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FLEXIBILITY OF SINGULAR EINSTEIN METRICS

by

Rafe Mazzeo

Dedicated to Jean Pierre Bourguignon on his 60th birthday.

Abstract. — This is a survey of a collection of related results about the deformation properties of Einstein metrics on a certain class of spaces with stratified singular structure. The results in low dimensions are particularly clean, and are motivated by applications in hyperbolic and convex geometry. The three-dimensional setting is related to an old conjecture by Stoker about flexibility of convex hyperbolic polyhedra, and we report on a partial answer. We also review some of the analytic methods used to prove these results.

***Résumé* (Flexibilité des métriques d'Einstein singulières).** — Cet article constitue un compte-rendu d'une collection de résultats autour des propriétés de déformation des métriques d'Einstein sur une certaine classe d'espaces à structure singulière stratifiée. Les résultats en basse dimension sont particulièrement intéressants, et ils sont motivés par des applications en géométrie hyperbolique et convexe. La configuration 3-dimensionnelle est reliée à une vieille conjecture de Stoker sur la flexibilité des polyèdres convexes hyperboliques et nous proposons une réponse partielle. Nous examinons également certaines méthodes analytiques utilisées pour démontrer ces résultats.

1. Introduction

The construction and study of canonical metrics on smooth Riemannian manifolds is a longstanding central theme in geometric analysis. The term 'canonical' can be interpreted in many ways; we shall take it here to mean Einstein, so we study metrics satisfying $\mathrm{Ric}^g = \lambda g$ for some constant λ. Beyond the basic existence questions, one of the main problems in this subject is to understand whether a given Einstein metric is rigid or flexible, i.e. admits nontrivial deformations amongst Einstein metrics.

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As a rule of thumb, negative curvature usually implies rigidity of while positive (or even just nonnegative) curvature often allows 'flexibility'. Our goal here is to discuss how Einstein metrics on a certain class of stratified singular spaces are sometimes flexible precisely because of the geometry of the singular set. There are several well-known instances of this, for example the classical problem of determining the flexibility of convex polyhedra in space forms, to which some of the theory discussed below is directly applicable. This provides one motivation for the more general study of Einstein metrics on stratified spaces proposed here.

This paper is intended as a brief survey of some small part of a broader subject, focusing on one interesting class of stratified spaces – the iterated cone-edge spaces – and presenting some recent results about the local deformation theory of Einstein metrics on these, particularly in low dimensions where it is closely related to many investigations in geometric topology concerning the class of 'cone-manifolds' introduced by Thurston. The new results reported here are parts of various ongoing collaborations by the author with Gregoire Montcouquiol, Frank Pacard and Hartmut Weiss, and are also very closely related to his work with Olivier Biquard. The intention is to indicate the beginnings of a coherent 'story', and one which seems worthy of further development, albeit from a very personal point of view. Due to limitations of space and the author's expertise, we do not touch on many interesting situations where singular Einstein metrics have already been studied by others, e.g. for metrics with special holonomy, particularly in complex geometry. Finally, we also do not discuss any global aspects of this moduli problem, in particular the compactification theory, though this is likely to be both important and very interesting.

Let us first mention a few facts about Einstein metrics on smooth manifolds. Recall that a deformation of an Einstein metric g is a (smooth) one-parameter family of metrics g_t with $g_0 = g$; it is called a trivial deformation if there exists a one-parameter family of diffeomorphisms ϕ_t of the underlying manifold such that $g_t = \phi_t^* g_0$. In other words, the moduli space $\mathcal{E}(M)$ of Einstein metrics on a given manifold M is the space of all metrics satisfying the Einstein condition modulo diffeomorphisms. (Just as for surfaces, one may mod out by all diffeomorphisms or by those isotopic to the identity, but since our focus is on local aspects of the deformation problem, we do not emphasize this here.) As usual, it is more convenient to study an auxiliary equation whose solution space yields all (nearby) Einstein metrics without the diffeomorphism redundancy; this is done by introducing an auxiliary gauge condition to make the problem elliptic; we describe this later. There is a well-known result due to Koiso [**22**] which states that if M is compact, then $\mathcal{E}(M)$ is a finite dimensional analytic set. (This means that it can be covered by neighbourhoods, each of which is identified by a real analytic diffeomorphism with the zero set of an analytic function in a finite dimensional Euclidean space.) The subtlety in proving this, and the reason that its conclusion is not more specific (e.g. with respect to the dimension or smoothness of the moduli space) is that when the manifold is compact, the deformation theory is either trivial or obstructed. Indeed, one standard approach to such a result is to apply an implicit function theorem, for which one needs surjectivity of the linearization of the relevant operator, and if this holds, then the space of solutions of the nonlinear geometric problem is locally parametrized by elements of the nullspace of this linearization. (We describe this in greater detail in §6 below.) This linearization is a self-adjoint elliptic operator, so when M is compact, its surjectivity is equivalent to its injectivity. Thus if the linearization is surjective, it is injective too and the Einstein metric is rigid; on the other hand, if the linearization has nontrivial nullspace, then it has cokernel too, so the implicit function theorem does not directly apply. There is a standard trick to handle situations of this sort, known as Ljapunov-Schmidt reduction, but one can then only deduce much less precise conclusions.

Despite the fact that the 'formal dimension' of the moduli space of Einstein metrics on a compact manifold is zero, there are many manifolds M for which $\mathcal{E}(M)$ is positive dimensional and sometimes even smooth. The best known examples in higher dimensions are the families of flat tori, and less trivially, the family of Calabi-Yau K3 surfaces, for which very detailed results may be obtained using algebraic geometric techniques. Also worthy of note are the recent results of [**11**] about the existence of smooth high dimensional families of Einstein metrics on the sphere, all far from the standard metric, obtained using an integrable systems approach. On the other hand, as suggested above, if M is compact and the sectional curvatures of g are everywhere nonpositive, and negative somewhere, then g is rigid; this can be proved using the Bochner technique. The special case where (M, g) is locally symmetric was proved in various settings of increasing generality by Weil, Calabi, and Matsushita-Murakami. For quite different reasons it is known that the standard metric on the sphere is also rigid. We refer to the outstanding expository monographs [**6**], [**21**], and the collection [**23**], for more about these facts and their proofs.

When the manifold is noncompact or incomplete, this rigidity or deformation theory has a very different flavour. At one end is the study of the asymptotic boundary problems associated to Einstein metrics with certain types of asymptotically symmetric geometries, in particular the much-studied case of asymptotically hyperbolic Einstein metrics (also called Poincaré-Einstein metrics), see [**17**], [**24**], [**2**], as well as the work by Biquard on complex and quaternionic analogues, [**8**], [**7**], and more recently, some 'higher rank' analogues studied by the author in collaboration wtih Biquard, [**9**], [**10**]. As the name 'asymptotic boundary problem' suggests, complete Einstein metrics with these various types of asymptotic conditions come in infinite dimensional families, and the emphasis changes to parametrizing them by some appropriate type of boundary data, which in these cases are the associated parabolic geometries on the ideal boundary at infinity. The parabolic geometry associated to an asymptotically real hyperbolic Einstein metric is a conformal structure on the boundary of the geodesic compactification. The classical analogue of this, when $\dim M = 3$ and M is a quasiFuchsian convex cocompact hyperbolic manifold, was developed by Ahlfors and Bers; here, hyperbolic structures satisfying these hypotheses are in bijective correspondence with the space of conformal structures on the boundary at infinity, which is then a compact surface with two components. In the asymptotically complex or quaternionic settings, the parabolic geometries are the CR and quaternionic CR structures; for the higher rank cases, the relevant asymptotic boundary structures are somewhat less familiar but quite explicit, see [**9**]. There is also some recent progress by Anderson on the boundary problem in the usual sense for incomplete Einstein metrics on manifolds with boundary [**3**].

Of a different nature is the study of Einstein metrics which are asymptotically locally Euclidean (ALE), or which satisfy other more intricate but related asymptotic conditions, but which in any case are complete and have polynomial volume growth. Almost all known examples of these are metrics with restricted holonomy group, e.g. Kähler-Einstein or even hyperKähler, and that extra structure provides a substantial key to unlocking their properties. These have only finite dimensional deformation spaces, which are in some cases very well understood; we refer again to [**21**] for more on this.

On the other hand, there does not seem to have been any systematic study of Einstein metrics on various classes of spaces with 'geometrically structured' singularities, e.g. manifolds with conic points, edges and iterated edges, or more general stratified spaces, despite their ubiquity 'in nature'. As indicated above, we focus on the local rigidity/flexibity question, and in particular how geometric data at the singular locus can provide at least some of the moduli parameters. There is nothing approaching a comprehensive understanding of this phenomenon yet; rather, we simply present several recent results in this area in order to explain what is possible with current techniques and to emphasize this as an interesting area of study.

To be more specific, we first recall a particular class of Riemannian stratified spaces obtained by an iterated coning procedure and a class of Riemannian metrics on their principal smooth strata which induce metrics on each of the substrata. The general problem we pose is to study Einstein metrics in this class of singular spaces. In successive sections we consider this problem in the two, three, and higher-dimensional settings. Not surprisingly, the results are of decreasing specificity. The case of conic surfaces is certainly well-motivated through its association with marked Teichmüller theory, and serves as an excellent test-case for refining techniques for the more general settings. The results on this discussed here are joint work with H. Weiss. The three-dimensional case has also appeared in many other guises before, in particular through the study of 'cone-manifolds' (or conifolds as we shall call them) by Thurston and many others, cf. the exposition in [**14**]. There is another application, however, concerning deformations of three-dimensional convex hyperbolic polyhedra; one result discussed here, obtained recently with G. Montcouquiol, answers the infinitesimal version of an old question due to Stoker in polyhedral geometry, and is of independent interest to that community. The local version of this same question has been treated more recently, using methods from geometric topology, independently by Montcouquiol and Weiss. There is quite a large literature about various aspects of these three-dimensional problems, however, and we shall mention only a few other related results. Finally, the situation in higher dimensions is much less complete; we discuss one result concerning isolated conic singularities, with Pacard and Weiss, and another (very special) set of examples of Einstein metrics bending along codimension two edges, but can mostly point to what are likely to be the tractable interesting directions. The final section contains some discussion of the analytic underpinnings of the proofs of these results: first, a reminder of one convenient gauge choice, and second, an overview of the analysis of elliptic operators in the conic and iterated edge settings. I am grateful to all these collaborators for allowing me to report on these ongoing projects here. I have also learned much from conversations with Steve Kerckhoff and Igor Rivin, and through my long-standing collaboration with Frank Pacard. Finally, the referee provided some very helpful comments about the exposition and relevant literature.

2. Iterated cone-edge spaces

Let (N, h) be a compact stratified Riemannian space with top-dimensional stratum an open dense subset. We refer to [**35**] for generalities on stratified spaces; by Riemannian we mean that each stratum S carries a Riemannian metric h_S, which extends smoothly to the closure of this stratum, and that this collection of metrics satisfies the obvious compatibility relationships: if S_1 and S_2 are any two strata with $\iota : S_1 \hookrightarrow \overline{S_2}$, then $\iota^* h_{S_2} = h_{S_1}$. We are interested in the subclass consisting of iterated cone-edge spaces; these are spaces obtained locally by an iterated coning process, starting from smooth compact manifolds. First, recall that the (complete) cone over N, $C(N)$, is the space $([0,\infty)_r \times N) \,/ \sim$, where $\sim$ is the equivalence relation collapsing $\{0\} \times N$ to a point, endowed with the metric $dr^2 + r^2 h$. The truncated cone where $r \leq 1$ is denoted $C_1(N)$. Any singular stratum $S \subset N$ induces a singular stratum $C(S)$ in $C(N)$, with $\dim C(S) = \dim S + 1$. Now we can make the

Definition 2.1. — *We define, for each $k \geq 0$, the class $\mathcal{I}_k$ of compact iterated cone-edge spaces of depth k. This is done by induction on k. An iterated cone-edge space of depth 0 is a compact smooth manifold. A stratified space M lies in $\mathcal{I}_k$ if for any $p \in M$, if S is the open singular stratum containing p and $\dim S = \ell$, then there exists a neighbourhood $\mathcal{U}$ of p in M diffeomorphic to the product $\mathcal{V} \times C_1(N)$ where $\mathcal{V} \subset \mathbb{R}^\ell$ is an open Euclidean ball diffeomorphic to a neighbourhood in S and $N \in \mathcal{I}_j$ for some $j < k$. We assume furthermore that the integer $n = \ell + \dim C(N)$ is independent of the point $p \in M$; this number is called the dimension of M.*

If $\dim S > 0$, then we say that the stratum S is an edge in M with link N; some neighbourhood of S in M is diffeomorphic to a bundle of cones over S with fibre $C(N)$. If $\dim S = 0$, then we call it a conic point, but note that if N is itself singular, then there are edges of lower depth which terminate at this point.

An iterated cone-edge metric g on M is by definition one which respects this diffeomorphism, i.e. is locally quasi-isometric to one of the form $g \sim dr^2 + r^2 h + \kappa$, where h is an iterated cone-edge metric on N and κ is a metric on S.

To simplify the name a bit, we shall often call these iterated edge spaces. They are much simpler than general stratified spaces, both geometrically and analytically. To our knowledge, they were first singled out for the tractability of analysis on them in Cheeger's famous paper [**12**].

We shall be discussing Einstein metrics on iterated edge spaces, but one should note that the precise definition of an Einstein metric on such a singular space is not necessarily clear. Obviously any such metric g should be Einstein on the principal open stratum, but it is not clear whether one should also require special conditions on the restrictions of these metrics to the lower dimensional strata. This might be clarified, for example, by examining what it means for a metric on an iterated edge space to be critical for the Einstein-Hilbert action. In the low dimensional cases we shall be focusing on mostly, this issue does not arise, while the higher dimensional examples discussed in §5 are so special that they are not necessarily indicative of the general case. In any case, this seems like an important issue to clarify.

3. Surfaces with conic singularities

The simplest setting for our general problem is the existence and deformation theory of compact constant curvature surfaces with isolated conic singularities. This can be approached by various different methods, but we follow one modelled on the presentation developed by Tromba [**41**] to study Teichmüller theory on compact surfaces without singular points since it generalizes to higher dimensions more readily.

We first recall some facts about 'marked Teichmüller theory'. Let M be a compact, oriented two-dimensional surface with genus γ. Any conformal class $[g]$ on M contains a constant curvature metric g_0 which is unique after some choice of normalization (when $\gamma > 1$, it is unique if we fix the curvature to be -1; on the other hand, requiring that the area equals 1, say, yields a unique solution for $\gamma \geq 1$; for $\gamma = 0$ there is the usual nonuniqueness due to the Möbius group). For $\gamma \geq 1$, the genus γ Riemann moduli space $\mathcal{R}_\gamma$ is thus identified with the space of all constant curvature metrics (completed in some Banach topology) with area 1 modulo the space of all diffeomorphisms (of appropriate regularity); the genus γ Teichmüller space $\mathcal{T}_\gamma$ is the quotient of the same space of metrics by the identity component of this group of diffeomorphisms, i.e. the subgroup of diffeomorphisms which are isotopic to the identity. Finally, the marked Teichmüller space $\mathcal{T}_{\gamma,k}$ is the quotient of the same space of metrics by the still smaller subgroup of diffeomorphisms which are isotopic to the identity and which fix a specified collection of points $\{p_1, \ldots, p_k\} \subset M$. When $\chi(M) - k < 0$, it again follows from the classical uniformization theorem that in each marked conformal class there is a unique complete, hyperbolic, finite area metric. When $\chi(M) - k = 0$, this uniformizing metric is flat. These metrics are the ones most commonly associated to marked conformal structures.

Another choice of canonical metric in this setting is obtained as follows: given a conformal class $\mathfrak{c}$ on M, a collection of distinct points $\{p_1, \ldots, p_k\} \subset M$, and a collection of positive numbers $\{\alpha_1, \ldots, \alpha_k\}$, find a metric with constant curvature K on $M \setminus \{p_1, \ldots, p_k\}$ which has an isolated conic singularity at each p_j, with specified cone angle $2\pi\alpha_j$ there.

In two dimensions, the local geometry of a constant curvature metric around a conic point is quite simple. Define the function $\operatorname{sn}_{\mathrm{K}}(r)$ to be the unique solutions to the initial value problem $f'' + \mathrm{K}f = 0$ satisfying $\operatorname{sn}_{\mathrm{K}}(0) = 0$, $\operatorname{sn}'_{\mathrm{K}}(0) = 1$. Then the metric

$$g = dr^2 + \operatorname{sn}^2_{\mathrm{K}}(r)\, dy^2, \qquad 0 < r < r_0, \quad y \in \mathbb{R}/2\pi\alpha$$

is a two-dimensional conic metric with curvature K and cone angle $2\pi\alpha$; when $\mathrm{K} \leq 0$ we can take $r_0 = \infty$, while $r_0 < \pi/\sqrt{\mathrm{K}}$ when $\mathrm{K} > 0$. There is another useful representation,

$$g = e^{2\phi}|z|^{2\beta}|dz|^2, \qquad \alpha = 1 + \beta,$$

in local holomorphic coordinates near 0 in the disk in $\mathbb{C}$; here ϕ is some explicit function (which equals 0 when $\mathrm{K} = 0$). More generally, if ϕ is any reasonably smooth function, we say that a metric of this form has isolated conic singularity at 0 with cone angle $2\pi\alpha = 2\pi(1 + \beta)$.

This existence problem translates into finding a solution of the following semilinear elliptic PDE. Let $\overline{g}$ be any fixed metric in the conformal class $\mathfrak{c}$, and fix a function

$Z(p_1, \dots, p_k, \beta_1, \dots, \beta_k)$ which depends smoothly on the p_j and β_j, is everywhere positive and smooth away from the p_j, and such that near each of these points, in a local holomorphic coordinate z, equals $|z|^{2\beta_j}|dz|^2$. Now write

$$\widehat{g} = Z(p_1, \dots, p_k, \beta_1, \dots, \beta_k)\overline{g};$$

the metric we seek can be written

$$g = e^{2\phi}\widehat{g}$$

and it has curvature $K_g \equiv \mathrm{K}$ if and only if

$$\Delta_{\widehat{g}}\phi - K_{\widehat{g}} + \mathrm{K}e^{2\phi} = 0. \tag{3.1}$$

The solvability of (3.1) in general, i.e. for arbitrary values of the cone angle parameters $\beta_j > -1$, is not known. One immediate constraint is obtained by applying the Gauss-Bonnet formula to the surfaces with boundary $M \setminus \cup_j B_\epsilon(p_j)$ and letting $\epsilon \searrow 0$; this shows that if a solution exists, then

$$\mathrm{K} \times \mathrm{Area}\,(M, g)) = 2\pi(\chi(M) + \sum_{j=1}^{k} \beta_j). \tag{3.2}$$

Since the term on the right, which we call the conic Euler characteristic, changes sign as the cone angles vary, it is more convenient to fix the area and let K be determined by (3.2). Solutions are obtained easily when $\mathrm{K} \leq 0$ using barrier techniques, but we pass out of this regime as soon as some of the β_j become sufficiently large. There is a complete existence theory when $\mathrm{K} > 0$ only if we restrict each β_j to lie in the interval $(-1, 0)$, corresponding to each cone angle lying in the interval $(0, 2\pi)$.

Theorem 3.1. — *Suppose that each $\beta_j \in (-1, 0)$. Then there is a solution of (3.1) if and only if for each $i = 1, \dots, k$,*

$$\chi(M) + \sum_{j \neq i} \beta_j < 2 + \beta_i, \tag{3.3}$$

and moreover, if we require its area to equal 1, *then this solution is unique. The Gauss curvature* K *of this solution is equal to the conic Euler characteristic $\chi(M) + \sum \beta_j$.*

Notice that by adding β_i to each side, (3.3) is equivalent to

$$\chi(M) + \sum_{j=1}^{k} \beta_j < 2(1 + \beta_i);$$

since $\beta_i \in (-1, 0)$, the right hand side is always positive, so this condition presents a genuine obstruction only when conic Euler characteristic is positive. Existence and uniqueness when $\mathrm{K} \leq 0$ is due to Troyanov [**42**] and also McOwen [**31**]; Troyanov used variational methods and was also able to obtain existence in the spherical case $(\mathrm{K} > 0)$, assuming (3.3). Later, Luo and Tian [**25**] proved that (3.3) is necessary and that the solution obtained by Troyanov is unique. We shall say that the k-tuple $(\beta_1, \dots, \beta_k) \in (-1,0)^k$ lies in the Troyanov region if it satisfies (3.3).

There are a few results concerning existence and uniqueness when some of the cone angles are larger than 2π, cf. [**16**], [**44**], but the situation is still far from being well understood. A very interesting recent survey paper by Troyanov [**43**] provides a lot of information about the flat case.

Although it is implicit in these existence proofs that the solutions in Theorem 3.1 depend smoothly on the underlying parameters, i.e. the marked conformal structure and cone angles, it is still of interest to understand the way in which all these metrics fit together. There are some analytic subtleties, and overcoming them in this context is good preparation for understanding the higher dimensional situation. Furthermore, it is hoped that these methods will eventually produce a much better picture of the existence theory when the cone angles are larger than 2π. This was carried out several years ago in joint work with Hartmut Weiss [**30**] (but only now finally being written). The basic result is the

Theorem 3.2. — *Let M be a compact orientable surface, as above. Let $\mathcal{T}^{\text{conic}}_{g,k}$ denote the space of all constant curvature metrics on M with area equal to 1 and with conic singularities at k distinct points on M with cone angles $2\pi(1+\beta_j)$, and $\mathcal{T}^{\text{conic}}_{g,k,o}$ the subset where the k-tuple $(\beta_1, \dots, \beta_k)$ satisfy the Troyanov constraint (3.3) and $\beta_j \in (-1,0)$ for all j. Then $\mathcal{T}^{\text{conic}}_{g,k,o}$ is a smooth open manifold of dimension $6g-6+3k$; it contains as an open submanifold the subset of metrics with negative curvature, and as a hypersurface the subset of flat conic metrics.*

The complete result contains other statements about the limiting behaviour of these metrics as $(\beta_1, \dots, \beta_k)$ approaches the boundary of the Troyanov region; we refer to [**30**] for more details.

The proof involves constructing coordinate charts for this space, which we do by regarding its elements as satisfying the Einstein equations (just the constant Gauss curvature equation in this dimension) along with an auxiliary gauge condition. The new feature, however, is that these equations are singular at the conic points, so one must substitute other techniques to handle them. The gauge condition and some discussion of elliptic theory adapted to conic spaces will be given at the end of this paper.

There are several intriguing open questions. First, although there is no constant curvature metric when the cone angles are still less than 2π but the β_j lie outside the Troyanov region, is there still some sort of canonical metric with these specified cone angles? Many years ago, Tian suggested that in these cases the canonical metric should be a Ricci soliton with prescribed cone angles; there has been no good progress on this yet. Second, when extending this result into the region where some of the cone angles are greater than 2π, it is likely that one will need to face the issue of bifurcations; for example a conic point with cone angle 3π may split into either two or three points. It will be interesting to put this on solid analytic footing.

4. Conifolds in dimension 3

Iterated cone-edge spaces with constant curvature (or more generally, with a (G, X) structure) were introduced by Thurston as a generalization of orbifolds. He called these 'cone-manifolds', but this is not a very satisfactory name, so we opt for the alternate moniker 'conifold'. Thus, for us, a conifold is an iterated cone-edge space (M, g) such that the induced metric g_S on any stratum has constant sectional curvature K, and each stratum is totally geodesic in an appropriate sense in all higher dimensional ones for which it lies in the frontier. As in the surface case, we call a conifold hyperbolic, flat or spherical depending on whether K is negative, zero or positive. We restrict attention to the 3-dimensional case, and mainly the hyperbolic case. This has been intensively studied due to many applications to the theory of smooth hyperbolic 3-manifolds, stemming from Thurston's proof of the orbifold theorem and various hopes to use similar methods to prove the full hyperbolization theorem. The monograph [**14**] provides a good introduction.

The singular locus of M, denoted Σ, is a union of 1- and 0-dimensional strata which constitute the edges and vertices of a graph (with the slightly nonstandard convention that it may have components which are closed loops). Near each edge of the singular locus, M is a bundle of cones with cone angle constant along that edge; this bundle is trivial unless the edge is a closed loop. Near each vertex, M is identified with the cone over a space Y, which is a copy of S^2 with k conic points, where k is the valence of that vertex. (Thus the edges of Σ are depth 1 singularities, while its vertices are depth 2 singular points.) We shall denote the vertex set of Σ by $\mathcal{V}$ and its edge set by $\mathcal{E}$. and we write the valence of a vertex v as the integer $n(v) \geq 3$.

In a neighbourhood of the interior of any edge of Σ, the metric g has a standard form; this involves the function $\mathrm{sn}_{\mathrm{K}}(\rho)$ used in the surface case, and its companion, $\mathrm{cs}_{\mathrm{K}}(\rho)$, which is the unique solution to $f'' + \mathrm{K}f = 0$, $\mathrm{cs}_{\mathrm{K}}(0) = 1$, $\mathrm{cs}'_{\mathrm{K}}(0) = 0$. Now, with ρ equal to the distance from that edge (in a sufficiently small neighbourhood), we have

$$g = d\rho^2 + \mathrm{sn}^2_{\mathrm{K}}(\rho)\, dy^2 + \mathrm{cs}^2_{\mathrm{K}}(\rho)\, dt^2, \qquad y \in \mathbb{R}/2\pi\alpha = S^1_{2\pi\alpha}, \quad t \in (-a, a). \tag{4.4}$$

We call this the constant curvature cylinder with cone angle $2\pi\alpha$. On the other hand, near a vertex $p \in \mathcal{V}$, M is a constant curvature cone over a spherical cone surface (N, h), so g has the form

$$dr^2 + \mathrm{sn}^2_{\mathrm{K}} r\, h, \tag{4.5}$$

where r is the distance to the vertex. Of course, h in turn has the form described in the previous section with $\mathrm{K} = +1$ near each one of its singular points. Note that the cone points of each link N correspond to edges of M.

Now let us identify natural geometric parameters. These are of two types: along each edge e there is the cone angle $2\pi\alpha(e)$ (or equivalently, the parameter $\beta(e) = \alpha(e) - 1$), the length $\ell(e)$ of the edge and also a certain twist parameter $\tau(e)$, which will be described below; at each vertex v the parameter is in fact the spherical cone metric on S^2 with $n(v)$ conic points, i.e. an element of $\mathcal{T}^{\mathrm{conic}}_{0,n(v)}$. Hence the total set of free parameters lies in some subset of the space

$$(0,\infty)^{|\mathcal{E}|}_{\alpha} \times (0,\infty)^{|\mathcal{E}|}_{\ell} \times (0,\infty)^{|\mathcal{E}|}_{\tau} \times \prod_{v\in\mathcal{V}} \mathcal{T}^{\mathrm{conic}}_{0,n(v)}.$$

There are some obvious constraints: the cone angle parameter $\alpha(e)$ associated to each edge e determines the angles at the cone points of the spherical links at the terminal vertices of that edge. The length and twist parameters do not satisfy any such 'local' constraints, nor apparently does the marked conformal structure on each spherical link. We denote the set of parameters satisfying these 'obvious' constraints by $\mathcal{P}$. For the same reasons as in the last section, namely the poor understanding of spherical cone surfaces with cone angles larger than 2π or outside the Troyanov region, we restrict attention to the subset $\mathcal{P}_o \subset \mathcal{P}$ where the cone angles satisfy (3.3) and are all less than 2π.

To each element $\zeta = (\alpha(e), \ell(e), \tau(e), h(v)) \in \mathcal{P}_o$, where $\alpha(e), \ell(e)$ and $\tau(e)$ are the cone angle, length and twist parameters associated to each edge $e \in \mathcal{E}$ and $h(v)$ is the spherical cone metric with $n(v)$ conic points in S^2 associated to each vertex v, we can associate a local conifold 'thickening' of the graph Σ as follows. First define the cones with constant curvature K over each spherical cone metric $h(v)$, $v \in \mathcal{V}$ by the formula (4.5), for $0 < r < r_0$. Next, over each edge $e \in \mathcal{E}$ construct the cylindrical metric (4.4), again only up to some small radius. Take the core geodesic of this cylinder to have length $\ell(e)$. The twist parameter $\tau(e)$ provides a way of measuring how these cylinders are attached at either end. It is only a relative parameter unless the edge e is a closed loop, but in that case it is equal to the holonomy around that loop. Let us call this thickened graph the singular germ associated to the data ζ (and with curvature K), and denote it by $\mathcal{S}(\zeta)$.

Here are the two main questions:

i) Which singular germs $\mathcal{S}(\zeta)$ arise as the restriction to a neighbourhood of the singular set Σ of a compact conifold; alternately, to which elements of $\mathcal{P}_o$ does $S(\zeta)$ extend to a compact conifold?
ii) Given a compact conifold (M, g) of curvature K, let $S(\zeta_0)$ denote the associated singular germ. Describe the local, or even just the infinitesimal, structure of

the space of nearby conifolds with the same curvature, or equivalently, of their geometric parameter sets.

Problem i) is much more subtle, and we do not have anything to say about it here. Problem ii), on the other hand, can be treated by analytic methods, much like for constant curvature conic surfaces.

Before proceeding, we describe one special case which is of interest in polyhedral geometry. Let A be a polyhedron in either the sphere S^3, Euclidean space $\mathbb{R}^3$ or hyperbolic space $\mathbb{H}^3$. Then A has a set of edges and vertices and its faces are totally geodesic. At each edge e we may associate a dihedral angle, $\delta(e)$, which is the inner angle between the two faces meeting at that point; similarly, at each vertex v we may associate a 'solid angle', which is a spherical polygon $B_v \subset S^2$ consisting of the set of interior normal directions at v (i.e. it is just the spherical link). The polyhedron A has other geometric parameters as well, namely the lengths of each edge, but in this context there is no twist parameter since there is a unique way of choosing a wedge with opening angle $\delta(e)$ along a geodesic of length $\ell(e)$, and the local structure at each vertex is obtained uniquely by intersecting these wedges associated to all incoming edges.

To pass from such a polyhedron to a conifold, double A simultaneously across all of its faces. Since these faces are totally geodesic, the resulting space M is singular only along a 1-skeleton. Its angle along each edge is given by angle $2\pi\alpha = 2\delta(e)$, while its link at each vertex v is the double of the spherical polygon B_v, hence a spherical cone surface. This conifold has a natural involution, for which A is a fundamental domain. Convexity is a natural condition; if A is a convex polyhedron, then each $\delta(e) < \pi$, hence the cone angles along each edge in the conifold M are all less than 2π. Furthermore, these cone angles lie in the Troyanov region simply because the link at each vertex is a spherical cone surface with all angles less than 2π, and by Luo-Tian [**25**], this can only exist when its cone angles satisfy (3.3). We have therefore proved that if the conifold M is the double of a convex polyhedron, then its geometric parameters lie in $\mathcal{P}_o$.

In an influential 1967 paper [**40**], J.J. Stoker studied the flexibility of convex polyhedra in $\mathbb{R}^3$, and made the conjecture that the dihedral angles of a convex polyhedron determine the angles in each face. (The polyhedron itself is not determined even up to homothety since translating any face parallel to itself leaves all dihedral angles unchanged.) This has become known as the Stoker conjecture. The analogous conjecture in hyperbolic space, that convex polyhedra in $\mathbb{H}^3$ are determined by their dihedral angles, was made explicit by Igor Rivin in his thesis. (Note the stronger statement than in the Euclidean setting; one does not have the same ambiguity from parallel translation of the faces.) Andreev [**4**] settled this when all dihedral angles are less than $\pi/2$, and it was proved by Rivin for ideal polyhedra [**37**] and later by Bao and

Bonahon for hyperideal polyhedra [**5**], with further extensions by Schlenker [**39**]. In the restricted setting of ideal and hyperideal polyhedra, the parameter space is convex, but one of the main difficulties in the general case is that this is no longer true, see [**15**]. There are counterexamples for spherical polyhedra due to Schlenker [**38**], and it is known that any corresponding assertion for conifolds will be more complicated, see [**20**].

One good place to start is to study the infinitesimal or local version of this conjecture, either for polyhedra or conifolds, and for this, analytic methods turn out to be very well suited. One can state the infinitesimal conjecture in the hyperbolic setting as follows:

> If (M, g_t) is a smooth one-parameter family of *hyperbolic* conifold structures with geometric parameters lying in $\mathcal{P}_o$ which preserves the cone angles at each edge to first order, then there is a one-parameter family of diffeomorphisms ϕ_t of the stratified space M such that $g_t - \phi_t^* g_0$ vanishes to second order.

Said more plainly, any nontrivial infinitesimal variation of conifold structures includes a nontrivial infinitesimal variation of some of the dihedral angles; likewise, in any nontrivial variation of convex polyhedra in $\mathbb{H}^3$, the set of dihedral angles must vary. The conjecture in the Euclidean setting is slightly more intricate since it must allow for the phenomenon of families of nonisometric polyhedra with the same dihedral angles which are obtained by parallel translations of the faces.

Several papers in the last decade have addressed special cases. The first, by Hodgson and Kerckhoff [**19**], concerns the case of hyperbolic conifolds with singular set a finite union of loops (hence, no vertices), and they settled the infinitesimal and local conjectures for cone angles less than 2π. More recently, Weiss [**45**] in his thesis generalized their methods to prove the same result for conifolds for which the singular set is allowed to have trivalent vertices and all cone angles are less than π. Some other nice results in this direction have been obtained by Porti and Weiss [**36**] and Huesener, Porti and Suarez [**18**].

The point of view of all of these is to study this from the point of view of deforming representations of the fundamental group into the Möbius group. However, it is also possible to approach these problems using methods from global analysis similar to those used in other dimensions, and this has led to the following result by the author in collaboration with G. Montcouquiol:

***Theorem 4.1* (Infinitesimal conifold Stoker conjecture [27]).** — *Let (M, g) be a hyperbolic conifold with parameters lying in $\mathcal{P}_o$. Then any nontrivial variation of g amongst hyperbolic conifolds changes at least one cone angle to first order.*

More recently still, Montcouquiol and Weiss, independently, have established a local (rather than infinitesimal) result. One formulation is that there is a local parametrization of the set of hyperbolic conifold structures in some neighbourhood of (M, g) by an analytic set (i.e. the zero set of an analytic function) in an ϵ-ball B_ϵ in the space of cone angle parameters $(\beta_1, \ldots, \beta_k)$ around those of g. In other words, all nearby conifolds are parametrized by letting β vary in some analytic subset of the space of cone angles. Both of these authors use techniques similar to the ones employed earlier by Hodgson-Kerckhoff, Weiss, et al.; it is quite likely that the approach of [**27**] can be extended to handle this as well, but this is still work in progress.

There are analogous results in the Euclidean case and also in both the infinitesimal and local setting for convex hyperbolic polyhedra, but we shall not state any of these explicitly here. One subtle point is that if A is a convex hyperbolic polyhedron and (M, g) its conifold double, then there may be conifold variations of (M, g) which are not doubles of hyperbolic polyhedra. This would be very interesting to understand better. The full Stoker conjecture in the polyhedral or conifold setting (either Euclidean or hyperbolic) remains open. A substantial new difficulty which must be faced in the global problem for conifolds is that as the cone angles vary, the topology of the singular set might be forced to change. For example, under a family of deformations, edges might shrink and disappear, or conversely, be generated and grow, or disjoint 'skew' edges might move toward each other and touch. As in the surface case, it is also important to try to push these techniques and results to when the cone angles are larger than 2π.

5. Higher dimensions and codimensions

It is possible to obtain reasonably explicit results about local deformation theory for singular Einstein metrics in low dimensions simply because these metrics have constant sectional curvature. This allows the analytic problem to be reduced to a finite dimensional one. In higher dimensions the situation is quite different. Even though the gauged Einstein equation seems formally well-posed, it becomes highly overdetermined on an iterated cone-edge space, at least near edges of positive dimension, and with codimension at least two. Because of this, very few singular Einstein spaces with interesting singular sets are known in higher dimensions. In this section we first describe the local structure theory of the space of Einstein metrics with isolated conic singularities in general dimensions, and then go on to discuss a few examples of Einstein metrics with higher dimensional singular sets. These examples have a lot of symmetry, and although it is reasonable to think that there might be many other Einstein metrics with similar singular structure, this is quite unknown and seems to be a very difficult problem. We do not discuss any examples where the singular set is itself stratified.

A standard computation, see [6], shows that the exact conic metric $g = dr^2 + r^2 h$ on $\mathbb{R}^+ \times N$ is Einstein if and only if the link (N, h) is itself Einstein, with $\mathrm{Ric}^h = (n-2)h$ (where $\dim N = n-1$). This generalizes the standard picture of $\mathbb{R}^n$ with its flat metric as a cone over the sphere with unit radius; cones over spheres of other radii are no longer even Ricci flat (except when $\dim N = 1$; in this case the condition on the link is satisfied by a circle of any radius). Based on this, we see immediately that Einstein deformations of the cone $C(N)$ can be obtained by deforming the link (N, h) in its own Einstein moduli space. If $\dim N = 3$, the link is either the sphere or a spherical space form, neither of which admits Einstein deformations; on the other hand, when $\dim N \geq 4$, it is sometimes possible to obtain a finite dimensional family of Einstein cones this way. More generally, if (M, g) is an Einstein space with isolated conic singularities $p_1, \ldots, p_k$, and if $n = \dim M \geq 5$, then denote by (N_j, h_j) the link at p_j and $\mathcal{E}(N_j)$ the Einstein deformation space of this link. We shall need to impose an extra integrability condition: for each such link, suppose that κ is an infinitesimal Einstein deformation on the entire cone $C(N_j)$ which is homogeneous of degree 0 with respect to radial dilations. Then we assume that κ is the derivative of a one-parameter family of conic Einstein metrics.

Theorem 5.1. — *Let (M, g) be as above, and suppose that the integrability condition is satisfied at each p_j. If the sectional curvature of (M, g) is nonpositive and negative somewhere, then the local Einstein deformation space can be identified with an analytic subset in the product $\prod_j \mathcal{E}(N_j)$. If this curvature condition is not satisfied, the local Einstein deformation space is contained in an analytic subset in the larger space $\prod_j \mathcal{E}(N_j) \times \mathbb{R}^\ell$, where ℓ is the dimension of the space of infinitesimal Einstein deformations which decay at each p_j.*

The integrability condition is a bit of a surprise. It holds in all known situations, but seems to be necessary, at least using our approach.

This theorem, joint with Frank Pacard and Hartmut Weiss [28], is a direct analog to the two-dimensional case, but with the important proviso that we know very little about the spaces $\mathcal{E}(N_j)$ beyond the fact that they too are finite dimensional analytic spaces. This last fact is a classical result due to Koiso, cf. [6]. The second result in this theorem, about the deformation space in the 'degenerate' situation where there are decaying infinitesimal Einstein deformations, follows by a standard adaptation of the proof of the first part, using Ljapunov-Schmidt reduction.

One motivation for studying this type of singular Einstein space is the fact that Einstein metrics with conic singularities arise naturally as limits in the compactification theory of the Einstein moduli space in four dimensions. This is due originally to

Anderson [**1**] and Nakajima [**33**], but see the more recent work by Cheeger and Tian [**13**]. The precise mechanism by which a conic singularity arises is that a Ricci flat ALE space 'pinches off' in the limit.

We finally turn to the case where (M, g) is an Einstein metric with higher dimensional singular set. Of particular interest is the case when the singular set has a stratum of codimension two, partly because this is quite natural in complex geometry, but also because this should correspond to the greatest flexibility. We mostly discuss this case. There are various examples of this phenomenon known; the simplest arise as quotients of smooth Einstein spaces. In particular, it is not hard to construct examples of hyperbolic manifolds singular along a codimension two edge. One may also construct cohomogeneity one metrics, for which the Einstein condition reduces to an ODE, and which have a singular edge. The paper [**29**] shows how to adapt an ansatz by Page and Pope to produce families of singular Einstein metrics with simple edge singularities along a smooth codimension two stratum. The examples emphasized there are actually noncompact (their other end is asymptotically hyperbolic), but this is immaterial for the present discussion. To write these down, fix a holomorphic line bundle L with Hermitian metric and connection 1-form θ over a compact Kähler-Einstein manifold $(X, \widehat{g})$ with $c_1 > 0$. The metrics are defined on the complement of a ball around the zero section in L by the formula

$$g = (r^2 - 1)^n P(r)^{-1} dr^2 + c^2 P(r)(r^2 - 1)^{-n} \theta^2 + c(r^2 - 1)\widehat{g};$$

here $P(r)$ satisfies the ODE

$$\frac{d}{dr}(r^{-1}P(r)) = r^{-2}\left(|\Lambda|(r^2 - 1)^{n+1} + c^{-1}\lambda(r^2 - 1)^n\right),$$

and Λ, c and λ are parameters. The issue is to show that there are choices for these parameters, including the initial condition for P, which yield metrics with the stated properties. We refer to [**29**] for more details.

These metrics have a number of interesting features, but their definition relies on many strong hypotheses and it is unclear whether these features are in any way necessary. Following the approach of this paper, one should be able to discern some of this from the local deformation theory. One attack on this is in the thesis by Montcouquiol [**32**], who proved that for higher dimensional hyperbolic conifolds with smooth codimension two singular set, all nontrivial infinitesimal deformations must vary the cone angle. However, this result does not in any obvious way imply a local rigidity statement: in the language explained in the final section of this paper, the defect space is infinite dimensional, and does not seem to integrate to families of Einstein metrics, even those just defined near the singular set.

6. Methods

After the geometric descriptions in the earlier parts of this paper, the reader is owed some indication of the methods used to prove these results. We begin with the fairly standard formalism of turning the Einstein deformation problem into an elliptic partial differential equation, and then discuss the extensions of ordinary elliptic theory to manifolds with conic and iterated cone-edge singularities needed for this problem.

6.1. The Einstein equation and Bianchi gauge. — Let M be a smooth compact manifold with $\dim M = n$ and define $\mathcal{M}^{k,\alpha}$ as the space of all $\mathcal{C}^{k,\alpha}$ metrics on M. The mapping

$$g \longmapsto \mathrm{Ric}^g$$

is a second order quasilinear differential operator which is polynomial in the components of g, g^{-1}, ∇g and $\nabla^2 g$, hence is a real analytic mapping $\mathcal{M}^{k+2,\alpha}(M) \to \mathcal{C}^{k,\alpha}(M, S^2T^*M)$. Fixing $\lambda \in \mathbb{R}$, the metrics which are Einstein with this given constant λ are the solutions of

$$\mathcal{E}_\lambda(g) \ := \ \mathrm{Ric}^g - \lambda g = 0. \tag{6.6}$$

Taking traces of both sides yields $\lambda = R^g/n$, where R^g is the scalar curvature. From now on we fix λ and drop it from the notation.

This equation is not elliptic because of its invariance under diffeomorphisms, i.e. if $\mathcal{E}(g) = 0$, then for any diffeomorphism ϕ of M, $\mathcal{E}(\phi^* g) = 0$. Equivalently, the gauge group $\mathcal{G}^{k+1,\alpha}(M)$ of $\mathcal{C}^{k+1,\alpha}$ diffeomorphisms acts on $\mathcal{M}^{k,\alpha}$ by pullback, and the zero set of $\mathcal{E}$ consists of the orbits of this action. This action is not $\mathcal{C}^1$, so the orbits are not in general smooth, which complicates the global analysis slightly.

Fix g with $\mathcal{E}(g) = 0$. To study the Einstein deformations of g, consider the mapping

$$h \mapsto E^g(h) := \mathrm{Ric}^{g+h} - \lambda(g + h). \tag{6.7}$$

From [**6**, p.63],

$$DE^g\,|_{h=0} = \frac{1}{2}(\nabla^*\nabla - 2\overset{o}{R}{}^g) - (\delta^g)^*(\delta + \frac{1}{2}d\mathrm{tr}^{\,g}); \tag{6.8}$$

here $\overset{o}{R}{}^g$ is the curvature operator acting as a symmetric endomorphism on symmetric two-tensors,

$$(\overset{o}{R}{}^g h)_{ij} = R_{ipjq}h^{pq}$$

and $((\delta^g)^*\omega)_{ij} = \frac{1}{2}(\omega_{i;j} + \omega_{j;i})$. For simplicity we set

$$L^g = \frac{1}{2}(\nabla^*\nabla - 2\overset{o}{R}{}^g), \qquad B^g = \delta^g + \frac{1}{2}d\mathrm{tr}^{\,g}.$$

so that (6.8) takes the simpler form

$$DE^g\,|_{h=0} = \frac{1}{2}L^g - (\delta^g)^* B^g. \tag{6.9}$$

The operator B^g is called the Bianchi operator, and appears in two important identities:

$$B^g(g) = 0, \qquad \text{and} \qquad B^g(\mathrm{Ric}^g) = 0. \tag{6.10}$$

The first is trivial, and the second follows from the contracted second Bianchi identity. Note that this yields

$$h \longmapsto B^{g+h}E^g(h) \equiv 0$$

for any g, h. Now suppose that g is Einstein; linearizing this identity at $h = 0$ gives

$$0 = B^g \ DE^g|_0 = B^g L^g - B^g(\delta^g)^* B^g. \tag{6.11}$$

This means in particular that

$$\mathrm{ran}\,(DE^g\,|_0) \subset \ker\,(B^g);$$

in other words, on any compact manifold, the Einstein equation is always obstructed since its linearization has range lying in a proper subspace (in fact, the nullspace of the underdetermined differential operator B^g).

The orbit of the diffeomorphism group has tangent space at g given by the range of the mapping $(\delta^g)^*$; the restriction of DE^g to the orthogonal complement of this subspace, i.e. to the nullspace of δ^g, is elliptic. We shall use a slight variant of this procedure, restricting DE^g instead to the nullspace of B^g. This 'Bianchi gauge', introduced in [**8**], is very convenient for calculations.

The system $h \mapsto (DE^g(h), B^g(h))$ is elliptic in the sense of Agmon-Douglis-Nirenberg, and so one can look for gauge group representatives for all Einstein metrics near to g as solutions of $E^g(h) = 0$, $B^g(h) = 0$. We consider instead the operator

$$h \longmapsto N^g(h) \ := \ E^g(h) + (\delta^{g+h})^* B^g(h). \tag{6.12}$$

Its linearization when g is Einstein is

$$DN^g|_{h=0} \ := \ L^g = \frac{1}{2}(\nabla^g)^*\nabla^g - \overset{o}{R}{}^g. \tag{6.13}$$

Clearly $(E^g(h), B^g(h)) = (0,0)$ implies $N^g(h) = 0$, and the converse is almost true as well:

Proposition 6.1. — *If $N^g(h) = 0$ and $\mathrm{Ric}^{g+h} < 0$, then $g+h$ is Einstein and h satisfies the gauge condition $B^g(h) = 0$.*

Proof. — Let $\gamma = B^g(h)$. Applying δ^{g+h} to $N^g(h) = 0$ gives $(\delta^{g+h}(\delta^{g+h})^* - \frac{1}{2}d\delta^{g+h})\gamma = 0$. Now recall the Weitzenböck formula on 1-forms

$$B^k(\delta^k)^* = \delta^k(\delta^k)^* - \frac{1}{2}d\delta^k = \frac{1}{2}\left((\nabla^k)^*\nabla^k - \mathrm{Ric}^k\right) \tag{6.14}$$

for any metric k (where the first equality uses $\operatorname{tr}^k(\delta^k)^* = -\delta^k$), and so the equation above becomes $((\nabla^{g+h})^*\nabla^{g+h} - \mathrm{Ric}^{g+h})\gamma = 0$. Because $\mathrm{Ric}^{g+h} < 0$, this operator is an isomorphism, and so $\gamma = 0$ as desired. □

As a final comment, if h is an arbitrary (small) solution of $N^g(h) = 0$, then the metric $g + h$ is a Ricci soliton: it satisfies the equation $E(g + h) = (\delta^{g+h})^*\omega$ where $\omega = -B^g(h)$. This suggests that a problem which may be somewhat less obstructed than the deformation problem for Einstein metrics is the deformation problem for Ricci solitons.

6.2. Conic and edge operators. — Implementing this analytic formalism for the Einstein deformation problem on singular spaces requires an understanding of the mapping properties for linear elliptic operators on such spaces. There is a good theory to draw upon for spaces with isolated conic or simple edge singularities, which we describe briefly below, and this can be extended to the depth 2 singularities which appear in the three-dimensional theory. However, its full extension to the general iterated cone-edge setting does not yet exist. Rather than presenting this linear theory in any sort of generality, we present the main results quickly in two and three-dimensions and then indicate the general picture. As a reference for this material we list [**26**]: it does contain all the results quoted below (at least when the singular set is either a point or a smooth submanifold), albeit in a very general form. There are other more accessible and direct approaches for some of this, which work particularly well in low dimensions, see [**34**] and [**45**], for example.

Let M be a surface with isolated conic points, and suppose that L is a 'generalized Laplacian' associated to the metric. In other words $L = \nabla^*\nabla + R$ where R is a naturally defined symmetric endomorphism depending only on the curvature tensor and its covariant derivatives; rather than being precise about this we turn always to the special operators which were described in the last subsection, e.g. the scalar Laplacian, the linearized gauged Einstein operator, etc. Near a conic point p we can choose coordinates (r, y) in terms of which

$$L = \frac{\partial^2}{\partial r^2} + \frac{A(r,y)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}H \tag{6.15}$$

where $A(r, y)$ is smooth on $r \geq 0$, $A(0, y) \equiv 1$, and $H = H(r) = \partial_y^2 + R_0(r, y)$ is a family of operators acting on (sections over the) circle, also depending smoothly on r. Associated to L is the set Λ of indicial weights. We say that γ is an indicial root of L if there exists a function $\phi(y)$ such that $L(r^\gamma\phi(y)) = \mathcal{O}(r^{\gamma-1})$. This is one order better than the expected rate of blowup or decay, $r^{\gamma-2}$, which indicates that there is some 'leading order' cancellation. Indeed, we see directly that

$$L(r^\gamma\phi(y)) = r^{\gamma-2}\left(\gamma^2 + \partial_y^2 + R_0(0, y)\right)\phi(y) + \mathcal{O}(r^{\gamma-1}),$$

so $-\gamma^2$ is equal to an eigenvalue of $\partial_y^2 + R_0(0,y)$ and ϕ its corresponding eigenfunction. A priori, these indicial roots may be real or complex, and we define the set Λ to consist of the real parts of these indicial roots. It is easy to see that Λ is infinite and discrete. For example, if L is the scalar Laplacian, or the linearized gauged Einstein operator acting on trace-free symmetric two-tensors, for a metric g which has a single conic point with cone angle $2\pi\alpha$, then $\Lambda = \{j/\alpha : j \in \mathbb{Z}\}$ and $\{\pm(2 \pm j/\alpha) : j \in \mathbb{Z}\}$, respectively. For generalized Laplacians in two dimensions, Λ is symmetric about 0. In all the examples of interest here, all indicial roots are real, so we think of Λ as being precisely equal to the set of indicial roots.

Consider the action of L on weighted Hölder spaces $r^\nu \mathcal{C}_g^{k,\alpha}(M)$ consisting of functions of the form $u = r^\nu v$ where v is in the 'geometric Hölder space' with respect to the metric g, i.e. computed using derivatives and distance functions for g. The basic result is

Proposition 6.2. — *The mapping*

$$L : r^\nu \mathcal{C}_g^{2,\alpha}(M) \longrightarrow r^{\nu-2}\mathcal{C}_g^{0,\alpha}(M) \tag{6.16}$$

is Fredholm if and only if $\nu \notin \Lambda$. This mapping is surjective for $\nu \notin \Lambda$, $\nu \ll 0$, and injective when $\nu \gg 0$. Finally, (6.16) is injective for some value $\nu \notin \Lambda$ if and only if the corresponding mapping with weight $-\nu$ is surjective.

If γ is an indicial root, then a sequence of appropriate cutoffs of the approximate solution $r^\gamma \phi(y)$ can be used to show that (6.16) does not have closed range when $\nu = \gamma$. The more subtle parts of this result are to show Fredholmness when $\nu \notin \Lambda$, and to prove the final statement. We comment on this last part especially a bit further. Both assertions are proved by constructing, for each $\nu \notin \Lambda$, a generalized inverse G for L. This is done using L^2 based methods, but the key is to show that the Schwartz kernel of this operator has a precise structure as a smooth (or rather, polyhomogeneous) function, which allows one to pass easily between weighted L^2 and weighted Hölder estimates. This is what makes it possible to prove the 'duality statement' in a Hölder setting.

For all the relevant operators in the Einstein deformation problem, one can prove that (6.16) is injective for $\nu > \nu_0 \geq 0$, hence surjective for $-\nu < -\nu_0 \leq 0$. In order to apply the inverse function theorem or any related contraction mapping arguments, the operator should be surjective, hence by this result we should be working on a Hölder space with negative weight. However, it is impossible to let the nonlinear PDE act on functions unbounded at the conic points. The resolution of this dilemma rests on the

Proposition 6.3. — *Let L be as above, and suppose that $\Lambda \ni \nu > 0$ is such that (6.16) is injective. List the indicial roots of L with real parts in the interval $(-\nu, \nu)$ by*

$\{\gamma_j : -N \leq j \leq N\}$ with the convention that $\gamma_{-j} = -\gamma_j$. Then for any $f \in r^{\nu-2}\mathcal{C}_g^{0,\alpha}$, there exists a solution $u \in r^{-\nu}\mathcal{C}_g^{2,\alpha}$ to $Lu = f$, and this u has a decomposition

$$u = \sum_{j=-N}^{N} u_j(y) r^{\gamma_j} + v, \qquad v \in r^{\nu}\mathcal{C}_g^{2,\alpha}, \tag{6.17}$$

where each $u_j(y)$ is an eigenfunction associated with that indicial root.

The finite dimensional span of terms $u_j(y)r^{\gamma_j}$ which appear in this partiale xpansion is called the defect space. This result is general, but the crucial observation is that for our particular problem, every element of this defect space can be identified with an infinitesimal variation of a one-parameter family of solutions of the nonlinear gauged Einstein operator. All the geometric moduli for the problem appear in this way: the underlying Teichmüller parameter on the compact surface, the location of the conic points and the cone angles. We can then 'solve' the problem $Lu = f$ with $f \in r^{\nu-2}\mathcal{C}_g^{0,\alpha}$, $\nu > \nu_0$, by first altering these geometric parameters and applying the operator L corresponding to the new metric to the remainder term v. In this way we can set up an iteration scheme to solve the nonlinear perturbation problem.

Suitable generalizations of this idea are behind all of the other deformation results discussed in the earlier parts of this paper. (Indeed, this type of idea has been applied in very many other circumstances.) The result about higher dimensional Einstein spaces with isolated conic singularities uses essentially the same linear theory, and there are direct analogues of the Propositions 6.2 and 6.3. The calculational aspects are substantially different, however, and unfortunately much more complicated. On the other hand, the main indicial term to understand is the one corresponding to the indicial root $\gamma = 0$. The integrability hypothesis we imposed is that this does indeed correspond to a one-parameter family of Einstein deformations of the cone $C(N)$. This appears to be generically true, and can be checked explicitly in several cases of interest, but it is unclear if it holds in general (chances are that it does not).

When the singular set is a submanifold Y of dimension $d > 0$, the linear theory is more complicated. It suffices to work in neighbourhoods diffeomorphic to $\mathcal{U} \times C_1(N)$, where $\mathcal{U} \subset Y$ is a coordinate neighbourhood. Each of the operators of interest have the form

$$L = \frac{\partial^2}{\partial r^2} + \frac{A(r,y)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}H + K;$$

where H is much as before, an elliptic operator acting the link N, while K restricts to an elliptic operator on Y. We can define the indicial roots of L exactly as in the previous case; the operator K does not appear at the lowest level in terms of a formal count of powers of r. It does play a role in a new model operator that we need to consider in this setting, called the normal operator of L. This is defined as

$$N(L) = \frac{\partial^2}{\partial r^2} + \frac{A(0,y)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}H(0) + \Delta_{\mathbb{R}^d},$$

acting on function (or sections) on $C(N) \times \mathbb{R}^d$. We roll the requisite results into the one

Proposition 6.4. — *The mapping*

$$L : r^\nu \mathcal{C}_g^{2,\alpha}(M) \longrightarrow r^{\nu-2}\mathcal{C}_g^{0,\alpha}(M)$$

is Fredholm if and only if ν does not lie in the indicial weight set Λ, and addition,

$$N(L) : r^\nu \mathcal{C}_g^{2,\alpha}(C(N) \times \mathbb{R}^d) \longrightarrow r^{\nu-2}\mathcal{C}_g^{0,\alpha}(C(N) \times \mathbb{R}^d)$$

is an isomorphism. If the nullspace of $N(L)$ is nontrivial, then it is automatically infinite dimensional and the same is true for the nullspace of L for the space with the same weight; the analogous statement is true for the cokernel. As in the conic setting, this mapping is surjective from $r^\nu \mathcal{C}_g^{2,\alpha}$ if and only if it is injective from $r^{-\nu}\mathcal{C}_g^{2,\alpha}$.

Suppose $\Lambda \ni \nu > 0$ is such that (6.16) is injective, and list the indicial roots of L in the interval $(-\nu, \nu)$ as $\{\gamma_j : |j| \leq N\}$ with $\gamma_{-j} = -\gamma_j$. Then for any $f \in r^{\nu-2}\mathcal{C}_g^{0,\alpha}$, there exists a solution $u \in r^{-\nu}\mathcal{C}_g^{2,\alpha}$ such that

$$u = \sum_{j=-N}^{N} u_j(y) r^{\gamma_j} + v. \tag{6.18}$$

with each $u_j(y)$ equal to a multiple of the eigenfunction associated with that indicial root.

It is no longer true that v or the coefficients $u_j(y)$ in this decomposition are as smooth as formal considerations would dictate, and this leads to some considerable, and perhaps insurmountable, analytic difficulties when attempting to apply this linear theory to our nonlinear problem. More plainly, when the codimension of Y in M is equal to 2, then the defect space corresponding to crossing the indicial root 0 is infinite dimensional, and its elements do not correspond in any reasonable way to geometric motions.

When M is 3-dimensional, it is possible to overcome this using the fact that there is a simple correspondence between the overall geometric parameters for the problem in a neighbourhood of the singular set and their 'traces' in the asymptotic expansions of infinitesimal Einstein deformations along the singular set.

The final issue to discuss is the general three-dimensional case, when M is a conifold whose singular set contains not only edges but also vertices. One now needs an extension of the theory of conic and edge operators discussed above. Fortunately, the generalization needed is the simplest one possible, where the depth 2 points are isolated vertices of cones where the links are themselves spaces with isolated conic singularities. The idea is to try to adapt the conic theory at these vertices, even though the links are not smooth; in general one would expect to have difficulties with lack of smoothness in the asymptotic expansions along the edges which terminate at this vertex, and the main new steps are to control these expansions uniformly on approach to these depth 2 vertices.

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