphism $(X', Z') \to (S', W')$ is log smooth and saturated (i.e. $X' \to S'$ is weakly semistable).*

3.10.1. Remark. — (i) In the case when X and S are integral proper varieties over an algebraically closed field k of characteristic zero, this theorem becomes the weak semistable reduction theorem of Abramovich-Karu. Our proof has many common lines with their arguments. In particular, the first step of their proof was to make f toroidal, and it was based on de Jong's theorem. (Note also that in a recent work [Abramovich et al., 2013] of Abramovich-Denef-Karu, the toroidalization theorem was extended to separated schemes of finite type over an arbitrary ground field of characteristic zero.) Our Theorem 3.9 can be viewed as a generalization of the toroidalization theorem of Abramovich-Karu.

(ii) The second stage in the proof of the weak semistable reduction theorem of Abramovich-Karu (the combinatorial stage) is analogous to Theorem 3.7. It obtains as an input a toroidal morphism $f\colon (X, Z) \to (S, W)$ between proper varieties of characteristic zero and outputs an alteration $h\colon S' \to S$ such that S' is regular, $W' = h^{-1}(W)$ is snc, and the saturated base change of f is weakly semistable. The proof is similar to the arguments used in the proofs of [Illusie et al., 2005, A.4.4 and A.4.3]. First, one constructs a toroidal blow up of the base that makes the fibers equidimensional (i.e. makes the log morphism integral), and then an appropriate normalized finite base change is used to make the fibers reduced.

Erratum. — Proof of Theorem 3.4, Step 1: First, one should take Z_1 to be the closure of the image of Z. Still, there is a gap since the preimage $\widetilde{Z} \subset X$ of Z_1 under $X \to X_1$ can be strictly larger than Z, while the argument proves the theorem for $(X, \widetilde{Z})$. This can be corrected as follows. In the beginning of the step, replace X with the blow up along Z achieving that Z is the support of an effective Cartier divisor. By the presented argument, if the theorem holds for X_1 and Z_1 then it holds for $(X, \widetilde{Z})$, i.e. there exist a, b such that $\widetilde{Z}' = c^{-1}(\widetilde{Z}) \cup f'^{-1}(W')$ is an snc divisor and $(X, \widetilde{Z}') \to (S', W')$ is a log smooth morphism. We claim that the same pair (a, b) works for (X, Z). Since $Z' = c^{-1}(Z) \cup f'^{-1}(W')$ is a subdivisor of $\widetilde{Z}'$, it is snc. The morphism $(X', Z') \to (S', W')$ is log smooth by Lemma 3.5.3 proved below.

Proof of Theorem 3.5, Step 2: Same patch as above. Start the step with blowing up X along Z so that Z becomes the support of an effective Cartier divisor; this is also needed for Step 3. The present argument of Step 2 shows that if the theorem holds for $(\overline{X}, \overline{Z})$ then it holds for $(X, \widetilde{Z})$ where $\widetilde{Z}$ is the preimage of $\overline{Z}$. But then Lemma 3.5.3 implies that the theorem also holds for (X, Z).