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MIRZAKHARNI'S RECURSION FORMULA IS EQUIVALENT TO THE WITTEN-KONTSEVICH THEOREM

by

Kefeng Liu & Hao Xu

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper, we give a proof of Mirzakhani's recursion formula of Weil-Petersson volumes of moduli spaces of curves using the Witten-Kontsevich theorem. We also describe properties of intersections numbers involving higher degree κ classes.

***Résumé* (La formule de récurrence de Mirzakhani est équivalente au théorème de Witten-Kontsevich)**

Dans cet article, nous démontrons la formule de récurrence de Mirzakhani sur les volumes de Weil-Petersson des espaces de module de courbes en utilisant le théorème de Witten-Kontsevich. Nous donnons aussi des propriétés des nombres d'intersection associées aux classes κ de degré supérieur.

1. Introduction

Following the notation of Mulase and Safnuk [**21**], let $\mathcal{M}_{g,n}(\mathbf{L})$ denote the moduli space of bordered Riemann surfaces with n geodesic boundary components of specified lengths $\mathbf{L} = (L_1, \ldots, L_n)$ and let $\mathrm{Vol}_{g,n}(\mathbf{L})$ denote its Weil-Petersson volume $\mathrm{Vol}(\mathcal{M}_{g,n}(\mathbf{L}))$. Using her remarkable generalization of the McShane identity, Mirzakhani [**19**] proved a beautiful recursion formula for these Weil-Petersson volumes

$$\begin{aligned}\mathrm{Vol}_{g,n}(\mathbf{L}) = \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \underline{n}=I\coprod J}} \int_0^{L_1}\int_0^{\infty}\int_0^{\infty} & xyH(t,x+y) \\ & \times \mathrm{Vol}_{g_1,n_1}(x,\mathbf{L}_I)\mathrm{Vol}_{g_2,n_2}(y,\mathbf{L}_J)dxdydt \\ + \frac{1}{2L_1}\int_0^{L_1}\int_0^{\infty}\int_0^{\infty} xyH(t,x+y)&\mathrm{Vol}_{g-1,n+1}(x,y,L_2,\ldots,L_n)dxdydt\end{aligned}$$

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$$+\frac{1}{2L_1}\sum_{j=2}^{n}\int_0^{L_1}\int_0^{\infty} x\big(H(x,L_1+L_j)+H(x,L_1-L_j)\big)$$
$$\times \operatorname{Vol}_{g,n-1}(x,L_2,\dots,\hat{L_j},\dots,L_n)dxdt,$$

where the kernel function

$$H(x,y)=\frac{1}{1+e^{(x+y)/2}}+\frac{1}{1+e^{(x-y)/2}}.$$

Using symplectic reduction, Mirzakhani [20] showed the following relation

$$\begin{aligned}\frac{\operatorname{Vol}_{g,n}(2\pi\mathbf{L})}{(2\pi^2)^{3g+n-3}} &= \frac{1}{(3g+n-3)!}\int_{\mathcal{M}_{g,n}}(\kappa_1+\sum_{i=1}^{n}L_i^2\psi_i)^{3g+n-3}\\ &= \sum_{\substack{d_0+\cdots+d_n\\=3g+n-3}}\prod_{i=0}^{n}\frac{1}{d_i!}\langle\kappa_1^{d_0}\prod\tau_{d_i}\rangle_{g,n}\prod_{i=1}^{\infty}L_i^{2d_i}.\end{aligned}$$

Combining with her recursion formula of Weil-Petersson volumes, Mirzakhani [20] found a new proof of the celebrated Witten-Kontsevich theorem.

By taking derivatives with respect to $\mathbf{L}=(L_1,\dots,L_n)$ in Mirzakhani's recursion, Mulase and Safnuk [21] obtained the following enlightening recursion formula of intersection numbers which is equivalent to Mirzakhani's recursion.

$$\begin{aligned}&(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g\\ &=\sum_{j=2}^{n}\sum_{b=0}^{a}\frac{a!}{(a-b)!}\frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!}\beta_b\langle\kappa_1^{a-b}\tau_{b+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g\\ &+\frac{1}{2}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{(a-b)!}(2r+1)!!(2s+1)!!\beta_b\langle\kappa_1^{a-b}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}\\ &\quad+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=b+d_1-2}\frac{a!}{c!c'!}(2r+1)!!(2s+1)!!\beta_b\\ &\qquad\times\langle\kappa_1^{c}\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},\end{aligned}$$

where

$$\beta_b=(2^{2b+1}-4)\frac{\zeta(2b)}{(2\pi^2)^b}=(-1)^{b-1}2^b(2^{2b}-2)\frac{B_{2b}}{(2b)!}.$$

Safnuk [23] gave a proof of the above differential form of Mirzakhani's recurson formula using localization techniques, but he also used the Mirzakhani-McShane formula. The relationship between Mirzakhani's recurson and matrix integrals has been studied by Eynard-Orantin [7] and Eynard [6].

Indeed, when $a = 0$, Mulase-Safnuk differential form of Mirzakhani's recursion is just the Witten-Kontsevich theorem [**14, 24**] in the form of DVV recursion relation [**4**]. There are several other new proofs of Witten-Kontsevich theorem [**3, 12, 13, 22**] besides Mirzakhani's proof [**20**].

More discussions about Weil-Petersson volumes from the point of view of intersection numbers can be found in the papers [**5, 10, 18, 26**].

In Section 2, we show that Mirzakhani's recursion formula is essentially equivalent to the Witten-Kontsevich theorem via a formula from [**11**] expressing κ classes in terms of ψ classes. In Section 3, we present certain results of intersection numbers involving higher degree κ classes.

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2. Proof of Mirzakhani's recursion formula

We first give three lemmas. The following lemma can be found in [**21**].

Lemma 2.1. — *The constants β_b in Mirzakhani's recursion satisfy the following:*

$$\sum_{k=0}^{\infty} \beta_k x^k = \frac{\sqrt{2x}}{\sin\sqrt{2x}}.$$

And its inverse:

$$\left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1} = \frac{\sin\sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{(2k+1)!} x^k.$$

Proof. — Since

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = \frac{x}{2} \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \frac{x}{2i} \cot \frac{x}{2i},$$

we have

$$\sum_{k=0}^{\infty} \beta_k x^k = \sqrt{2x}\left(\cot\sqrt{\frac{x}{2}} - \cot\sqrt{2x}\right) = \frac{\sqrt{2x}}{\sin\sqrt{2x}}.$$ □

The following elementary result is crucial to our proof.

Lemma 2.2. — *Let $F(m,n)$ and $G(m,n)$ be two functions defined on $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of nonnegative integers. Let α_k and β_k be real numbers that satisfy*

$$\sum_{k=0}^{\infty} \alpha_k x^k = \left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1}.$$

Then the following two identities are equivalent:

$$G(m,n) = \sum_{k=0}^{m} \alpha_k F(m-k, n+k), \quad \forall\ (m,n) \in \mathbb{N} \times \mathbb{N},$$

$$F(m,n) = \sum_{k=0}^{m} \beta_k G(m-k, n+k), \quad \forall\ (m,n) \in \mathbb{N} \times \mathbb{N}.$$

Proof. — Assume the first identity holds, then we have

$$\begin{aligned}\sum_{i=0}^{m} \beta_i G(m-i, n+i) &= \sum_{i=0}^{m} \beta_i \sum_{j=0}^{m-i} \alpha_j F(m-i-j, n+i+j) \\ &= \sum_{k=0}^{m} \sum_{i+j=k} (\beta_i \alpha_j) F(m-k, n+k) \\ &= \sum_{k=0}^{m} \delta_{k0} F(m-k, n+k) \\ &= F(m,n).\end{aligned}$$

So we proved the second identity. The proof of the other direction is the same. □

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [**2**], Faber [**8**] and Kaufmann-Manin-Zagier [**11**] into a nice combinatorial formalism.

Lemma 2.3 **([11])**. — *For* $m > 0$,

$$\langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^m \rangle_g = \sum_{k=1}^{m} \frac{(-1)^{m-k}}{k!} \sum_{\substack{m_1+\cdots+m_k=m \\ m_i>0}} \binom{m}{m_1, \ldots, m_k} \langle \prod_{j=1}^{n} \tau_{d_j} \prod_{j=1}^{k} \tau_{m_j+1} \rangle_g.$$

Proof. — (sketch) Let $\pi_{n+p,n} : \overline{\mathcal{M}}_{g,n+p} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1} \cdots \psi_{n+p}^{a_p+1})$ by $R(a_1, \ldots, a_p)$, then we have the formula from [**2**]

$$R(a_1, \ldots, a_p) = \sum_{\sigma \in \mathbb{S}_p} \prod_{\substack{\text{each cycle } c \\ \text{of } \sigma}} \kappa_{\sum_{j \in c} a_j},$$

where we write any permutation σ in the symmetric group $\mathbb{S}_p$ as a product of disjoint cycles.

A formal combinatorial argument [**11**] leads to the following inversion equation

$$\kappa_{a_1} \cdots \kappa_{a_p} = \sum_{k=1}^{p} \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1,\ldots,p\} = S_1 \coprod \cdots \coprod S_k \\ S_k \neq \emptyset}} R(\sum_{j \in S_1} a_j, \ldots, \sum_{j \in S_k} a_j),$$

from which the result follows easily. □

Proposition 2.4. — *We have*

$$\sum_{b=0}^{a}(-1)^b\binom{a}{b}\frac{(2(d_1+b)+1)!!}{(2b+1)!!}\langle\tau_{d_1+b}\prod_{i=2}^{n}\tau_{d_i}\kappa_1^{a-b}\rangle_g$$
$$=\sum_{j=2}^{n}\frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!}\langle\kappa_1^a\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^a\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{c+c'=a\\ I\coprod J=\{2,\dots,n\}}}\binom{a}{c}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Proof. — Let LHS and RHS denote the left and right hand side of the equation respectively. By Lemma 2.3 and the Witten-Kontsevich theorem, we have

$$(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g$$
$$=(2d_1+1)!!\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{k}\tau_{m_j+1}\rangle_g$$
$$=\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}$$
$$\times\left(\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_g\right.$$
$$+\sum_{j=1}^{k}\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\langle\tau_{d_1+m_j}\prod_{i=2}^{n}\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{I\coprod J=\{2,\dots,n\}\\ I'\coprod J'=\{1,\dots,k\}}}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!$$
$$\left.\times\langle\tau_r\prod_{i\in I}\tau_{d_i}\prod_{i\in I'}\tau_{m_i+1}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\prod_{i\in J'}\tau_{m_i+1}\rangle_{g-g'}\right)$$

$$
\begin{aligned}
&= \sum_{j=2}^{n} \frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \rangle_g \\
&+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^a \tau_r \tau_s \prod_{i\neq 1} \tau_{d_i} \rangle_{g-1} \\
+ \frac{1}{2} &\sum_{\substack{c+c'=a \\ I\coprod J=\{2,\ldots,n\}}} \binom{a}{c} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^c \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i\in J} \tau_{d_i} \rangle_{g-g'} \\
&+ \sum_{k=0}^{a} \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1+\cdots+m_k=a \\ m_i>0}} \binom{a}{m_1,\ldots,m_k} \\
&\times \sum_{j=1}^{k} \frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^{n} \tau_{d_i} \prod_{i\neq j} \tau_{m_i+1} \rangle_g \\
&= RHS + \sum_{k\geq 0} \frac{(-1)^{a-k-1}}{(k+1)!} \sum_{b=1}^{a} \sum_{\substack{m_1+\cdots+m_k=a-b \\ m_i>0}} \binom{a}{b} \binom{a-b}{m_1,\ldots,m_k} \\
&\times (k+1) \frac{(2(d_1+b)+1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^{n} \tau_{d_i} \prod_{i=1}^{k} \tau_{m_i+1} \rangle_g \\
&= RHS - \sum_{b=1}^{a} (-1)^b \binom{a}{b} \frac{(2(d_1+b)+1)!!}{(2b+1)!!} \langle \tau_{d_1+b} \prod_{i=2}^{n} \tau_{d_i} \kappa_1^{a-b} \rangle_g \\
&= RHS - LHS + (2d_1+1)!! \langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^a \rangle_g .
\end{aligned}
$$

So we have proved $RHS = LHS$. □

Proposition 2.4 is also implicitly contained in the arguments of Mulase and Safnuk [**21**].

Theorem 2.5. — *We have*

$$
\begin{aligned}
&\frac{(2d_1+1)!!}{a!} \langle \prod_{j=1}^{n} \tau_{d_j} \kappa_1^a \rangle_g \\
&= \sum_{b=0}^{a} \sum_{j=2}^{n} \frac{(2(b+d_1+d_j)-1)!!}{(a-b)!(2d_j-1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i\neq 1,j} \tau_{d_i} \rangle_g \\
&+ \frac{1}{2} \sum_{b=0}^{a} \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{(a-b)!} \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i\neq 1} \tau_{d_i} \rangle_{g-1}
\end{aligned}
$$

$$+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=b+d_1-2}\frac{(2r+1)!!(2s+1)!!}{c!c'!}\beta_b$$

$$\times\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

where the constants β_k are given by

$$\Big(\sum_{k=0}^{\infty}\beta_k x^k\Big)^{-1}=\frac{\sin\sqrt{2x}}{\sqrt{2x}}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(2k+1)!!}x^k.$$

Proof. — Denote the LHS by $F(a,d_1)$. Let

$$G(a,d_1)=\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{a!(2d_j-1)!!}\langle\kappa_1^a\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$

$$+\frac{1}{2}\sum_{r+s=d_1-2}\frac{(2r+1)!!(2s+1)!!}{a!}\langle\kappa_1^a\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$

$$+\frac{1}{2}\sum_{\substack{c+c'=a\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=d_1-2}\frac{(2r+1)!!(2s+1)!!}{c!c'!}\times\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

Note that Proposition 2.4 is just

$$\sum_{b=0}^{a}\frac{(-1)^b}{b!(2b+1)!!}F(a-b,d_1+b)=G(a,d_1).$$

By Lemmas 2.1 and 2.2, we have

$$F(a,d_1)=\sum_{b=0}^{a}\beta_b G(a-b,d_1+b)=RHS.$$

So we conclude the proof. □

3. Higher Weil-Petersson volumes

Mirzakhani's formula provides a recursive way of computing the following Weil-Petersson volumes of moduli spaces of curves

$$WP(g):=\int_{\overline{\mathcal{M}}_{g,n}}\kappa_1^{3g-3+n}.$$

Mirzakhani's formula resorts to intersection numbers of mixed ψ and κ classes.

A natural question is whether there exist an explicit formula expressing $WP(g)$ in terms of those $WP(g')$ with $g'<g$. Recall the following beautiful formula due to Itzykson-Zuber [**9**].

Proposition 3.1 **(Itzykson-Zuber).** — *Let $g \geq 0$. Then*

$$\phi_{g+1} = \frac{25g^2-1}{24}\phi_g + \frac{1}{2}\sum_{m=1}^{g} \phi_{g+1-m}\phi_m,$$

where $\phi_0 = -1, \phi_1 = \frac{1}{24}$ and

$$\phi_g = \frac{(5g-5)(5g-3)}{2^g(3g-3)!}\langle \tau_2^{3g-3}\rangle_g, \quad g \geq 2.$$

By projection formula, we have

$$\langle \tau_2^{3g-3}\rangle_g = \langle \kappa_1^{3g-3}\rangle_g + \cdots,$$

where $\cdots$ denote terms involving higher degree kappa classes. Also note that $\langle \kappa_1^{3g-3}\rangle_g$ is conjecturally [**16**] the largest term in the right hand side.

To our disappointment, so far, all recursion formulae for $WP(g)$ stemming from the Witten-Kontsevich theorem involve either ψ class or higher degree κ classes inevitably.

Mirzakhani, Mulase and Safnuk's arguments use Wolpert's formula [**25**]

$$\kappa_1 = \frac{1}{2\pi^2}\omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. We have no similar formulae for higher degree κ classes. So a priori κ_1 may be rather special in the intersection theory. However, as we will see, this is not the case.

First we fix notations as in [**11**]. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \dots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i.

Let $\mathbf{m}, \mathbf{t}, \mathbf{a_1}, \dots, \mathbf{a_n} \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a_i}$, $\mathbf{m} \geq \mathbf{t}$ and $\mathbf{s} := (s_1, s_2, \dots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i\geq 1} im(i), \quad ||\mathbf{m}|| := \sum_{i\geq 1} m(i), \quad \mathbf{s}^\mathbf{m} := \prod_{i\geq 1} s_i^{m(i)}, \quad \mathbf{m}! := \prod_{i\geq 1} m(i)!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i\geq 1}\binom{m(i)}{t(i)}, \quad \binom{\mathbf{m}}{\mathbf{a_1}, \dots, \mathbf{a_n}} := \prod_{i\geq 1}\binom{m(i)}{a_1(i), \dots, a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i\geq 1}\kappa_i^{b(i)}.$$

We are interested in the following intersection numbers

$$\langle \kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \kappa(\mathbf{b})\psi_1^{d_1}\cdots\psi_n^{d_n}.$$

When $d_1 = \cdots = d_n = 0$, these intersection numbers are called higher Weil-Petersson volumes of moduli spaces of curves. The details of the following discussions are contained in [**17**].

The following lemma is a direct generalization of Lemma 2.2.

Lemma 3.2. — *Let $F(\mathbf{L},n)$ and $G(\mathbf{L},n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N}=\{0,1,2,\ldots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L}\in N^\infty$ that satisfy $\alpha_{\mathbf{0}}\beta_{\mathbf{0}}=1$ and*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}}\beta_{\mathbf{L}'}=0, \qquad \mathbf{b}\neq 0.$$

Then the following two identities are equivalent:

$$G(\mathbf{b},n)=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} F(\mathbf{L}',n+|\mathbf{L}|), \quad \forall\ (\mathbf{b},n)\in N^\infty\times\mathbb{N},$$

$$F(\mathbf{b},n)=\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_{\mathbf{L}} G(\mathbf{L}',n+|\mathbf{L}|), \quad \forall\ (\mathbf{b},n)\in N^\infty\times\mathbb{N}.$$

We may generalize Mirzakhani's recursion formula to include higher degree κ classes.

Theorem 3.3. — *There exist (uniquely determined) rational numbers $\alpha_{\mathbf{L}}$ depending only on $\mathbf{L}\in N^\infty$, such that for any $\mathbf{b}\in N^\infty$ and $d_j\geq 0$, the following recursion relation of mixed ψ and κ intersection numbers holds.*

$$\begin{aligned}
&(2d_1+1)!!\langle\kappa(\mathbf{b})\prod_{j=1}^n \tau_{d_j}\rangle_g\\
&=\sum_{j=2}^n \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g\\
&+\frac{1}{2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\ \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{L}')\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}\\
&\quad+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\ldots,n\}}}\ \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}}(2r+1)!!(2s+1)!!\\
&\qquad\qquad\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}$$

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}=0, \qquad \mathbf{b}\neq 0,$$

namely

$$\alpha_{\mathbf{b}}=\mathbf{b}!\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \mathbf{L}'\neq\mathbf{0}}}\frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}, \qquad \mathbf{b}\neq 0,$$

with the initial value $\alpha_{\mathbf{0}}=1$.

Theorem 3.4. — *We have*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2d_1+2|\mathbf{L}|+1)!!}{(2|\mathbf{L}|+1)!!}\langle\kappa(\mathbf{L}')\tau_{d_1+|\mathbf{L}|}\prod_{j=2}^{n}\tau_{d_j}\rangle_g$$
$$=\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{b})\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=|d_1|-2}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{b})\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=d_1-2}\binom{\mathbf{b}}{\mathbf{e}}(2r+1)!!(2s+1)!!$$
$$\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Theorem 3.3 and Theorem 3.4 implies each other through Lemma 3.2.

Both Theorems 3.3 and 3.4 are effective recursion formulae for computing higher Weil-Petersson volumes with the three initial values

$$\langle\tau_0\kappa_1\rangle_1=\frac{1}{24},\qquad \langle\tau_0^3\rangle_0=1,\qquad \langle\tau_1\rangle_1=\frac{1}{24}.$$

From the following Proposition 3.4, we have

$$\langle\kappa(\mathbf{b})\rangle_g=\frac{1}{2g-2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\kappa(\mathbf{L}')\rangle_g.$$

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}|\leq 15$ and have written a Maple program [**1**] implementing Theorems 3.3 and 3.4.

In fact, we find that ψ and κ classes are compatible in the sense that recursions of pure ψ classes can be neatly generalized to recursions including both ψ and κ classes by the same proof as Proposition 2.4. In view of Theorem 3.8 below, this can be rephrased as differential equations governing generating functions of ψ classes also govern generating functions of mixed ψ and κ classes.

We present some examples below.

Proposition 3.5. — *Let* $\mathbf{b}\in N^\infty$ *and* $d_j\geq 0$. *Then*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}(-1)^{||\mathbf{L}||}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_{|\mathbf{L}|+1}\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{L}')\rangle_g=(2g-2+n)\langle\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\rangle_g.$$

The above proposition is a generalization of the dilaton equation. In the special case $\mathbf{b}=(m,0,0,\ldots)$, it has been proved by Norman Do and Norbury [**5**].

Proposition 3.6. — *Let* $\mathbf{b}\in N^\infty$. *Then*

$$\langle\tau_0\tau_1\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\rangle_g=\frac{1}{12}\langle\tau_0^4\prod_{j=1}^{n}\tau_{d_j}\kappa(\mathbf{b})\rangle_g$$
$$+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b}\\ \underline{n}=I\coprod J}}\binom{\mathbf{b}}{\mathbf{L}}\langle\tau_0^2\prod_{i\in I}\tau_{d_i}\kappa(\mathbf{L})\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\kappa(\mathbf{L}')\rangle_{g-g'}.$$

The above proposition, together with the projection formula, can be used to derive an effective recursion formula for higher Weil-Petersson volumes [**17**] (without ψ classes).

Let $\mathbf{s}:=(s_1,s_2,\dots)$ and $\mathbf{t}:=(t_0,t_1,t_2,\dots)$, we introduce the following generating function

$$G(\mathbf{s},\mathbf{t}):=\sum_g\sum_{\mathbf{m},\mathbf{n}}\langle\kappa_1^{m_1}\kappa_2^{m_2}\cdots\tau_0^{n_0}\tau_1^{n_1}\cdots\rangle_g\frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}\prod_{i=0}^{\infty}\frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}}=\prod_{i\geq 1}s_i^{m_i}$.

Following Mulase and Safnuk [**21**], we introduce the following family of differential operators for $k\geq -1$,

$$V_k=-\frac{1}{2}\sum_{\mathbf{L}}(2(|\mathbf{L}|+k)+3)!!\frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}\mathbf{s}^{\mathbf{L}}\frac{\partial}{\partial t_{|\mathbf{L}|+k+1}}$$
$$+\frac{1}{2}\sum_{j=0}^{\infty}\frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}}+\frac{1}{4}\sum_{d_1+d_2=k-1}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}}$$
$$+\frac{\delta_{k,-1}t_0^2}{4}+\frac{\delta_{k,0}}{48}.$$

Theorem 3.7 **([17, 21])**. — *The recursion of Theorem 3.4 implies*

$$V_k\exp(G)=0.$$

Moreover, we can check directly that the operators V_k, $k\geq -1$ satisfy the Virasoro relations

$$[V_n,V_m]=(n-m)V_{n+m}.$$

The Witten-Kontsevich theorem states that the generating function for ψ class intersections

$$F(t_0,t_1,\dots)=\sum_g\sum_{\mathbf{n}}\langle\prod_{i=0}^{\infty}\tau_i^{n_i}\rangle_g\prod_{i=0}^{\infty}\frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy.

Theorem 3.8 **([17, 21])**. — *We have*

$$G(\mathbf{s},t_0,t_1,\dots)=F(t_0,t_1,t_2+p_2,t_3+p_3,\dots),$$

where p_k are polynomials in $\mathbf{s}$ *given by*

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{||\mathbf{L}||-1}}{\mathbf{L}!}\mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of $\mathbf{s}$, $G(\mathbf{s},\mathbf{t})$ *is a τ-function for the KdV hierarchy.*

At a final remark, it would be interesting to prove that $\alpha_{\mathbf{L}}$ in Theorem 3.3 are positive for all $\mathbf{L} \in N^{\infty}$. This problem is kindly pointed out to us by a referee.

More generally the question can be formulated as following: two sequences $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ with $\alpha_{\mathbf{0}} = \beta_{\mathbf{0}} = 1$ are said to be inverse to each other if they satisfy

$$\left(\sum_{\mathbf{L}} \alpha_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\right) \cdot \left(\sum_{\mathbf{L}} \beta_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}\right) = 1.$$

Find sufficient conditions on $\beta_{\mathbf{L}}$ such that $\alpha_{\mathbf{L}} > 0$ for all $\mathbf{L}$.

We conjecture that $\alpha_{\mathbf{L}}$ are positive when $\sum_{\mathbf{L}} \beta_{\mathbf{L}}\mathbf{s}^{\mathbf{L}}$ equals any of the following.

$$\sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|+1)!!}\mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!(2|\mathbf{L}|-1)!!}\mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{||\mathbf{L}||}}{\mathbf{L}!|\mathbf{L}|!}\mathbf{s}^{\mathbf{L}}.$$

The latter two arise when we consider Hodge integrals involving λ classes [**17**].

For works on the positivity criteria of coefficients of reciprocal power series of a single variable, see for example [**15**]. However it seems there is no literature dealing with the coefficients of reciprocal series of several variables.

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