Astérisque

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Astérisque, tome 249 (1998)
<http://www.numdam.org/item?id=AST_1998__249__R1_0>

ASTÉRISQUE 249

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Société Mathématique de France 1998

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1991 *Mathematics Subject Classification*. — Primary: 35J15, 47A60, 42B25. Secondary: 42B20, 35K10, 47F05, 46G20.

***Key words and phrases*.** — Second order complex elliptic operators, maximal accretive operators, H^∞-functional calculus, multilinear expansions, Gaussian estimates, Littlewood-Paley-Stein quadratic functionals, Carleson measures, absolutely bounded mean oscillation, Riesz transforms, Calderón-Zygmund operators.

SQUARE ROOT PROBLEM FOR DIVERGENCE OPERATORS AND RELATED TOPICS

Pascal Auscher, Philippe Tchamitchian

Abstract. — We present in this work recent progress on the square root problem of Kato for differential operators in divergence form on $\mathbb{R}^n$. We discuss topics on functional calculus, heat and resolvent kernel estimates, square function estimates and Carleson measure estimates for square roots. In the first chapter, we show in a quantitative way how the theorems of Aronson-Nash and of De Giorgi are equivalent. In the central chapters, we take advantage of recent development in functional calculus and in harmonic analysis to propose a new point of view on Kato's problem which allows us to unify previous results and extend them. In the last chapter we study the associated Riesz transforms, their relation to Calderón-Zygmund operators and their behavior on L^p-spaces.

Résumé **(Problème de la racine carrée pour les opérateurs sous forme divergence et sujets connexes)**. — Ce travail a pour thème principal le problème de Kato concernant la racine carrée des opérateurs différentiels elliptiques sous forme divergence dans $\mathbb{R}^n$. Pour mener à bien cette étude, nous nous intéressons à des questions relatives au calcul fonctionnel, aux estimations de noyaux, aux fonctionnelles quadratiques et aux mesures de Carleson associées aux racines carrées. Dans le premier chapitre, nous montrons en un sens précis comment les théorèmes d'Aronson-Nash et de De Giorgi sont équivalents. Dans les deux chapitres centraux, nous tirons parti de développements récents sur le calcul fonctionnel et en analyse harmonique pour proposer un nouveau point de vue sur le problème de Kato qui permet d'unifier les résultats antérieurs et de les généraliser. Enfin, dans le dernier chapitre, nous étudions les transformées de Riesz associées, leur relation aux opérateurs de Calderón-Zygmund et leur comportement sur les espaces L^p.

Contents

INTRODUCTION

The main goal of this memoir is the study of square roots of divergence form differential operators $L = -\operatorname{div}(A\nabla)$ on $\mathbb{R}^n$, $n \geq 2$, where $A(x)$ is a matrix with complex-valued bounded entries and satisfying a uniform ellipticity condition. This operator is defined using the theory of maximal accretive operators on $L^2(\mathbb{R}^n)$ via the associated variational form on the Sobolev space $H^1(\mathbb{R}^n)$. We denote by $L^{1/2}$ the square root of L.

We are interested in the following questions.

Question 1. — *Does the domain of $L^{1/2}$ agree with $H^1(\mathbb{R}^n)$?*

When L is a selfadjoint operator or has smooth coefficients (*e.g.*, Lipschitz), this is true. However, the answer is not known in general.

Question 2. — *If A_0 is such that the domain of $L_0^{1/2}$ is $H^1(\mathbb{R}^n)$, where $L_0 = -\operatorname{div}(A_0\nabla)$, is the map $A \to L^{1/2}$, valued in $\mathcal{B}(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n))$, complex analytic at A_0?*

Classical complex function theory tells us that analyticity of a Taylor expansion follows from boundedness in complex balls. This is the same principle that partly justifies the use of complex-valued coefficients here.

Question 3. — *If L is such that the domain of $L^{1/2}$ agrees with $H^1(\mathbb{R}^n)$, what can we say about L^p-boundedness, $p \neq 2$, of the Riesz transforms associated to L, namely $\nabla L^{-1/2}$? More generally, how do $\|L^{1/2}f\|_p$ and $\|\nabla f\|_p$ compare?*

In this work we bring new answers to the first two questions and, under a technical hypothesis that the kernel of the semigroup generated by $-L$ has Gaussian upper bounds and regularity, we completely elucidate the third one. Some results were announced in **[15]**.

Questions 1 and 2 were raised by T. Kato in the 60's. They were motivated by perturbation theory for some hyperbolic partial differential equations (see **[46]**,**[56]**). They turned out to be profound by their connections to other topics.

These questions have been studied in the abstract Hilbert space framework for a long time, until counterexamples (which are not differential operators) were found by McIntosh **[54, 55]**.

As mentioned by Journé **[45]**, McIntosh observed that his results are related to counterexamples for the commutator inequality

$$||T|S| - |S|T|| \leq C||TS - ST||$$

for arbitrary selfadjoint operators. In the case where $S = -id/dx$ and T is multiplication by a Lipschitz function on $\mathbb{R}$, however, it becomes the celebrated Calderón first commutator inequality, proved to be true by deep techniques of harmonic analysis **[17]**.

The relation between question 1 and Calderón's work opened the door to new developments. This crystallized in the collaboration between McIntosh, and Coifman and Meyer who, at this time, were working on Calderón's conjecture concerning the Cauchy integral on Lipschitz curves. The multilinear analysis which was developed by these authors brought positive answers to both Calderón's conjecture and Kato's first question in dimension 1 **[20]**.

Multilinear analysis consists in expanding in a formal Taylor series operator-valued functionals that depend non-linearly on their argument in a suitable Banach space (called the space of holomorphy) and, then, in controlling each term individually together with norm growth estimates in order to obtain a radius of convergence. See **[21]** for an overview. Here $A \to L^{1/2}$ is expanded in the variable $A - I$ or $A^{-1} - I$ in L^∞. In the case of the Cauchy integral the variable is related to the derivative of a parametrization. Harmonic analysis comes into play when proving boundedness by making use of Littlewood-Paley-Stein quadratic functionals, the control of which requiring Carleson measures.

This breakthrough led to important results in real harmonic analysis, and the $T1$-theorem of David and Journé for Calderón-Zygmund singular integrals is probably one of the best examples **[26]**.

The square root problem in dimension $n \geq 2$ was also studied by the above multilinear expansions. It was shown in **[19]** and **[35]** that questions 1 and 2 have positive answers for matrices satisfying $||A^{-1} - I||_\infty \leq \varepsilon(n)$ for some small $\varepsilon(n)$. The best known value of this constant is given in **[45]** by refinements of the method. On the other hand, it is shown in **[57]** that the answer to question 1 is in the affirmative when the matrix entries are pointwise multipliers of a Sobolev space with positive regularity index.

Recent progress in operator theory and in harmonic analysis allowed us to develop a different approach which we built in 4 steps, each one corresponding to Preliminaries, Chapters 1, 2 and 3 of this work.

1. The equivalence between the coincidence of the domain of $L^{1/2}$ with $H^1(\mathbb{R}^n)$ and the boundedness of some quadratic functionals: this uses the works of Yagi **[81]** and McIntosh **[58]** on functional calculus, and interpolation theory **[8]**.
2. Precise estimates for operator kernels occuring in these quadratic functionals: this led us to study the semigroup and resolvent kernels of L when it has complex coefficients. We show that the well-known estimates of Aronson and Nash (**[4]**, **[66]**), valid in the case of real coefficients, are not always true **[7]**. We give sufficient conditions for such estimates to hold, based on **[5]**.
3. The elaboration of an adapted $T1$-theorem: by this we mean a criterion which relates the boundedness of a class of quadratic functionals to a Carleson measure estimate. The model case is a result by Christ and Journé **[18]**, but which is not directly applicable because we are dealing with "rough" kernels. We overcome this difficulty by an adapted smoothing technique (which uses the classical Riesz transforms).
4. The last step sheds light on relations between the study of $L^{1/2}$ and the properties of weak solutions to the inhomogeneous equation $Lu = f$ with nice f.

We cover and extend this way all the aforementioned results concerning questions 1 and 2. For example, we show that question 2 has a positive answer when A_0 belongs to a class of matrices enjoying a very mild regularity condition (weaker than the one imposed by McIntosh in **[57]**), and that the space of holomorphy for $A \to L^{1/2}$ near A_0 is BMO instead of L^∞ (this is in spirit with commutators estimates between BMO functions and Calderón-Zygmund operators). Still, we do not fully answer these questions.

Let us come to a discussion on question 3, which is treated in Chapter 4, and begin with its relation to boundary value problems.

Consider the elliptic operator $M = \partial^2/\partial t^2 - L$ on $\mathbb{R}^{n+1}_+ = (0,\infty) \times \mathbb{R}^n$. The Dirichlet problem $Mu = 0$ with data u_0 at $t = 0$ (and data 0 at ∞) is formally solved by $u_t(x) = P_t f(x)$, where $P_t = e^{-tL^{1/2}}$ is the Poisson semigroup associated to L. The assumption made in Chapter 4 implies that the kernel of this operator satisfies natural size estimates. Hence, for all $1 \le p \le \infty$, the Dirichlet problem with data in L^p has a solution with $\|u_t\|_p \le c\|u_0\|_p$ uniformly for all $t > 0$, and u_t converges to u_0 a.e. and in $L^p(\mathbb{R}^n)$ as $t \to 0$ (weak star convergence only if $p = \infty$).

Fix $1 < p < \infty$. Consider the Neumann problem with data $v_0 \in L^p(\mathbb{R}^n)$:

$$\begin{cases} Mu = 0 & \text{on } \mathbb{R}^{n+1}_+, \\ \lim\limits_{t\to 0} \dfrac{\partial u_t}{\partial t} = v_0, \end{cases}$$

where the limit occurs in $L^p(\mathbb{R}^n)$. An inequality of the type $||\nabla f||_p \le c||L^{1/2}f||_p$ insures that the formal solution $u_t = -e^{-tL^{1/2}}L^{-1/2}v_0$ satisfies

$$\sup_{t>0}\left(||\frac{\partial u_t}{\partial t}||_p + ||\nabla_x u_t||_p\right) \le c||v_0||_p.$$

In the same vein, consider for any $u_0 \in W^{1,p}(\mathbb{R}^n)$ the regularity problem with data u_0:

$$\begin{cases} Mu = 0 & \text{on } \mathbb{R}^{n+1}_+, \\ \lim_{t\to 0} u_t = u_0, \end{cases}$$

where the limit occurs in $L^p(\mathbb{R}^n)$. Then, a comparison $||L^{1/2}f||_p \sim ||\nabla f||_p$ insures that the solution $u_t = e^{-tL^{1/2}}u_0$ satisfies

$$\sup_{t>0}\left(||\frac{\partial u_t}{\partial t}||_p + ||\nabla_x u_t||_p\right) \le c||\nabla u_0||_p.$$

This type of problems (formulated from the point of view of non-tangential limits at the boundary) have been recently studied by Kenig and Pipher for the class of real symmetric operators on the unit ball of $\mathbb{R}^{n+1}$ **[51]**. They prove that the regularity and Neumann problems are solvable for $p \in (1, 2+\varepsilon)$ whenever they are solvable for $p = 2$, and that this range is optimal.

The study of Riesz transforms on L^p is also of interest when L is the Laplace-Beltrami operator on a Riemannian manifold because it is related to geometry. In this case, the L^2 theory is granted from selfadjointness. Under quite general assumptions on the manifold the inequality $||\nabla f||_p \le c||L^{1/2}f||_p$ is valid when $1 < p \le 2$ (the reverse inequality holding in the dual range) and this is sharp. See **[24]** and the references therein.

In our particular situation, less general than the one in **[51]** if we restrict ourselves to real symmetric operators (but we allow complex coefficients), or not linked to any kind of geometry, we obtain that $||\nabla f||_p \le c||L^{1/2}f||_p$ holds for $p \in (1, 2+\varepsilon)$, while the converse holds for all $p \in (1, \infty)$, provided that we have $||\nabla f||_2 \sim ||L^{1/2}f||_2$ and that we make assumptions on the semigroup kernel for L. This is optimal by an example of Kenig. The methods rely on Calderón-Zygmund theory, on the gradient estimates for the semigroup kernel established in Chapter 1, and on smoothing techniques for "rough" operators as used in Chapter 2.

To conclude, let us indicate that the square root problem can be studied for operators defined on a domain. In the case of a Lipschitz domain, assuming Dirichlet or Neumann boundary condition, we are able to obtain similar results on the L^2 and L^p theories. This work will be presented in a subsequent paper.

Reading help. — In this memoir each chapter focuses on a different topic. It has its own introduction where a summary of ideas and results is presented. For the sake of fluidity, but at the expense of being repetitive, we have made the different chapters as much independent as possible. One exception though, Chapter 3, the corner stone of our work, heavily depends on Chapter 2. Developments on functional calculus are in Preliminaries; on kernel estimates in Chapter 1; on square functions and Carleson measures in Chapter 2; on positive answers to the square root problem in Chapter 3; and on Riesz transforms and singular integrals on Chapter 4. The four appendices contain further results or technical proofs.

The formulae are labelled by numbers that are reset to 0 at the beginning of each chapter. The same thing holds for definitions, lemmas, propositions, corollaries and theorems, all taken in the same group. We refer to a formula or a result from the same chapter by its label only. As usual, we use the floating constant principle that the same symbol c, C etc change values from line to line.

Acknowledgements. — This work is the fruit of several years of research. We have had the privilege of many conversations with several people and/or collaborators on the topics of this work. We wish to thank T. Coulhon, X.T. Duong, G. David, E.B. Davies, L. Escauriaza, E. Fabes, S. Hofmann, N. Lerner, C. Li, G. Métivier, A. Nahmod, J. Pipher and N. Trudinger. We are grateful to C. Kenig for letting us include a result of his.

Special thanks go to A. McIntosh from whom we learned what we know on functional calculus (any mistake remains of course our responsibility) and whose influence can be found at many places in this work. He also suggested several improvements in our presentation and provided us with historical references.

We wish to thank Brown University for hospitality and support for both of us. Support from several other institutions during completion of this work is also gratefully acknowledged: the University of Rennes I and the CNRS, the Australian National University at Canberra, the ICMS at Edinburgh and Macquarie University in Sydney.

PRELIMINARIES

The operators considered in this work are differential operators built using the theory of maximal accretive operators. Although they are not necessarily selfadjoint, such operators have a holomorphic functional calculus which coincides with the Borel functional calculus in case of selfadjointness. This preliminary chapter is devoted to introduce basic material on operator theory in order to prepare the grounds for our study of square roots. For further considerations consult, *e.g.*, **[48]**, **[77]**, **[81]**, **[58]**, **[1]** or **[8]**. The results of Section 0.7 seem new in this generality.

0.1. Maximal accretive operators and their functional calculus

Let H be a Hilbert space with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$. All operators are assumed to be linear.

A maximal accretive operator on H is an operator L on H with the following properties:

(i) its domain $\mathcal{D}(L)$ is dense in H;
(ii) $\operatorname{Re}\langle Lu, u\rangle \geq 0$ for all $u \in \mathcal{D}(L)$;
(iii) $L + \lambda$ is onto for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.

Under these conditions, $(L+\lambda)^{-1}$ is a bounded operator with

$$\|(L+\lambda)^{-1}u\| \leq \frac{\|u\|}{\operatorname{Re}\lambda}\ . \tag{1}$$

Denote by $\arg z$ the argument in $(-\pi, \pi]$ of $z \in \mathbb{C}$ and by Γ_μ the open sector $\{z \in \mathbb{C}\setminus\{0\}; |\arg z| < \mu\}$ for $\mu \in [0, \pi)$.

Let $\omega \leq \pi/2$. An ω-accretive operator on H is an operator, L, such that

(iv) $L + 1$ is invertible on H;
(v) $|\arg\langle Lu, u\rangle| \leq \omega$ for all $u \in \mathcal{D}(L)$.

Under these conditions, one has

$$\|(L+\lambda)^{-1}u\| \leq \frac{\|u\|}{\operatorname{dist}(-\lambda, \Gamma_\omega)}, \qquad \lambda \in \Gamma_{\pi-\omega}. \tag{2}$$

It is easy to show that an operator is $\pi/2$-accretive if and only if it is maximal accretive. Also, an ω-accretive operator is bounded if and only if its domain is closed in H. Moreover, a well-known theorem (see **[48]**, Theorem 1.24 of Chapter IX) asserts that, given $\omega < \pi/2$ and an operator L, then L is ω-accretive if and only if $-L$ is the generator of a contraction semigroup on H that is holomorphic in $\Gamma_{\pi/2-\omega}$. Denoting this semigroup by e^{-zL}, we have

$$||e^{-zL}u|| \leq ||u||, \quad |\arg z| < \frac{\pi}{2} - \omega. \tag{3}$$

Consider a one-one ω-accretive operator L on H for some $\omega < \pi/2$ (we exclude the case $\omega = \pi/2$ to simplify the presentation). We list some of the important facts relative to its functional calculus.

For D a domain in $\mathbb{C}$, call $H^\infty(D)$ the algebra of bounded holomorphic functions $f\colon D \to \mathbb{C}$ and $||f||_\infty = \sup\{|f(\zeta)|\,;\, \zeta \in D\}$.

a) *L has a bounded holomorphic functional calculus.* By this we mean that for all $\mu \in (\omega, \pi)$ there is a linear mapping h_μ from $H^\infty(\Gamma_\mu)$ to $\mathcal{B}(H)$, the space of bounded operators on H, with the following properties:

1. $||h_\mu(f)|| \leq c_\mu ||f||_\infty$;
2. $h_\mu(f) = (L+\lambda)^{-1}$ whenever $f(\zeta) = (\zeta+\lambda)^{-1}$;
3. $h_\mu(f)h_\mu(g) = h_\mu(fg)$;
4. If f_n is a uniformly bounded sequence in $H^\infty(\Gamma_\mu)$ that converges to $f \in H^\infty(\Gamma_\mu)$ uniformly on compact subsets of Γ_μ as n tends to ∞, then $h_\mu(f_n)$ converges to $h_\mu(f)$ strongly.

Because of property 2., the functional calculi defined on different sectors are consistent: $h_\mu(f) = h_\nu(f)$ whenever $f \in H^\infty(\Gamma_\nu)$ and $\omega < \mu \leq \nu < \pi$. Hence, as is customary, we set $f(L) = h_\mu(f)$ whenever it makes sense.

An important example is the exponential function $f(\zeta) = e^{-z\zeta}$ which belongs to $H^\infty(\Gamma_\mu)$ when $|\arg z| \leq \pi/2 - \mu$. In particular, it follows from the properties of the functional calculus that $f(L)$ agrees with e^{-zL}.

Also, if $\mu \in (\omega, \pi)$ and ψ is a holomorphic function in Γ_μ such that

$$|\psi(\zeta)| \leq c|\zeta|^s(1+|\zeta|)^{-2s}$$

for some c, $s > 0$, then $\psi(L)$ can be computed using the Cauchy formula

$$\psi(L) = \frac{1}{2\pi i}\int_\gamma (\zeta - L)^{-1}\psi(\zeta)\,d\zeta,$$

where the path γ is made of two rays $re^{\pm i\theta}, r \geq 0$ and $\omega < \theta < \mu$, described counterclockwise.

b) *L satisfies quadratic estimates.* Let ψ be as above and not identically 0. Set $\psi_t(\zeta) = \psi(t\zeta)$. Then, there exists $c = c(\psi) > 0$ such that (**[81, 58]**)

$$c||u|| \leq \left(\int_0^\infty ||\psi_t(L)u||^2\,\frac{dt}{t}\right)^{1/2} \leq c^{-1}||u||, \quad u \in H. \tag{4}$$

Particular choices for ψ are $e^{-\zeta}\zeta^{1/2}$ or $(1+\zeta)^{-1}\zeta^{1/2}$ where $\zeta^{1/2}$ is the principal determination of the square root of ζ. The first choice relates the quadratic functional to the parabolic operator $d/dt + L$; the second choice to the elliptic operator $1 + L$.

Note that because of the homogeneity of the measure dt/t, t can be changed to t^α for any $\alpha > 0$ without affecting the equivalence.

c) *The functional calculus can be extended to more general holomorphic functions so as to include fractional powers of L.* In particular, $L^{1/2}$ is the unique maximal accretive operator on H such that $\left(L^{1/2}\right)^2 = L$. The operator $L^{1/2}$ is called the maximal accretive square root of L ([**48**], p. 281). It is $\omega/2$-accretive. To compute this operator, we use two classical formulae:

$$L^{1/2}u = \frac{2}{\pi}\int_0^\infty (1+t^2L)^{-1}Lu\,dt, \quad u \in \mathcal{D}(L), \tag{5}$$

$$L^{1/2}u = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2L}Lu\,dt, \quad u \in \mathcal{D}(L). \tag{6}$$

Both integrals converge normally in H since

$$||(1+t^2L)^{-1}Lu|| + ||e^{-t^2L}Lu|| \le c\inf(||Lu||, t^{-2}||u||) \tag{7}$$

when $u \in \mathcal{D}(L)$. The last inequality follows from a fact we frequently use. If $f(\zeta), \zeta f(\zeta) \in H^\infty(\Gamma_\mu)$, then

$$f(L)Lu = Lf(L)u = (\zeta f)(L)u, \quad u \in \mathcal{D}(L). \tag{8}$$

0.2. Maximal accretive operators and forms

We turn to a systematic construction of maximal accretive operators via sesquilinear forms and establish abstract resolvent and semigroup estimates.

The first result is the classical representation theorem on regularly accretive forms which we formulate without proof in a slightly less general way that is more practical for our purpose. See [**48**], Chapter VI.

Proposition 1. — *Let H_0 and H_1 be two Hilbert spaces. Let $D : H_0 \to H_1$ be a densely defined closed operator with domain $\mathcal{V}$ and let $A : H_1 \to H_1$ be a bounded operator such that*

$$Re\,\langle ADu, Du\rangle_1 \ge \delta||Du||_1^2, \quad u \in \mathcal{V}, \tag{9}$$

for some $\delta > 0$. Then there is a unique maximal accretive operator L with domain contained in $\mathcal{V}$ such that

$$\langle Lu, v\rangle_0 = \langle ADu, Dv\rangle_1, \quad u \in \mathcal{D}(L),\ v \in \mathcal{V}. \tag{10}$$

Furthermore, $\mathcal{D}(L)$ is a dense subspace of $\mathcal{V}$,

$$\mathcal{D}(L) = \{\, u \in \mathcal{V};\ ADu \in \mathcal{D}(D^*)\,\},$$

and L is of type $\omega < \pi/2$ where

$$\omega = \sup\{|\arg\langle Au, u\rangle_1|; u \in H_1\}.$$

In addition, L is one-one if D is one-one.

We have denoted by $\langle , \rangle_k$ and $\| \ \|_k$ the inner product and the associated norm in H_k.

Notation. — From now on, we consistently use the notation $L = D^*AD$ to refer to operators built as above.

Remarks

1. If $L = D^*AD$, then one can show that L^* is also maximal accretive and $L^* = D^*A^*D$.
2. The inequality (9) is weaker than

$$\mathrm{Re}\,\langle Av, v\rangle_1 \geq \delta\|v\|_1^2, \quad v \in H_1,$$

as $\overline{\mathcal{R}(D)}$, the closure of the range of D, may be a proper subspace of H_1. In other words, A may not be invertible on H_1.

Here are some additional resolvent and semigroup estimates.

Proposition 2. — *Let H_0, H_1, D, A, L and ω be as in the Proposition 1. Let $\mu \in (\omega, \pi/2)$, $\lambda \in \Gamma_{\pi-\mu}$ and $z \in \Gamma_{\pi/2-\mu}$. Then we have the following inequalities:*

$$\begin{aligned} \|D(L+\lambda)^{-1}u\|_1 &\leq \frac{c\|u\|_0}{|\lambda|^{1/2}}, \\ \|(L+\lambda)^{-1}D^*u\|_0 &\leq \frac{c\|u\|_1}{|\lambda|^{1/2}}, \\ \|D(L+\lambda)^{-1}D^*u\|_1 &\leq c\|u\|_1, \end{aligned}$$

and

$$\begin{aligned} \|Le^{-zL}u\|_0 = \left\|\frac{d}{dz}e^{-zL}u\right\|_0 &\leq \frac{c\|u\|_0}{|z|}, \\ \|De^{-zL}u\|_1 &\leq \frac{c\|u\|_0}{|z|^{1/2}}, \\ \|e^{-zL}D^*u\|_0 &\leq \frac{c\|u\|_1}{|z|^{1/2}}, \\ \|De^{-zL}D^*u\|_1 &\leq \frac{c\|u\|_1}{|z|}, \end{aligned}$$

for u in the appropriate space, where c depends only on $\delta, \|A\|, \omega, \mu$.

Proof. — We give parts of the argument, beginning with the estimates for the semigroup. The estimate for the derivative follows from the Cauchy contour formula

$$\frac{d}{dz}e^{-zL} = \frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon|z|} e^{-\zeta L}\frac{d\zeta}{(\zeta - z)^2},$$

for $\varepsilon > 0$ small enough, and the integral does not depend on ε. (The choice $\varepsilon = \frac{\sin(\mu - \omega)}{2}$ insures that $\{\zeta\,;\,|\zeta - z| \le \varepsilon|z|\}$ is contained in $\Gamma_{\pi/2-\omega}$.)

Next, by holomorphy, we have

$$-Le^{-zL}u = \frac{d}{dz}e^{-zL}u, \quad u \in H_0.$$

Hence, we have just shown that Le^{-zL} is bounded on H_0 with norm bounded by $c_\mu/|z|$. Using this, and (9) and (10), we have

$$\begin{aligned} \|De^{-zL}u\|_1^2 &\le \delta^{-1}\,\mathrm{Re}\,\langle ADe^{-zL}u, De^{-zL}u\rangle_1 \\ &= \delta^{-1}\,\mathrm{Re}\,\langle Le^{-zL}u, e^{-zL}u\rangle_0 \\ &\le \frac{c_\mu\delta^{-1}}{|z|}\|u\|_0^2. \end{aligned}$$

The inequality for $e^{-zL}D^*$ now follows by duality on applying the above with L^* replacing L. Finally, by using the semigroup property we obtain $De^{-zL}D^*u = De^{-zL/2}e^{-zL/2}D^*u$, when $u \in \mathcal{D}(D^*)$ and the desired estimate for $De^{-zL}D^*$ follows.

Let us turn to the last estimate for the resolvent. Let $u \in \mathcal{D}(D^*)$ and set $v = (L+\lambda)^{-1}D^*u$. Then $v \in \mathcal{D}(D)$ and

$$\langle ADv, Dv\rangle_1 + \lambda\langle v, v\rangle_0 = \langle D^*u, v\rangle_0 = \langle u, Dv\rangle_1.$$

Since $|\arg\langle ADv, Dv\rangle_1| \le \omega$ and $|\arg\lambda| \le \pi - \mu < \pi - \omega$, we can choose $\theta \in \mathbb{R}$ such that

$$\mathrm{Re}\left[e^{i\theta}\langle ADv, Dv\rangle_1\right] > 0 \quad \text{and} \quad \mathrm{Re}\left[e^{i\theta}\lambda\right] = 0.$$

In such a case

$$\mathrm{Re}\left[e^{i\theta}\langle ADv, Dv\rangle_1\right] \ge c\|Dv\|_1^2\,, \qquad c = c(\mu, \omega, \delta, \|A\|) > 0.$$

Hence

$$c\|Dv\|_1^2 \le \mathrm{Re}\left[e^{i\theta}(\langle ADv, Dv\rangle_1 + \lambda\langle v, v\rangle_0)\right] \le \|u\|_1\|Dv\|_1$$

which yields $\|Dv\|_1 \le c^{-1}\|u\|_1$ as desired. We leave the other estimates to the reader. □

Remark. — Let us first recall a classical construction. Equip $\mathcal{V}$, the domain of D, with the norm $(\|Du\|_1^2 + \|u\|_0^2)^{1/2}$ which makes $\mathcal{V}$ a Hilbert space. Then $D\colon \mathcal{V} \to H_1$ has an adjoint $D^\sharp\colon H_1 \to \mathcal{V}'$. Since the inclusion $\mathcal{V} \subset H_0$ is dense, $D^\sharp$ is an extension of D^* (defined as the adjoint of the unbounded operator D). Making the confusion of notations, the Riesz representation theorem shows that $\mathcal{V}'$ is the set of vectors of the form $u = u_0 + D^*u_1$, $u_i \in H_i$, and its norm is defined by $\|u\| = \inf(\|u_0\|_0^2 + \|u_1\|_1^2)^{1/2}$ where the infimum is taken over all representations of u. This makes H_0 a subspace of $\mathcal{V}'$ and the inclusion is dense. With this formalism, the estimates above, together with (2) and (3), mean that $(L+\lambda)^{-1}$ and e^{-zL} have bounded extensions from $\mathcal{V}'$ to $\mathcal{V}$. In particular, the identity

$$(I+L)^{-1}Lu = u - (I+L)^{-1}u, \quad u \in \mathcal{D}(L),$$

extends to $\mathcal{V}$.

0.3. The square root problem from an abstract point of view

The square root problem for an operator $L = D^*AD$, as defined in Proposition 1, is the following problem posed by Kato in [**47**] (see [**60**]): prove that $\mathcal{D}(L^{1/2}) = \mathcal{D}(D)$ with either the equivalence

$$||L^{1/2}u|| + ||u|| \sim ||Du|| + ||u|| \tag{11}$$

or the stronger equivalence

$$||L^{1/2}u|| \sim ||Du||. \tag{12}$$

Note that since $\mathcal{D}(L)$ is dense in $\mathcal{D}(D)$, it suffices to establish these equivalences *a priori* for $u \in \mathcal{D}(L)$. For simplicity, we have dropped the subscripts 0 and 1 in the notation for the norms.

If A is selfadjoint on H_1 then L is selfadjoint on H_0 and (12) follows from the representation formula

$$||L^{1/2}f||^2 = \langle ADf, Df \rangle, \quad f \in \mathcal{D}(L).$$

When A is no longer selfadjoint, (12) is not automatically true.

From the point of view of operator theory, the square root problem is an interpolation problem. Let us quote a result of J.L. Lions in this particular setting [**52**]. More can be found in [**8**] and the references therein.

Proposition 3. — *Let $L = D^*AD$ be as in Proposition 1.*

(i) *$\mathcal{D}(L^{1/2})$ is the complex interpolation space midway between $\mathcal{D}(L)$ and H_0.*
(ii) *$\mathcal{D}(D) \subset \mathcal{D}(L^{1/2})$ with $||L^{1/2}u|| \le c||Du||$ if and only if $\mathcal{D}((L^*)^{1/2}) \subset \mathcal{D}(D)$ with $||Du|| \le c||(L^*)^{1/2}u||$.*
(iii) *$||L^{1/2}u|| \sim ||Du||$ if and only if $||L^{1/2}u|| \le c||Du||$ and $||(L^*)^{1/2}u|| \le c||Du||$.*

When the identification of the domain of L is possible this is a powerful tool. See [**42, 12**], where square root problems are tackled using this interpolation result.

Another tool is the use of the quadratic estimates (4). To this end, let us describe another representation of the operators built in Proposition 1.

Lemma 4. — *Any maximal accretive operator of the form D^*AD on a Hilbert space H_0 can be written in the form SBS, where S is a positive selfadjoint operator on H_0 and B is a bounded invertible ω-accretive operator on H_0 for some $\omega < \pi/2$.*

Proof. — By the polar representation, one can write $D = US$, where $S = (D^*D)^{1/2}$ has same domain as D (this is $S = (D^*AD)^{1/2}$ with $A = I$) and $U\colon H_0 \to H_1$ is a partial isometry. Set $B = U^*AU$ on $\overline{\mathcal{R}(S)}$, and $B = I$ on $\mathcal{R}(S)^\perp$ if $\overline{\mathcal{R}(S)} \neq H_0$. The hypotheses on A easily imply that B is bounded, invertible on H_0 and ω-accretive for some $\omega < \pi/2$. Furthermore, the representation $L = SBS$ holds in the sense of unbounded operators on H_0. □

Proposition 5. — *Let S be a one-one selfadjoint operator in H_0 and B be a bounded invertible ω-accretive operator in H_0 for some $\omega < \pi/2$. Let $T = SBS$ be the associated maximal accretive operator. Then the following inequalities are equivalent.*

$$||T^{1/2}u|| \leq c||Su||, \quad u \in \mathcal{D}(T). \tag{13}$$

$$\left(\int_0^\infty ||e^{-t^2T}tSBu||^2 \,\frac{dt}{t}\right)^{1/2} \leq c||u||, \quad u \in H_0. \tag{14}$$

$$\left(\int_0^\infty ||(1+t^2T)^{-1}tSBu||^2 \,\frac{dt}{t}\right)^{1/2} \leq c||u||, \quad u \in H_0. \tag{15}$$

Remark. — Note that S is not necessarily positive. Also, the injectivity hypothesis is unnecessary but will be satisfied in our applications.

Proof. — We only show that (13) and (15) are equivalent, the argument for the equivalence between (13) and (14) being analogous. Let $\psi(\zeta) = (1+\zeta^2)^{-1}\zeta$. Then $\psi_t(T^{1/2}) = (1+t^2T)^{-1}tT^{1/2}$ and $\psi_t(T^{1/2})T^{1/2}u = (1+t^2T)^{-1}tTu$ for $u \in \mathcal{D}(T)$ by (8) applied to $T^{1/2}$. Hence, using the quadratic inequality (4) we have

$$||T^{1/2}u|| \sim \left(\int_0^\infty ||(1+t^2T)^{-1}tSBSu||^2 \,\frac{dt}{t}\right)^{1/2}.$$

That (15) implies (13) is then immediate. For the converse, we obtain (15) on the range of S. Observe that since S is one-one and selfadjoint, it has dense range. We conclude the proof by a density argument and (15) holds on H_0. □

As we shall see, this proposition is one key of our approach to the square root problem for differential operators. Still, more work is needed and we have to take into account the concrete nature of differential operators. Indeed, in its full generality, Kato's conjecture has been disproved by McIntosh [**54**], and this sheds light on the limitations of abstract methods from operator theory to solve the problem.

Theorem 6. — *There exists a Hilbert space and a maximal accretive operator L built as in Proposition 1 for which $\mathcal{D}(L^{1/2}) \neq \mathcal{D}(D)$.*

Proof. — We adapt the construction of McIntosh. On $H = \ell^2(\mathbb{Z})$, define an unbounded selfadjoint operator D by $De_j = 2^j e_j$ and a bounded operator B by $Be_j = \sum_{n\in\mathbb{Z}} a_n e_{j+n}$, where (e_j) is the natural hilbertian basis of H and (a_n) is a sequence of complex numbers such that $\hat{a}(\theta) = \sum a_n e^{in\theta}$ satisfies $||\hat{a}||_\infty = 1$. Clearly, the operator

B has norm equal to $||B|| = ||\hat{a}||_\infty = 1$. For $z \in \mathbb{C}$ with $|z| < 1$, $A_z = \mathrm{Id} + zB$ satisfies (9) so that $L_z = DA_zD$ is maximal accretive by Proposition 1. Let $R_z = (L_z)^{1/2}$.

Assume that $||R_z u|| \leq c||Du||$ for all $u \in \mathcal{D}(D)$ and uniformly for $|z| \leq r < 1$. As a function of z, R_z is an operator-valued holomorphic function so that $R_0' D^{-1}$ is bounded on H. Differentiating at $z = 0$ the equation $R_z R_z = L_z$, we find

$$R_0' D + D R_0' = DBD.$$

Solving for R_0' one finds that

$$R_0' e_j = 2^j \sum b_n e_{j+n}\ , \quad b_n = \frac{a_n 2^n}{1+2^n}.$$

Hence, $||R_0' D^{-1}|| = ||\hat{b}||_\infty$ with evident notation. Now take $a_n = i/\pi n$, then

$$\hat{a}(\theta) = -\frac{2}{\pi} \sum_{n>0} \frac{\sin(n\theta)}{n} = \frac{\theta}{\pi} - 1, \quad 0 < \theta < 2\pi,$$

so that $||\hat{a}|| = 1$. But $\hat{b}(\theta) \sim -\frac{i}{\pi} \ln|\sin(\theta/2)|$ near 0 so that $\hat{b}$ is not bounded. This is a contradiction, hence $||R_z u|| \leq c||Du||$ fails for some z. □

0.4. The square root problem for differential operators on $\mathbb{R}^n$

Our main interest in this work is about some questions for second order elliptic operators, including the square root problem. We also introduce higher order elliptic operators in divergence form. We have two motivations for doing so. In the one hand, most of our results are valid independently of the order of the operators. In the other hand, higher order operators will be instrumental in some arguments for the study of second order operators.

We use the classical notation for multiindices $\alpha \in \mathbb{N}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, their length $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the powers $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $\xi \in \mathbb{C}^n$, and the associated partial derivatives $\partial^\alpha = (\partial_{x_1})^{\alpha_1} \cdots (\partial_{x_n})^{\alpha_n}$.

Let $a_{\alpha\beta} \in L^\infty(\mathbb{R}^n; \mathbb{C})$, where α and β are multiindices of length $m \in \mathbb{N}^*$. Define a sesquilinear form on the Sobolev space $H^m(\mathbb{R}^n)$ by

$$J(f,g) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \partial^\beta f(x) \overline{\partial^\alpha g(x)}\, dx. \tag{16}$$

Setting $\nabla^m f = (\partial^\alpha f)_{|\alpha|=m}$ and $||\nabla^m f||_2 = (\int \sum |\partial^\alpha f|^2)^{1/2}$, we have that J is bounded on $H^m(\mathbb{R}^n)$ with

$$|J(f,g)| \leq ||(a_{\alpha\beta})||_\infty ||\nabla^m f||_2 ||\nabla^m g||_2, \tag{17}$$

where $(a_{\alpha\beta})$ is matrix-valued. We assume that the Gårding inequality

$$\mathrm{Re}\, J(f,f) \geq \delta ||\nabla^m f||_2^2, \quad f \in H^m(\mathbb{R}^n), \tag{18}$$

holds for some $\delta > 0$.

Let $H_0 = L^2(\mathbb{R}^n;\mathbb{C})$ and $H_1 = L^2(\mathbb{R}^n;\mathbb{C}^p)$ where $p = \binom{m+n-1}{n-1}$ is the number of those multiindices of length m. Setting $A\colon H_1 \to H_1$ by

$$\langle Au, v\rangle = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)u_\beta(x)\overline{v_\alpha(x)}\, dx, \tag{19}$$

the Gårding inequality (18) reads

$$\operatorname{Re}\langle ADf, Df\rangle \geq \delta\|Df\|_2^2$$

with $D = \nabla^m$. Hence, Proposition 1 applies: one can define $L = D^*AD$ as a maximal accretive operator on $L^2(\mathbb{R}^n)$. Classically, we write

$$L = \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|}\partial^\alpha(a_{\alpha\beta}\partial^\beta) \tag{20}$$

and L is a homogeneous elliptic operator of order $2m$ in divergence form.

When $m = 1$, we identify the multiindices of length 1 with the integers in $\{1,\dots,n\}$. We simply write $L = -\operatorname{div}(A\nabla)$ where A is the matrix (a_{jk}) and $\nabla = \nabla^1$ is the ordinary gradient operator. The Gårding inequality is equivalent to

$$\operatorname{Re} A(x)\xi\cdot\bar{\xi} = \operatorname{Re}\sum_{1\leq j,k\leq n} a_{jk}(x)\xi_k\overline{\xi_j} \geq \delta|\xi|^2 \quad a.e. \quad \xi\in\mathbb{C}^n, \tag{21}$$

which is to say that $A + A^* \geq 2\delta$ a.e. in the sense of selfadjoint operators.

Let us fix some terminology. Given an integer $N \geq 1$, we call $\mathcal{A}(\delta)$ the set of all matrix-valued functions $A(x) \in L^\infty(\mathbb{R}^n, M_N(\mathbb{C}))$ such that $\|A\|_\infty \leq \delta^{-1}$ and $A + A^* \geq 2\delta$ a.e. in the sense of selfadjoint operators. We denote by $\mathcal{A}$ the union of all $\mathcal{A}(\delta)$, $\delta > 0$. We shall say that the matrices in $\mathcal{A}$ are accretive. The context will make the value of N clear. The case where $N = 1$ is that of complex-valued functions: a function $a(x)$ on $\mathbb{R}^n$ is accretive provided $\operatorname{Re} a(x) \geq \delta$ a.e. for some $\delta > 0$.

The equivalence above is a specific feature of second order operators. When $m > 1$ and L has constant coefficients, an argument via the Fourier transform shows that (18) is equivalent to

$$\operatorname{Re}\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi^\beta\xi^\alpha \geq \delta|\xi|^{2m}, \quad \xi\in\mathbb{R}^n,$$

which is weaker than imposing the matrix $(a_{\alpha\beta})$ to be accretive.

When $m > 1$, the Gårding inequality (18) often takes the weaker form

$$\operatorname{Re} J(f,f) \geq \delta\|\nabla^m f\|_2^2 - C\|f\|_2^2, \quad f\in H^m(\mathbb{R}^n), \tag{22}$$

for some $C, \delta > 0$. (See, *e.g.*, **[29, 38]**.) In such a case, Proposition 1 can be used to construct $L + \lambda$ as a maximal accretive operator on $L^2(\mathbb{R}^n)$ for all $\lambda > C$ (see Section 0.7).

Let us now formulate Kato's first conjecture for the square root of differential operators:

Conjecture 1. — *Let L be given by (20) and assume that (18) holds. Then one has that $\mathcal{D}(L^{1/2}) = H^m(\mathbb{R}^n)$.*

By Proposition 3, it is enough to consider either the (homogeneous) inequality

$$\|L^{1/2}f\|_2 \leq c\|\nabla^m f\|_2 \tag{K}$$

or its local (inhomogeneous) version

$$\|L^{1/2}f\|_2 \leq c(\|\nabla^m f\|_2 + \|f\|_2) \tag{K$_{loc}$}$$

with their analogs for L^*.

If (18) is replaced by (22), the conjecture becomes $\mathcal{D}((L+\lambda)^{1/2}) = H^m(\mathbb{R}^n)$ for all (or, equivalently, some) $\lambda > C$.

0.5. The square root problem and perturbation theory

One of the applications of (K) or (K)$_{loc}$ is analytic perturbation theory for partial differential equations.

Let us come back to the general situation of Section 0.3 and consider the following question.

For $s \in (-1,1)$, let $L_s = D^*A_sD$ be the selfadjoint operator associated with D and A_s, where A_s is selfadjoint with $0 < \delta I \leq A_s \leq \delta^{-1}I$ uniformly over $(-1,1)$. What are the regularity properties of $s \to L_s^{1/2}$ in terms of the regularity of $s \to A_s$?

Proposition 7. — *Assume that $s \to A_s$ is continuous into the space of bounded operators on H_1. Then $s \to L_s^{1/2}$ is strongly continuous into the space of bounded operators from $\mathcal{D}(D)$ into H_0.*

Proof. — Observe that there exists a constant $c = c(\delta)$ such that for all s the domain of $L_s^{1/2}$ is $\mathcal{D}(D)$ and that if $f \in \mathcal{D}(D)$,

$$\sup_{0<\alpha<\beta<\infty} \left\| \int_\alpha^\beta L_s e^{-t^2L_s} f\, dt \right\| \leq c\|Df\|.$$

Indeed, the first fact follows from the selfadjointness of L_s and the second one from the functional calculus. It follows from (6) that $\|L_s^{1/2}f - L_0^{1/2}f\|$ will converge to 0 as s tends to 0 provided we can show that, for fixed $0 < \alpha < \beta < \infty$ and $f \in \mathcal{D}(D)$,

$$\left\| \int_\alpha^\beta L_s e^{-t^2L_s} f\, dt - \int_\alpha^\beta L_0 e^{-t^2L_0} f\, dt \right\| \to 0, \quad s \to 0.$$

To this end, observe that $L_s e^{-t^2L_s} = -\frac{1}{2t}\frac{d}{dt}(e^{-t^2L_s})$ and integrate by parts in each integral. The conclusion follows easily by invoking a result of Kato [**48**], p. 504, which asserts that $e^{-t^2L_s}$ converges strongly to $e^{-t^2L_0}$ as s tends to 0. □

Remark. — Let us observe that [**48**], Chapter VII, contains regularity results of a different nature, imposing monotonicity conditions on A_s.

One can ask whether more is true. If $s \to A_s$ is of class C^k, $k \geq 1$, into the space of bounded operators on H_1, can we conclude that $s \to L_s^{1/2}$ is strongly or weakly C^k?

McIntosh proved in **[55]** that this last question has a negative answer.

In the specific case of second order differential operators $-\operatorname{div}(A_s\nabla)$, the above question is still open when $n \geq 2$. Our approach to answering this question in the affirmative is based on the study of the topological properties of the set

$$\mathcal{K} = \{A \in \mathcal{A};\ -\operatorname{div}(A\nabla) \text{ satisfies (K)}\},$$

which is a motivation for using complex coefficients.

Theorem 8. — *Let $O \subset \mathcal{K}$ be open for the $L^\infty(\mathbb{R}^n, M_n(\mathbb{C}))$ topology. Then the mapping $A \to (-\operatorname{div}(A\nabla))^{1/2}$ is norm analytic from O into the space $\mathcal{B}(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n))$ of bounded operators from $H^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Proof. — Pick $A_0 \in O$ and $M \in L^\infty(\mathbb{R}^n, M_n(\mathbb{C}))$, and consider the holomorphic map $z \to A_z = A_0 + zM$ from a sufficiently small complex ball $|z|\|M\|_\infty < \varepsilon$ into O. Set $L_z = -\operatorname{div}(A_z\nabla)$. By the remark at the end of Section 0.2, we may write,

$$(I + t^2L_z)^{-1} - (I + t^2L_0)^{-1} = (I + t^2L_0)^{-1} t \operatorname{div} zMt\nabla(I + t^2L_z)^{-1},$$

so that iterating this equality into (5), one finds that $L_z^{1/2} = (-\operatorname{div}(A_z\nabla))^{1/2}$ has a formal Taylor series expansion

$$L_z^{1/2} = L_0^{1/2} + \sum_{j=1}^{\infty} z^j T_j. \tag{23}$$

On the other hand, the T_j's can be computed by the Cauchy integrals

$$T_j = \frac{1}{2\pi i}\int_{|z|=r} L_z^{1/2}\,\frac{dz}{z^{j+1}}, \quad r\|M\|_\infty < \varepsilon. \tag{24}$$

Hence, by (K) applied to $L_z^{1/2}$ uniformly for $|z| = r$, $\|T_jf\|_2 \leq cr^{-j}\|\nabla f\|_2$. This shows that (23) converges normally in $\mathcal{B}(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n))$ when $|z| < r$. □

Remark. — The control of the T_j's when $A_0 = Id$ is precisely the aim of the multilinear theory used in **[20]**, **[35, 37, 36]**, **[19]**, **[26]**, **[57]**, **[59]**, **[45]**, **[50]**. See also **[22]**, **[23]** for related works. We shall not use multilinear estimates in this work.

In view of Theorem 8, it is natural to formulate the following conjecture.

Conjecture 2. — *$\mathcal{K}$ is open in the L^∞ topology.*

If we can prove this, then in particular, $\mathcal{K}$ will be a neighborhood of any uniformly positive definite selfadjoint matrix $A_0(x)$ and the regularity problem posed above will be solved.

0.6. Connections between second and higher order operators

Higher order operators are naturally involved, even when dealing with the square root problem for second order operators. To explain why, let us state an abstract result proved in [**8**]. A similar statement is used in [**61**] to prove the first Tb-Theorem.

Proposition 9. — *Let $S : H_0 \to H_0$ be a positive selfadjoint operator with domain $\mathcal{D}(S)$ and $B\colon H_0 \to H_0$ be bounded invertible and ω-accretive on H_0. Construct $T = SBS$ and for $r > 0$, $T_r = S^r BS^r$ as in the Proposition 1. Then the square root problems for T and T_r are equivalent. More precisely, $\mathcal{D}(S) \subset \mathcal{D}(T^{1/2})$ with $\|T^{1/2}u\| \leq c\|Su\|$ if and only if $\mathcal{D}(S^r) \subset \mathcal{D}(T_r^{1/2})$ with $\|T_r^{1/2}u\| \leq c\|S^r u\|$.*

In this statement, u is taken in the appropriate space and the constants do not depend on u.

Remark. — One can drop the positivity assumption on S when r is restricted to being a non negative integer.

This proposition applies to the class of maximal accretive operators D^*AD, provided we use the representation in Lemma 4.

Proposition 9 allows us to increase the order of operators (while in [**61**], it was used to lower the order). An example is the following important result for us (see Chapter 2). Let $L = -\operatorname{div}(A\nabla)$ be defined as in Section 0.4, where $A \in \mathcal{A}$ and, for all $k \geq 1$, define

$$L_k = -\Delta^k \operatorname{div}(A\nabla)\Delta^k$$

by the method of sesquilinear forms. The Gårding inequality for the form associated with L clearly implies the corresponding inequality (18) for the form associated with L_k, so that L_k is a maximal accretive differential operator on $L^2(\mathbb{R}^n)$ of order $4k+2$.

Proposition 10. — *Fix $k \geq 1$. Then (K) for L is equivalent to (K) for L_k.*

Proof. — The polar decomposition of ∇ is $\nabla = R(-\Delta)^{-1/2}$, where R is the array of Riesz transforms $R_j = \dfrac{\partial}{\partial x_j}(-\Delta)^{-1/2}$, $j = 1, \ldots, n$. It follows from Lemma 4 that $L = SBS$ with $B = R^*AR$ and $S = (-\Delta)^{1/2}$.

Now, $L_k = S^{2k+1}BS^{2k+1}$. Applying Proposition 9 yields

$$\|L^{1/2}f\|_2 \leq c\|Sf\|_2 \quad \text{if and only if} \quad \|L_k^{1/2}f\|_2 \leq c\|S^{2k+1}f\|_2.$$

To conclude the argument, it remains to observe that $\|Sf\|_2 = \|\nabla f\|_2$ and that $\|S^{2k+1}f\|_2 = \|\nabla\Delta^k f\|_2 \sim \|\nabla^{2k+1}f\|_2$. The last equivalence follows from Plancherel theorem for the Fourier transform. □

0.7. Perturbations with lower order terms

In this section, we consider inhomogeneous elliptic operators of arbitrary order. They are obtained by perturbing with lower order terms the homogeneous operators defined in Section 0.4. A precise definition is given after we prove an abstract result that, as far as the square root problem is concerned, allows us to dispose of these terms. In other words, it will be sufficient to study the square root problem for homogeneous differential operators.

Here is the abstract setting. Consider a one-one selfadjoint operator S acting on a Hilbert space H_0. Let $m \in \mathbb{N}^*$ and for $0 \leq k, \ell \leq m$, let $B_{k\ell}$ be a bounded operator on H_0 so that Proposition 1 applies to the matrix of operators $B = (B_{k\ell})_{0\leq k,\ell\leq m}$ acting on $H_1 = H_0^{m+1}$ and to $D = (I, S, \dots, S^m)^T$ (where ^{T} stands for transpose in the sense of vectors) with domain $\mathcal{D}(S^m)$. Call $L = D^*BD$ the maximal operator thus obtained. Expanding L formally yields

$$L = \sum_{0\leq k,\ell\leq m} S^k B_{k\ell} S^\ell. \tag{25}$$

Set $L_0 = S^m B_{mm} S^m$ where B_{mm} is assumed, in addition, to be bounded and ω-accretive on H_0 for some $\omega < \frac{\pi}{2}$.

One can view L as a perturbation of L_0 with lower order terms. The perturbation result takes the following form.

Proposition 11. — *The inequality $\|L^{1/2}f\| \leq c\|Df\| \sim \|S^m f\| + \|f\|$ is a consequence of $\|L_0^{1/2}f\| \leq c\|S^m f\|$.*

Proof. — We begin with an application of Proposition 5 to L_0. Changing t to t^m for reasons of homogeneity, and using the invertibility of B_{mm}, we get

$$\left(\int_0^\infty \|(1+t^{2m}L_0)^{-1}t^m S^m u\|^2 \, \frac{dt}{t}\right)^{1/2} \leq c\|u\| \tag{26}$$

for all $u \in H_0$.

Now, in view of the following result which we admit for the moment, we replace the resolvent of L_0 by the resolvent of L.

Lemma 12. — *There exist $t_0 > 0$ and $c_0 > 0$ such that for all $0 < t \leq t_0$ and $u \in H_0$,*

$$\|\big[(1+t^{2m}L)^{-1} - (1+t^{2m}L_0)^{-1}\big]t^m S^m u\| \leq c_0 t\|u\|.$$

From (26) and the lemma, we obtain

$$\left(\int_0^{t_0} \|(1+t^{2m}L)^{-1}t^m S^m u\|^2 \, \frac{dt}{t}\right)^{1/2} \leq c\|u\|.$$

Next, if $k = 0, \dots, m-1$, interpolation and the resolvent estimates of Proposition 2 yield $||(1+t^{2m}L)^{-1}t^kS^ku|| \leq c||u||$, hence

$$\left(\int_0^{t_0} ||(1+t^{2m}L)^{-1}t^mS^ku||^2 \frac{dt}{t}\right)^{1/2} \leq c\left(\int_0^{t_0} t^{2m-2k}\frac{dt}{t}\right)^{1/2} ||u|| \leq c||u||.$$

Therefore, writing $L = \sum S^kB_{k\ell}S^\ell$ and using the boundedness of each $B_{k\ell}$, we have

$$\left(\int_0^{t_0} ||(1+t^{2m}L)^{-1}t^mLu||^2 \frac{dt}{t}\right)^{1/2} \leq c\sum_{\ell=0}^{m} ||S^\ell u|| \sim ||Du||.$$

Finally, since $||(I+t^{2m}L)t^mLu|| \leq ct^{-m}||u||$ when $u \in \mathcal{D}(L)$ by (7), we obtain

$$\begin{aligned} ||L^{1/2}u|| &\leq c\left(\int_0^{\infty} ||(1+t^{2m}L)^{-1}t^mLu||^2\frac{dt}{t}\right)^{1/2} \\ &\leq c\left(\int_0^{t_0} ||(1+t^{2m}L)^{-1}t^mLu||^2\frac{dt}{t}\right)^{1/2} + c\left(\int_{t_0}^{\infty} t^{-2m}\frac{dt}{t}\right)^{1/2} ||u|| \\ &\leq c||Du|| + c||u||. \end{aligned}$$

□

Remark. — A slight modification of the proof shows that the hypothesis can be replaced by $||L_0^{1/2}f|| \leq c(||S^mf|| + ||f||)$. A result of this type appears first in **[13]**, with the limitation that $m = 1$ and stated in a weaker form with equivalences instead of inequalities. The argument given there, relying on Proposition 3, does not seem to generalize.

It remains to prove Lemma 12. Thanks to the remark at the end of Section 0.2, we may compare the resolvents of L and L_0. By the classical Neumann series expansion, writing $L = L_0 - R$ we obtain,

$$(1+t^{2m}L)^{-1} = \sum_{j=0}^{\infty}((1+t^{2m}L_0)^{-1}t^{2m}R)^j(1+t^{2m}L_0)^{-1}.$$

Hence,

$$[((1+t^{2m}L)^{-1} - (1+t^{2m}L_0)^{-1}]t^mS^m = \sum_{j=1}^{\infty} M_j,$$

where

$$M_j = ((1+t^{2m}L_0)^{-1}t^{2m}R)^j(1+t^{2m}L_0)^{-1}t^mS^m.$$

The operator R is the sum of $(m+1)^2 - 1$ operators $S^kB_{k\ell}S^\ell$ with $k+\ell < 2m$, so that M_j is the sum of $(m^2+2m)^j$ operators of the form

$$(1+t^{2m}L_0)^{-1}t^mS^{k_1}\left\{\prod_{s=1}^{j} B_{k_s\ell_s}t^mS^{\ell_s}(1+t^{2m}L_0)^{-1}t^mS^{k_{s+1}}\right\}, \tag{27}$$

where $0 \leq k_s, \ell_s \leq m$, $k_{j+1} = m$ and $k_s + \ell_s < 2m$. The resolvent estimates of Proposition 2 and interpolation yield

$$||t^k S^k (1 + t^{2m} L_0)^{-1} t^\ell S^\ell u|| \leq c||u||$$

uniformly over $0 \leq k, \ell \leq m$. Therefore, using also that $||B_{k\ell}|| \leq ||B||$, we have if $t \leq 1$,

$$\begin{aligned} ||(27)|| &\leq c^{j+1} ||B||^j \prod_{s=1}^{j} t^{2m-k_s-\ell_s} \\ &\leq c(c||B||t)^j, \end{aligned}$$

which implies that

$$\sum_{j=1}^{\infty} ||M_j u|| \leq c \sum_{j=1}^{\infty} (c(m^2 + 2m)||B||t)^j ||u||.$$

This series converges when, in addition, $c(m^2 + 2m)||B||t \leq 1/2$. This ends the proof of Lemma 12.

An inhomogeneous elliptic operator of order $2m$ in divergence form is defined via the method of sesquilinear forms and is formally written as

$$L = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_{\alpha\beta} \partial^\beta). \tag{28}$$

All coefficients $a_{\alpha\beta}$ are bounded complex-valued functions. The Gårding inequality (18) or (22) is assumed on the homogeneous part, which we denote by L_0, so that the Gårding inequality for L reads

$$\operatorname{Re} \langle Lf, f \rangle \geq \delta ||\nabla^m f||_2^2 + C||f||_2^2 \tag{29}$$

for some $C \in \mathbb{R}$ and $\delta > 0$. Up to changing the 0th order term in L we can make $C \geq \delta$ in (29), which we do.

Let us see why L is maximal accretive on $L^2(\mathbb{R}^n)$. Once again, this follows on applying Proposition 1. With the notation in Section 0.4, one has the following representations, which are rigourously justified by systematically going back to the corresponding sesquilinear forms. Letting $A_{k\ell}$ be the matrix with entries $a_{\alpha\beta}$, $|\alpha| = k$, $|\beta| = \ell$, one first obtains

$$L = \sum_{0 \leq k, \ell \leq m} (-1)^k \nabla^k \cdot A_{k\ell} \nabla^\ell \tag{30}$$

(by convention, $\nabla^0 = I$). Regrouping terms, one obtains $L = D^* A D$ with A being the matrix with matrix-valued entries $A_{k\ell}$, and D being the array of higher gradients $(\nabla^k)_{k=0}^m$, which is a closed linear operator with domain $H^m(\mathbb{R}^n)$.

To apply Proposition 11 to L and L_0, we keep transforming (30), by factoring out powers of the Laplacian. Set $S = (-\Delta)^{1/2}$ and

$$B_{k\ell} = (-1)^k (-\Delta)^{-k/2} \nabla^k \cdot A_{k\ell} \nabla^\ell (-\Delta)^{-\ell/2}.$$

From Plancherel theorem, $(-\Delta)^{-k/2}\nabla^k$ is bounded on L^2, hence $B_{k\ell}$ is a bounded operator. We obtain a representation in the form (25) for L and a similar one for L_0, and the properties needed for Proposition 11 to apply are easily verified. Consequently, we have

Corollary 13. — *The inequality (K) for L_0 implies $(K)_{loc}$ for L.*

A similar result was proved by Evans under the additional assumption that the leading term L_0 be selfadjoint and is described in **[55]** with a completely different argument.

Remarks

1. It can be observed that the nature of the perturbation to L_0 is irrelevant: the multiplication by $a_{\alpha\beta}$ entering in the perturbation can be replaced by the action of any bounded operator on $L^2(\mathbb{R}^n)$.
2. In fact, inequalities (11) and (12) are of the same type. More precisely, the inhomogeneous equivalence (11) applied to L corresponds to the homogeneous equivalence (12) applied to $L+I$. Indeed, if $L = D^*AD$, one can write $L+I = \widetilde{D}^*\widetilde{A}\widetilde{D}$ where $\widetilde{D}u = (Du, u)^T$ and $\widetilde{A}$ is the matrix of operators
$$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$
Start from the homogeneous equation (12) for $L+I$, that is $\|(L+I)^{1/2}u\| \sim \|\widetilde{D}u\|$. Now, by functional calculus, $\|(L+I)^{1/2}u\| \sim \|L^{1/2}u\| + \|u\|$ (observe that the function $\zeta \to |(\zeta+1)(\zeta^2+1)^{-1/2}|$ is bounded away from 0 and ∞ in a conic neighborhood of the spectrum of $-L^{1/2}$, which is contained in $|\arg\zeta| \leq \omega/2 < \pi/4$). Since $\|\widetilde{D}u\| \sim \|Du\| + \|u\|$, we see that (12) rewrites $\|L^{1/2}u\| + \|u\| \sim \|Du\| + \|u\|$ which is (11) for L.

0.8. Change of variables

We restrict our discussion to second order operators.

Consider $L_A = -\operatorname{div}(A\nabla)$ where $A \in \mathcal{A}(\delta)$ as in Section 0.4. Let $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz change of variables on $\mathbb{R}^n$. Denote by J_ϕ its jacobian matrix. For $u, v \in H^1(\mathbb{R}^n)$, putting $x = \phi(y)$, we have
$$\int_{\mathbb{R}^n} A(x)\nabla u(x) \cdot \nabla v(x)\, dx = \int_{\mathbb{R}^n} B(y)\nabla(u\circ\phi)(y) \cdot \nabla(v\circ\phi)(y)\, dy, \tag{31}$$
where
$$B(y) = |\det J_\phi(y)|\, {}^tJ_\phi^{-1}(y) A(\phi(y)) J_\phi^{-1}(y). \tag{32}$$
Clearly $B \in \mathcal{A}$: consider analogously $L_B = -\operatorname{div}(B\nabla)$.

Define an isomorphism on $L^2(\mathbb{R}^n)$ by setting $Vu = u \circ \phi$. In terms of operators the equality (31) means that

$$L_A = V^* L_B V$$

with the equality of domains. Since V is also an isomorphism on $H^1(\mathbb{R}^n)$ with $\|\nabla u\|_2 \sim \|\nabla(Vu)\|_2$, the Kato square root problem is invariant under conjugation by V: in particular, we have $\|L_B^{1/2} f\|_2 \leq c\|\nabla f\|_2$ if and only if $\|(V^{-1}L_B V)^{1/2} f\|_2 \leq c\|\nabla f\|_2$. Now, a straightforward calculation yields

$$V^*u(x) = |\det J_{\phi^{-1}}(x)| V^{-1}u(x).$$

Hence,

$$V^{-1}L_B V = mL_A,$$

where m denotes the multiplication by $m(x) = |\det J_{\phi^{-1}}(x)|^{-1}$. Thus, $\|L_B^{1/2} f\|_2 \leq c\|\nabla f\|_2$ if and only if

$$\|(mL_A)^{1/2} f\|_2 \leq c\|\nabla f\|_2. \tag{33}$$

Let us explain the meaning of $(mL_A)^{1/2}$. Since m is a bounded and non-negative function with bounded inverse, mL_A is the maximal-accretive operator built on $H = L^2(\mathbb{R}^n, dx/m(x))$ via the equality

$$\langle mL_A u, v\rangle_H = \langle L_A u, v\rangle.$$

Hence $(mL_A)^{1/2}$ is well-defined on H. Moreover, the study of its domain on H or on $L^2(\mathbb{R}^n)$ is the same, these two spaces being equal with equivalence of norms. As we now see, this domain does not depend on m.

Lemma 14. — *Let $L = -\operatorname{div}(A\nabla)$ where $A \in \mathcal{A}(\delta)$ and m be a non-negative function with $m, m^{-1} \in L^\infty(\mathbb{R}^n)$. Then for $f \in \mathcal{D}(L)$,*

$$c_1\|L^{1/2} f\|_2 \leq \|(mL)^{1/2} f\|_2 \leq c_2\|L^{1/2} f\|_2, \tag{34}$$

where c_1 and c_2 depend only on $\|m\|_\infty$, $\|m^{-1}\|_\infty$.

Proof. — From the discussion above and Section 0.1, we can use quadratic functionals to compute $\|(mL)^{1/2} f\|_2^2$ when $f \in \mathcal{D}(L)$ to obtain

$$\begin{aligned}\|(mL)^{1/2} f\|_2^2 \sim \|(mL)^{1/2} f\|_H^2 &\sim \int_0^\infty \|(1 + t^2 mL)^{-1} tmLf\|_H^2 \frac{dt}{t} \\ &\sim \int_0^\infty \|(m^{-1} + t^2 L)^{-1} tLf\|_2^2 \frac{dt}{t}.\end{aligned} \tag{35}$$

To end the proof, it remains to compare $(m^{-1} + t^2 L)^{-1}$ with $(1 + t^2 L)^{-1}$ for each $t > 0$. If $g \in L^2$, we have

$$c_1\|(1 + t^2 L)^{-1} g\|_2 \leq \|(m^{-1} + t^2 L)^{-1} g\|_2 \leq c_2\|(1 + t^2 L)^{-1} g\|_2$$

with $c_1^{-1} = 1 + \|m^{-1} - 1\|_\infty$ and $c_2 = 1 + \|m\|_\infty \|m^{-1} - 1\|_\infty$. The first inequality can be seen by writing

$$(1 + t^2L)^{-1}g = (m^{-1} + t^2L)^{-1}g - (1 + t^2L)^{-1}(m^{-1} - 1)(m^{-1} + t^2L)^{-1}g$$

and using the fact that $(1+t^2L)^{-1}$ is a contraction on L^2. The second one is obtained similarly. □

Proposition 15. — *Let $A \in \mathcal{A}(\delta)$ and B be related to A by (32) under a bilipschitz change of variables. Then (K) holds for L_B if and only if (K) holds for L_A. In particular, $\mathcal{D}(L_B^{1/2}) = H^1(\mathbb{R}^n)$ if and only if $\mathcal{D}(L_A^{1/2}) = H^1(\mathbb{R}^n)$.*

The proof is clear from the above discussion and Lemma 14. Also changing A to A^* makes B become B^*, so that the same argument applies for adjoints.

Remarks

1. Let φ be some holomorphic function for which $\varphi(L_B)$ makes sense. If $K(x, y)$ denotes the Schwartz kernel of $\varphi(L_B)$, then the Schwartz kernel of $\varphi(mL_A) = V^{-1}\varphi(L_B)V$ is

   $$K(\phi^{-1}(x), \phi^{-1}(y))|\det J_{\phi^{-1}}(y)| = K(\phi^{-1}(x), \phi^{-1}(y))m^{-1}(y).$$

   Since ϕ^{-1} is Lipschitz, estimates such as pointwise upper bounds and Hölder regularity with exponent between 0 and 1 are the same for the kernels $K(x, y)$ and $K(\phi^{-1}(x), \phi^{-1}(y))$. In particular, the property (G) which is defined in Chapter 1 will be stable under the change of variables, provided the function $m^{-1}(y)$ is considered as a weight. See Chapter 1.
2. For any bounded and accretive function m on $\mathbb{R}^n$ and any L as above, one can make sense of mL using the theory of operators of type ω and prove that $\mathcal{D}((mL)^{1/2}) = H^1(\mathbb{R}^n)$ if and only if $\mathcal{D}(L^{1/2}) = H^1(\mathbb{R}^n)$. The argument presented here breaks down, because (35) is not obviously true as mL is no longer maximal accretive. A quite sophisticated argument which is out of the scope of this work follows from interpolation results and the (non-trivial) fact that $m(-\Delta)^{1/2}$ has an H^∞-functional calculus on $L^2(\mathbb{R}^n)$. See Section 9 and Theorem 10.1 in **[8]** for more details, and also **[9]** and **[1]**.
3. The statement in Lemma 14 and its generalization discussed in the previous remark are specific to the concrete nature of the operators. Consider a bounded invertible ω-accretive operator M on a Hilbert space H with $0 \le \omega < \pi/2$ and $L = D^*AD = SBS$ as in Proposition 1 and Lemma 4. Then **[8]** shows that for M as above $\mathcal{D}((ML)^{1/2}) = \mathcal{D}(D) = \mathcal{D}(S)$ if, and only if, MS and BS have an H^∞-functional calculus on H. It also applies to L, in which case $M = I$. But, since S is selfadjoint, S has such a functional calculus. Hence if BS has an H^∞-functional calculus on H but MS does not, then $\mathcal{D}((ML)^{1/2})$ and $\mathcal{D}(L^{1/2})$ do not agree. See **[62]** for examples of such operators M.

0.9. Further comments

Should one use the representation D^*AD or the representation SBS in studying the square root problem for differential operators? Recall that the first one comes from Proposition 1 and the second one from Lemma 4.

From the point of view of operator theory, the second one is certainly the easiest to work with because S is selfadjoint (its positiviness being a marginal feature) and B is bounded and ω-accretive. Also, because of Proposition 5, there is a direct connection with square function estimates.

From the point of view of PDE's and harmonic analysis, the second one seems to be the worst because B is an unfriendly operator in dimensions larger than two: a pointwise multiplication sandwiched between non invertible Calderón-Zygmund operators (see Chapter 2). In the first representation, the action of B is replaced by the action of A which is a much nicer operator: a pointwise multiplication. As we see in Chapter 2, this representation leads to square function estimates of a different nature. However, a drawback of this representation is that D^* has a large kernel space, in other words the partial isometry U such that $D = U(D^*D)^{1/2}$ is non invertible (using PDE terminology, we would say that the divergence operator has characteristics).

The conclusion is that both representations have specific features which complement each other. As we mix techniques from operator theory, PDE's and harmonic analysis, our idea is to take advantage of both.

In one dimension, where $D = d/dx$, the partial isometry U in the polar decomposition of D is the Hilbert transform, an operator that is invertible. Hence, there is only a superficial difference between the two representations. The moral is that any representation enjoys the properties of both, which is an indication on why this case is favorable.

To conclude these Preliminaries, let us observe that the representations (5) and (6) lead to studying a class of singular integrals that is most efficiently analyzed using real variable analysis. This is the content of Chapter 4. Similarly, the quadratic functionals arising from (4) as in Proposition 5 are of Littlewood-Paley-Stein type. Their analysis led us to extend the modern theory of quadratic functionals and, in particular, their connections with Carleson measures. This aspect is studied in Chapter 2. Applications towards positive answers to the square root problem are in Chapter 3. To make this program work, we need estimates on the distributional kernel of functions of L, and especially on the heat kernel, which is the main topic of Chapter 1.

CHAPTER 1

GAUSSIAN ESTIMATES

1.1. Introduction

In this chapter, we make a thorough analysis of the heat kernels of the elliptic operators introduced in Preliminaries. Our motivation for doing so is that estimates on heat kernels are instrumental in studying the square root problem. Nevertheless, this chapter has its own interest and can be read independently of the square root problem.

For uniformly elliptic operators of order 2 in divergence form with real measurable coefficients, elliptic and parabolic regularity theories are well understood. Elliptic theory relies on the fundamental work of De Giorgi [**31**], and on that of Morrey [**64**]. Parabolic regularity was done by Nash [**66**]; in addition, Moser established Harnack inequalities [**65**]. The link with Gaussian estimates of heat kernels was done by Aronson [**4**], while Fabes and Stroock [**34**] proved that pursuing further ideas of Nash already leads to Gaussian upper and lower bounds and to Harnack inequalities.

It is impossible for us to draw a complete list of the consequences that these results had in many fields in analysis. Let us just mention a few. The elliptic theory is related to questions in the calculus of variations and in non-linear elliptic equations [**38**], [**40**]. It also has strong connection with harmonic analysis via the study of harmonic measure (see, *e.g.*, [**49**] for a general overview). Parabolic theory and semigroup techniques are important toward spectral theory on manifolds, the study of geometry on groups and manifolds, the study of Markov processes and random walks on graphs. See [**28, 67, 68, 80**] for updates until the end of the 80's, though these topics are evolving quite rapidly.

It is a natural question (and, for us, an important one towards the study of the square root problem) to wonder what properties such as the Gaussian decay and the regularity properties of heat kernels remain valid in the case of complex coefficients. It turns out that these two properties may fail, as well as the maximum principle [**7**]. To obtain positive results, we cannot then rely on the techniques found in [**28, 67, 80**].

We use instead a quantitative formulation of the general principle that parabolic and elliptic theories be strongly related, following [**5**].

There are multiple consequences of Gaussian decay and regularity properties for heat kernels of operators with complex coefficients. Among them are estimates on their spatial derivatives which are essential in many aspects of our study of the square root problem. Let us also mention a new analytic pertubation result for heat kernels of operators with real coefficients.

Section 0.6 of Preliminaries shows that in the study of the square root problem, we can replace $-\operatorname{div}(A\nabla)$ by a higher order elliptic operator. This is of particular interest when the heat kernel for $-\operatorname{div}(A\nabla)$ does not have good properties. For this reason, we also study decay and regularity properties for heat kernels of complex elliptic operators of order $2m$, $m \geq 1$. We prove that decay and regularity hold when $2m \geq n$ (the case $2m > n$ is also treated in [**29**]).

1.2. Gaussian estimates for second order operators

For a measurable function $A : \mathbb{R}^n \to M_n(\mathbb{C})$, set

$$||A||_\infty : = \sup\{|A(x)\xi \cdot \bar{\eta}|, x \in \mathbb{R}^n, \ \xi, \eta \in \mathbb{C}^n, |\xi| = |\eta| = 1\}.$$

Recall that for $\delta > 0$, $\mathcal{A}(\delta)$ denotes the class of uniformly elliptic matrices with ellipticity constant δ, that is

$$||A||_\infty \leq \delta^{-1} \quad \text{and} \quad \operatorname{Re} A(x)\xi \cdot \bar{\xi} \geq \delta|\xi|^2, \quad a.e., \quad \xi \in \mathbb{C}^n,$$

and that $\mathcal{A}$ is the union of all $\mathcal{A}(\delta)$, $\delta > 0$. Next, define $\mathcal{E}$ as the union of all $\mathcal{E}(\delta)$, the latter being the class of maximal accretive operators $L = -\operatorname{div}(A\nabla)$ on $L^2(\mathbb{R}^n)$ for some $A \in \mathcal{A}(\delta)$, given by

$$\langle Lf, g\rangle = \int A\nabla f \cdot \overline{\nabla g}, \quad f \in \mathcal{D}(L), \quad g \in H^1(\mathbb{R}^n), \tag{1}$$

and constructed as in Section 0.4 of Preliminaries.

For convenience, let us summarize some properties of L in this specific situation. Recall that $\Gamma_\mu = \{z \in \mathbb{C}^*; |\arg z| < \mu\}$ and set $\partial\Gamma_\mu$ the boundary of Γ_μ.

(i) $\mathcal{D}(L)$ is a dense subspace in $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ for the respective topologies.

(ii) L is one-one and ω-accretive with $\omega = \sup\{|\arg A(x)\xi \cdot \bar{\xi}|; x \in \mathbb{R}^n, \xi \in \mathbb{C}^n\} < \pi/2$ and, if $\lambda \in \Gamma_{\pi-\omega}$,

$$||(L+\lambda)^{-1}f||_2 \leq \frac{1}{\operatorname{dist}(\lambda, \partial\Gamma_{\pi-\omega})}||f||_2, \quad f \in L^2(\mathbb{R}^n). \tag{2}$$

(iii) $-L$ generates a holomorphic contraction semigroup on $L^2(\mathbb{R}^n)$:

$$||e^{-zL}f||_2 \leq ||f||_2, \qquad |\arg z| < \frac{\pi}{2} - \omega. \tag{3}$$

(iv) The resolvent and semigroup satisfy the basic L^2 estimates which are stated in the next result.

Proposition 1. — *Let $L \in \mathcal{E}(\delta)$ and ω be as above. Fix $\nu \in (\omega, \pi/2)$. There exists $c = c(n, \delta, \nu)$ such that for all $\lambda \in \Gamma_{\pi-\nu}$, $j, k \in \{1, \dots, n\}$ and $f \in H^1(\mathbb{R}^n)$,*

$$|\lambda|^{1/2}\|D_j(L+\lambda)^{-1}f\|_2 + |\lambda|^{1/2}\|(L+\lambda)^{-1}D_k f\|_2 + \|D_j(L+\lambda)^{-1}D_k f\|_2 \le c\|f\|_2 \tag{4}$$

and for all $z \neq 0$ with $|\arg z| \le \pi/2 - \nu$, $j, k \in \{1, \dots, n\}$ and $f \in H^1(\mathbb{R}^n)$,

$$\left\|\frac{d}{dz}e^{-zL}f\right\|_2 \le \frac{c}{|z|}\|f\|_2 \tag{5}$$

and

$$|z|^{1/2}\|D_j e^{-zL}f\|_2 + |z|^{1/2}\|e^{-zL}D_k f\|_2 + |z|\|D_j e^{-zL}D_k f\|_2 \le c\|f\|_2. \tag{6}$$

We have set $D_j = \partial/\partial x_j$.

(v) $L^* \in \mathcal{E}(\delta)$, and it is associated with the matrix A^* (that is $\mathcal{E}(\delta)$ is stable under taking adjoints).

(vi) Defining $(U_{x_0,s}f)(x) = f(\frac{x - x_0}{s})$ the operator of dilation by $s > 0$ and translation by $x_0 \in \mathbb{R}^n$, then $U_{x_0,s}^{-1}LU_{x_0,s} \in \mathcal{E}(\delta)$ and is associated with the matrix $A(sx + x_0)$ (that is, $\mathcal{E}(\delta)$ is invariant under translations and dilations).

If $L \in \mathcal{E}(\delta)$, we denote by $K_t(x, y) \in \mathcal{D}'(\mathbb{R}^{2n})$ the Schwartz kernel of e^{-tL}. As is customary, we use the terminology "heat kernel" when speaking of $K_t(x, y)$.

Definition 2. — L has the Gaussian property (G) if $K_t(x, y)$ is, for each $t > 0$, a Hölder continuous function in x and in y and if there exist constants c, $\mu > 0$ and $\beta > 0$ such that for all $t > 0$ and $x, y, h \in \mathbb{R}^n$,

$$|K_t(x,y)| \le \frac{c}{t^{n/2}} \exp\left\{-\frac{\beta|x-y|^2}{t}\right\}, \tag{7}$$

$$|K_t(x,y) - K_t(x+h,y)| \le \frac{c}{t^{n/2}} \left(\frac{|h|}{t^{1/2} + |x-y|}\right)^{\mu} \exp\left\{-\frac{\beta|x-y|^2}{t}\right\} \tag{8}$$

and

$$|K_t(x,y+h) - K_t(x,y)| \le \frac{c}{t^{n/2}} \left(\frac{|h|}{t^{1/2} + |x-y|}\right)^{\mu} \exp\left\{-\frac{\beta|x-y|^2}{t}\right\} \tag{9}$$

whenever $2|h| \le t^{1/2} + |x - y|$.

Definition 3. — L has the local Gaussian property $(\mathrm{G})_{loc}$ if the same inequalities hold for $0 < t \le 1$.

Remarks

1. Observe that the property (G) is preserved under taking adjoints. It is also preserved by scaling and translation. Indeed, fix $s > 0$ and $x_0 \in \mathbb{R}^n$ and let $A \in \mathcal{A}(\delta)$ and $K_t(x, y)$ be the heat kernel of $-\operatorname{div}(A\nabla)$. Then the heat kernel of $U_{x_0,s}^{-1}LU_{x_0,s}$ is $s^{-n/2}K_{t/s^{1/2}}(\frac{x - x_0}{s^{1/2}}, \frac{y - x_0}{s^{1/2}})$.

2. Assume that (7) is valid. Then (8) is equivalent to the following: there exist constants c and $\nu > 0$ such that for all $t > 0$, $x, y, h \in \mathbb{R}^n$,

$$|K_t(x+h,y) - K_t(x,y)| \leq \frac{c}{t^{n/2}} \left(\frac{|h|}{t^{1/2}}\right)^{\nu}. \tag{10}$$

Inequality (10) is easier to prove but (8) is more useful in pratice. Another way of writing (10) is by means of the $\dot{C}^{\nu}(\mathbb{R}^n)$ semi-norm:

$$|f|_{\dot{C}^{\nu}} := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\nu}} = \sup_{x,h\neq 0} \frac{|f(x+h)-f(x)|}{|h|^{\nu}}. \tag{11}$$

Then (10) is equivalent to

$$\sup_{y\in\mathbb{R}^n} |K_t(\cdot,y)|_{C^{\nu}} \leq ct^{-(n+\nu)/2}. \tag{12}$$

3. Assume that L has the property $(\mathrm{G})_{loc}$. Using the semigroup property

$$e^{-tL}e^{-sL} = e^{-(t+s)L}$$

and the fact that the convolution of two Gaussian functions is again a Gaussian function, one sees that the inequalities (7-9) hold for all $t > 0$ with c replaced by $c\Phi(t)$, where $\Phi\colon [0,\infty) \to [1,\infty)$ is continuous with $\Phi(0) = 1$ and non decreasing. Reciprocally, if (7-9) hold with c replaced by $c\Phi(t)$ with Φ as above, then L has the property $(\mathrm{G})_{loc}$. Typically $\Phi(t) = e^{at}$ for some $a > 0$, but it is also possible to obtain polynomial growth (see Section 1.4.4). Finally, note that if L has the property $(\mathrm{G})_{loc}$ then for $\lambda > 0$ large enough, $L + \lambda$ has the property (G) with c replaced by ce^{-at} for some $a > 0$ (see also Theorem 18).

Of course, (G) holds for constant coefficient elliptic operators, and when the coefficients are smooth the local Gaussian property is obtained by classical means such as the point freezing technique of Korn. Here, we seek estimates under minimal regularity assumptions.

The results of Nash and Aronson mentioned in the introduction show the following.

Theorem 4. — *If $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ where A has real entries then L has the Gaussian property (G).*

Unfortunately, in the case of complex entries, Theorem 4 is no longer true.

Theorem 5. — *If $n \geq 5$, there is a complex elliptic $L = -\operatorname{div}(A\nabla) \in \mathcal{E}$ for which e^{-tL} is not bounded on $L^{\infty}(\mathbb{R}^n)$ for any $t > 0$. In particular, L does not have the Gaussian property (G).*

We give in Section 1.3 a proof of the failure of (7) in (G) and refer the reader to **[7]** for a complete argument (see also **[30]** for a simplified argument).

Still, there are some positive results. In order to state them, we introduce the following notation.

For a measurable function f defined on $\mathbb{R}^n$ and $\rho > 0$, set

$$\omega_\infty(f,\rho) = \sup\{|f(x) - f(y)| \,;\, |x-y| \le \rho\}$$

and

$$\omega_2(f,\rho) = \sup_{B_r\,;\,0<r\le\rho} \left(\frac{1}{|B_r|} \int_{B_r} |f - m_r f|^2 \right)^{1/2},$$

where the supremum is taken over all Euclidean balls B_r with radius $r \le \rho$ and $m_r f$ is the mean of f over B_r.

Denote by BUC the space of bounded uniformly continuous functions on $\mathbb{R}^n$. It is characterized by $\|f\|_\infty < \infty$ and $\lim_{\rho\to 0} \omega_\infty(f,\rho) = 0$. Observe then that $\inf_{\rho>0} \omega_\infty(f,\rho)$ is equivalent to the distance of f to BUC in $L^\infty(\mathbb{R}^n)$.

The BMO norm of f is $\|f\|_{BMO} = \sup_{\rho>0} \omega_2(f,\rho)$. A particular subspace of BMO is defined by $\|f\|_\infty < \infty$ and $\inf_{\rho>0} \omega_2(f,\rho) = 0$: this is $L^\infty(\mathbb{R}^n) \cap vmo$ (see **[41]** for a definition of vmo). More generally, one can show that for f bounded, $\inf_{\rho>0} \omega_2(f,\rho)$ is equivalent to the distance of f to BUC in BMO.

Theorem 6. — *Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$.*

(i) *L has the Gaussian property (G) when $n = 1$ and $n = 2$.*

(ii) *If L has the Gaussian property (G) then so does $-\operatorname{div}(A'\nabla)$ when $\|A-A'\|_\infty < \varepsilon$ for some $\varepsilon = \varepsilon(n,\delta) > 0$.*

(iii) *There exists an $\varepsilon = \varepsilon(n,\delta) > 0$ such that if $\|A\|_{BMO} < \varepsilon$ then L has the Gaussian property (G).*

(iv) *There exists an $\varepsilon = \varepsilon(n,\delta) > 0$ such that if $\inf_{\rho>0} \omega_\infty(A,\rho) < \varepsilon$ then L has the Gaussian property $(G)_{loc}$.*

(v) *There exists an $\varepsilon = \varepsilon(n,\delta) > 0$ such that if $\inf_{\rho>0} \omega_2(A,\rho) < \varepsilon$ then L has the Gaussian property $(G)_{loc}$.*

Part (i) is taken from **[10, 11]**. Parts (ii) and (iv) are due to one of us **[5]**. The refinement to obtain part (iii) and (v) arose in discussions with L. Escauriaza.

Let us comment on this theorem. Its proof is in Section 1.4.6.

Because of (i) and of Theorem 4, the items (ii) to (v) are of interest only when $n \ge 3$ and the coefficients are complex-valued. For example, in terms of matrices (assumed to be in $\mathcal{A}(\delta)$), the following cases are covered: L^∞ perturbations of real matrices (part (ii)), L^∞ pertubations of constant complex matrices (part (ii) or (iii)), BUC and vmo matrices and their BMO pertubations (parts (iv) and (v)).

Note that (iv) is a consequence of (v) since $\omega_2 \le \omega_\infty$.

An equivalent formulation of part (ii) is that the set of matrices $A \in \mathcal{A}$ for which $-\operatorname{div}(A\nabla)$ has the Gaussian property is open in L^∞ (see Section 1.6 for more about this). We claim that this cannot be true in BMO. Indeed, consider the operator $L = -\operatorname{div}(A\nabla)$ of the counterexample stated in Theorem 5. By part (ii), if $\varepsilon > 0$ is small $-\operatorname{div}(I + \varepsilon A)\nabla$ has the Gaussian property (G). But $I + \varepsilon A$ and εA are equal in BMO, which proves the claim.

Remark. — If the coefficients have further regularity such as Hölder continuity it is shown in **[10, 11]** that not only L has the property $(G)_{loc}$ but the heat kernel has also pointwise gradients in x and y each satisfying equations (7-9) for $0 < t \leq 1$ up to a normalizing factor $t^{-1/2}$. As for the Green's kernel of the elliptic equation $Lu = 0$, uniform continuity is not enough for the gradient of the heat kernel to be bounded (see **[43]**).

Gradient estimates on heat kernels play an important role in this chapter and in the next ones.

Theorem 7. — *Let $n \geq 2$. Assume that $L \in \mathcal{E}(\delta)$ has the Gaussian property (G). Then, there are constants $c, \alpha > 0, \eta > 0$ depending only on the constants in (G), n and δ, such that for all $y_0, h \in \mathbb{R}^n$, $t > 0$ and $r > 0$ with $2|h| \leq r + t^{1/2}$, we have*

$$\left(\int_{r \leq |x-y_0| \leq 2r} |\nabla_x K_t(x, y_0)|^2 \, dx \right)^{1/2} \leq \frac{c}{t^{1/2+n/4}} \left(\frac{r}{t^{1/2}} \right)^{(n-2)/2} e^{-\alpha r^2/t} \tag{13}$$

and

$$\left(\int_{r \leq |x-y_0| \leq 2r} |\nabla_x K_t(x, y_0 + h) - \nabla_x K_t(x, y_0)|^2 \, dx \right)^{1/2} \leq \frac{c}{t^{1/2+n/4}} \left(\frac{|h|}{t^{1/2} + r} \right)^{\eta} \left(\frac{r}{t^{1/2}} \right)^{(n-2)/2} e^{-\alpha r^2/t}. \tag{14}$$

Note that the above statement does not contain any regularity estimate in the y-variable. For its proof see Section 1.4.7.

Remark. — In dimension 1, pointwise Gaussian estimates for $t\dfrac{\partial K_t}{\partial x}$, $t\dfrac{\partial K_t}{\partial y}$ and $t^2 \dfrac{\partial^2 K_t}{\partial x \partial y}$ are always valid ([**11**], Theorem 2.21).

1.3. A counterexample

Before going into details, let us give a quick proof of the failure of (7) in (G) **[7]**, adapting the argument in **[30]**.

By **[53]**, if $n \geq 5$ there exists $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ for some $\delta > 0$ and u weak solution on $\mathbb{R}^n$ of $Lu = 0$ with the following properties:

(a) A is homogeneous of degree 0 and C^∞ on $x_n \neq 0$ with a jump across $x_n = 0$ except at $x = 0$.

(b) $u(x) = |x|^s G(x/|x|)$ where $s \in \mathbb{C}$ with $-1/2 < \operatorname{Re} s < 0$ and G is a Lipschitz function on the unit sphere, and $u(x) = 0$ on the hyperplane $x_n = 0$.

In particular, u is not L^p-integrable near 0 if $p > -\operatorname{Re} s/n$.

Assume that the heat kernel of L satisfies (7). Since $L^{-1} = \int_0^\infty e^{-tL}\, dt$, the Green's kernel of L is controlled above by $c|x-y|^{2-n}$ so that L^{-1} maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ whenever $1 < p < n/2$ and $1/p - 1/q = 2/n$.

Pick ϕ a test function with $\phi = 1$ in a neighborhood of 0 and set $v = u\phi$. An easy computation gives $Lv = -\operatorname{div}(Au\nabla\phi) + A\nabla u.\nabla\phi = f$ and f is bounded and compactly supported (note that $Au\nabla\phi$ is Lipschitz since it is supported away from 0 and $u = 0$ where A jumps). Hence f belongs to all L^p spaces and $v \in L^p(\mathbb{R}^n)$ for large $p < \infty$. Since $u = v$ near 0, we have a contradiction.

Remark. — The situation regarding the validity of the Gaussian property when $n = 3$ or $n = 4$ is open.

1.4. The Dirichlet property and the Gaussian property

This is the main section of this chapter. Roughly speaking, we prove that the Gaussian estimates for the semigroup kernel are equivalent to an appropriate growth of the Dirichlet integral for solutions of the corresponding elliptic equation.

1.4.1. Dirichlet integrals and the Dirichlet property. — In what follows, we assume $n \geq 2$. We begin with a review on the properties of weak solutions.

Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ with complex coefficients. Let Ω be an open bounded subset of $\mathbb{R}^n$ and u be a weak solution of $\operatorname{div}(A\nabla)$ on Ω. This means that u is a (complex-valued) function in the Sobolev space $H^1(\Omega)$ such that for all $\varphi \in H_0^1(\Omega)$

$$\int_\Omega A\nabla u \cdot \nabla\varphi = 0.$$

Recall that $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of $\mathcal{C}_0^\infty(\Omega)$. The Lebesgue measure is omitted in the integral. We shall write $\operatorname{div}(A\nabla u) = 0$ on Ω.

The following estimates hold with constants independent of u (see **[38]**).

i) Cacciopoli estimate: there exists $C = C(n, \delta)$ such that,

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{r^2} \int_{B_{2r}} |u|^2, \tag{15}$$

whenever $\overline{B_{2r}} \subset \Omega$. Here and thereafter in this section B_r denotes a Euclidean ball of radius r. By $B_{\lambda r}$ we mean the ball having the same center as B_r and radius equal to λr. Remark that u can be replaced by $u - c$ for any constant c, and in particular $c = u_{2r}$, the mean of u on B_{2r}.

ii) The gradient estimate of C.B. Morrey: there exist $C = C(n, \delta)$ and $\alpha = \alpha(n, \delta) > 0$ such that,

$$\int_{B_r} |\nabla u|^2 \leq C r^\alpha \|u\|^2_{H^1(\Omega)} \tag{16}$$

whenever $\overline{B_{2r}} \subset \Omega$.

iii) The L^p estimates of N. Meyers: there exists $\varepsilon = \varepsilon(n,\delta) > 0$ and $C = C(n,\delta) > 0$ such that, if $p \in [2, 2+\varepsilon)$, then $u \in W^{1,p}_{loc}(\Omega)$ and

$$\left(\int_{B_r} |\nabla u|^p\right)^{1/p} \le C r^{n/p - n/2} \left(\int_{B_{2r}} |\nabla u|^2\right)^{1/2} \tag{17}$$

whenever $\overline{B_{2r}} \subset \Omega$.

The integrals $\int |\nabla u|^2$ are often called Dirichlet integrals. If the exponent α in (16) satisfies $\alpha > n-2$ then u is locally Hölder continuous by a result of Morrey (see Lemma 11). This is the case for real equations by the famous regularity theorem of De Giorgi **[31]**. For complex equations, this is also true in dimension 2 by Morrey estimate, but it fails in general by the counterexample of **[53]** described above. Hence, it is natural to look at the class of elliptic operators whose solutions have Dirichlet integrals with appropriate growth.

Definition 8. — Let $L \in \mathcal{E}(\delta)$. We say that L has the Dirichlet property (D) if there are constants $\mu \in (0,1)$ and C depending on L and dimension only such that for any ball B_R, if $Lv = 0$ in B_R, then for all $0 < \rho \le R$

$$\int_{B_\rho} |\nabla v|^2 \le C\left(\frac{\rho}{R}\right)^{n-2+2\mu} \int_{B_R} |\nabla v|^2. \tag{18}$$

Again B_ρ has same center as B_R. The estimate is uniform over the position of the center. This formulation of the growth of Dirichlet integral with $\int_{B_R} |\nabla v|^2$ on the right hand side is invariant under scaling and translation: it will prove useful. In particular, one can reformulate the gradient estimate of Morrey in this way, which shows that any elliptic $L = -\operatorname{div}(A\nabla)$ in $\mathbb{R}^2$ has the Dirichlet property (D).

A reformulation of the above mentioned theorem of De Giorgi is that any real operator $L \in \mathcal{E}(\delta)$ has the Dirichlet property (D).

Definition 9. — Let $L \in \mathcal{E}(\delta)$. We say that L has the local Dirichlet property $(D)_{loc}$ if the same estimates hold with the restriction that balls have radii less than 1.

The main result is

Theorem 10. — *Assume that $n \ge 2$ and that $L \in \mathcal{E}(\delta)$. Then L has the Gaussian property (G) if and only if L and L^* have the Dirichlet property (D). Similarly, L has the local Gaussian property $(G)_{loc}$ if and only if L and L^* have the local Dirichlet property $(D)_{loc}$.*

That De Giorgi theorem and the estimates of Aronson-Nash are qualitatively similar is folk result in PDEs. The interest is in the quantitative equivalence, which holds for complex operators.

There are other regularity conditions that are equivalent to (D) and we shall use them in the course of the proof of Theorem 10. The advantage of property (D) over all the other ones, including (G), is that it is easily seen to be stable under various perturbations, essentially because of the L^2 character of its formulation.

1.4.2. From the Gaussian property to the Dirichlet property. — Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$. We prove the implications (G) $\Rightarrow$ (H) $\Rightarrow$ (D) for L where the intermediate property (H) is related to a weak form of Harnack inequalities and is defined as follows.

We say that L has the property (H) if there are $\mu \in (0,1)$ and a constant C depending on L such that for any ball B_R, if $Lu = 0$ on B_R,

$$\sup_{B_{R/4}} |u| + R^\mu \sup_{(x,y)\in B_{R/4}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\mu} \leq C\left(\frac{1}{|B_R|}\int_{B_R} |u|^2\right)^{1/2}. \tag{19}$$

Assume first that L has the property (G). The argument to prove (H) relies on the gradient estimates of Theorem 7.

Let $u \in H^1(B_R)$ be a weak solution of $Lu = 0$ on B_R. Let $\chi \in C_0^\infty(\mathbb{R}^n)$, supported in $B_{3R/4}$ with $\chi = 1$ on $B_{R/2}$. Let $v = u\chi$. Since $v = u$ on $B_{R/2}$ it suffices to show that for any $\varphi \in C_0^\infty(B_{R/4})$ and any $h \in B_{R/4}$,

$$\left|\int v(x)\bar\varphi(x)dx\right| \leq CR^{-n/2}\|\varphi\|_1\|u\|_2 \tag{20}$$

and

$$\left|\int (v(x+h)-v(x))\bar\varphi(x)dx\right| \leq C|h|^\mu R^{-\mu-n/2}\|\varphi\|_1\|u\|_2, \tag{21}$$

where we extend u by 0 outside of B_R.

Set $\langle v,\varphi\rangle = \int v\bar\varphi$ and $T_t = e^{-tL}$. Observing that $T_t^*\varphi \in H^1(\mathbb{R}^n)$ and $v \in H^1(\mathbb{R}^n)$, then

$$\begin{aligned}\langle v,\varphi\rangle &= \langle u\chi, T_{R^2}^*\varphi\rangle - \int_0^{R^2} \langle u\chi, \frac{dT_t^*}{dt}\varphi\rangle\, dt\\ &= \langle u\chi, T_{R^2}^*\varphi\rangle + \int_0^{R^2} \langle \nabla(u\chi), A^*\nabla T_t^*\varphi\rangle\, dt.\end{aligned}$$

First, using (7)

$$|\langle v, T_{R^2}^*\varphi\rangle| \leq \|v\|_2\|T_{R^2}^*\varphi\|_2 \leq CR^{-n/2}\|\varphi\|_1\|v\|_2.$$

Second we compute the inner product: we have

$$\begin{aligned}\langle \nabla(u\chi), A^*\nabla T_t^*\varphi\rangle &= \langle A\nabla u, \nabla(\chi T_t^*\varphi)\rangle + \langle A\nabla\chi\, u, \nabla T_t^*\varphi\rangle - \langle A\nabla u, \nabla\chi\, T_t^*\varphi\rangle\\ &= I + II + III.\end{aligned}$$

The term I vanishes since $Lu = 0$ and the other terms can be estimated by expanding out the integrals.

For the term II, we have,

$$|II| \leq C \int |\nabla\chi(x)||u(x)||\nabla_x K_t^*(x,y)||\varphi(y)|\,dydx,$$

where $K_t^*(x,y)$ is the kernel of T_t^*. On the support of the integral, we have $R/4 \leq |x-y| \leq R$. Thus using (13) for $K_t^*(x,y)$ and $\|\nabla\chi\|_\infty \leq C/R$, we have

$$\begin{aligned}|II| &\leq CR^{-1}\|u\|_2 \frac{c}{t^{1/2+n/4}}\left(\frac{R}{t^{1/2}}\right)^{(n-2)/2} e^{-\alpha R^2/t}\|\varphi\|_1 \\ &\leq CR^{-n/2}t^{-1}w(R^2/t)\|u\|_2\|\varphi\|_1,\end{aligned}$$

where $w(u) = u^{n-2}e^{-\alpha u^2}$.

Now,

$$|III| \leq C \int |\nabla\chi(x)||\nabla u(x)||K_t^*(x,y)||\varphi(y)|\,dydx.$$

Again $R/4 \leq |x-y| \leq R$ on the support of the integral. Using (7) we get

$$|III| \leq CR^{-1}R^{-n/2}\|\nabla u\|_2 t^{-n/2}e^{-\alpha R^2/t}\|\varphi\|_1,$$

and by the elliptic Cacciopoli estimate (15),

$$|III| \leq CR^{-n/2}t^{-1}w(R^2/t)\|u\|_2\|\varphi\|_1.$$

Since $\int_0^{R^2} t^{-1}w(R^2/t)\,dt = \int_1^\infty u^{-1}w(u)\,du < \infty$, we obtain that

$$\left|\int_0^{R^2} \langle A\nabla(u\chi), \nabla T_t^*\varphi\rangle\,dt\right| \leq CR^{-n/2}\|u\|_2\|\varphi\|_1$$

as desired and (20) is proved.

Next to prove (21), we begin with the identity

$$\int (v(x+h)-v(x))\bar{\varphi}(x)\,dx = \int v(x)\bar{\varphi}_h(x)\,dx,$$

where $\varphi_h(x) = \varphi(x-h) - \varphi(x)$, and then, we follow the same representation, replacing φ by φ_h. To obtain the desired estimate, remark that $\int K_t^*(x,y)\varphi_h(y)\,dy = \int (K_t^*(x,y+h) - K_t^*(x,y))\varphi(y)\,dy$ and use (9) and (14) instead of (7) and (13). The conclusion readily follows.

We turn to the proof of (H) $\Rightarrow$ (D). Fix a ball B_R with center x_0 and let $v \in H^1(B_R)$ be a weak solution in B_R of $\operatorname{div}(A\nabla v) = 0$. Let $0 < \rho \leq R/8$. Using successively Cacciopoli inequality (15), (H) applied to $v - c$, $c = v(x_0)$ (since v is continuous by

(H), $v(x_0)$ is well-defined), and Poincaré inequality, we obtain

$$\begin{aligned}\int_{B_\rho} |\nabla v|^2 &\leq C\rho^{-2}\int_{B_{2\rho}} |v(x)-c|^2\,dx \\ &\leq C\rho^{-2}\rho^{n+2\mu}R^{-2\mu}R^{-n}\int_{B_R}|v(x)-c|^2dx \\ &\leq C\rho^{-2}\rho^{n+2\mu}R^{-2\mu}R^{-n}R^2\int_{B_R}|\nabla v|^2.\end{aligned}$$

Thus, we have

$$\int_{B_\rho}|\nabla v|^2 \leq C\left(\frac{\rho}{R}\right)^{n-2+2\mu}\int_{B_R}|\nabla v|^2$$

when $0<\rho\leq R/8$ and hence when $0<\rho\leq R$ as it is obvious when $R/8<\rho\leq R$.

1.4.3. From the Dirichlet property to the Gaussian property. — We turn to the main part of the equivalence, which is to establish the estimates in the property (G) assuming L and its adjoint to have the property (D).

We proceed by using elliptic regularity in Morrey-Campanato spaces. Then we get estimates on high power of the resolvent of L which we transfer to the semigroup by a contour formula.

Introduce the Morrey-Campanato norms as follows. Full details can be found, e.g., in **[39]**. In order to simplify the exposition we use a different notation.

For $0\leq\gamma\leq n$, define the Morrey space M^γ by $f\in L^2_{loc}(\mathbb{R}^n)$ and

$$||f||_{M^\gamma}:=\sup_{B_\rho,0<\rho\leq 1}\left(\rho^{-\gamma}\int_{B_\rho}|f|^2\right)^{1/2}<\infty. \tag{22}$$

For $0\leq\gamma\leq n+2$, define the Campanato space by $f\in L^2_{loc}(\mathbb{R}^n)$ and

$$||f||_{M_1^\gamma}:=\sup_{B_\rho,0<\rho\leq 1}\left(\rho^{-\gamma}\int_{B_\rho}|f-m_\rho f|^2\right)^{1/2}<\infty, \tag{23}$$

where $m_\rho f$ is the mean of f on B_ρ. The estimates being independent of the center of the balls, these (semi-)norms are translation invariant.

The expression $||f||_{M_1^\gamma}$ is a semi-norm since $||f||_{M_1^\gamma}=0$ if and only if f is constant.

Lemma 11. — *Let $0\leq\gamma<n$. We have the following embeddings and estimates:*

$$M^\gamma\hookrightarrow M_1^\gamma \quad \text{with} \quad ||f||_{M_1^\gamma}\leq 2||f||_{M^\gamma}, \tag{24}$$

$$M_1^\gamma\cap M^0\hookrightarrow M^\gamma \quad \text{with} \quad ||f||_{M^\gamma}\leq C||f||_{M^0}+C||f||_{M_1^\gamma}, \tag{25}$$

$$\text{if } \nabla f\in M^\gamma \text{ then } f\in M_1^{\gamma+2} \quad \text{with} \quad ||f||_{M_1^{\gamma+2}}\leq C||\nabla f||_{M^\gamma}, \tag{26}$$

if $0 < \eta < 1$, *then* $L^\infty \cap \dot{C}^\eta = M^0 \cap M_1^{n+2\eta}$ *and*

$$\|f\|_\infty + |f|_{\dot{C}^\eta} \sim \|f\|_{M_1^{n+2\eta}} + \|f\|_{M^0}. \tag{27}$$

The constants C *depend only on* n, γ.

(The Hölder semi-norm in the last statement has been defined in (11).)

The first two inequalities imply that $M_1^\gamma \cap M^0 = M^\gamma$ with equivalence of norms. Inequality (26) is a simple application of Poincaré inequality. Formula (27) means that functions in M_1^γ are Hölder continuous when $n < \gamma < n+2$. When $\gamma = n$, the Campanato space is apparented to BMO. Let us finally note that the Morrey spaces and the Campanato spaces interpolate by the complex method.

We assume that $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ and L^* have the property (D) with constants C_0 and μ: for any $R > 0$ and $v \in H^1(B_R)$ weak solution of L on B_R we have

$$\int_{B_\rho} |\nabla v|^2 \leq C_0 \left(\frac{\rho}{R}\right)^{n-2+2\mu} \int_{B_R} |\nabla v|^2 \tag{28}$$

when $0 < \rho \leq R$, and the same for L^*.

Step 1: regularity theory for inhomogeneous elliptic equations

Lemma 12. — *Assume that*

$$\operatorname{div}(A\nabla u) = f + \operatorname{div} g,$$

where

$$\nabla u \in M^0, \quad f \in M^\alpha, \quad g \in M^\beta,$$

with $0 \leq \alpha, \beta < n$. *Then, for any* $\gamma \geq 0$ *with*

$$\begin{cases} \gamma \leq \inf(\alpha+2, \beta) & \text{if} \quad \inf(\alpha+2,\beta) < n-2+2\mu \\ \gamma < n-2+2\mu & \text{if} \quad \inf(\alpha+2,\beta) \geq n-2+2\mu, \end{cases}$$

$\nabla u \in M^\gamma$ *and for all* $0 < \rho \leq 1$,

$$\int_{B_\rho} |\nabla u|^2 \leq C\rho^\gamma \int_{B_1} |\nabla u|^2 + C\rho^\gamma (\|f\|_{M^\alpha} + \|g\|_{M^\beta})^2, \tag{29}$$

C *depending only on* $n, \delta, \alpha, \beta, \mu, C_0$ *(here, the balls are concentric).*

In the case $\inf(\alpha+2, \beta) > n-2$, this is a theorem due to Morrey **[64]**, for it gives $u \in C^\eta$ for some $\eta > 0$ as Lemma 11 shows. The full range of indices in this result will be exploited. The proof does not contain any new idea and is mostly taken from the proof of Theorem 1.1, Chapter VI of **[38]**.

Remark. — The Morrey norms are monotonically increasing with γ.

We need a key lemma due to Campanato (see **[38]**).

Lemma 13. — *Let $\Phi, w\colon [0,R] \to [0,R]$ be two non-decreasing functions. Suppose that for $0<\rho\le r\le R$*

$$\Phi(\rho) \le a\Big[\Big(\frac{\rho}{r}\Big)^{\alpha} + w(r)\Big]\Phi(r) + br^{\beta}$$

where a, b, α and β are constants with $a \ge 1$, $b \ge 0$ and $\alpha > \beta > 0$. Set

$$\varepsilon_0 := \varepsilon_0(a,\alpha,\beta) = \sup\{\, a^{-1}\tau^{\gamma} - \tau^{\alpha}\,;\, 0\le\tau<1\,, \alpha<\gamma<\beta\,\}.$$

If $I = \{0 < r \le R\,;\, w(r) < \varepsilon_0\}$ is not empty, choose $R_0 < \sup I$ if $R \notin I$ or $R_0 = R$ otherwise. Then for $0<\rho\le r\le R_0$,

$$\Phi(\rho) \le c\Big[\Big(\frac{\rho}{r}\Big)^{\beta}\Phi(r) + b\rho^{\beta}\Big],$$

where c depends only on a, α, β (in fact, one can take $c = (2a)^s$ with $s = s(\alpha,\beta) \ge 1$).

We begin the proof of Lemma 12. By Lemma 13, it suffices to prove that there are constants a, b with $b \sim (\|f\|_{M^\alpha} + \|g\|_{M^\beta})^2$ such that for $0<\rho\le r\le 1$

$$\int_{B_\rho} |\nabla u|^2 \le a\Big(\frac{\rho}{r}\Big)^{n-2+2\mu}\int_{B_r}|\nabla u|^2 + br^{\gamma_1}, \tag{30}$$

where $\gamma_1 = \inf(\alpha+2,\beta)$.

Fix $r\le 1$ and let $0<\rho\le r$. Let $v \in H^1(B_r)$ solve

$$\operatorname{div}(A\nabla v) = 0 \quad \text{in } B_r,$$
$$v - u \in H^1_0(B_r).$$

From ellipticity, we have

$$\int_{B_r} |\nabla v|^2 \le c(n,\delta)\int_{B_r}|\nabla u|^2.$$

Combining this with (28) gives us

$$\int_{B_\rho}|\nabla u|^2 \le c_1\Big(\frac{\rho}{r}\Big)^{n-2+2\mu}\int_{B_r}|\nabla u|^2 + c_2\int_{B_r}|\nabla(u-v)|^2.$$

Now, $w = u - v$ satisfies

$$\int A\nabla w\cdot\nabla\phi = -\int f\phi + \int g\cdot\nabla\phi$$

for all $\phi \in H^1_0(B_r)$. Taking $\phi = \bar{w}$, using ellipticity, Schwarz inequality and the hypothesis on f and g we obtain

$$\int_{B_r}|\nabla w|^2 \le c_3 r^{\alpha/2}\|f\|^2_{M^\alpha}\Big(\int_{B_r}|w|^2\Big)^{1/2} + c_4 r^{\beta/2}\|g\|^2_{M^\beta}\Big(\int_{B_r}|\nabla w|^2\Big)^{1/2}.$$

Since, by Poincaré inequality, $(\int_{B_r}|w|^2)^{1/2} \le c(n)r(\int_{B_r}|\nabla w|^2)^{1/2}$, we obtain after simplification

$$\int_{B_r}|\nabla w|^2 \le br^{\gamma_1},$$

with $b \sim (||f||_{M^\alpha} + ||g||_{M^\beta})^2$ and (30) follows. This finishes the proof of Lemma 12.

Step 2: Regularity for inhomogeneous elliptic operators. — Consider

$$\widetilde{L} = -\partial_{x_i}(a_{ij}(x)\partial_{x_j} + b_i(x)) + c_j(x)\partial_{x_j} + d(x), \tag{31}$$

where $a_{ij}(x), 1 \leq i, j \leq n$, are the coefficients of $A(x)$ and the other coefficients are complex-valued measurable and bounded functions on $\mathbb{R}^n$. Set

$$\kappa = \sup(||b_j||_\infty, ||c_j||_\infty, ||d||_\infty, 1 \leq j \leq n).$$

Lemma 14. — *Assume that $\widetilde{L}u = h$, where $u, \nabla u \in M^0$ and $h \in M_1^s \cap M^0$ with $s + 2 < n - 2 + 2\mu$. Then $u \in M_1^{s+4}$ and for $0 \leq \gamma \leq s + 4$,*

$$||u||_{M_1^\gamma} \leq C(||\nabla u||_{M^0} + ||u||_{M^0}) + C||h||_{M_1^s} + C||h||_{M^0}, \tag{32}$$

where C depends on $n, \delta, s, \mu, C_0, \kappa$.

Before proving Lemma 14, let us state an immediate consequence.

Corollary 15. — *Assume that L has the Dirichlet property and that $\widetilde{L}^{-1}$ exists as a bounded operator on $L^2(\mathbb{R}^n)$. Then, $\widetilde{L}^{-1}$ extends to a bounded operator from $M^0 \cap M_1^s$ to $M^0 \cap M_1^\sigma$ for all $0 \leq s \leq \sigma < n + 2\mu$ with $\sigma - s \leq 4$.*

Proof of Lemma 14. — Using

$$\widetilde{L} = -\text{div}(A\nabla + b) + c \cdot \nabla + d$$

to abbreviate (31), the equation $\widetilde{L}u = h$ becomes

$$\text{div}(A\nabla u) = -h + du + c \cdot \nabla u - \text{div}(bu) \equiv f + \text{div} g.$$

Let $2 \leq \gamma \leq s + 2$ and apply (29) with $\alpha = \gamma - 2$ and $\beta = \gamma$. Then

$$\begin{aligned} ||\nabla u||_{M^\gamma} &\leq C||\nabla u||_{M^0} + C||h||_{M^s} + C(||du||_{M^{\gamma-2}} + ||c\nabla u||_{M^{\gamma-2}} + ||bu||_{M^\gamma}) \\ &\leq C||\nabla u||_{M^0} + C||h||_{M^s} + C\kappa(||\nabla u||_{M^{\gamma-2}} + ||u||_{M^\gamma}) \end{aligned}$$

where we have used $||du||_{M^{\gamma-2}} \leq ||d||_\infty ||u||_{M^{\gamma-2}}$ etc, and $||u||_{M^{\gamma-2}} \leq ||u||_{M^\gamma}$.

After simple calculations using Lemma 11, we obtain

$$||\nabla u||_{M^\gamma} \leq C(||\nabla u||_{M^0} + ||u||_{M^0}) + C||h||_{M^s} + C||\nabla u||_{M^{\gamma-2}}.$$

Therefore, we see by induction that $||\nabla u||_{M^{s+2}} < \infty$ and

$$||\nabla u||_{M^{s+2}} \leq C(||\nabla u||_{M^0} + ||u||_{M^0}) + C||h||_{M^s}.$$

Next, using (26) for u and (25) for h, we conclude that

$$||u||_{M_1^{s+4}} \leq C(||\nabla u||_{M^0} + ||u||_{M^0}) + C||h||_{M_1^s} + C||h||_{M^0},$$

which yields (32) by the monotonicity of the Morrey norms. □

Step 3: iteration and operator estimates. — Let $u_0 \in L^2(\mathbb{R}^n)$ and define u_k, $k = 1, 2, \ldots$ by

$$u_{k+1} = (L+1)^{-1} u_k.$$

Since $(L+1)^{-1}$ is bounded on $L^2(\mathbb{R}^n)$, we may apply Corollary 15 to see that either $u_1 \in M_1^s$ for all $0 \le s < n + 2\mu$ if $n + 2\mu \le 4$ or that $u_1 \in M_1^4$ if $n + 2\mu > 4$. In the former case, we stop, while in the latter we iterate applying successively Corollary 15. We obtain that there is an integer $k_0 \le 1 + n/4$ such that for all $\eta \in [0, \mu)$, $u_{k_0} \in M_1^{n+2\eta}$ and

$$\|u_{k_0}\|_{M_1^{n+2\eta}} \le C^{k_0+1} \|u_0\|_2, \quad C = C(n, \delta, C_0, \eta, \mu). \tag{33}$$

Note that $\|u_k\|_2 \le C^k \|u_0\|_2$ for all $k \ge 1$ with $C = C(n, \delta)$. Combining this with (27) in Lemma 11, and since k_0 is bounded by $1 + n/4$, we obtain

$$\|u_{k_0}\|_\infty + |u_{k_0}|_{\dot{C}^\eta} \le C \|u_0\|_2, \quad C = C(n, \delta, C_0, \eta, \mu). \tag{34}$$

In other words, the operator $(L+1)^{-k_0}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n) \cap \dot{C}^\eta(\mathbb{R}^n)$ with

$$\|(L+1)^{-k_0} u\|_\infty + |(L+1)^{-k_0} u|_{\dot{C}^\eta} \le C \|u\|_2. \tag{35}$$

One can do the same thing for $L + \lambda$ and obtain (35) uniformly for λ in a strict subsector $\Gamma_{\pi-\nu}$ of $\Gamma_{\pi-\omega}$ (see Section 1.2) and $|\lambda| = 1$.

We have observed that (D) is an invariant property under scaling. Since

$$V_s^{-1}(-\operatorname{div}(A\nabla) + \lambda) V_s = \frac{1}{s^2}(-\operatorname{div}(A_s \nabla) + s^2 \lambda), \tag{36}$$

where $V_s f(x) = f(\frac{x}{s})$ and $A_s(x) = A(sx)$, we see by choosing $s = |\lambda|^{-1/2}$ that

$$|\lambda|^{-n/4+k_0} \|(L+\lambda)^{-k_0} u\|_\infty \le C \|u\|_2, \tag{37}$$

$$|\lambda|^{-n/4-\eta/2+k_0} |(L+\lambda)^{-k_0} u|_{\dot{C}^\eta} \le C \|u\|_2. \tag{38}$$

Note that the constant $C = C(n, \delta, C_0, \eta, \mu)$ does not depend on λ.

Now, we convert these last estimates to the semigroup using the identity

$$e^{-tL} u = \frac{(k_0 - 1)!}{2\pi i t^{k_0 - 1}} \int_\gamma e^{t\lambda} (\lambda + L)^{-k_0} u \, d\lambda, \tag{39}$$

obtained by integrating by parts k_0 times the Cauchy formula. Here, γ consists of two half-rays $\gamma_{\pm 1} = \{\lambda = re^{\pm i\nu}, r \ge R\}$ and of the arc $\gamma_0 = \{\lambda = Re^{i\theta}, |\theta| \le \nu\}$. The number ν is chosen in $(\pi/2, \pi - \omega)$ and $R = 1/t$. By direct estimates from (37) and (38), it is then easy to derive

$$\|e^{-tL} u\|_\infty \le C t^{-n/4} \|u\|_2, \quad t > 0, \tag{40}$$

$$|e^{-tL} u|_{\dot{C}^\eta} \le C t^{-n/4-\eta/2} \|u\|_2, \quad t > 0. \tag{41}$$

Step 4: kernel estimates

Lemma 16. — *If $L \in \mathcal{E}(\delta)$ is such that (40) and (41) hold for L and L^* then L has the Gaussian property (G).*

Note that we just proved that (D) implies (40) and (41) for L. This and the above lemma conclude the proof of Theorem 10.

It remains to prove this lemma. Let us first recall a standard result.

Lemma 17. — *Let T be a continuous operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ with distributional kernel $K(x,y)$. Let $0 < \mu < 1$.*

(i) *T extends to a bounded operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ if and only if*

$$\sup_{x,y\in\mathbb{R}^n} |K(x,y)| < \infty.$$

(ii) *T extends to a bounded operator from $L^1(\mathbb{R}^n)$ to $\dot{C}^\mu(\mathbb{R}^n)$ if and only if*

$$\sup_{y\in\mathbb{R}^n} |K(\cdot,y)|_{\dot{C}^\mu} < \infty.$$

In each case, the supremum agrees with the operator norm.

Proof. — Part (i) is well-known. To prove part (ii), define T_h with kernel $K_h(x,y) = |h|^{-\mu}(K(x+h,y) - K(x,y))$. Then

$$|Tf|_{\dot{C}^\mu} = \sup\{\|T_h f\|_\infty\,;\, h \in \mathbb{R}^n, h \neq 0\}.$$

It suffices to apply (i) to T_h to complete the proof. □

Let us come back to the proof of Lemma 16. We have to prove that the heat kernel $K_t(x,y)$ satisfies (7-9) of Definition 2. Let us admit for a moment that (7) holds and turn to the proof of (8). By Remark 2 after Definition 2, we only have to prove that for some $\eta > 0$,

$$\sup_{y\in\mathbb{R}^n} |K_t(\cdot,y)|_{\dot{C}^\eta} \le ct^{-(n+\eta)/2}, \tag{42}$$

and by Lemma 17, this is equivalent to the boundedness of e^{-tL} from $L^1(\mathbb{R}^n)$ to $\dot{C}^\eta(\mathbb{R}^n)$. Now, using (40) and duality we see that $e^{-tL/2}$ maps $L^1(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. We thus obtain the desired boundedness by combining this fact, (41) and the semigroup formula.

Next, since the assumptions are stable under taking adjoints we also obtain (9).

It remains to prove (7). By an idea of Davies **[28]**, consider $K_t^\phi = e^{-\phi}e^{-tL}e^\phi$ the semigroup generated by $-e^{-\phi}Le^\phi$ where $\phi \in C_0^\infty$ is real-valued and e^ϕ is the operator of multiplication by $e^{\phi(x)}$. Then, (7) is equivalent to the existence of constants $a > 0$ and c such that for all $t > 0$ and ϕ as above,

$$\|K_t^\phi u\|_\infty \le ct^{-n/2}e^{a\rho^2 t}\|u\|_1, \quad \text{with} \quad \rho = \|\nabla\phi\|_\infty. \tag{43}$$

Indeed, by Lemma 17, (43) means that

$$|K_t(x,y)| \le ct^{-n/2}e^{a\rho^2 t}e^{\phi(x)-\phi(y)}$$

and for t, x, y fixed it suffices to pick ϕ with $\phi(x) - \phi(y) = -\rho|x-y|/2$ and to optimize over $\rho > 0$.

By duality and the semigroup formula, it suffices to prove half of (43), namely the boundedness of K_t^ϕ from $L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ with

$$||K_t^\phi u||_\infty \leq ct^{-n/4}e^{a\rho^2 t}||u||_2, \quad \text{where } \rho = ||\nabla\phi||_\infty. \tag{44}$$

Without loss of generality, we may assume that $t = 1$, the general case following by scaling. Fix $\phi \in C_0^\infty$, real-valued, with $||\nabla\phi||_\infty = 1$ and set $T_\rho = e^{-\rho\phi}e^{-L}e^{\rho\phi}$ for $\rho \in \mathbb{C}$. Then (T_ρ) is a complex family of operators.

For $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho = 0$, (40) and (41) together with (27) in Lemma 11 imply that

$$||T_\rho u||_{M_1^{n+2\eta}} \leq c||u||_2. \tag{45}$$

On the other hand, it is classical consequence of the Gårding inequality that T_ρ is bounded on $L^2(\mathbb{R}^n)$ with for all $\rho \in \mathbb{C}$

$$||T_\rho u||_2 \leq e^{a(\operatorname{Re}\rho)^2}||u||_2 \tag{46}$$

for some $a > 0$ depending only on n and δ. In particular,

$$||T_\rho u||_{M_1^0} \leq 2e^{a(\operatorname{Re}\rho)^2}||u||_2. \tag{47}$$

Applying the Stein interpolation theorem (see **[75]**) to (45) and (47), we find that for all $\eta' < \eta$, there exists $a' > 0$ such that for all $\rho \in \mathbb{C}$,

$$||T_\rho u||_{M_1^{n+2\eta'}} \leq c'e^{a'(\operatorname{Re}\rho)^2}||u||_2. \tag{48}$$

This estimate, (45) and (27) in Lemma 11 imply that T_ρ maps $L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and (44) follows. This concludes the proof of Lemma 16.

Remarks

1. If L has real coefficients, then (40) and (41) hold as a consequence of the contracting property of the semigroup on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$ **[28, 80]**. Actually, if L has complex coefficients and enjoys this contracting property, then it must have real coefficients **[6]**.
2. Assume that $L \in \mathcal{E}(\delta)$ is complex and that the semigroup is uniformly bounded on $L^\infty(\mathbb{R}^n)$, that is, the operator norms of e^{-tL} are bounded uniformly by a constant that may exceed 1 (Theorem 5 shows that it is not always true). Then (40) holds. This can be shown by applying the argument of Theorem II.3.2 in **[80]**. But we do not know how to deduce (41) and the Gaussian decay (7). None of the arguments in the real case seem to work. A proof of this fact would mean that the Gaussian property (including regularity estimates) is equivalent to uniform boundedness on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$ for the semigroup. In particular, this would encompass Nash's theorem.

1.4.4. Equivalence of the local properties. — The equivalence between the local properties $(\mathrm{G})_{loc}$ and $(\mathrm{D})_{loc}$ is proved similarly: the argument in Section 1.4.2 yields that $(\mathrm{G})_{loc}$ implies $(\mathrm{D})_{loc}$ and, for the converse, the argument in Section 1.4.3 applies with some modifications due to the lack of scale invariance. We wish to briefly explain the polynomial behavior of some constants in the large time estimate for the heat kernel.

The starting point is (28) which is satisfied with constants C_0 and μ only for balls of radii less than some given R_0. Let us study the effect of scaling. With the notation in (36), the operator $-\operatorname{div}(A_s\nabla)$ satisfies (28) on the same set of balls with constants $C_0\inf(1,s^{n-2+2\mu})$ and μ. Now, a careful checking of the argument tells us that the constant C in (35) grows polynomially as a function of C_0 (see Lemma 13).

Thus, performing the same scaling as in (36) with $s=|\lambda|^{-1/2}$ yields

$$\inf(1,|\lambda|)^{-M}|\lambda|^{-n/4+k_0}\|(L+\lambda)^{-k_0}u\|_\infty \le C\|u\|_2, \tag{49}$$

$$\inf(1,|\lambda|)^{-M}|\lambda|^{-n/4-\eta/2+k_0}|(L+\lambda)^{-k_0}u|_{\dot{C}^\eta} \le C\|u\|_2, \tag{50}$$

where M is a non-negative number that depends only on n and μ and which value we do not know.

Now, using (39) again yields estimates comparable to (40) and (41) with a constant C that blows up polynomially for large time. Hence the proof of Lemma 16 shows that L satisfies $(\mathrm{G})_{loc}$ with a polynomial behavior for the constant c in (7-9).

Remark. — As we shall see (D) holds for operators L with uniformly continuous coefficients or *vmo* coefficents. See Remark 3 after Definition 3 for the interest of the polynomial growth just described.

1.4.5. Inhomogeneous operators. — What we just did for $L+1$ applies to $\widetilde{L}=-\operatorname{div}(A\nabla+b)+c\cdot\nabla+d$ as defined in (36) provided $\widetilde{L}^{-1}$ exists on $L^2(\mathbb{R}^n)$. For example, this is the case if $\widetilde{L}$ satisfies the Gårding inequality

$$\operatorname{Re}\langle\widetilde{L}f,f\rangle \ge \widetilde{\delta}(\|f\|_2^2+\|\nabla f\|_2^2). \tag{51}$$

Let us state the result and leave the verifications to the reader.

Theorem 18. — *Let $L=-\operatorname{div}(A\nabla)\in\mathcal{E}(\delta)$ on $\mathbb{R}^n$ and $\widetilde{L}=-\operatorname{div}(A\nabla+b)+c\cdot\nabla+d$ be such that (51) holds and set $\kappa=\sup(\|b\|_\infty,\|c\|_\infty,\|d\|_\infty)<\infty$. Assume that L has the property $(\mathrm{D})_{loc}$ with constants C_0 and μ. Then $\widetilde{L}$ has the Gaussian property (G). More precisely, its heat kernel $\widetilde{K}_t(x,y)$ satisfies for all $t>0$, $x,y\in\mathbb{R}^n$,*

$$|\widetilde{K}_t(x,y)| \le \frac{ce^{-at}}{t^{n/2}}\exp\left\{-\frac{\beta|x-y|^2}{t}\right\},$$

$$|\widetilde{K}_t(x,y)-\widetilde{K}_t(x+h,y)| \le \frac{ce^{-at}}{t^{n/2}}\left(\frac{|h|}{t^{1/2}+|x-y|}\right)^\eta\exp\left\{-\frac{\beta|x-y|^2}{t}\right\}$$

and

$$|\widetilde{K}_t(x, y+h) - \widetilde{K}_t(x,y)| \leq \frac{ce^{-at}}{t^{n/2}} \left(\frac{|h|}{t^{1/2} + |x-y|} \right)^\eta \exp\left\{ -\frac{\beta|x-y|^2}{t} \right\}$$

whenever $2|h| \leq t^{1/2} + |x-y|$. *Here,* $0 < \eta$, $c = c(n, \delta, \widetilde{\delta}, \eta, C_0, \kappa)$, $a = a(n, \widetilde{\delta}) > 0$ *and* $\beta = \beta(n, \widetilde{\delta})$.

1.4.6. Stability of (D) and proof of Theorem 6. — Our goal in this section is to prove Theorem 6, with the exception of the one dimensional case for which we refer the reader to **[14]** and **[11]**.

When $n = 2$, we already observed that the property (D), hence the property (G), is always satisfied. A different argument is otherwise presented in Section 1.7.

Thus, only parts (ii), (iii), (iv) and (v) are considered. To prove each of them, by Theorem 10 it suffices to prove the analogous statements by replacing systematically (G) by (D) (respectively $(G)_{loc}$ by $(D)_{loc}$). Let us observe that some of the techniques shown below are well known in the study of regularity properties of the solutions of variational problems **[39]**. In particular, we use Lemma 13 in a crucial way.

Proof of (ii). — Let $A \in \mathcal{A}(\delta)$ such that $L = -\operatorname{div}(A\nabla)$ has the Dirichlet property (D). Let $A' \in \mathcal{A}$ be another matrix-valued function: we have to show that L' has the Dirichlet property (D) provided $\|A - A'\|_\infty$ is small enough.

Fix a ball B_R. Let $u \in H^1(B_R)$ be a weak solution of $\operatorname{div}(A'\nabla u) = 0$ in B_R. Let $0 < \rho \leq r \leq R$ and define $v \in H^1(B_r)$ by solving the elliptic problem

$$\operatorname{div}(A\nabla v) = 0 \quad \text{in } B_r,$$
$$v - u \in H^1_0(B_r).$$

(Again all balls have same center.) From (D) for L, we have

$$\int_{B_\rho} |\nabla v|^2 \leq C \left(\frac{\rho}{r}\right)^{n-2+2\mu} \int_{B_r} |\nabla v|^2, \tag{52}$$

hence

$$\int_{B_\rho} |\nabla u|^2 \leq c_0 \left(\frac{\rho}{r}\right)^{n-2+2\mu} \int_{B_r} |\nabla u|^2 + c_1 \int_{B_r} |\nabla(u-v)|^2.$$

By definition of u and v, the function $w = u - v$ satisfies

$$\int_{B_r} A\nabla w \cdot \nabla\varphi = \int_{B_r} (A - A')\nabla u \cdot \nabla \varphi$$

for all $\varphi \in H^1_0(B_r)$. Using $\varphi = \bar{w}$ and the ellipticity condition for A, we obtain

$$\int_{B_r} |\nabla w|^2 \leq \delta^{-2} \|A' - A\|^2_\infty \int_{B_r} |\nabla u|^2.$$

Thus,

$$\int_{B_\rho} |\nabla u|^2 \leq c_2 \left(\left(\frac{\rho}{r}\right)^{n-2+2\mu} + \|A' - A\|^2_\infty \right) \int_{B_r} |\nabla u|^2.$$

By Lemma 13, for any $0 < \nu < \mu$ we have

$$\int_{B_\rho} |\nabla u|^2 \leq c_3 \left(\frac{\rho}{r}\right)^{n-2+2\nu} \int_{B_r} |\nabla u|^2,$$

for all $0 < \rho \leq r \leq R$ provided $\|A' - A\|_\infty$ is small enough. By taking $r = R$, we have proved that L' has the property (D). □

Remark. — We may wonder about the size of the perturbation, especially when the dimension is large. Consider $A = Id$ and $A' = Id - M$. Following the same argument, since $|\nabla v|^2$ is subharmonic, (52) becomes

$$\int_{B_\rho} |\nabla v|^2 \leq \left(\frac{\rho}{r}\right)^n \int_{B_r} |\nabla v|^2.$$

Setting $\Phi(\rho) = (\int_{B_\rho} |\nabla u|^2)^{1/2}$, we see that

$$\Phi(\rho) \leq \left[\left(\frac{\rho}{r}\right)^{n/2} + 2\|M\|_\infty\right] \Phi(r).$$

Applying Lemma 13 with $a = 1, b = 0, \alpha = \dfrac{n}{2}$ and $\beta = \dfrac{n}{2} - 1 + \nu$ where $0 < \nu < 1$, we have

$$\Phi(\rho) \leq c \left(\frac{\rho}{r}\right)^\beta \Phi(r)$$

provided

$$2\|M\|_\infty \leq \epsilon_0 = (1-s)s^{s/(1-s)} = f(s), \quad s = \frac{\beta}{\alpha} = 1 + \frac{2(1-\nu)}{n}.$$

Since $f(s)$ is a non-increasing function, the smaller ν, the larger ε_0. Further, for fixed ν, letting n grow to infinity, ε_0 is asymptotic to $2(1-\nu)/en$ where e is the base of the exponential function. In conclusion, this technique allows perturbations of the size c/n as $n \to \infty$. The value of c cannot be arbitrary large. This can be seen by considering the counterexample in **[53]** already discussed in Section 1.3. The matrix A there is the sum of a real symmetric matrix and of a bounded matrix whose L^∞ norm is asymptotic to c'/n as $n \to \infty$.

Proof of (iv). — We want to prove that $L \in \mathcal{E}(\delta)$ has the local Dirichlet property $(\mathrm{D})_{loc}$ provided the modulus of continuity $\omega_\infty(r)$ of the matrix A is small for small r.

Fix a ball B_R. Let $u \in H^1(B_R)$ be a weak solution of $\operatorname{div}(A\nabla u) = 0$ in B_R. Let $0 < \rho \leq r \leq R$. Define $v \in H^1(B_r)$ by solving the constant coefficients elliptic problem

$$\operatorname{div}(A(x_0)\nabla v) = 0 \quad \text{in } B_r,$$
$$v - u \in H_0^1(B_r),$$

where x_0 is the center of B_r. By classical regularity theory for constant elliptic operators (e.g., **[38]**, Chapter 3, Theorem 2.1), we have

$$\int_{B_\rho} |\nabla v|^2 \leq C\left(\frac{\rho}{r}\right)^n \int_{B_r} |\nabla v|^2,$$

where C depends only on ellipticity and dimension. Hence

$$\int_{B_\rho} |\nabla u|^2 \leq c_0\left(\frac{\rho}{r}\right)^n \int_{B_r} |\nabla u|^2 + c_1 \int_{B_r} |\nabla (u-v)|^2.$$

Now, the function $w = u - v$ satisfies

$$\int_{B_r} A(x_0)\nabla w \cdot \nabla \varphi = \int_{B_r} (A(x_0) - A)\nabla u \cdot \nabla \varphi$$

for all $\varphi \in H_0^1(B_r)$. Using $\varphi = \bar{w}$ and the ellipticity condition for A, we obtain

$$\int_{B_r} |\nabla w|^2 \leq \delta^{-2}\omega_\infty(r)^2 \int_{B_r} |\nabla u|^2.$$

Thus,

$$\int_{B_\rho} |\nabla u|^2 \leq c_2\left(\left(\frac{\rho}{r}\right)^n + \omega_\infty(r)^2\right) \int_{B_r} |\nabla u|^2,$$

for all $0 < \rho \leq r \leq R$ where c_2 depends only on dimension and ellipticity.

Pick $\mu \in (0,1)$ and apply Lemma 13 with $a = c_2$, $b = 0$, $\alpha = n$ and $\beta = n - 2 + 2\mu$. If R_0 is such that $\omega_\infty^2(R_0) < \varepsilon_0$, then we have

$$\int_{B_\rho} |\nabla u|^2 \leq c_3\left(\frac{\rho}{r}\right)^{n-2+2\mu} \int_{B_r} |\nabla u|^2,$$

for all $0 < \rho \leq r \leq R_0$. □

Proof of (iii). — Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ and assume that the BMO norm of A is small.

Fix a ball B_R. Let $u \in H^1(B_R)$ be a weak solution of $\operatorname{div}(A\nabla u) = 0$ in B_R. Let $0 < \rho \leq r \leq R/2$. Start the proof of (iii) as the preceding one, replacing $A(x_0)$ by the mean $m_r A$ of A on B_r. The only change is the derivation of the estimate of

$$I = \int_{B_r} (A - m_r A)\nabla u \cdot \nabla \bar{w},$$

which we owe to L. Escauriaza. Choose p for which Meyers estimate (17) applies and use Hölder inequality with exponents $2, p$ and q where $1/2 + 1/p + 1/q = 1$. Then

$$\begin{aligned} |I| &\leq \left(\int_{B_r} |A - m_r A|^q\right)^{1/q} \left(\int_{B_r} |\nabla u|^p\right)^{1/p} \left(\int_{B_r} |\nabla w|^2\right)^{1/2} \\ &\leq C r^{n(\frac{1}{p} - \frac{n}{2})} \left(\int_{B_r} |A - m_r A|^q)^{1/q} \left(\int_{B_{2r}} |\nabla u|^2\right)^{1/2} \left(\int_{B_r} |\nabla w|^2\right)^{1/2}. \end{aligned}$$

Now, the John-Nirenberg inequality gives us

$$r^{-n/q}\left(\int_{B_r} |A - m_r A|^q\right)^{1/q} \leq C(q,n)\ \omega_2(A,r),$$

where $\omega_2(A,r)$ is defined just before Theorem 6 and $\omega_2(A,r) \leq ||A||_{BMO}$. Working out the details, we find that

$$\int_{B_\rho} |\nabla u|^2 \leq c_2\left(\left(\frac{\rho}{r}\right)^n + ||A||^2_{BMO}\right)\int_{B_{2r}} |\nabla u|^2$$

provided $0 < \rho \leq r$ and $r \leq R/2$. If $0 < r \leq \rho \leq 2r$ and $r \leq R/2$, the above inequality is trivially satisfied. Hence, changing $2r$ to r we obtain

$$\int_{B_\rho} |\nabla u|^2 \leq c_3\left(\left(\frac{\rho}{r}\right)^n + ||A||^2_{BMO}\right)\int_{B_r} |\nabla u|^2 \tag{53}$$

whenever $0 < \rho \leq r \leq R$, and we conclude as usual with Lemma 13 provided $||A||_{BMO}$ is small enough. □

Proof of (v). — Let $L \in \mathcal{E}(\delta)$. We prove that L has the property $(\mathrm{D})_{loc}$ provided the L^2-modulus of continuity $\omega_2(A,r)$ is small for small r.

The only change from the preceeding argument is in the refinement of (53) where $||A||_{BMO}$ is replaced by $\omega_2(A,r)$. Thus we obtain for any $\mu \in (0,1)$ the existence of $\varepsilon_0 > 0$ such that if $R_0 > 0$ satisfies $\omega_2(R_0)^2 < \varepsilon_0$ then

$$\int_{B_\rho} |\nabla u|^2 \leq c_4\left(\frac{\rho}{r}\right)^{n-2+2\mu}\int_{B_r} |\nabla u|^2$$

for all $0 < \rho \leq r \leq R_0$. This proves that L has the property $(\mathrm{D})_{loc}$ when $\inf_{t>0} \omega_2(t)^2 < \varepsilon_0$. □

Remark. — The proofs of (iv) and (v) combined with the argument that $(\mathrm{D})_{loc}$ implies $(\mathrm{G})_{loc}$ show that, in this case, the regularity estimates (8) and (9) hold for every $\mu \in (0,1)$. See also Lemma 28 in Chapter 4, where a related result is stated.

1.4.7. Gradient estimates on the heat kernel. — In this section, we prove the gradient estimates of Theorem 7, which were already used in a previous argument. We need two intermediate results. The first one is about Gaussian estimates for complex time heat kernels.

Lemma 19. — *Assume that $n \geq 2$ and that $L \in \mathcal{E}(\delta)$ has the Gaussian property (G). Then the kernels of $t\frac{d}{dt}e^{-tL}$ satisfies (7-9). Moreover, for any $\gamma < \pi/2 - \omega$ there are constants c, $\mu > 0$ and $\beta' > 0$ depending only on the constants in (G), n, δ and γ such that for $|\arg z| \leq \gamma$*

$$|K_z(x,y)| \leq c\,|z|^{-n/2}\exp\left\{-\frac{\beta'|x-y|^2}{|z|}\right\} \tag{54}$$

and

$$|K_z(x+h,y)-K_z(x,y)| \leq \frac{c}{|z|^{n/2}}\left(\frac{|h|}{|z|^{1/2}+|x-y|}\right)^{\mu} \exp\left\{-\frac{\beta'|x-y|^2}{|z|}\right\} \tag{55}$$

and

$$|K_z(x,y+h)-K_z(x,y)| \leq \frac{c}{|z|^{n/2}}\left(\frac{|h|}{|z|^{1/2}+|x-y|}\right)^{\mu} \exp\left\{-\frac{\beta'|x-y|^2}{|z|}\right\} \tag{56}$$

whenever $2|h| \leq |z|^{1/2}+|x-y|$. *We have set* $K_z(x,y)$ *the distributional kernel of* e^{-zL}.

Remark. — In fact, μ is the same as in (G).

Proof. — Assume that the statement on $K_z(x,y)$ is proved. Then, using Cauchy formula applied to the holomorphic function $z \to K_z(x,y)$, we obtain the desired estimates for $t\frac{\partial}{\partial t}K_t(x,y)$.

It remains to prove the statement on $K_z(x,y)$. This is done in **[28]** when L is real and selfadjoint. There is no substantial change but we include an argument for completeness.

First we prove that $|K_z(x,y)| \leq c|z|^{-n/2}$ for $|\arg z| \leq \gamma$. By Lemma 17, this is equivalent to the $L^1 - L^\infty$ boundedness of e^{-zL}.

For $1 \leq p \leq q \leq \infty$, denote by $||T||_{q,p}$ the operator norm of T from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. We deduce from (7) and Lemma 17 that $||e^{-tL}||_{\infty,1} \leq ct^{-n/2}$, $||e^{-tL}||_{1,1} \leq c$ and $||e^{-tL}||_{\infty,\infty} \leq c$. Hence, by interpolation

$$||e^{-tL}||_{q,p} \leq ct^{n/2(1/q-1/p)}, \quad 1 \leq p \leq q \leq \infty.$$

Write $z = t+t+\zeta$ where $t > 0$, $|\arg \zeta| < \pi/2 - \omega$ and $|z| \sim t \sim |\zeta|$. Then using $||e^{-\zeta L}||_{2,2} \leq 1$ we get

$$||e^{-zL}||_{\infty,1} \leq ||e^{-tL}||_{\infty,2}||e^{-\zeta L}||_{2,2}||e^{-tL}||_{2,1} \leq ct^{-n/2} \leq c|z|^{-n/2}.$$

Next, we prove that $|h|^{-\mu}|K_z(x+h,y)-K_z(x,y)| \leq c|z|^{-(n+\mu)/2}$. Once we have proved (54), it will imply (55) (see Remark 2 after Definition 3). Replacing L by L^* will also give (56).

Using Lemma 17 again, this inequality is equivalent to the $L^1 - \dot{C}^\mu$ boundedness of e^{-zL}, its right hand side being the operator norm. We easily deduce from (8) that

$$|e^{-tL}f|_{\dot{C}^\mu} \leq ct^{-(n+\mu)/2}||f||_1$$

and

$$|e^{-tL}f|_{\dot{C}^\mu} \leq ct^{-\mu/2}||f||_\infty.$$

Hence, by interpolation,

$$|e^{-tL}f|_{\dot{C}^\mu} \leq ct^{-n/2p-\mu/2}||f||_p, \quad 1 \leq p \leq \infty.$$

We conclude by taking the same decomposition of e^{-zL} as before, and by using from the right to the left the $L^1 - L^2$, $L^2 - L^2$ and $L^2 - \dot{C}^\mu$ boundedness for each operator respectively.

We now prove the Gaussian decay for $K_z(x,y)$. Fix $x \neq y$ and $0 < \gamma < \pi/2 - \omega$. Apply the three lines theorem to the holomorphic function

$$f(z) = z^{n/2} \exp\left(-\frac{\rho^2 z}{\beta}\right) K_z(x,y)$$

for $z \in \Gamma_\gamma$, $z \neq 0$, where β is the same constant as in (7) and $\rho > 0$ is to be chosen.

For $\arg z = 0$, *i.e.*, $z = t > 0$,

$$|f(t)| \le c_0 \exp\left(-\frac{\rho^2 t}{\beta} - \frac{\beta|x-y|^2}{t}\right) \le c_0 \exp(-2\rho|x-y|).$$

For $|\arg z| \le \gamma$, $f(z)$ is continuous and

$$|f(z)| \le c_1.$$

Thus for $\theta \in (0,\gamma)$ and $\arg z = \pm\theta$,

$$|f(z)| \le c_0^a c_1^{1-a} \exp(-2a\rho|x-y|), \quad a = 1 - \frac{|\theta|}{\gamma}.$$

Hence,

$$|K_z(x,y)| \le \max(c_0, c_1)|z|^{-n/2} \exp\left(-2a\rho|x-y| + \frac{\rho^2|z|\cos\theta}{\beta}\right).$$

Optimizing over $\rho > 0$ yields a bound of the form $c|z|^{-n/2}\exp(-\beta'|x-y|^2/|z|)$ that is uniform in every subsector of Γ_γ. Since γ was chosen arbitrarily in $(0, \pi/2 - \omega)$, we have established (54). The proof is complete. □

The next result is a general inequality which is the parabolic analog of (15).

***Lemma 20* (Parabolic Cacciopoli inequality).** — *Let $n \ge 2$ and let $L \in \mathcal{E}(\delta)$. For $f \in L^2(\mathbb{R}^n)$, let $u_t = e^{-tL}f$. Then, for all $\varphi \in C_0^1(\mathbb{R}^n)$,*

$$||t^{1/2}\varphi\nabla u_t||_2^2 \le c(n,\delta)(||t^{1/2}u_t\nabla\varphi||_2^2 + ||u_t\varphi||_2||t\frac{\partial u_t}{\partial t}\varphi||_2). \tag{57}$$

Proof. — Since u_t satisfies the parabolic equation $\dfrac{\partial u_t}{\partial t} + Lu_t = 0$, we have

$$\langle A\nabla u_t, (\nabla u_t)\,\varphi^2\rangle = -\langle\frac{\partial u_t}{\partial t}, u_t\varphi^2\rangle - 2\langle A\varphi\nabla u_t\,, u_t\nabla\varphi\,\rangle. \tag{58}$$

Set $M = ||t^{1/2}\varphi\nabla u_t||_2$ and observe that $M \le c||t^{1/2}\nabla u_t||_2 < \infty$ since $u_t \in \mathcal{D}(L) \subset H^1(\mathbb{R}^n)$. Multiplying (58) by t, using Cauchy-Schwarz inequality and ellipticity we obtain

$$M^2 \le cM||t^{1/2}u_t\nabla\varphi||_2 + c||u_t\varphi||_2||t\frac{\partial u_t}{\partial t}\varphi||_2,$$

and (57) follows using the elementary inequality $2ab \le \varepsilon a^2 + \varepsilon^{-1}b^2$ with $a = M$, $2b = c||t^{1/2}u_t\nabla\varphi||_2$ and $\varepsilon = 1/2$. □

We are now in position to prove Theorem 7. We begin with the proof of (13). To this end, we specify the choice of f and φ in the parabolic Cacciopoli inequality.

Fix $y_0 \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$ with support in the ball $B(y_0, r/4)$ centered in y_0 and of radius $r/4$. Then, from estimate (7) and Lemma 19, there exist non negative constants c and a such that whenever $|x - y_0| \geq 7r/8$,

$$|u_t(x)| + \left| t\frac{\partial u_t(x)}{\partial t} \right| \leq ct^{-n/2}e^{-a|x-y_0|^2/t} \ \|f\|_1. \tag{59}$$

Next, pick φ with $\varphi(x) = 0$ if $|x - y_0| \leq 7r/8$ or $|x - y_0| \geq 17r/8$, $\varphi(x) = 1$ if $r \leq |x - y_0| \leq 2r$ and such that $\|\varphi\|_\infty \leq 1$ and $\|\nabla\varphi\|_\infty \leq c/r$. Then, using (59) on the support of φ

$$\|u_t\varphi\|_2 + \|t\frac{\partial u_t}{\partial t}\varphi\|_2 \leq c\|\varphi\|_\infty t^{-n/4}\left(\frac{r}{t^{1/2}}\right)^{n/2} e^{-ar^2/t}\|f\|_1$$

and

$$\begin{aligned} \|t^{1/2}u_t\nabla\varphi\|_2 &\leq \frac{ct^{1/2}}{t^{n/2}}e^{-ar^2/t}\|f\|_1\|\nabla\varphi\|_\infty r^{n/2} \\ &\leq \frac{c}{t^{n/4}}\left(\frac{r}{t^{1/2}}\right)^{(n-2)/2} e^{-ar^2/t}\|f\|_1. \end{aligned}$$

Inserting these estimates in (57), we easily obtain

$$\|t^{1/2}\varphi\nabla u_t\|_2 \leq \frac{c}{t^{n/4}}\left(\frac{r}{t^{1/2}}\right)^{(n-2)/2} e^{-ar^2/t}\|f\|_1,$$

with appropriate constants c and $a > 0$. Hence,

$$\begin{aligned} &\left(\int_{r\leq|x-y_0|\leq 2r}\left|\int_{\mathbb{R}^n} t^{1/2}\nabla_x K_t(x,y)f(y)\,dy\right|^2 dx\right)^{1/2} \\ &\qquad\qquad \leq \frac{c}{t^{n/4}}\left(\frac{r}{t^{1/2}}\right)^{(n-2)/2} e^{-ar^2/t}\|f\|_1 \end{aligned}$$

holds for all $f \in L^1(B(y_0, r/4))$. Letting f be an approximation of the Dirac mass at y_0, we see that $\nabla_x K_t(x,y)$ exists as a measurable function, and that (13) holds.

To prove (14), take $|h| \leq r/2$, f and φ as above and apply the parabolic Cacciopoli inequality to $v_t = e^{-tL}(f - f(\cdot - h))$. Then, rewrite v_t as

$$v_t(x) = \int_{\mathbb{R}^n}(K_t(x,y) - K_t(x,y+h))f(y)\,dy$$

so that one obtains the desired estimates by using (9) instead of (7). We leave the remaining details to the reader.

Remark. — The conclusion of Theorem 7 is valid for the complex time heat kernel uniformly in appropriate sectors. Then t is replaced by $|z|$ in the estimates.

1.5. Further consequences of the Gaussian property

This section is devoted to establishing some further local and global estimates for the heat kernel and the resolvent kernel and also the conservation property (*i.e.*, $e^{-tL}(1) = 1$) under the Gaussian property. Of course, there are corresponding estimates under the local Gaussian property $(\mathrm{G})_{loc}$ which we do not state.

It is worth remarking that many of the results stated for the heat kernel have counterparts in terms of the resolvent kernel, and vice versa. This is due to the Laplace formula

$$(L+\lambda)^{-1} = \int_0^\infty e^{-z(L+\lambda)}\,dz \tag{60}$$

and the Cauchy formula

$$e^{-zL} = \frac{1}{2\pi i}\int_\gamma e^{z\lambda}(L+\lambda)^{-1}\,d\lambda \tag{61}$$

on suitable paths. For the Laplace formula, take a ray $re^{i\theta}$, $r>0$, on which $\lambda e^{i\theta}$ has non negative real parts. For the Cauchy formula, take the path made of two rays on which $z\lambda$ has non positive real part and of an arc of circle, as in (39), Section 1.4.3.

1.5.1. Green kernel estimates. — A first application of (G) is that the resolvent kernel satisfies integrable estimates. For example, denoting by $R_t(x,y)$ the kernel of $(1+t^2L)^{-1}$, $t>0$, we deduce from the Laplace formula after straightforward calculations that

$$|R_t(x,y)| \le c\left(\frac{t^2}{|x-y|^{n-2}}\right)e^{-\alpha|x-y|/t}$$

for some constants c and $\alpha>0$ when $n\ge 3$, and the usual modification with a logarithmic singularity at $x=y$ applies when $n=2$. We could also write Hölder type estimates. Let us rather state some gradient estimates. These are useful in the next chapters.

Theorem 21. — *Assume that $L\in\mathcal{E}(\delta)$ on $\mathbb{R}^n$, $n\ge 3$, has the Gaussian property (G). Then we have the following estimates: there are constants $c,\alpha>0,\eta>0$ depending only on the constants in (G), n and δ, such that for all $y_0,h\in\mathbb{R}^n$, $t>0$ and $r>0$ with $2|h|\le r+t$, we have*

$$\int_{r\le|x-y_0|\le 2r}|\nabla_x R_t(x,y_0)|\,dx \le \frac{c}{t}\left(\frac{r}{t}\right)e^{-\alpha r/t} \tag{62}$$

and

$$\int_{r\le|x-y_0|\le 2r}|\nabla_x R_t(x,y_0+h)-\nabla_x R_t(x,y_0)|\,dx \le \frac{c}{t}\left(\frac{|h|}{t+r}\right)^\eta\left(\frac{r}{t}\right)e^{-\alpha r/t}. \tag{63}$$

The proof of this result follows by combining Theorem 7 and the Laplace formula (60). Details are left to the reader.

Remarks

1. When $n = 2$ these estimates hold too. Starting from (13) gives us $\frac{r}{t}\left(|\ln\frac{r}{t}|+1\right)$ instead of r/t in front of the exponential. However, one can get rid of the logarithm by using the stronger estimate (65) below.
2. There are also L^2 estimates instead of L^1 estimates, but they blow up as r/t tends to 0, which is in agreement with the general fact that the Green kernel of a general second order operator is not globally L^2.
3. Estimates (62) and (63) hold for $(m + t^2L)^{-1}$, where m is some non-negative bounded function with bounded inverse. Here is a sketch of the argument. By rescaling, it is no loss of generality to assume $t = 1$. Then, by the Laplace formula, it is enough to study the heat kernel of $L+m$. Now, Theorem 18 gives estimates on the heat kernel; estimates on its gradient as in Theorem 7 follow from the analysis similar to the one of Section 1.4.7. This remark is used in Chapter 4, Section 4.7.3.

1.5.2. L^p estimates. — We have obtained L^2 estimates for the gradient of heat kernels. In fact, there is always a slight improvement and we can get L^p estimates for some $p > 2$. This is the parabolic version of Meyers inequality (17).

Proposition 22. — *Let $n \geq 2$ and let $L \in \mathcal{E}(\delta)$. Then there exists $\varepsilon = \varepsilon(n,\delta) > 0$ such that for all p with $|1/2 - 1/p| < \varepsilon$, e^{-tL} is bounded from $L^p(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ with*

$$\|e^{tL}f\|_p + \|t^{1/2}\nabla e^{-tL}f\|_p \leq c_p\|f\|_p.$$

Proof. — We claim that $L+1$ is invertible from $W^{1,p}(\mathbb{R}^n)$ onto $W^{-1,p}(\mathbb{R}^n)$ for p in a neighborhood of 2. There are several ways to see this. A direct way is in **[11]** adapting an earlier argument in **[16]** (this way gives a numerical value of ε in terms of $\|A - \mathrm{Id}\,\|_\infty$). Another way is to use the following abstract result of Sneiberg **[70]**.

Lemma 23. — *Let X^s, Y^s, $s \in [0,1]$ be two scales of complex interpolation Banach spaces. If $T\colon X^s \to Y^s$ is bounded for each $s \in [0,1]$, then the set of $s \in (0,1)$ for which there exists $C > 0$ such that $\|Tf\|_{Y_s} \geq C\|f\|_{X_s}$ holds for all $f \in X^s$ is open.*

Indeed, observe that $L+1$ is bounded from $W^{1,p}(\mathbb{R}^n)$ into $W^{-1,p}(\mathbb{R}^n)$ for all $1 < p < \infty$ and that, by the L^2-estimates (4) of Proposition 1, it is invertible for $p = 2$.

Therefore, there is a neighborhood of 2 such that $L+1$ and its adjoint are one-one with closed range from $W^{1,p}(\mathbb{R}^n)$ into $W^{-1,p}(\mathbb{R}^n)$ for all p in this neighborhood. The claim follows easily.

Next, one can clearly change $L+1$ to $L+\lambda$ and using scaling we deduce that

$$\|(L+\lambda)^{-1}f\|_p + |\lambda|^{-1/2}\|\nabla(L+\lambda)^{-1}f\|_p \leq c_p\|f\|_p,$$

for any $\lambda \neq 0$ inside a closed subsector of $\Gamma_{\pi-\omega}$, and $|1/2 - 1/p| < \varepsilon$, ε depending on the aperture of the chosen subsector. We conclude the proof of Proposition 22 by inserting these estimates in the Cauchy formula (61). □

Proposition 24. — *Let $n \geq 2$ and assume that $L \in \mathcal{E}(\delta)$ has the Gaussian property (G). Then, there exists $\varepsilon > 0$ such that ∇e^{-tL} is bounded from $L^1(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ if $1 \leq p < 2 + \varepsilon$, and for all $t > 0$ and $y_0 \in \mathbb{R}^n$,*

$$\left(\int_{\mathbb{R}^n} |\nabla_x K_t(x, y_0)|^p \, dx \right)^{1/p} \leq ct^{-1/2-(1-1/p)n/2}. \tag{64}$$

Proof. — Let p with $|1/2 - 1/p| < \varepsilon$ where ε is the same as in Proposition 22. As already seen e^{-tL} is $L^1 - L^p$ bounded. Thus $\nabla e^{-2tL} = \nabla e^{-tL} e^{-tL}$ is also $L^1 - L^p$ bounded with norm not exceeding $ct^{-1/2-(1-1/p)n/2}$. Thus

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \nabla_x K_t(x, y) f(y) \, dy \right|^p dx \right)^{1/p} \leq ct^{-1/2} t^{-(1-1/p)n/2} \|f\|_1$$

and we deduce (64) for this range of p's by letting f approximate the Dirac mass at y_0.

If $1 \leq p \leq 2$ and $|1/2 - 1/p| \geq \varepsilon$, then using Hölder inequality and (13)

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla_x K_t(x, y_0)|^p \, dx &= \sum_{j=-\infty}^{\infty} \int_{2^j t^{1/2} \leq |x-y_0| \leq 2^{j+1} t^{1/2}} |\nabla_x K_t(x, y_0)|^p \, dx \\
&\leq c \sum_{j=-\infty}^{\infty} (2^j t^{1/2})^{n(1-p/2)} \left(t^{-1/2-n/4} \, 2^{j(n-2)/2} \, e^{-\beta^2 4^j} \right)^p \\
&\leq ct^{\alpha p} \sum_{j=-\infty}^{\infty} 2^{js} e^{-\beta^2 p 4^j},
\end{aligned}$$

where $\alpha = -1/2 - (1 - 1/p)n/2$ and $s = n(1 - p/2) + p(n-2)/2 > 0$, so that the series converge. Thus (64) holds and the boundedness of ∇e^{-tL} from $L^1(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ follows. □

Remarks

1. Note that the last series diverges when $n = 2$ and $p = 2$. However, once (64) holds for $p > 2$, Hölder inequality and interpolation imply an improvement of (13) when $n = 2$:

$$\left(\int_{r \leq |x-y_0| \leq 2r} |\nabla_x K_t(x, y_0)|^2 \, dx \right)^{1/2} \leq ct^{-1} \left(\frac{r}{t^{1/2}} \right)^{\varepsilon} e^{-\beta r^2/t} \tag{65}$$

for some $\varepsilon > 0, \beta > 0$ and $c \geq 0$. This is the parabolic analog of Morrey estimate (16): it is valid in full generality since L has the Gaussian property (G) when $n = 2$.

2. By using the Laplace formula (60), one can see that the global L^p estimates for $\nabla_x R_t(x,y)$ (see Theorem 21) hold only for p in the range $1 \leq p < n/(n-1)$, with

$$\left(\int_{\mathbb{R}^n} |\nabla_x R_t(x,y)|^p \, dx\right)^{1/p} \leq ct^{-1} t^{-n(1-1/p)} \|f\|_1.$$

1.5.3. The conservation property. — This is the conservation constants by the semigroup. Recall that without the property (G), e^{-tL} can be unbounded on $L^\infty(\mathbb{R}^n)$ **[7]**.

***Proposition 25* (Conservation Property).** — *Let $n \geq 2$. Assume that $L \in \mathcal{E}(\delta)$ satisfies (7) in the property (G). Then $e^{-tL}(1) = 1$ for all $t > 0$.*

The argument in **[11]**, Lemma 5.8, applies here. We present a different one.

Proof. — Define

$$b_t(x) = e^{-tL}(1)(x) = \int_{\mathbb{R}^n} K_t(x,y) \, dy.$$

From (G) and Lemma 19, $b_t(x)$ and its partial derivative

$$\frac{\partial b_t}{\partial t}(x) = \int_{\mathbb{R}^n} \frac{\partial K_t(x,y)}{\partial t} \, dy$$

are bounded functions for each $t > 0$.

Let $\chi, \varphi \in C_0^1(\mathbb{R}^n)$ with χ supported in $B(0,2)$ and $\chi = 1$ on $B(0,1)$ (the balls centered at 0 and with radii 2 and 1 respectively). Set $\chi_r(x) = \chi(x/r)$ for $r > 0$. Then, using (7) and the dominated convergence theorem we have

$$\frac{d}{dt}\langle b_t, \varphi\rangle = \langle \frac{\partial b_t}{\partial t}, \varphi\rangle = \lim_{r\to\infty} \langle \chi_r, \frac{de^{-tL^*}}{dt}\varphi\rangle.$$

To compute this limit, we use the parabolic equation to write

$$\langle \chi_r, \frac{de^{-tL^*}}{dt}\varphi\rangle = -\int A(x)\nabla(\chi_r)(x)\overline{\nabla_x K_t^*(x,y)\varphi(y)} \, dy \, dx.$$

Let r_0 be such that the support of φ is contained in $B(0, r_0)$ and choose $r \geq 2r_0$. Then $r/2 \leq |x-y| \leq 5r/2$ on the domain of integration. Using (13) for the kernel of e^{-tL^*} it is easy to see that, for fixed $t > 0$, the double integral is $o(1)$ as r tends to ∞. We have obtained that $\frac{d}{dt}\langle b_t, \varphi\rangle = 0$, which means that b_t is a distribution independent of t. Denoting by b this distribution, it remains to show that $b = 1$.

Let φ as above. Then

$$|\langle b-1, \varphi\rangle| = |\langle b_t - 1, \varphi\rangle| = |\lim_{t\to 0}\langle 1, e^{-tL^*}\varphi - \varphi\rangle| \leq \lim_{t\to 0} \|e^{-tL^*}\varphi - \varphi\|_1.$$

Now, for r_0 as above,

$$\begin{aligned}||e^{-tL^*}\varphi - \varphi||_1 &\leq \int_{B(0,2r_0)} |e^{-tL^*}\varphi - \varphi| \\ &\quad + c\int_{|x|\geq 2r_0}\int_{|y|\leq r_0} |t^{-n/2}e^{-\beta|x-y|^2/t}\varphi(y)|\,dy\,dx \\ &\leq c_n(2r_0)^{n/2}||e^{-tL^*}\varphi - \varphi||_2 + c_{n,\beta}e^{-\beta r_0^2/t}||\varphi||_2,\end{aligned}$$

which tends to 0 with t since $||e^{-tL^*}\varphi - \varphi||_2$ tends to 0 by right continuity at 0 of the semigroup. Therefore $b = 1$. □

Remark. — This proposition holds under the weaker assumption that e^{-tL^*} extends to a C^0-semigroup on $L^1(\mathbb{R}^n)$ **[6]**.

1.6. Analytic perturbation

The purpose of this section is to present an analytic perturbation result for heat kernels as a corollary of (ii) in Theorem 6. Let us begin by indicating that the usual ways of doing perturbation theory are inappropriate.

The usual perturbation theory of semigroup can be attempted via the Duhamel formula

$$e^{-tL'} - e^{-tL} = \int_0^t e^{-sL'} \operatorname{div} M\nabla e^{(s-t)L}\,ds,$$

where $L = -\operatorname{div}(A\nabla)$, $L' = -\operatorname{div} A'\nabla$ and $M = A' - A$. By the semigroup estimate (6) of Proposition 1, we see that the operator in the right hand side is bounded on $L^2(\mathbb{R}^n)$ with a norm controlled by

$$c\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\,||M||_\infty = c\pi||M||_\infty.$$

Iterating the Duhamel formula we get a second term of the form

$$\int_{0\leq r\leq s\leq t} e^{-rL'} \operatorname{div} M\nabla e^{(r-s)L} \operatorname{div} M\nabla e^{(s-t)L}\,dsdr.$$

Using again (6) in Proposition 1, an estimate for the norm is

$$\int_{0\leq r\leq s\leq t} r^{-1/2}(s-r)^{-1}(t-s)^{-1/2}\,dsdr = \infty.$$

Hence the Duhamel formula is of limited interest for this kind of perturbation.

Another way is to iterate the resolvent formula

$$(\lambda + L')^{-1} - (\lambda + L)^{-1} = (\lambda + L)^{-1}(L - L')(\lambda + L')^{-1}$$

in the Cauchy formula (61), so that

$$e^{-tL'} = e^{-tL} + \sum_{k=1}^{\infty} Z_{t,k}(M,\dots,M), \tag{66}$$

where

$$Z_{t,k}(M,\dots,M)=\frac{1}{2\pi i}\int_\gamma e^{t\lambda}T_{\lambda,k}(M,\dots,M)\,d\lambda$$

and

$$T_{\lambda,k}(M_1,\dots,M_k)=(\lambda+L)^{-1}\operatorname{div}M_1\nabla(\lambda+L)^{-1}\cdots\operatorname{div}M_k\nabla(\lambda+L)^{-1}.$$

Using the resolvent estimates (4) of Proposition 1, one sees that

$$\|Z_{t,k}(M_1,\dots,M_k)f\|_2\le c^{k+1}\|M_1\|_\infty\cdots\|M_k\|_\infty\|f\|_2$$

for some $c>0$. Further, $Z_{t,k}(M_1,\cdots,M_1)$ is k-linear as a function of $(M_1,\dots,M_k)$.

In other words, we have shown that the mapping $A\to e^{t\operatorname{div}(A\nabla)}$ is analytic from $\mathcal{A}$ to the space of bounded operators on $L^2(\mathbb{R}^n)$.

But this is about all we can get from such a representation in this generality. For example, when A and A' are real-valued, we know that L and L' have the property (G), but (66) fails to provide an estimate of the form $\|A'-A\|_\infty t^{-n/2}e^{-\beta|x-y|^2/t}$ for the kernel of $e^{-tL'}-e^{-tL}$.

This drawback is taken care of thanks to (ii) in Theorem 6. Indeed this implies that the semigroup kernel depends analytically on the coefficients for the topologies described below.

For $\alpha>0$ and $\mu\in(0,1)$, let $\mathcal{K}_{\alpha,\mu}$ be the Banach space of complex-valued functions $p_t(x,y)$ defined on $E=(0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$ such that

$$N_\alpha(p)\colon=\sup_{(t,x,y)\in E}|p_t(x,y)|t^{n/2}\exp\left\{\frac{\alpha|x-y|^2}{t}\right\}<\infty$$

and

$$N'_\mu(p)\colon=\sup_{(t,x,y)\in E}\sup_{h\neq0}\frac{t^{\mu/2}}{|h|^\mu}(|p_t(x,y)-p_t(x+h,y)|+|p_t(x,y+h)-p_t(x,y)|)<\infty.$$

Define a subclass in $L^\infty(\mathbb{R}^n;M_n(\mathbb{C}))$ by

$$\mathcal{G}=\{A\in\mathcal{A};\ -\operatorname{div}(A\nabla)\text{ has the Gaussian property (G)}\}.$$

By Theorems 6 and 4, $\mathcal{G}$ is an open subset of $L^\infty(\mathbb{R}^n;M_n(\mathbb{C}))$ and a neighborhood of the class of real symmetric elliptic matrices.

For $A\in\mathcal{A}$, denote by $K^A\colon(t,x,y)\to K^A_t(x,y)$ the heat kernel of $L=-\operatorname{div}(A\nabla)$.

With the notations above we have

Theorem 26. — *For each $A_0\in\mathcal{G}$, there are constants $\alpha>0$ and $\mu\in(0,1)$ such that $K^{A_0}\in\mathcal{K}_{\alpha,\mu}$ and that $A\to K^A$ is analytic from a neighborhood of A in $L^\infty(\mathbb{R}^n;M_n(\mathbb{C}))$ into $\mathcal{K}_{\alpha,\mu}$. In particular, there are constants $c\ge0$, $\varepsilon>0$ and $\alpha>0$, depending on A_0 such that for all $A\in L^\infty(\mathbb{R}^n;M_n(\mathbb{C}))$ with $\|A-A_0\|_\infty<\varepsilon$ then $A\in\mathcal{G}$ and*

$$|K^A_t(x,y)-K^{A_0}_t(x,y)|\le c\|A-A_0\|_\infty\,t^{-n/2}\exp\left\{-\frac{\alpha|x-y|^2}{t}\right\}.$$

Proof. — Let $M \in L^\infty(\mathbb{R}^n; M_n(\mathbb{C}))$ with norm 1 and set $f(z) = K^{A_z}$ where $A_z = A_0 + zM$, $z \in \mathbb{C}$. By (ii) of Theorem 6, f is bounded from a neighborhood of 0 into $\mathcal{K}_{\alpha,\mu}$ for some $\alpha > 0$ and $\mu > 0$. We have to show that f is analytic at 0. By (66), we obtain an expansion

$$K_t^{A_z}(x,y) = K_t^A(x,y) + \sum_{k=1}^{\infty} z^k Z_{t,k}(M,\dots,M)(x,y),$$

where $Z_{t,k}(M,\dots,M)(x,y)$, the kernel of $Z_{t,k}(M,\dots,M)$, can be computed by the Cauchy formulæ. The boundedness of f and the Cauchy estimates imply that this series converges in $\mathcal{K}_{\alpha,\mu}$ for $|z|$ small enough. Hence, f is analytic at 0. □

1.7. Higher order elliptic operators

In this section, we consider elliptic homogeneous operators of any order $2m$, $m \geq 1$, as defined in Section 0.4 of Preliminaries by

$$L = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha\beta} \partial^\beta) \tag{67}$$

where $a_{\alpha\beta} \in L^\infty(\mathbb{R}^n; \mathbb{C})$, and where we assume the Gårding inequality

$$\operatorname{Re} \langle Lf, f \rangle \geq \delta \|\nabla^m f\|_2^2, \tag{68}$$

for some $\delta > 0$ independent of $f \in \mathcal{D}(L)$. Recall that ∇^m denotes the array of all partial derivatives of order m.

As for second order operators, one can define the property (G) for L. The usual Gaussian function is replaced by $G_{m,a}(u) = e^{-au^{2m/(2m-1)}}$.

Definition 27. — L has the Gaussian property (G) if for each $t > 0$, the heat kernel $K_t(x,y)$ is a Hölder continuous function in each x and y and if there exist constants c, $\mu > 0$ and $a > 0$ such that for all $t > 0$ and $x, y, h \in \mathbb{R}^n$,

$$|K_t(x,y)| \leq \frac{c}{t^{n/2m}} G_{m,a}\left(\frac{|x-y|}{t^{1/2m}}\right), \tag{69}$$

$$|K_t(x,y) - K_t(x+h,y)| \leq \frac{c}{t^{n/2m}} \left(\frac{|h|}{t^{1/2m} + |x-y|}\right)^\mu G_{m,a}\left(\frac{|x-y|}{t^{1/2m}}\right) \tag{70}$$

and

$$|K_t(x,y+h) - K_t(x,y)| \leq \frac{c}{t^{n/2m}} \left(\frac{|h|}{t^{1/2m} + |x-y|}\right)^\mu G_{m,a}\left(\frac{|x-y|}{t^{1/2m}}\right) \tag{71}$$

whenever $2|h| \leq t^{1/2m} + |x-y|$.

It seems possible to prove a result similar to Theorem 10. We refer the reader to the forthcoming thesis of Qafsaoui. Here we only prove that (G) holds when $2m \geq n$ and we also state and prove gradient estimates on $K_t(x,y)$ whenever (G) holds.

Proposition 28. — *Any operator L as above has the Gaussian property (G) when $2m \geq n$. In particular the constants $c, a > 0, \mu > 0$ depend only on m, n, δ and $\|a_{\alpha\beta}\|_\infty$.*

Remarks

1. Note that this covers the case $m = 1$ and $n \leq 2$, which is part (i) of Theorem 6.
2. When $2m > n$ this theorem is proved in **[29]**; the argument here is similar. When $2m = n$ the argument follows that of Theorem 3.5 in **[11]**. When $2m < n$, the property (G) may fail. Section 1.3 gives a counterexample and further counterexamples are presented in **[30]**.

Proof. — We first deal with the case $2m > n$. By the final remark in Section 0.2 of Preliminaries, $L + 1$ is invertible from $H^m(\mathbb{R}^n)$ onto its dual $H^{-m}(\mathbb{R}^n)$. Since $m > n/2$, the Sobolev embedding theorem gives us

$$H^m(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \cap C^\mu(\mathbb{R}^n)$$

for all $\mu \in (0, \inf(1, m - n/2))$. Therefore, $(L+1)^{-1}$ extends to a bounded operator from $L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n) \cap C^\mu(\mathbb{R}^n)$.

Next, $L + 1$ can be replaced by $L + \lambda$ for λ chosen in an appropriate sector: the same argument applies when $|\lambda| = 1$ and the general situation follows by a rescaling argument. Doing this, we see that for all ν in some interval $(\pi/2, \pi - \omega)$ there is a constant C depending only on $m, n, \nu, \delta, \|a_{\alpha\beta}\|_\infty$, such that

$$|\lambda|^{-n/4m+1}\|(L+\lambda)^{-1}u\|_\infty \leq C\|u\|_2 \tag{72}$$

$$|\lambda|^{-n/4m-\mu/2m+1}|(L+\lambda)^{-1}u|_{\dot{C}^\mu} \leq C\|u\|_2 \tag{73}$$

whenever $\lambda \in \overline{\Gamma_\nu}$, $\lambda \neq 0$.

Integrating these inequalities in the Cauchy formula (61), we obtain

$$\|e^{-tL}u\|_\infty \leq Ct^{-n/4m}\|u\|_2, \quad t > 0, \tag{74}$$

$$|e^{-tL}u|_{\dot{C}^\mu} \leq Ct^{-n/4m-\mu/2m}\|u\|_2, \quad t > 0. \tag{75}$$

Then, we finish the argument by an extension of Lemma 16, referring to **[29]** for the necessary changes in the exponential perturbation technique for higher order operators.

Now, we turn to the case where $2m = n$. This is the critical case for the Sobolev embeddings and the embedding of $H^m(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ fails. To get around this difficulty we use Lemma 23. Since $L+1$ is bounded from $W^{m,p}(\mathbb{R}^n)$ into $W^{-m,p}(\mathbb{R}^n)$ for all $p \in (1, \infty)$ and invertible when $p = 2$ for $W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$, it is invertible from $W^{m,p}(\mathbb{R}^n)$ onto $W^{-m,p}(\mathbb{R}^n)$ for p in a neighborhood of 2. Fix $p > 2$ in such a neighborhood so that the Sobolev embedding $L^2(\mathbb{R}^n) \subset W^{-m,p}(\mathbb{R}^n)$ holds and observe that $m - n/p > 0$. Then $(L+1)^{-1}$ maps $L^2(\mathbb{R}^n)$ into $W^{m,p}(\mathbb{R}^n)$ which embeds into $L^\infty(\mathbb{R}^n) \cap C^\mu(\mathbb{R}^n)$ for all $\mu \in (0, \inf(1, m - n/p))$. From now on, the argument is identical to the previous one, and is left to the reader. □

Our next task is to obtain gradient estimates analogous to the ones in Theorem 7.

Theorem 29. — *Let $n \geq 2$ and $m \geq 1$ and assume that L given by (67) satisfies (68) and has the property (G). Then, there are constants $c, \varepsilon > 0, a > 0, \eta > 0$ depending only on the constants in (G), n, m, δ and $\|a_{\alpha\beta}\|_\infty$, such that for all $y_0, h \in \mathbb{R}^n$, $t > 0$ and $r > 0$ with $2|h| \leq r + t^{1/2m}$, we have*

$$\left(\int_{r\leq|x-y_0|\leq 2r} |\nabla_x^m K_t(x,y_0)|^2\,dx\right)^{1/2} \leq \frac{c}{t^{1/2+n/4m}}\left(\frac{r}{t^{1/2m}}\right)^{\varepsilon} G_{m,a}\left(\frac{r}{t^{1/2m}}\right) \tag{76}$$

and

$$\left(\int_{r\leq|x-y_0|\leq 2r} |\nabla_x^m K_t(x,y_0+h) - \nabla_x^m K_t(x,y_0)|^2\,dx\right)^{1/2} \leq$$
$$\frac{c}{t^{1/2+n/4m}}\left(\frac{|h|}{t^{1/2m}+r}\right)^{\eta}\left(\frac{r}{t^{1/2m}}\right)^{\varepsilon} G_{m,a}\left(\frac{r}{t^{1/2m}}\right). \tag{77}$$

The proof of this result is an adaptation of the one for second order operators. We begin with the proof of (76). By limiting arguments, it clearly suffices to show the following lemma.

Lemma 30. — *Under the assumptions of Theorem 29, there are constants $c, \varepsilon > 0$, $a > 0$, such that for all $f \in L^1(\mathbb{R}^n)$ supported in the ball centered at y_0 of radius $r/4$,*

$$\left(\int_{r\leq|x-y_0|\leq 2r} |\nabla^m e^{-tL} f|^2\,dx\right)^{1/2} \leq \frac{c}{t^{1/2+n/4m}}\left(\frac{r}{t^{1/2m}}\right)^{\varepsilon} G_{m,a}\left(\frac{r}{t^{1/2m}}\right)\|f\|_1. \tag{78}$$

The main ingredients are a Cacciopoli inequality in the spirit of the one in Lemma 20 and some L^p estimates on $\nabla^m e^{-tL}$ generalizing those of Section 1.5.2 to the case of higher order operators.

We begin with the case where $r \geq t^{1/2m}$. The first step is to obtain the parabolic Cacciopoli inequality. It takes the following form: if $u_t = e^{-tL}f$, $f \in L^2(\mathbb{R}^n)$, then for all real valued $\varphi \in C_0^m(\mathbb{R}^n)$,

$$\|t^{1/2}\nabla^m(u_t\varphi)\|_2^2 \leq c\left(\|u_t\varphi\|_2\|t\frac{\partial u_t}{\partial t}\varphi\|_2 + \sum_{|\alpha|=|\beta|=m} t\|v_{t,\alpha\beta}\|_1\right), \tag{79}$$

$c = c(\delta, \|a_{\alpha\beta}\|_\infty, n, m)$, where

$$v_{t,\alpha\beta} = \partial^\beta u_t\ \partial^\alpha(\overline{u_t}\,\varphi^2) - \partial^\beta(u_t\,\varphi)\ \partial^\alpha(\overline{u_t}\,\varphi).$$

Indeed, since u_t satisfies the parabolic equation $\dfrac{\partial u_t}{\partial t} + Lu_t = 0$, we have

$$\langle\frac{\partial u_t}{\partial t}, u_t\varphi^2\rangle = -\sum_{|\alpha|=|\beta|=m}\langle a_{\alpha\beta}\partial^\beta u_t, \partial^\alpha(u_t\,\varphi^2)\rangle.$$

Therefore,

$$\sum_{|\alpha|=|\beta|=m} \langle a_{\alpha\beta}\partial^\beta(u_t\,\varphi), \partial^\alpha(u_t\,\varphi)\rangle = -\langle \frac{\partial u_t}{\partial t}\varphi, u_t\varphi\rangle - \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta} v_{t,\alpha\beta},$$

and (79) follows from the Gårding inequality (68) and the Cauchy-Schwarz inequality.

Now, assume that (G) holds. Fix $y_0 \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$ with support in the ball centered at y_0 and of radius $r/4$. By adapting Lemma 19, one sees that $t\frac{\partial}{\partial t}K_t(x,y)$ also satisfies (69). Thus, there exist non negative constants c and a such that when $|x-y_0| \geq r/2$, we have,

$$|u_t(x)| + \left|t\frac{\partial u_t(x)}{\partial t}\right| \leq \frac{c}{t^{n/2m}} G_{m,a}\left(\frac{|x-y_0|}{t^{1/2m}}\right) \ ||f||_1. \tag{80}$$

Next, pick φ with support defined by $c_0 r \leq |x-y_0| \leq 2r/c_0$ for some $c_0 < 1$ such that $1-c_0$ is small and $\varphi(x) = 1$ if $r \leq |x-y_0| \leq 2r$ and such that $||\partial_\alpha\varphi||_\infty \leq cr^{-|\alpha|}$ for all multiindex α with $|\alpha| \leq m$. Then, using (80) on the support of φ, we have

$$||u_t\varphi||_2 + ||t\frac{\partial u_t}{\partial t}\varphi||_2 \leq \frac{c}{t^{n/4m}} \left(\frac{r}{t^{1/2m}}\right)^{n/2} G_{m,a}\left(\frac{r}{t^{1/2m}}\right) \ ||f||_1, \tag{81}$$

which yields the correct estimate for the first term in the right hand side of (79).

It remains to estimate $t||v_{t,\alpha\beta}||_1$. By the Leibniz rule, we have

$$v_{t,\alpha\beta} = \partial^\beta u_t \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \partial^\gamma(\overline{u_t}\,\varphi)\,\partial^{\alpha-\gamma}\varphi - \partial^\alpha(\overline{u_t}\varphi) \sum_{\gamma'\leq\beta} \binom{\beta}{\gamma'} \partial^{\gamma'} u_t\,\partial^{\beta-\gamma'}\varphi, \tag{82}$$

where $\gamma \leq \alpha$ means that $\gamma_i \leq \alpha_i$ for all $1 \leq i \leq n$. Observe that the $\gamma = \alpha$ and $\gamma' = \beta$ terms cancel, hence both sums restrict to multiindices of length at most $m-1$. Cancellation is no longer used and we estimate each term separately.

Lemma 31. — *Assume that f and φ are functions with the properties specified above. For $|\beta| = m$, $|\gamma| \leq m-1$ and $r \geq t^{1/2m}$, we have*

$$||\partial^\beta(u_t\varphi)||_2 \leq ct^{-1/2-n/4m} \ ||f||_1 \tag{83}$$

and

$$||\partial^\gamma(u_t\,\varphi)||_2 \leq ct^{-|\gamma|/2m-n/4m}\, G_{m,a}\left(\frac{r}{t^{1/2m}}\right) \ ||f||_1, \tag{84}$$

for some $a > 0$ and $c \geq 0$.

Assume that this lemma holds. Consider $\partial^\beta u_t\,\partial^\gamma(\overline{u_t}\,\varphi)\,\partial^{\alpha-\gamma}\varphi$ which is a generic term in (82). Introduce a smooth function ϕ with the same properties as φ and $\phi = 1$ on the support of φ so that $\partial^\beta u_t = \partial^\beta(u_t\phi)$ on the support of φ. By Lemma 31 one obtains

$$t||\partial^\beta u_t\,\partial^\gamma(\overline{u_t}\,\varphi)\,\partial^{\alpha-\gamma}\varphi||_1 \leq \left(\frac{r}{t^{1/2m}}\right)^{-m+|\gamma|} t^{-n/2m} G_{m,a}\left(\frac{r}{t^{1/2m}}\right) ||f||_1.$$

This gives us an estimate of $v_{t,\alpha\beta}$ and (78) follows in this case. The argument is complete provided we prove Lemma 31.

To this end, write $\nabla^m u_t = \nabla^m e^{-tL/2} u_{t/2}$ so that by Proposition 2 of Preliminaries and (69) we obtain

$$t^{1/2}\|\nabla^m u_t\|_2 \le c\|u_{t/2}\|_2 \le ct^{-n/4m}\|f\|_1.$$

Using the interpolation inequality

$$\|\partial^\gamma g\|_2 \le \|\nabla^m g\|_2^{|\gamma|/m}\|g\|_2^{1-|\gamma|/m} \tag{85}$$

valid for all $g \in H^m(\mathbb{R}^n)$, we find that

$$\|\partial^\gamma u_t\|_2 \le ct^{-|\gamma|/2m-n/2m}\|f\|_1$$

for all multiindices γ with $|\gamma| \le m-1$. Now, writing the Leibniz rule, using these estimates, the properties of φ and $r \ge t^{1/2m}$, it is easy to obtain (83).

To see (84), apply the interpolation inequality (85) to $g = u_t\varphi$, and use (83) and (81) for $\|u_t\varphi\|_2$ to obtain

$$\|\partial^\gamma(u_t\,\varphi)\|_2 \le ct^{-|\gamma|/2m-n/4m}\left(G_{m,a}\left(\frac{r}{t^{1/2m}}\right)\right)^{(m-|\gamma|)/m}\|f\|_1.$$

The conclusion follows by noticing that $G_{m,a}^b = G_{m,ab}$ for all $a, b > 0$.

Let us come back to the proof of Lemma 30.

It remains to study the case $r \le t^{1/2m}$. Curiously, the Cacciopoli inequality is useless and we rely instead on some L^p estimates. First, we have observed that $t^{1/2}\|\nabla^m u_t\|_2 \le ct^{-n/4m}\|f\|_1$. Thus, by Hölder inequality,

$$\int_{r\le|x-y_0|\le 2r}|\nabla^m e^{-tL}f|\,dx \le \frac{c}{t^{1/2}}\left(\frac{r}{t^{1/2m}}\right)^{n/2}\|f\|_1. \tag{86}$$

Now, completely analogous arguments to those in Section 1.5.2 show that there exists an $\varepsilon > 0$ such that, if $p \in [1, 2+\varepsilon)$, $\nabla^m e^{-tL}$ is bounded from $L^1(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with

$$\|\nabla^m e^{-tL}f\|_p \le ct^{-1/2-(1-1/p)n/2m}\|f\|_1. \tag{87}$$

It is now easy to obtain (78) by interpolating (86) and (87) with a given $p > 2$.

Lemma 30 is thus proved, and hence (76).

To prove (77) it suffices to establish

$$\left(\int_{r\le|x-y_0|\le 2r}|\nabla^m e^{-tL}f_h|^2\,dx\right)^{1/2} \le ct^{-1/2-n/4m}\left(\frac{|h|}{r}\right)^\eta\left(\frac{r}{t^{1/2m}}\right)^\varepsilon\|f\|_1 \tag{88}$$

for all $f \in L^1(\mathbb{R}^n)$ supported in the ball centered at y_0 of radius $r/4$ and all h with $2|h| \le r \le t^{1/2m}$ and where $f_h(x) = f(x-h) - f(x)$. Indeed, (88) implies (77) in the case $r \le t^{1/2m}$ by a limiting argument and the Gaussian decay in the case $r \ge t^{1/2m}$ comes from (76).

The proof of (88) basically uses the same method as above. From the L^p estimate (87) we have

$$\|\nabla^m e^{-tL} f_h\|_p \leq c t^{-1/2-(1-1/p)n/2m} \|f\|_1,$$

while the L^1 estimate (86) becomes

$$\int_{r \leq |x-y_0| \leq 2r} |\nabla^m e^{-tL} f_h|\, dx \leq \frac{c}{t^{1/2}} \left(\frac{|h|}{r}\right)^{\varepsilon} \left(\frac{r}{t^{1/2m}}\right)^{n/2} \|f\|_1$$

by using (71) and

$$e^{-tL} f_h(x) = \int_{\mathbb{R}^n} (K_t(x, y+h) - K_t(x,y)) f(y)\, dy.$$

Interpolation finishes the argument.

Remarks

1. One can write down L^p estimates for the higher gradient of the resolvent kernel similar to the L^2 estimates in Theorem 29. The range of p's depends on n and m.
2. The L^p estimates generalizing the ones in Section 1.5.2 are valid: we used them in the argument.
3. The conservation property for e^{-tL} holds when L has the property (G). The proof is an adaptation of that of Proposition 25.
4. Lower order terms can be added to L; if the leading part has order $2m \geq n$ then local Gaussian estimates are valid. The proof is the same as the one of Proposition 28.
5. Let $L = (-1)^m \frac{d^m}{dx^m} a(x) \frac{d^m}{dx^m}$ in $L^2(\mathbb{R})$ assumed to be maximal accretive. When $m = 1$, then (G) holds and pointwise estimates exist for the first derivatives of the heat kernel (see **[14]**, and **[11]** where lower order terms are added). If $m \geq 2$, that (G) holds is in Proposition 28. Also it is possible to show with a suitable generalization of Lemma I.4 in **[14]** that all x (or y) derivatives up to order m of the heat kernel have pointwise Gaussian decay as in (69) with the natural scaling in terms of t.

CHAPTER 2

QUADRATIC FUNCTIONALS, CARLESON MEASURES AND SQUARE ROOTS OF DIFFERENTIAL OPERATORS

2.1. Introduction

We know from Preliminaries that we can study the square root problem for differential operators by considering some quadratic functionals. A classical theorem in harmonic analysis, using the paraproducts of Coifman, Meyer and Bony, roughly asserts that the boundedness of a quadratic functional amounts to the control of a Carleson measure. However, the hypotheses that are needed to apply this result are not fulfilled in our case as we deal with more singular operators. The purpose of this chapter is to make the conclusion of this theorem valid and to prepare the ground for the next chapter in which we control the Carleson measures that appear in the case of square roots of differential operators.

To make the discussion accessible to non-experts of square function estimates, Section 2.2 presents a review on the known theory of quadratic functionals. For example, we show that the one-dimensional square root problem falls under its scope. In Section 2.3, an extension of this theory is given by modifying and somehow weakening the hypotheses for which it works. This applies to square roots of elliptic operators with a special structure. The counterexamples of Section 2.4 show that further considerations are needed in order to treat general square roots: this is the content of Section 2.5. Section 2.6 contains miscellaneous material such as the modifications to handle inhomogeneous elliptic operators and localization techniques.

2.2. Classical quadratic functionals

2.2.1. A review. — We begin with some notations. We denote by $|T|_{2,2}$ the norm of an operator that is bounded on $L^2(\mathbb{R}^n)$. Consider a family of linear operators $U_t\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ that is uniformly bounded and that depends measurably on $t \in (0,\infty)$ (in what follows, measurability is always satisfied in applications and thus assumed throughout without mention). The quadratic functionals of our study have the form

$$\int_0^\infty \|U_t f\|_2^2 \frac{dt}{t}.$$

Definition 1. — We denote by $|U_t|_S$ the smallest constant C such that

$$\left(\int_0^\infty \|U_t f\|_2^2 \frac{dt}{t}\right)^{1/2} \le C\|f\|_2, \tag{1}$$

and we say that the family $(U_t)_{t>0}$ is bounded whenever $|U_t|_S < \infty$.

If $H = L^2(\mathbb{R}^+; L^2(\mathbb{R}^n), dt/t)$ is equipped with the norm $\left(\int_0^\infty \|f_t\|_2^2\, dt/t\right)^{1/2}$, this boundedness property says nothing but the continuity of the operator $f \to (U_t f)_{t>0}$ acting from $L^2(\mathbb{R}^n)$ into H.

Quadratic inequalities such as (1) arose in the work of Littlewood and Paley (see **[82]**). These inequalities have played an important role in the development of harmonic analysis. See, *e.g.*, the works by Stein **[72, 73]**.

The classical situation is when U_t is a convolution operator with symbol $\hat{\psi}(t\xi)$ for some function ψ, in which case, by Plancherel theorem, we have

$$|U_t|_S = \sup_{\xi\in\mathbb{R}^n} \left(\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t}\right)^{1/2}. \tag{2}$$

This supremum is finite when, for example, $|\hat{\psi}(\xi)| \le c|\xi|^s(1+|\xi|)^{-2s}$ for some $c, s > 0$.

The holomorphic functional calculus that has been described in Preliminaries, Section 0.1, is an extension of this situation.

In the absence of a functional calculus or when the Fourier transform is not available, these inequalities can be handled by making size and regularity assumptions on the kernels of U_t. Let us describe a typical set of assumptions taken from **[18]**.

Definition 2. — Let $(U_t)_{t>0}$ be a family of operators acting on $L^2(\mathbb{R}^n)$. We say that it is an ε-family when the operators are uniformly bounded on $L^2(\mathbb{R}^n)$ and their kernels are measurable functions with the estimates

$$|U_t(x,y)| \le c_0 \frac{t^\varepsilon}{(|x-y|+t)^{n+\varepsilon}}, \tag{3}$$

$$|\nabla_y U_t(x,y)| \le c_0 \frac{t^\varepsilon}{(|x-y|+t)^{n+1+\varepsilon}}, \tag{4}$$

uniformly in $t > 0, x, y$, for some $\varepsilon > 0$.

Note that (3) implies that U_t extends to a bounded operator on $L^p(\mathbb{R}^n)$ by

$$U_t f(x) = \int_{\mathbb{R}^n} U_t(x,y) f(y)\, dy.$$

No regularity in the x-variable is assumed. The aim here is to state a kind of $T(1)$ theorem relating the boundedness of quadratic functionals and Carleson measure estimates. We first recall some basic results. The first one is a consequence of a variation on the Cotlar-Knapp-Stein lemma. See **[74]**.

***Lemma 3* (Almost orthogonality).** — *Let $(U_t)_{t>0}$ be an ε-family. If $U_t 1 = 0$, that is $\int U_t(x,y)\,dy = 0$ a.e. for all $t>0$, then $(U_t)_{t>0}$ is bounded and one has*

$$|U_t|_S \leq c(n,\varepsilon)c_0, \tag{5}$$

where the constant c_0 appears in (3-4).

Proof. — We give a sketchy argument. Since we have

$$\int_0^\infty \|U_t f\|_2^2 \frac{dt}{t} = \int_0^\infty \langle U_t^* U_t f, f\rangle \frac{dt}{t},$$

it is enough to estimate the norm on $L^2(\mathbb{R}^n)$ of the selfadjoint operator

$$V = \int_\delta^{1/\delta} U_t^* U_t \frac{dt}{t}$$

for fixed $\delta > 0$ with a bound that is independent of $\delta > 0$. By the spectral radius theorem, we have that $|V|_{2,2} = \lim |V^N|_{2,2}^{1/N}$ when $N \to \infty$. Now, for $N \geq 1$, we write

$$V^N = \int_\delta^{1/\delta} \cdots \int_\delta^{1/\delta} U_{t_1}^* U_{t_1} U_{t_2}^* U_{t_2} \cdots U_{t_N}^* U_{t_N} \frac{dt_1 \cdots dt_N}{t_1 \cdots t_N},$$

so that

$$|V^N|_{2,2} \leq \int_\delta^{1/\delta} \cdots \int_\delta^{1/\delta} |U_{t_1}^*|_{2,2} |U_{t_1} U_{t_2}^*|_{2,2} \cdots |U_{t_{N-1}} U_{t_N}^*|_{2,2} |U_{t_N}|_{2,2} \frac{dt_1 \cdots dt_N}{t_1 \cdots t_N}.$$

The hypotheses on $U_t(x,y)$ enter at this point and, using $U_t 1 = 0$ and the inequalities (3) and (4), one shows as in **[27]** that $|U_{t_i} U_{t_j}^*|_{2,2} \leq C\, h(t_i/t_j)$, where $h(t) = \inf(t, t^{-1})$. Using this and the uniform L^2-boundedness of U_t one has that

$$|V^N|_{2,2} \leq C \int_\delta^{1/\delta} \cdots \int_\delta^{1/\delta} C^{N-1} h\left(\frac{t_1}{t_2}\right) \cdots h\left(\frac{t_{N-1}}{t_N}\right) \frac{dt_1 \cdots dt_N}{t_1 \cdots t_N} \leq 2C^N |\ln \delta\,|.$$

Here C depends on c_0 in (3-4), n and ε, but not on δ. Taking Nth root and letting $N \to \infty$ yields $|V|_{2,2} \leq C$ as desired. □

The second important result is the following localization using the notion of a Carleson measure.

Definition 4. — A Carleson measure on $\mathbb{R}^n \times \mathbb{R}^+$ is a positive Radon measure $d\mu(x,t)$ on $\mathbb{R}^n \times \mathbb{R}^+$ such that

$$\sup\left\{\frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} d\mu(x,t)\right\} < \infty. \tag{6}$$

The supremum is taken over the collection $\mathcal{Q}$ of all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes, $\ell(Q)$ and $|Q|$ denoting respectively the sidelength and the volume of such cubes. The quantity in (6) is denoted by $|d\mu|_c$.

Definition 5. — A Carleson function is a measurable function $b(x,t) = b_t(x)$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, such that $d\mu(x,t) = |b(x,t)|^2 dxdt/t$ is a Carleson measure. We set $|b_t|_c = \sqrt{|d\mu|_c}$.

***Lemma 6* (Localization).** — *Let $(U_t)_{t>0}$ be a bounded family whose kernels satisfy (3). Then, for all $f \in L^\infty(\mathbb{R}^n)$, $U_t f$ is a Carleson function and*

$$|U_t f|_c \leq c|U_t|_S \|f\|_\infty + cc_0 \|f\|_\infty, \tag{7}$$

where c_0 is the constant in (3) and $c = c(n,\varepsilon)$. In particular, $|U_t 1|_c < \infty$.

We do not include a proof as it is similar to that of Lemma 15 later on.

That $|U_t 1|_c < \infty$ is also sufficient for the boundedness of ε-families follows from the Carleson inequality which we now recall.

For a function $\varphi \in C_0^\infty(\mathbb{R}^n)$, supported in the unit ball, and $t > 0$, set $\varphi_t(x) = t^{-n}\varphi(x/t)$. Define the convolution operator P_t by $P_t f = f * \varphi_t$. Then the maximal operator defined by

$$P^* f(x) = \sup_{\{(y,t);|x-y|\leq t\}} |P_t f(y)|$$

is bounded on $L^2(\mathbb{R}^n)$ as a consequence of the L^2 boundedness of the Hardy-Littlewood maximal operator, see **[71]**. Carleson inequality asserts that

$$\int_{\mathbb{R}^n} \int_0^\infty |P_t f(x)|^2 \, d\mu(x,t) \leq c(n) \, |d\mu|_c \int_{\mathbb{R}^n} |P^* f(x)|^2 \, dx, \quad f \in L^2(\mathbb{R}^n),$$

(see **[74]**) so that

$$\int_{\mathbb{R}^n} \int_0^\infty |P_t f(x)|^2 \, d\mu(x,t) \leq c(n,\varphi) \, |d\mu|_c \int_{\mathbb{R}^n} |f(x)|^2 \, dx, \quad f \in L^2(\mathbb{R}^n). \tag{8}$$

In terms of quadratic inequalities, this gives the following result.

Lemma 7. — *Define P_t as above. Let b_t be a Carleson function with* $\sup_{t>0} \|b_t\|_\infty < \infty$ *and let M_t be the pointwise multiplication operator by b_t. Then $(M_t P_t)_{t>0}$ is a bounded ε-family for any $\varepsilon > 0$ and*

$$|M_t P_t|_S \leq c(n,\varphi)|b_t|_c. \tag{9}$$

Operator families such as the ones of Lemma 3 or of Lemma 7 are the basic building blocks of ε-families. More precisely, we have

Lemma 8. — *Let $(U_t)_{t>0}$ be an ε-family. Let $P_t f = f * \varphi_t$, φ being defined as above with, in addition, $\int \varphi = 1$. Then*

$$U_t = M_t P_t + Q_t, \tag{10}$$

where M_t is pointwise multiplication by $U_t 1$ and $(Q_t)_{t>0}$ is an ε-family with $Q_t 1 = 0$.

Proof. — It is clear that the equality $Q_t = U_t - M_t P_t$ defines an ε-family. Moreover, $Q_t 1 = 0$ follows from $P_t 1 = 1$ and the definition of M_t. □

As a consequence of this chain of results is the following $T1$-theorem for Carleson measures as stated by Christ and Journé **[18]**.

Theorem 9. — *Let $(U_t)_{t>0}$ be an ε-family.*

(i) *$(U_t)_{t>0}$ is bounded if and only if U_t1 is a Carleson function.*

(ii) *Moreover, when $|U_t1|_c < \infty$, one has*

$$|U_tf|_c \leq |U_t1|_c\|f\|_\infty + cc_0\|f\|_\infty, \qquad f \in L^\infty(\mathbb{R}^n), \tag{11}$$

where $c = c(n,\varepsilon)$ and c_0 is the constant in (3-4).

A major point is the precise value of the constant, namely 1, in front of $|U_t1|_c$ in (11). This is observed and used efficiently in **[18]** toward polynomial growth in norm estimates for multilinear expansions, including the one used for the one dimensional square root problem. We shall exploit later this value in a different way.

Proof. — The equivalence in (i) follows readily from the previous lemmas. The proof of (11) exploits a fact that has not been used so far: P_t is contractive on $L^\infty(\mathbb{R}^n)$ provided we choose $\varphi \geq 0$, which we may. This fact yields

$$|(U_t1)(P_tf)|_c \leq |U_t1|_c\|f\|_\infty.$$

This and (10) yields

$$|U_tf|_c \leq |(U_t1)(P_tf)|_c + |Q_tf|_c \leq |U_t1|_c\|f\|_\infty + |Q_tf|_c$$

and we conclude using $|Q_tf|_c \leq c\|f\|_\infty$ by Lemma 6 and Lemma 3. □

Remark. — All of the above goes through if (4) is replaced by the weaker Hölder continuity estimate

$$|U_t(x,y') - U_t(x,y)| \leq c_0\frac{t^\varepsilon}{(|x-y|+t)^{n+\varepsilon}}\left[\frac{|y'-y|}{|x-y|+t}\right]^\varepsilon.$$

See **[18]**.

2.2.2. Application to the one dimensional square root problem. — Even though we are mostly concerned with the higher dimensional case, it is of interest to present a simple argument for the case $n = 1$. Here, $L = -\frac{d}{dx}(a\frac{d}{dx})$, where $a(x)$ is a bounded and accretive function on $\mathbb{R}$. It is always possible to normalize $a(x)$ so that $a^{-1} = 1 - m$ where $\|m\|_\infty < 1$ (see Appendix C).

By Proposition 5 of Preliminaries, establishing

$$\|L^{1/2}f\|_2 \leq c\|\frac{df}{dx}\|_2,$$

which was first proved in **[20]**, amounts to proving that $|U_t|_S < \infty$, where

$$U_tf = \left(I - t^2\frac{d}{dx}(a\frac{d}{dx})\right)^{-1} t\frac{d}{dx}(af).$$

That $(U_t)_{t>0}$ is an ε-family is proved in **[14]** (there, an exponential decay is obtained). By Theorem 9, we are reduced to estimating $|U_t 1|_c$: this is done using (11).

The key observation is the cancellation property $U_t\, a^{-1} = 0$ which implies $U_t 1 = U_t\, m$. Thus, by (11)

$$|U_t 1|_c = |U_t\, m|_c \leq \|m\|_\infty |U_t 1|_c + cc_0 \|m\|_\infty,$$

and solving for $|U_t 1|_c$ gives us

$$|U_t 1|_c \leq \frac{cc_0 \|m\|_\infty}{1 - \|m\|_\infty} < \infty \tag{12}$$

since $\|m\|_\infty < 1$.

To finish the proof, we have to justify the use of (11). Indeed, it requires the *a priori* knowledge that $|U_t 1|_c$ be finite, which is to be proved. To overcome this vicious circle, we use truncations. As this trick is needed at various places without mention, we describe it in detail here.

Let χ be any non-negative measurable function with compact support in $(0, \infty)$ and such that $\|\chi\|_\infty \leq 1$. If $(U_t)_{t>0}$ is any ε-family, then the truncated family $(\chi(t)\, U_t)_{t>0}$ is also an ε-family, with the same constant c_0.

Because of the support assumption, there is a constant $c = c(\chi) > 0$ such that $|\chi(t)\, U_t|_S \leq c \sup |U_t|_{2,2} < \infty$. Hence, $|\chi(t)\, U_t 1|_c < \infty$ by Theorem 9. As the truncation by χ does not affect the cancellation property, the same argument as above applies and we have

$$|\chi(t)\, U_t 1|_c \leq \frac{cc_0 \|m\|_\infty}{1 - \|m\|_\infty} = c_1 < \infty.$$

This means that for a given cube $Q \in \mathcal{Q}$,

$$\frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\chi(t)\, (U_t 1)(x)|^2 \, \frac{dt dx}{t} \leq c_1^2.$$

Fix Q and for $k \geq 1$, let $\chi = 1$ on $[1/k, k]$ and 0 elsewhere. By letting k tend to ∞, the monotone convergence theorem of Beppo Levi yields

$$\frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |(U_t 1)(x)|^2 \, \frac{dt dx}{t} \leq c_1^2.$$

Taking the supremum over all cubes gives the desired inequality and the proof of (12) is finished.

2.2.3. Quadratic functionals of weakly regular families. — Our goal in this chapter is to generalize Theorem 9 to the class of quadratic functionals arising from the square root problem in higher dimensions. As a first step, we study a weakened version of this result.

Before going into statements, it is worth taking a closer look at the decomposition (10) in Lemma 8 to show when the different hypotheses are used. This argument borrows ideas from **[69]** and **[21]**.

To avoid unnecessary technicalities, we consider a model case where $U_t(x,y)$ is supported in $|x-y| \le t$ and satisfies

$$|U_t(x,y)| \le t^{-n} \qquad \text{and} \qquad |\nabla_y U_t(x,y)| \le t^{-n-1}. \tag{13}$$

Denoting by M_t the operator of multiplication by $U_t 1$, one can write (with the notation of Lemma 8)

$$U_t = M_t P_t + (U_t - M_t)P_t + U_t(I - P_t). \tag{14}$$

In other words, the operator Q_t in (10) is broken up in two parts, $(U_t - M_t)P_t$ and $U_t(I - P_t)$. That these operators send 1 to 0 is obvious and each part eventually has the same property as Q_t. The interesting point we want to make concerns the kernel analysis of each part.

The kernel of $(U_t - M_t)P_t$ is

$$K_t^1(x,y) = \int U_t(x,z)\big(\varphi_t(z-y) - \varphi_t(x-y)\big)\,dz, \tag{15}$$

This kernel is supported in $|x-y| \le 2t$ where, up to a numerical constant, it satisfies (13). Here, $U_t(x,z)$ is used as an averaging integrable function and only a size estimate is needed.

The kernel of $U_t(I - P_t)$ is

$$K_t^2(x,y) = \int (U_t(x,z) - U_t(x,y))\varphi_t(z-y)\,dz. \tag{16}$$

Since φ is supported in the unit ball, this kernel also has support in $|x-y| \le 2t$ where, up to a numerical constant, it satisfies (13). This is a consequence of the mean value theorem and this indicates the importance of regularity assumptions in the y-variable on $U_t(x,y)$ for this term.

We are now ready for introducing the notation for the next result.

Definition 10. — Let $(U_t)_{t>0}$ be a family of operators acting on $L^2(\mathbb{R}^n)$. We say that it is a weakly regular family (with constant c_0) when the operators U_t are uniformly bounded on $L^2(\mathbb{R}^n)$ with $|U_t|_{2,2} \le c_0$ and when there are constants $\varepsilon > 0$ and $s > 0$ such that:

(i) the kernels $U_t(x,y)$ are measurable functions with the estimate

$$\int_{r \le |x-y| \le 2r} |U_t(x,y)|\,dy \le c_0 \inf\left(\left(\frac{r}{t}\right)^{\varepsilon}, \left(\frac{t}{r}\right)^{n+\varepsilon}\right), \tag{17}$$

uniformly in $t > 0$, $r > 0$ and $x \in \mathbb{R}^n$;

(ii) $U_t(-\Delta)^{s/2}$ is bounded on $L^2(\mathbb{R}^n)$ with

$$\|U_t(-\Delta)^{s/2} f\|_2 \le c_0 t^{-s} \|f\|_2, \quad t > 0. \tag{18}$$

Remarks

1. Condition (ii) is a smoothing condition for U_t. It means that U_t maps the homogeneous Sobolev space $\dot{H}^{-s}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

2. It is easy to see that ε-families are weakly regular families provided $\varepsilon > n$. Clearly, (3) implies (17) and this is where $\varepsilon > n$ is needed. That (4) implies (18) with $s = 1$ goes as follows. The decay estimate in (4) implies that $U_t\partial/\partial x_j$ is bounded on $L^2(\mathbb{R}^n)$ by interpolation, and its norm does not exceed ct^{-1}. Using that

$$(-\Delta)^{1/2} f = -\sum_{j=1}^{n} \frac{\partial(R_j f)}{\partial x_j},$$

where R_j is the jth Riesz transform, we obtain

$$||U_t(-\Delta)^{1/2} f||_2 \le \sum_{j=1}^{n} ||U_t \frac{\partial(R_j f)}{\partial x_j}||_2 \le ct^{-1} \sum_{j=1}^{n} ||R_j f||_2 \le ct^{-1}||f||_2$$

from the boundedness of the Riesz transforms.

3. One can also prove that the Hölder estimate stated in the remark concluding Section 2.2.1 is stronger than (18). The proof proceeds by duality and one uses the semi-norm

$$\left(\iint \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\, dxdy\right)^{1/2}$$

as an equivalent semi-norm on $\dot{H}^s(\mathbb{R}^n)$ for $0 < s < 1$.

As we shall see, (17) is only a technical refinement of (3). But, the use of (18) leads to a different analysis of the term $U_t(I - P_t)$. Still the statement of Theorem 9 goes through.

Theorem 11. — *Let $(U_t)_{t>0}$ be a weakly regular family with constant c_0.*

(i) *$(U_t)_{t>0}$ is bounded if and only if $U_t 1$ is a Carleson function.*

(ii) *When $|U_t 1|_c < \infty$, one has*

$$|U_t f|_c \le |U_t 1|_c ||f||_\infty + cc_0 ||f||_\infty, \quad f \in L^\infty(\mathbb{R}^n), \tag{19}$$

where $c = c(n, \varepsilon, s)$.

We begin the proof with a series of lemmas.

Lemma 12. — *Assume that $U_t(x,y)$ satisfies (17). Then $\int_{\mathbb{R}^n} |U_t(x,y)|dy \le c(n,\varepsilon)c_o$ uniformly over $x \in \mathbb{R}^n$ and $t > 0$. In particular, U_t extends to a bounded operator on $L^\infty(\mathbb{R}^n)$ with norm bounded by $c(n,\varepsilon)c_0$ and $U_t f(x) = \int_{\mathbb{R}^n} U_t(x,y) f(y)\, dy$ a.e. when $f \in L^\infty(\mathbb{R}^n)$.*

Proof. — It suffices to prove the integral estimate and the rest of the statement follows from standard arguments. By (17),

$$\int_{\mathbb{R}^n} |U_t(x,y)|\,dy \leq \sum_{j=-\infty}^{+\infty} \int_{2^j t\leq|x-y|\leq 2^{j+1}t} |U_t(x,y)|\,dy$$
$$\leq \sum_{j=-\infty}^{+\infty} c_0 \inf(2^{j\varepsilon}, 2^{-(n+\varepsilon)j}) = c(n,\varepsilon)c_0,$$

which finishes the proof. □

Lemma 13. — *Assume that $U_t(x,y)$ satisfies (17) and that $\varphi \in L^\infty$ has support in the unit ball. Then*

$$\left|\int_{\mathbb{R}^n} U_t(x,z)\varphi_t(z-y)\,dz\right| \leq c\|\varphi\|_\infty c_0 \frac{t^\varepsilon}{(|x-y|+t)^{n+\varepsilon}}.$$

Proof. — Without loss of generality one can take $t=1$ after rescaling and assume also that $\|\varphi\|_\infty = 1$. Calling $K(x,y)$ the integral, we have

$$|K(x,y)| \leq \int |U_1(x,z)|\,dz \leq c(n,\varepsilon)c_0$$

by Lemma 12. This is enough if $|x-y|\leq 4$, while if $|x-y|\geq 4$ then $3|x-y|/4 \leq |x-z| \leq 5|x-y|/4$ on the support of the integral and, by (17),

$$|K(x,y)| \leq c_0\left(\frac{4}{3|x-y|}\right)^{n+\varepsilon}.$$

This ends the proof. □

Remark. — This is the only place where the term $(t/r)^{n+\varepsilon}$ is used in (17), while a term $(t/r)^\varepsilon$ is enough elsewhere.

Corollary 14. — *Assume that $U_t(x,y)$ satisfies (17). Let M_t be the pointwise multiplication operator by U_t1 and let P_t be a convolution operator with φ_t, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ has support in the unit ball. Then $(M_tP_t)_{t>0}$ and $(U_tP_t)_{t>0}$ are ε-families and $((M_t-U_t)P_t)_{t>0}$ is bounded when, in addition, $\int\varphi = 1$.*

Proof. — The kernel of M_tP_t is $(U_t1)(x)\varphi_t(x-y)$, so that estimates (3-4) are immediate.

The kernel of U_tP_t is given by the integral in Lemma 13, and (3) follows from Lemma 13. The inequality (4) is obtained similarly on replacing φ_t by its gradient.

That $((M_t-U_t)P_t)_{t>0}$ is bounded follows on applying Theorem 9 since

$$(M_t - U_t)(P_t1) = 0$$

when $\int\varphi = 1$. The proof is complete. □

The next result is a generalization of Lemma 6, where, for subsequent use, we obtain an explicit constant.

Lemma 15. — *Assume that $(U_t)_{t>0}$ is a bounded family of operators that satisfies $|U_t|_{2,2} \leq c_0$ and (17). Then for all $Q \in \mathcal{Q}$, one has*

$$\left(\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)} |U_t f|^2 \frac{dxdt}{t}\right)^{1/2} \leq |U_t|_S \left(\frac{1}{|Q|}\int_Q |f|^2\right)^{1/2} + c(n,\varepsilon)c_0 \sup_{\mathbb{R}^n\setminus Q} |f|. \tag{20}$$

In particular, for all $f \in L^\infty(\mathbb{R}^n)$, one has

$$|U_t f|_c \leq (|U_t|_S + c(n,\varepsilon)c_0)\|f\|_\infty. \tag{21}$$

Proof. — It is clear that (21) follows from (20).

To prove (20), write $f = f_1 + f_2$, where $f_1 = f$ on the cube Q and $f_1 = 0$ elsewhere. On the one hand,

$$\int_Q\int_0^{\ell(Q)} |U_t f_1(x)|^2 \frac{dxdt}{t} \leq \int_{\mathbb{R}^n}\int_0^{\infty} |U_t f_1(x)|^2 \frac{dxdt}{t} \leq |U_t|_S^2 \int_Q |f(x)|^2\, dx,$$

and this gives us the first term in the right hand side of (20).

On the other hand, it remains to prove

$$\int_Q\int_0^{\ell(Q)} |U_t f_2(x)|^2 \frac{dxdt}{t} \leq cc_0^2 \|f_2\|_\infty^2 |Q|.$$

For fixed $t > 0$, decompose further f_2 as $f_2 = f_3 + f_4$, where $f_3(y) = f_2(y)$ if $d(y) \leq t$ and $f_3(y) = 0$ otherwise. We have denoted by $d(y)$ the distance of y to the boundary of Q in the norm $|y|_\infty = \max\{\,|y_i|, 1 \leq i \leq n\,\}$.

First, using $|U_t|_{2,2} \leq c_0$ we have,

$$\begin{aligned}\int_Q |U_t f_3(x)|^2\, dx &\leq c_0^2 \int |f_3|^2\\ &\leq c_0^2 \|f_3\|_\infty^2 |\{\, y \in \mathbb{R}^n; d(y) \leq t\,\}|\\ &\leq cc_0^2 \|f_2\|_\infty^2 \ell(Q)^{n-1} t.\end{aligned}$$

Integrating this inequality against dt/t over $[0, \ell(Q)]$ gives us the desired control for this term.

Next, for $x \in Q$ and y in the support of f_4 we have

$$|x-y|_\infty \geq t + \frac{\ell(Q)}{2} - |x - x_Q|_\infty = r(t,x) = r,$$

where x_Q is the center of Q. Thus, by splitting the range of integration into the dyadic annuli defined by $2^j r \leq |x-y|_\infty \leq 2^{j+1} r$, $j = 0, 1, \ldots$ and using (17) we have

$$\int_{\mathbb{R}^n} |U_t(x,y) f_4(y)|\, dy \leq c(n) c_0 \sum_{j=0}^{\infty} \left(\frac{t}{2^j r}\right)^{n+\varepsilon} \|f_4\|_\infty = cc_0 \left(\frac{t}{r}\right)^{n+\varepsilon} \|f_2\|_\infty.$$

Now, by making the change of variables $u = (\ell(Q)/2 - |x - x_Q|_\infty)/t$ we have

$$\int_0^{\ell(Q)} \left(\frac{t}{r(t,x)}\right)^{2n+2\varepsilon} \frac{dt}{t} \leq c(n,\varepsilon)\left(\left|\ln\left(1 - \frac{2|x - x_Q|_\infty}{\ell(Q)}\right)\right| + 1\right).$$

Hence,

$$\int_Q \int_0^{\ell(Q)} |(U_t f_4)(x)|^2 \frac{dxdt}{t} \leq cc_0^2 \|f_2\|_\infty^2 \int_Q \left(\left| \ln \left(1 - \frac{2|x - x_Q|_\infty}{\ell(Q)}\right) \right| + 1 \right) dx.$$

By scaling, it is easily seen that the last integral is bounded above by $c(n)|Q|$ and this ends the argument. □

We conclude this series of preparatory lemmas with the following result.

Lemma 16. — *Assume that U_t satisfies (18) for some $s > 0$. Let P_t be a convolution operator by $t^{-n}\varphi(x/t)$ for some smooth function $\varphi \in L^1(\mathbb{R}^n)$ with $\int \varphi = 1$. Then*

$$|U_t(I - P_t)|_S \leq c_0 c(n, s, \varphi), \tag{22}$$

where

$$c(n, s, \varphi) = \sup_{\xi \in \mathbb{R}^n} \left(\int_0^\infty \frac{|1 - \hat{\varphi}(t\xi)|^2}{|t\xi|^{2s}} \frac{dt}{t} \right)^{1/2}.$$

Proof. — Write

$$U_t(I - P_t) = U_t(-t^2\Delta)^{s/2} Q_t$$

where

$$Q_t = (-t^2\Delta)^{-s/2}(I - P_t).$$

This operator is a convolution operator with symbol $|t\xi|^{-s}(1 - \hat{\varphi}(t\xi))$, so that $|Q_t|_S = c(n, s, \varphi)$ by (2).

We conclude the proof of Lemma 16 on applying the following remark to $L_t = U_t(-t^2\Delta)^{s/2}$, $M_t = Q_t$ and $N = I$, whose proof is left to the reader. □

Lemma 17. — *For operators L_t, M_t and N, assumed to be uniformly bounded on $L^2(\mathbb{R}^n)$, one has*

$$|L_t M_t N|_S \leq \left(\sup_{t>0} |L_t|_{2,2} \right) |M_t|_S |N|_{2,2}. \tag{23}$$

We now turn to the proof of Theorem 11, first establishing the equivalence between the boundedness of $(U_t)_{t>0}$ and the Carleson measure estimate for $U_t 1$.

If $|U_t|_S$ is finite then, by Lemma 15, $U_t 1$ is a Carleson function.

Conversely, choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball and such that $|\hat{\varphi}(\xi) - 1| \leq C|\xi|^{s+1}$ for $|\xi| \leq 1$. Again, P_t denotes the convolution operator with $\varphi_t(x) = t^{-n}\varphi(x/t)$. Write as before, with M_t being multiplication by $U_t 1$,

$$U_t = M_t P_t + (U_t - M_t) P_t + U_t(I - P_t). \tag{24}$$

We have to prove that each term in the right hand side defines a bounded family.

For the first one, apply the Carleson estimate of Lemma 7 to obtain

$$|M_t P_t|_S \leq c |U_t 1|_c.$$

For the second one, notice that $\int \varphi = \hat{\varphi}(0) = 1$ and apply Corollary 14.

For the last one, observe that the constant $c(n,s,\varphi)$ in (22) of Lemma 16 is finite with our choice of φ.

It remains to prove (19). Assume that U_t1 is a Carleson function. The starting point is again equality (24) which gives

$$|U_tf|_c \leq |M_tP_tf|_c + |(U_t - M_t)P_tf|_c + |U_t(I-P_t)f|_c.$$

It follows from what we just established and Lemma 15 that

$$|(U_t - M_t)P_tf|_c + |U_t(I-P_t)f|_c \leq cc_o\|f\|_\infty.$$

With our choice of φ, P_t may not be contractive on L^∞, so we introduce an L^∞-contractive operator $\widetilde{P}_t$ of convolution with $\phi_t(x) = t^{-n}\phi(x/t)$, with ϕ smooth, supported in the unit ball, $\phi \geq 0$, $\int \phi = 1$. Then as in the proof of (11) in Theorem 9, $|M_t\widetilde{P}_tf|_c \leq |U_t1|_c\|f\|_\infty$. The error term can be handled as follows: since $|M_t|_{2,2} = \|U_t1\|_\infty$ is uniformly bounded by cc_0 by Lemma 12, we have

$$|M_t(P_t - \widetilde{P}_t)f|_c \leq cc_0|(P_t - \widetilde{P}_t)f|_c \leq c'c_0\|f\|_\infty,$$

where in the last inequality we used the fact that $P_t - \widetilde{P}_t$ defines a bounded family of convolution type since $(P_t - \widetilde{P}_t)1 = 0$. This completes the proof of Theorem 11.

Remark. — Let us quickly discuss the vector-valued extension of this theory. We restrict ourselves to the case where U_t are operators acting from $L^2(\mathbb{R}^n;\mathbb{C}^p)$ into $L^2(\mathbb{R}^n;\mathbb{C}^q)$, where p,q are two non-negative integers. Here, $\mathbb{C}^p$ and $\mathbb{C}^q$ are equipped with their natural Hilbert space structure. In the formula $\int U_t(x,y)f(y)\,dy$, $U_t(x,y)$ takes values in $\mathcal{B}(\mathbb{C}^p,\mathbb{C}^q)$, the space of linear operators from $\mathbb{C}^p$ into $\mathbb{C}^q$, equipped with the induced norm. On the canonical orthonormal bases of $\mathbb{C}^p$ and $\mathbb{C}^q$, $U_t(x,y)$ becomes a $q \times p$ complex matrix with entries $U_t^{k\ell}(x,y)$. Then, $(U_t1)(x)$ should be understood as the matrix with entries $\int U_t^{k\ell}(x,y)\,dy$ and $|(U_t1)(x)|$ as its operator norm. With these precautions, all the results above extend straightforwardly. In the arguments, P_t or $(-\Delta)^{1/2}$ are scalar operators, which means they act componentwise on vector-valued functions.

2.2.4. Square roots of operators with special structures.

— There is a class of differential operators to which the theorem on quadratic functionals of weakly regular families may be applied. An example is $\Delta(a(x)\Delta)$. Such operators enjoy more regularity properties than the general differential operators as presented in Preliminaries.

Fix a real $m > 0$. Let $a(x)$, $x \in \mathbb{R}^n$, be a complex-valued accretive function. Let $\sigma(\xi)$ be a real-valued homogeneous polynomial of degree m that does not vanish on the unit sphere. Let $\sigma(D)$ be the self-adjoint differential operator on $L^2(\mathbb{R}^n)$ with symbol $\sigma(\xi)$. Observe that $\|\sigma(D)f\|_2 \sim \|(-\Delta)^{m/2}f\|_2$.

Define $L = \sigma(D)(a(x)\sigma(D))$, the associated maximal-accretive differential operator of order $2m$ associated with multiplication by $a(x)$ and $\sigma(D)$ as in Proposition 1 of Preliminaries.

Theorem 18. — *With the notation above,*

$$||L^{1/2}f||_2 \sim ||(-\Delta)^{m/2}f||_2. \tag{25}$$

This result is not new since it can be deduced from multilinear estimates similar to the ones in **[20]** as in **[59]**, but it is a good illustration of the method of quadratic functionals that we have just developed.

Proof. — By Proposition 10 of Preliminaries, we may assume that $m \geq n/2$. Indeed, (25) is equivalent to

$$||\{(\sigma(D)^r(a(x)\sigma(D)^r)\}^{1/2}f||_2 \sim ||(-\Delta)^{mr/2}f||_2$$

for any integer $r > 0$ and $\sigma(\xi)^r$ is a homogeneous polynomial of degree mr.

We show that $||L^{1/2}f||_2 \leq c||(-\Delta)^{m/2}f||_2$. The argument for obtaining this inequality with L^* replacing L is similar and we conclude using Proposition 3 of Preliminaries.

By Proposition 5 of Preliminaries the above inequality is equivalent to the quadratic estimate

$$\left(\int_0^\infty ||(1+t^{2m}L)^{-1}t^m\sigma(D)(af)||_2^2 \,\frac{dt}{t}\right)^{1/2} \leq c||f||_2, \tag{26}$$

which we obtain by applying Theorem 11 to the operator family defined by $U_tf = (1+t^{2m}L)^{-1}t^m\sigma(D)(af)$. Indeed, the estimate (18) holds for $s = m$ by the global L^2-estimates (Proposition 2 of Preliminaries) and, since $m \geq n/2$, it follows from Theorem 29 in Chapter 1 that the kernel of U_t satisfies (17). Thus, (26) holds if and only if U_t1 is a Carleson function. The latter is proved following the argument presented for $-\frac{d}{dx}(a(x)\frac{d}{dx})$ in dimension 1 since the cancellation property $U_t\, a^{-1} = 0$ holds here too. □

2.3. Counterexamples

We do not claim that the conditions on weakly regular families are the weakest ones for the conclusion of Theorem 11 to hold. However, one cannot drop the regularity assumption (18). Here is a counterexample in $L^2(\mathbb{R})$ where the size condition (17) is satisfied, yet the conclusion of Theorem 11 fails.

Consider operators U_t given by their kernels

$$U_t(x,y) = \sum_{k=-\infty}^{\infty} \overline{a_{jk}(y)}\, h_{jk}(x),$$

when $2^{-j} \le t < 2^{-j+1}$, the functions $h_{jk}(x) = 2^{j/2}h(2^j x - k)$, $j,k \in \mathbb{Z}$, generating the Haar system, and the functions $a_{jk}(y)$ being of the form $2^{j/2}a(2^j y - k)$ where a is some square integrable function with support in $[0,1]$ and $\int_0^1 a = 0$.

These assumptions imply that for all $t > 0$,

$$\int_{r \le |x-y| \le 2r} |U_t(x,y)|\,dy \le 2\|a\|_2 \sqrt{r/t}\,\chi_{r \le t}$$

and $U_t 1 = 0$. Observing that

$$\int_0^\infty \|U_t f\|_2^2 \frac{dt}{t} = \ln 2 \sum_{jk} |\langle f, a_{jk}\rangle|^2, \tag{27}$$

it remains to choose a so that the last sum is infinite for some $f \in L^2(\mathbb{R})$.

Let (a_n) and (f_n) be two non negative sequences in $\ell^2(\mathbb{N})$ and set

$$a(x) = \sum_{n \ge 0} a_n e^{2i\pi 2^n x}\, \chi_{[0,1]}(x)$$

and

$$f(x) = \sum_{n \ge 0} f_n e^{2i\pi 2^n x}\, \chi_{[0,1]}(x).$$

Restricting the summation in (27) to $j \ge 0$ and $0 \le k < 2^j$, we find after an explicit calculation that

$$\langle f, a_{jk}\rangle = \sum_{n \ge 0} a_n f_{j+n} 2^{-j/2} + \sum_{n \ge 0} \sum_{0 \le m < j} a_n f_m \frac{e^{(2k+1)i\pi 2^{m-j}}}{\pi(2^m - 2^{n+j})} \sin(\pi 2^{m-j}) 2^{j/2}.$$

Calling c_j and d_{jk} respectively the series above, we have

$$|d_{jk}| \le 2 \sum_{n \ge 0} \sum_{0 \le m < j} a_n f_m 2^{-n-j} 2^{m-j} 2^{j/2},$$

hence, with the above restriction on j and k,

$$\begin{aligned} \sum_{j,k} |d_{jk}|^2 &\le 4 \sum_{j \ge 0} \Big(\sum_{n \ge 0} \sum_{0 \le m < j} a_n f_m 2^{-n} 2^{m-j} \Big)^2 \\ &\le \frac{16}{3} \sum_{j \ge 0} \sum_{n \ge 0} \sum_{0 \le m < j} a_n^2 f_m^2 2^{m-j} \\ &= \frac{16}{3} \sum_{n \ge 0} a_n^2 \sum_{m \ge 0} f_m^2 = \frac{16}{3} \|a\|_2^2 \|f\|_2^2. \end{aligned}$$

Thus, (27) is infinite as soon as $\sum_{j \ge 0} |c_j|^2 2^j$ diverges. Choosing $a_n = (n+1)^{-\alpha}$, $1/2 < \alpha < 1$, and $f_n = (n+1)^{-\beta}$, $1/2 < \beta \le 3/2 - \alpha$, gives us the desired conclusion.

It turns out that the smoothing property (18) may fail for the operator families arising from the square root problem.

Proposition 19. — *For each $n \geq 2$ there is a real symmetric elliptic operator $L = -\operatorname{div}(A\nabla)$ on $\mathbb{R}^n$ such that, if $\theta_t = e^{-t^2L}t \operatorname{div} A$, then $\theta_t(-\Delta)^{s/2}$ is unbounded on $L^2(\mathbb{R}^n)$ for any $t > 0$ and $s \in (0,1)$.*

Proof. — The idea is based on the fact that weak solutions of $Lu = 0$ do not satisfy $A\nabla u \in \cup_{s>0} H^s_{loc}$ in general. The first step of the construction is the reduction to weak solutions.

Since the norm on the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ is given by $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_2$, by duality the conclusion of the proposition is that $At\nabla e^{-t^2L}$ is not bounded from $L^2(\mathbb{R}^n)$ into $\dot{H}^s(\mathbb{R}^n)$.

We argue by contradiction. Assume that L is such that there exists $T > 0$ such that $AT\nabla e^{-T^2L}$ is bounded from $L^2(\mathbb{R}^n)$ into $\dot{H}^s(\mathbb{R}^n)$. If, in addition, the coefficients of L are homogeneous of degree 0, then L is invariant under dilation on $\mathbb{R}^n$ and this implies that for all $t > 0$, $At\nabla e^{-t^2L}$ is bounded from $L^2(\mathbb{R}^n)$ into $\dot{H}^s(\mathbb{R}^n)$ with

$$\|At\nabla e^{-t^2L}\|_{\dot{H}^s} \leq ct^{-s}\|f\|_2.$$

Furthermore, since e^{-t^2L} is bounded from $H^{-1}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ (see Chapter 1) with norm bounded by $c \sup(t^{-1}, 1)$, we obtain using the semigroup property that

$$\|At\nabla e^{-t^2L}\|_{\dot{H}^s} \leq ct^{-s} \sup(t^{-1}, 1)\|f\|_{H^{-1}}.$$

Since $s < 1$, this estimate can be used in the Laplace transform formula (60) of Chapter 1 to obtain that

$$\|A\nabla(1+L)^{-1} f\|_{\dot{H}^s} \leq c\|f\|_{H^{-1}}.$$

Now, let u be any weak solution of L on an open set Ω and $v = \chi u$ where χ is a test function on Ω. Since

$$(1+L)v = v - A\nabla u \cdot \nabla\chi - \operatorname{div}(uA\nabla\chi) \in H^{-1}(\mathbb{R}^n),$$

we have that $A\nabla v \in \dot{H}^s(\mathbb{R}^n)$. Also, $\nabla v \in L^2(\mathbb{R}^n)$, so that $A\nabla v \in H^s(\mathbb{R}^n)$, hence $A\nabla u \in H^s_{loc}(\mathbb{R}^n)$. It remains to construct a counterexample to this last fact.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a bilipschitz homeomorphism that is homogeneous of degree 1 and orientation preserving. Such a map is easily built from its restriction to the unit sphere: in polar coordinates, any map $e^{i\theta} \mapsto e^{ib(\theta)}$, where b is a non-decreasing bijection on $[0, 2\pi]$ that is Lipschitz with b' and $1/b'$ bounded, gives us such an F. In particular, b can be chosen so that $\nabla F \notin \cup_{0<s<1} H^s_{loc}(\mathbb{R}^2)$.

To see this, observe that $\nabla F \in H^s_{loc}(\mathbb{R}^2)$ would imply

$$\int_K \int_K \frac{|\nabla F(x) - \nabla F(y)|^2}{|x-y|^{2+2s}}\, dxdy < \infty$$

for any compact set K of $\mathbb{R}^2$. Choosing K defined by $1/2 \leq r \leq 3/2$ in polar coordinates, an elementary computation would yield to

$$\int_{[0,2\pi]} \int_{[0,2\pi]} \frac{|b'(\theta) - b'(\theta')|^2}{|\theta - \theta'|^{1+2s}}\, d\theta d\theta' < \infty,$$

which, together with $b' \in L^\infty([0,2\pi])$ would mean that $b' \in H^s([0,2\pi])$. Hence it suffices to have $b' \notin \cup_{0<s<1} H^s([0,2\pi])$.

Next, the jacobian matrix of F, J_F, and its inverse are bounded so that $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ for some $\delta > 0$, where $A = |\det J_F|^t J_F^{-1} J_F^{-1}$. Denoting by F_1, F_2 the coordinate functions of F, it is easy to see that $LF_1 = 0$ in the weak sense and that $A\nabla F_1 = (\partial F_2/\partial x_2, -\partial F_2/\partial x_1)$. We have obtained the desired conclusion when $n = 2$.

For dimensions $n \geq 3$, it suffices to add to the above operator acting on the first two variables the Laplacian in the other $n-2$ variables. □

2.4. Quadratic functionals of irregular families

A consequence of Proposition 19 is that in dimension $n \geq 2$, the quadratic functionals arising from the square root problem for divergence form operators do not fall under the scope of Theorem 11.

The counterexample presented in Section 2.3, which is not related to a differential equation, suggests that there is no abstract result to handle $|\theta_t|_S$ unless we make use of the differential character of $\theta_t = e^{-t^2L} t \operatorname{div} A$.

There are two features expressing this character. The first one is a special form of regularity, namely that θ_t is smoothing of order 1 only when acting on gradient vector fields.

Lemma 20. — *With the notation above,*

$$|\theta_t \nabla|_{2,2} \leq ct^{-1}.$$

Proof. — Since

$$\theta_t \nabla f = -e^{-t^2L} tLf = -tLe^{-t^2L} f, \quad f \in \mathcal{D}(L),$$

the estimate follows from the uniform boundedness of $t^2 Le^{-t^2L}$ on $L^2(\mathbb{R}^n)$. □

The second one is what we call the structure of θ_t and is a consequence of ellipticity; this structural feature tells us that the study of θ_t can be restricted to gradient vector fields. Unfortunately, this is at some expense as far as the control of constants is concerned.

Let us come to precise statements, formulated in such a way that they apply also to various situations, including higher order operators (and even systems, see Section 2.6.1).

Consider $V_t \colon L^2(\mathbb{R}^n; \mathbb{C}^p) \to L^2(\mathbb{R}^n; \mathbb{C}^q)$, $t > 0$, a family of uniformly bounded operators with $|V_t|_{2,2} \leq c_0$. Here, p, q are two non-negative integers and $\mathbb{C}^p, \mathbb{C}^q$ are equipped with their respective Hilbert space structure. We make the following assumptions.

(i) *Size condition*: there exists $\varepsilon > 0$ such that the distributional kernels $V_t(x, y)$ of V_t, taking values in $\mathcal{B}(\mathbb{C}^p, \mathbb{C}^q)$, are locally integrable and satisfy

$$\int_{r \leq |x-y| \leq 2r} |V_t(x,y)|\, dy \leq c_0 \inf\left(\left(\frac{r}{t}\right)^{\varepsilon}, \left(\frac{t}{r}\right)^{n+\varepsilon}\right), \tag{28}$$

uniformly in $t > 0$, $r > 0$ and $x \in \mathbb{R}^n$.

(ii) *Partial regularity condition*: there exist a bounded convolution operator, Π, on $L^2(\mathbb{R}^n; \mathbb{C}^p)$ and a constant $s > 0$ such that

$$\|V_t \Pi (-\Delta)^{s/2} f\|_2 \leq c_0 t^{-s} \|f\|_2, \quad t > 0. \tag{29}$$

(iii) *Structural condition*: V_t factors through $V_t\Pi$ in the sense that there exists a bounded operator X on $L^2(\mathbb{R}^n; \mathbb{C}^p)$ such that

$$V_t = V_t \Pi X. \tag{30}$$

The operator Π can be represented by a $p \times p$ matrix with entries being scalar convolution operators. In applications, they will be Calderón-Zygmund singular integral operators of convolution type acting on $L^2(\mathbb{R}^n)$. See **[71]** or **[63]** for definitions.

Under the size condition, the vector-valued version of Lemma 12 shows that V_t maps bounded vector-valued functions to bounded vector-valued functions. In particular, note that for each $t > 0$ and $x \in \mathbb{R}^n$, $(V_t f)(x)$ belongs to $\mathbb{C}^q$, while $(V_t 1)(x)$ belongs to $\mathcal{B}(\mathbb{C}^p; \mathbb{C}^q)$ (see a prior remark in Section 2.2.3), so that their norms should be taken in the respective spaces.

Theorem 21. — *With the assumptions above, we have*
(i) *$(V_t)_{t>0}$ is a bounded family if and only if $V_t 1$ is a Carleson function.*
(ii) *When $|V_t 1|_c < \infty$, one has*

$$|V_t f|_c \leq c_1 |V_t 1|_c \|f\|_\infty + c_2 c_0 \|f\|_\infty, \quad f \in L^\infty(\mathbb{R}^n; \mathbb{C}^p), \tag{31}$$

where $c_1 = c|\Pi|_{2,2}|X|_{2,2}$ and $c_2 = c(n, |\Pi|_{2,2}, |X|_{2,2}, s)$.

Proof. — By the remark at the end of Section 2.2.3, the $\mathbb{C}^p$-valued version of (21) holds, hence for some constant c,

$$|V_t 1|_c \leq c|V_t|_S + cc_0 < \infty$$

and this uses only the size condition (28) and the uniform L^2-boundedness of V_t. Thus, $|V_t|_S < \infty$ implies $|V_t 1|_c < \infty$.

We turn to the converse implication. First, the structural condition and (23) implies that

$$|V_t|_S \leq |V_t \Pi|_S |X|_{2,2}. \tag{32}$$

Next, mimicking the proof of Theorem 11, we have

$$V_t \Pi = M_t P_t \Pi + (V_t - M_t) P_t \Pi + V_t (I - P_t) \Pi, \tag{33}$$

where P_t acts componentwise on $L^2(\mathbb{R}^n;\mathbb{C}^p)$ and is appropriately chosen, and where M_t is the pointwise multiplication by the matrix $(V_t1)(x)$.

Using (28), the first two operators in the right hand side of (33) are analyzed as in the proof of Theorem 11 and Π plays no other role than being bounded. Thus,

$$|M_tP_t\Pi|_S \leq c|\Pi|_{2,2}|V_t1|_c \quad \text{and} \quad |(V_t - M_t)P_t\Pi|_S \leq c_0c(n,\varepsilon)|\Pi|_{2,2}.$$

In the last term, we cannot control $V_t(I-P_t)$ since the operators V_t do not satisfy (18). This is where Π plays its best role. We have $V_t(I-P_t)\Pi = V_t\Pi(I-P_t)$ since both $I-P_t$ and Π are convolution operators. Applying now Lemma 16 to $U_t = V_t\Pi$ gives us

$$|V_t(I-P_t)\Pi|_S \leq c_0c(n,s).$$

Thus, we have shown that

$$|V_t\Pi|_S \leq C|\Pi|_{2,2}|V_t1|_c + c_0c(n,\varepsilon,s), \tag{34}$$

and together with (32) we obtain

$$|V_t|_S \leq C|\Pi|_{2,2}|X|_{2,2}|V_t1|_c + c_0c(n,\varepsilon,s). \tag{35}$$

We turn to proving (31). Let $f \in L^\infty(\mathbb{R}^n;\mathbb{C}^p)$ with norm 1. Then, by Lemma 15 and (35), we obtain,

$$|V_tf|_c \leq |V_t|_s + cc_o \leq C|\Pi|_{2,2}|X|_{2,2}|V_t1|_c + cc_0,$$

which is (31) and the proof of Theorem 21 is finished. □

Remark. — We do not know whether $c_1 = 1$ in (31) (compare with (19)). For application to the square root problem, a good description of c_1 is desirable towards the best perturbative results. This involves optimizing the various inequalities used in the course of the proof. We postpone this matter until Appendix C.

2.5. The heart of the matter

In Section 2.2, we have treated square roots of differential operators that enjoy a special structure. Let us now consider the general case.

With the notation of Preliminaries, we are given a homogeneous differential operator of order $2m$ in $\mathbb{R}^n$, $n \geq 2$,

$$L = (-1)^m\partial^\alpha\ (a_{\alpha\beta}\ \partial^\beta) \tag{36}$$

assumed to satisfy the Gårding inequality (18) in Preliminaries, that is

$$\operatorname{Re}\int_{\mathbb{R}^n} a_{\alpha\beta}\ \partial^\beta f\ \overline{\partial^\alpha f} \geq \delta\|\nabla^m f\|_2^2.$$

To simplify the exposition, we use the summation convention for repeated indices. Here, $|\alpha| = |\beta| = m$. Recall that $\nabla^m f$ is the array of all mth order partial derivatives of f, valued in $\mathbb{C}^p$, $p = \binom{m+n-1}{n-1}$.

2.5.1. Characterizations of (K) in terms of quadratic functionals. — The next result turns out to be more useful for our purpose than Proposition 5 of Preliminaries.

Theorem 22. — *Assume that L is given by (36). Then, the following are equivalent.*

(i) $\|L^{1/2}f\|_2 \le C\|\nabla^m f\|_2$;

(ii) $\left(\int_0^\infty \|e^{-t^{2m}L}t^m\partial^\alpha(a_{\alpha\beta}F_\beta)\|_2^2\,dt/t\right)^{1/2} \le c\|F\|_2$;

(iii) $\left(\int_0^\infty \|(1+t^{2m}L)^{-1}t^m\partial^\alpha(a_{\alpha\beta}F_\beta)\|_2^2\,dt/t\right)^{1/2} \le c\|F\|_2$.

Here, $F = (F_\beta) \in L^2(\mathbb{R}^n;\mathbb{C}^p)$.

Proof. — We only consider the equivalence between (i) and (ii) as the argument for the equivalence between (i) and (iii) is the same.

Defining $\theta_t\colon L^2(\mathbb{R}^n;\mathbb{C}^p) \to L^2(\mathbb{R}^n)$ by

$$\theta_t F = e^{-t^{2m}L}t^m\partial^\alpha(a_{\alpha\beta}F_\beta), \quad F = (F_\beta) \in L^2(\mathbb{R}^n;\mathbb{C}^p), \tag{37}$$

then (ii) means that $|\theta_t|_S < \infty$.

Let A be the operator of multiplication by $(a_{\alpha\beta})$ on $L^2(\mathbb{R}^n;\mathbb{C}^p)$.

As in Preliminaries, we begin with the polar decomposition of ∇^m. To simplify the notation, write $D = \nabla^m$. Then $D = RS$, where S is the one-one positive selfadjoint operator on $L^2(\mathbb{R}^n)$ given by

$$S = \Big(\sum_{|\alpha|=m}(-1)^m(\partial^\alpha)^2\Big)^{1/2}$$

and R is the partial isometry defined by $R = DS^{-1}$. Define next, $B = R^*AR$. We have $L = D^*AD = SBS$.

The operator B is bounded and it follows from the Gårding inequality that

$$\operatorname{Re}\langle BSf, Sf\rangle \ge \delta\|Sf\|_2^2, \quad f \in \mathcal{D}(S).$$

Since S has a dense range in $L^2(\mathbb{R}^n)$, this implies that B is invertible and ω-accretive on $L^2(\mathbb{R}^n)$ for some $\omega < \pi/2$. By Proposition 5 of Preliminaries, assertion (i) is, therefore, equivalent to

$$\left(\int_0^\infty \|e^{-t^{2m}L}t^m SBf\|_2^2\,\frac{dt}{t}\right)^{1/2} \le c\|f\|_2, \quad f \in L^2(\mathbb{R}^n), \tag{38}$$

where we have changed t to t^m for convenience.

Next, observe that

$$\theta_t Rf = e^{-t^{2m}L}t^m SBf.$$

We conclude that (38) holds if and only if $|\theta_t\Pi|_S < \infty$, where $\Pi = RR^*$ is the orthogonal projector onto the range of R.

The next result shows that $|\theta_t\Pi|_S \sim |\theta_t|_S$, which ends the proof of Theorem 22. □

Lemma 23. — *We have $\theta_t = \theta_t \Pi X$, where $X = \Pi(\Pi A \Pi)^{-1}\Pi A$. Hence,*

$$|\theta_t \Pi|_{\mathcal{S}} \leq |\theta_t|_{\mathcal{S}} \leq \|(a_{\alpha\beta})\|_\infty \delta^{-1} |\theta_t \Pi|_{\mathcal{S}}. \tag{39}$$

Proof. — Let us first explain the meaning of X. We have $\Pi A \Pi = RBR^*$, hence $\Pi A \Pi$ is invertible on the range of Π, and it is not hard to see that the norm of its inverse is controlled by the inverse of the ellipticity constant δ in the Gårding inequality. Thus, X is a well-defined and bounded operator on $L^2(\mathbb{R}^n; \mathbb{C}^p)$ and

$$|X|_{2,2} \leq \|(a_{\alpha\beta})\|_\infty \delta^{-1}.$$

Note that, Π being a projection, $\Pi A \Pi X = \Pi A X = \Pi A$. Since $R^* = R^* \Pi$, we have

$$R^* A \Pi X = R^* \Pi A \Pi X = R^* \Pi A = R^* A.$$

Hence,

$$\theta_t \Pi X = e^{-t^{2m}L} t^m S R^* A \Pi X = e^{-t^{2m}L} t^m S R^* A = \theta_t.$$

We have shown that θ_t factors through X, and this implies $|\theta_t|_{\mathcal{S}} \leq |X|_{2,2} |\theta_t \Pi|_{\mathcal{S}}$. Moreover $|\theta_t \Pi|_{\mathcal{S}} \leq |\theta_t|_{\mathcal{S}}$ since Π has norm 1. The lemma is proved. □

2.5.2. Characterizations of (K) in terms of Carleson measures. — In the next result, we assume that L has the property (G) defined in Chapter 1. Recall that this means that the heat kernel for L has a Gaussian decay (together with Hölder estimates which are not used here).

Let us check the hypotheses of Theorem 21 when $V_t = \theta_t$ defined by (37).

The vector-valued kernel of θ_t is $-t^m a_{\alpha\beta}(y)(\partial_y)^\alpha K_{t^{2m}}(x,y)$ where $K_t(x,y)$ is the heat kernel for L. Thus, the size condition (28) is a consequence of Theorem 29 in Chapter 1.

Next, the partial regularity condition (29) is obtained as in Lemma 20 for second order operators. First, one sees via the Fourier transform that Π is a Calderón-Zygmund singular integral operator of convolution type in $L^2(\mathbb{R}^n; \mathbb{C}^p)$. Then, using the notation introduced in the proof of Theorem 22, we write for $f \in \mathcal{D}(L)$,

$$\theta_t \Pi S f = e^{-t^{2m}L} t^m \partial^\alpha (a_{\alpha\beta} \partial^\beta f) = (-1)^m e^{-t^{2m}L} t^m L f = (-1)^m t^m L e^{-t^{2m}L} f,$$

and the uniform boundedness of $t^{2m} L e^{-t^{2m}L} f$ on $L^2(\mathbb{R}^n)$ gives us

$$\|\theta_t \Pi S f\|_2 \leq c t^{-m} \|f\|_2.$$

Hence,

$$\begin{aligned}\|\theta_t \Pi (-\Delta)^{m/2} f\|_2 &= \|\theta_t \Pi S S^{-1} (-\Delta)^{m/2} f\|_2 \\ &\leq c t^{-m} \|S^{-1}(-\Delta)^{m/2} f\|_2 \sim t^{-m} \|f\|_2,\end{aligned}$$

where the last equivalence is obtained from the Plancherel theorem.

Lastly, the structural condition has already been checked in Lemma 23. Thus, combining Theorem 21 and Theorem 22, we have obtained

Theorem 24. — *Assume that L given by (36) has the Gaussian property (G). Then, the following assertions are equivalent.*

(i) $\|L^{1/2}f\|_2 \leq C\|\nabla^m f\|_2$;

(ii) *For* $|\beta| = m$, $|e^{-t^{2m}L}t^m(\partial^\alpha a_{\alpha\beta})(x)|^2 dxdt/t$ *are Carleson measures on* $\mathbb{R}^n \times \mathbb{R}^+$;

(iii) *For* $|\beta| = m$, $|(1+t^{2m}L)^{-1}t^m(\partial^\alpha a_{\alpha\beta})(x)|^2 dxdt/t$, *are Carleson measures on* $\mathbb{R}^n \times \mathbb{R}^+$.

Again, the argument with $(1+t^{2m}L)^{-1}$ replacing $e^{-t^{2m}L}$ is completely similar and is omitted.

In the general case where the property (G) is not fulfilled, we have the following characterizations, which we only write for operators of order 2.

Theorem 25. — *Assume that* $L = -\operatorname{div}(A\nabla)$ *is maximal accretive on* $L^2(\mathbb{R}^n)$. *Set* $L_k = -\Delta^k \operatorname{div}(A\nabla)\Delta^k$, *with k integer such that* $4k+2 \geq n$. *Then, the following assertions are equivalent.*

(i) $\|L^{1/2}f\|_2 \leq C\|\nabla f\|_2$;

(ii) $|e^{-t^{4k+2}L_k}t^{2k+1}(\Delta^k \operatorname{div} A)(x)|^2 dxdt/t$ *is a Carleson measure on* $\mathbb{R}^n \times \mathbb{R}^+$;

(iii) $|(1+t^{4k+2}L_k)^{-1}t^{2k+1}(\Delta^k \operatorname{div} A)(x)|^2 dxdt/t$ *is a Carleson measure on* $\mathbb{R}^n \times \mathbb{R}^+$.

Proof. — By Proposition 10 of Preliminaries, (i) for L is equivalent to (i) in Theorem 24 for L_k. By Proposition 28 of Chapter 1, L_k has the property (G) since k is large enough. The proof is complete by applying Theorem 24. □

Remarks

1. It looks as though checking (ii) or (iii) in Theorem 25 is more difficult because we are dealing with a higher order operator instead of a second order operator. But this is only superficial: the conditions we can produce to verify (ii) or (iii) do not depend on k but only on the structure of A.
2. We have stated these results using the heat semigroup and the resolvent for L by simplicity and for their connections with parabolic and elliptic equations. However, one can use more general functions of L of the form $\varphi(t^{2m}L)$.
3. Assume that L is of order 2 in Theorem 24. An examination of the argument shows that one can replace in the proof the resolvent by $(m^{-1}+t^2L)^{-1}$, where m is any bounded and accretive function on $\mathbb{R}^n$, since the kernel estimates are not affected by the choice of m (see Chapter 1, Section 2.5.1). Using the discussion in Section 0.8 of Preliminaries, this shows that Theorem 24 applies to mL. Since the space of Carleson functions on $\mathbb{R}^n \times \mathbb{R}^+$ is invariant under bilipschitz transformations in $\mathbb{R}^n$, this also proves that the statement of Theorem 24 for second order operators is invariant under a bilipschitz change of variables.

2.6. Miscellaneous results

2.6.1. The inequality $(\mathrm{K})_{loc}$. — So far we have only considered equivalent formulations of the inequality (K), namely $\|L^{1/2}f\|_2 \le C\|\nabla^m f\|_2$, where L is as in (36). We indicate in this section how to deal with its local version:

$$\|L^{1/2}f\|_2 \le C(\|\nabla^m f\|_2 + \|f\|_2). \qquad (\mathrm{K})_{loc}$$

As explained in Section 0.7 of Preliminaries, functional calculus shows that $(\mathrm{K})_{loc}$ is a consequence of

$$\|(L+\lambda)^{1/2}f\|_2 \le C(\|((-\Delta)^m+\lambda)^{1/2}f\|_2 \qquad (40)$$

for any $\lambda > 0$. Recall that this is obtained by factoring $L+\lambda$ as $D^*\tilde{A}D$, where $Df = (\nabla^m f, f)$ and $\tilde{A}$ is the operator of multiplication by the matrix

$$\begin{pmatrix} (a_{\alpha\beta}) & 0 \\ 0 & \lambda \end{pmatrix}.$$

If L has the local Gaussian property $(\mathrm{G})_{loc}$ and if λ is large enough, $L+\lambda$ has the property (G), with in addition, an extra factor $e^{-at^{2m}}$ for some $a > 0$ in the estimates. This is where the choice of λ is made. Therefore, the same results on quadratic functionals apply and (40) is equivalent to

$$e^{-t^{2m}(L+\lambda)}t^m(\partial^\alpha a_{\alpha\beta})(x), \quad |\beta| = m, \quad \text{and} \quad e^{-t^{2m}(L+\lambda)}(t^m\lambda^{1/2})(x)$$

being Carleson functions on $\mathbb{R}^n \times \mathbb{R}^+$. As we have the bound

$$|e^{-t^{2m}(L+\lambda)}(t^m\lambda^{1/2})(x)| \le ct^m e^{-at^{2m}},$$

the Carleson measure estimate is readily seen. Now, the first function can be written $e^{-at^{2m}}g(x,t)$ for some $a > 0$ and the following simple lemma applies.

Lemma 26. — *Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be bounded and such that $\int_\tau^\infty f(t)\,dt/t$ is finite for all $\tau > 0$. Let $g(x,t)$ be a bounded function defined on $\mathbb{R}^n \times \mathbb{R}^+$. Then $f(t)g(x,t)$ is a Carleson function if*

$$\inf_{\rho>0} \sup_{Q\in\mathcal{Q};\ell(Q)\le\rho} \frac{1}{|Q|}\int_Q\int_0^{\ell(Q)} |g(x,t)|^2\frac{dxdt}{t} < \infty. \qquad (41)$$

Calling local Carleson functions the functions $g(x,t)$ satisfying (41), we have obtained the local version of Theorem 24.

Theorem 27. — *Assume that L given by (36) has the local Gaussian property $(G)_{loc}$. If $(e^{-t^{2m}L}t^m\partial^\alpha a_{\alpha\beta})(x)$ are local Carleson functions for all $|\beta| = m$, then $(K)_{loc}$ is valid.*

Remarks

1. Of course the analogous statement holds with the resolvent replacing the semigroup.

2. Note that the above theorems on quadratic functionals also have local analogs in which the variable t is restricted to bounded intervals $(0,T]$. Alternately the measure $\frac{dt}{t}$ may be multiplied by any non-negative bounded function of t. We leave to the reader the care of stating them.
3. Theorem 27 remains valid if L is perturbed with lower order terms as in Section 0.7 of Preliminaries. This can be seen either by combining Theorem 27 for the leading part and Proposition 11 of Preliminaries or by adapting *mutatis mutandis* the arguments on quadratic functionals of this chapter. This, in particular, is useful if $L+\lambda$ is maximal-accretive for $\operatorname{Re}\lambda$ large enough, which is the case when L satisfies the Gårding inequality

$$\operatorname{Re}\langle Lf, f\rangle \geq \delta\|\nabla^m f\|_2^2 - c\|f\|_2^2$$

for some $\delta > 0$ and $c \geq 0$.

If L does not have the local Gaussian property, then we use Proposition 9 of Preliminaries and change $L+\lambda$ to $(-\Delta+1)^k(L+\lambda)(-\Delta+1)^k$ for some k. Expanding terms one finds

$$(-\Delta)^k L(-\Delta)^k + \lambda + \text{ perturbations}$$

and these perturbations can be disposed of by adapting Proposition 11 in Preliminaries. We find that (40) is equivalent to

$$\|((-\Delta)^k L(-\Delta)^k + \lambda)^{1/2} f\|_2 \leq C(\|((-\Delta)^{m+2k} + \lambda)^{1/2} f\|_2. \tag{42}$$

If k is large enough then $(-\Delta)^k L(-\Delta)^k$ has the property (G) and Theorem 27 applies to this operator. Specializing to second order operators, we have obtained the following result.

Theorem 28. — *Let $L = -\operatorname{div}(A\nabla)$, k be an integer with $4k+2 \geq n$ and set $L_k = (-\Delta)^k L(-\Delta)^k$. If $e^{-t^{4k+2}L_k} t^{2k+1}(\Delta^k \operatorname{div} A)(x)$ is a local Carleson function then $(K)_{loc}$ is valid.*

Remark. — The results presented in this section extend to elliptic systems satisfying a *bona fide* Gårding inequality. The reader can consult **[38]** for a presentation of the theory of elliptic systems. The extra work needed here is purely algebraic.

2.6.2. Localization principles. — When reformulating Theorem 21 for differential operators, one can obtain local statements. The main observation is the following lemma that comes after reexamination of (33) and its use.

Lemma 29. — *Let L be as in (36) and assume that it has the property (G). Let θ_t be defined by (37). There is a constant c such that for $f \in H^m(\mathbb{R}^n)$ one has*

$$(\theta_t \nabla^m f)(x) = (\theta_t 1)(x)\cdot(P_t\nabla^m f)(x) + b_t(x)$$

with

$$\int_0^\infty \int_{\mathbb{R}^n} |b_t(x)|^2 \, \frac{dxdt}{t} \le c\|\nabla^m f\|_2^2, . \tag{43}$$

Corollary 30. — *With the assumptions above, let $Q_0 \in \mathcal{Q}$ be a cube such that*

$$\sup \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |(\theta_t 1)(x)|^2 \frac{dxdt}{t} < \infty,$$

where the supremum is taken over all cubes $Q \subset Q_0$. Then, for all $f \in H^m(\mathbb{R}^n)$ with support contained in $\frac{1}{2}Q_0$, we have

$$\|L^{1/2} f\|_2 \le C\|\nabla^m f\|_2.$$

Proof. — We have to estimate $\int_0^\infty \int_{\mathbb{R}^n} |(\theta_t \nabla^m f)(x)|^2 dxdt/t$. By the previous lemma, we are reduced to control $\int_0^\infty \int_{\mathbb{R}^n} |(\theta_t 1)(x) \cdot (P_t \nabla^m f)(x)|^2 dxdt/t$. The integration over $x \in Q_0$ and $t \le \ell(Q_0)$ can be controlled by the hypothesis on $(\theta_t 1)(x)$ and Carleson inequality. The integration over the remaining part is taken care of using the support condition on $\nabla^m f$ and straightforward estimates on $P_t \nabla^m f$. □

In order to use this result efficiently, we note the following lemma.

Lemma 31. — *Let L_0 and L_1 be as in Lemma 29, such that their coefficients agree on an open set Ω. Then, there exists $c \ge 0$ such that for all cube $Q \in \mathcal{Q}$ with $\overline{2Q} \subset \Omega$, we have*

$$\int_Q |\theta_t^1 F - \theta_t^0 F|^2 \le C \frac{t}{\ell(Q)} \|F\|_2^2, \quad F \in L^2(\mathbb{R}^n; \mathbb{C}^p), \quad \mathit{Supp}\,(F) \subset \Omega, \tag{44}$$

and

$$\int_Q |\theta_t^1 F - \theta_t^0 F|^2 \le C \frac{t}{\ell(Q)} \|F\|_\infty^2 |Q|, \quad F \in L^\infty(\mathbb{R}^n; \mathbb{C}^p). \tag{45}$$

Proof. — We have used the evident notation that θ_t^i corresponds to L_i, where we replace the heat semigroup by the resolvent for convenience.

Let us prove (44) first. Since the coefficients of L_i agree on Ω, $u_t = \theta_t^1 F - \theta_t^0 F$ is a weak solution of the equation

$$\int u_t \, \phi + t^{2m} \int a_{\alpha\beta} \, \partial^\beta u_t \, \partial^\alpha \phi = 0$$

for all $\phi \in H_0^m(\Omega)$. Using $\phi = \overline{u_t} \, \varphi^2$ where φ is a smooth test function equal to 1 on Q, supported in $2Q$ and scaled to the cube Q, we obtain

$$\int |u_t|^2 \varphi^2 + t^{2m} \int a_{\alpha\beta} \, \partial^\beta (u_t \varphi) \, \partial^\alpha (\overline{u_t} \varphi) = t^{2m} \int a_{\alpha\beta} \, v_{t,\alpha\beta}$$

where

$$v_{t,\alpha\beta} = -\partial^\beta u_t \, \partial^\alpha (\overline{u_t} \, \varphi^2) + \partial^\beta (u_t \, \varphi) \, \partial^\alpha (\overline{u_t} \, \varphi).$$

Hence, by the Gårding inequality,

$$\int_Q |u_t|^2 \leq \int |u_t|^2\varphi^2 \leq ct^{2m} \int |v_{t,\alpha\beta}|.$$

Calculating $v_{t,\alpha\beta}$ using Leibniz formula, we see that the terms $\partial^\beta u_t\ \partial^\alpha \overline{u_t}\ \varphi^2$ cancel. For the other terms, since $\|\partial^\gamma u_t\|_2 \leq ct^{-|\gamma|}\|F\|_2$ by the basic L^2 estimates and using size estimates on the derivatives of φ, we obtain that $t^{2m}\int |v_{t,\alpha\beta}| \leq ct/\ell(Q)$.

To prove (45), we localize F as $F_1 + F_2$ where $F_1 = 0$ outside of $2Q$. Then we use (44) for F_1 and the method of Lemma 15 for F_2 where the difference plays no role in $\theta_t^1 F_2 - \theta_t^0 F_2$. This lemma applies since the kernels of θ_t^i satisfy estimates (76) of Chapter 1. This finishes the proof. □

Theorem 32 **(Localization principle for square roots).** — *Let L_0 and L_1 be given by (36). Assume that they both have the property (G) and that their coefficients agree on an open set Ω. If (K) holds for L_0 then for all $f \in H^m(\mathbb{R}^n)$ with support contained in a fixed compact subset of Ω, we have*

$$\|L_1^{1/2} f\|_2 \leq C\|\nabla^m f\|_2.$$

The proof goes as follows. If (K) holds for L_0, then $\theta_t^0 1$ is a Carleson function. Using Lemma 31, we obtain that $\theta_t^1 1$ satisfy the hypothesis of Corollary 30 for every cube Q_0 such that $2Q_0 \subset \Omega$. The conclusion follows by using a Whitney decomposition on Ω (see [**71**]).

Remark. — It is not clear whether one can drop the hypothesis on Gaussian estimates in this result.

CHAPTER 3

POSITIVE ANSWERS TO THE SQUARE ROOT PROBLEM

3.1. Introduction

In this chapter, we present some positive answers to the square root problem, that is, situations where the inequalities (K) or $(\mathrm{K})_{loc}$ are valid. As we rely on the criteria and methods developed in Chapter 2, it cannot be read independently. For simplicity, we restrict our discussion to second order operators but it will be clear that many analogous results hold for higher order operators.

The most original part of this chapter is the introduction and the study of a class, denoted by S (resp. S_{loc}), of complex-valued matrices for which the inequality (K) (resp. $(\mathrm{K})_{loc}$) is valid. The main property of S and S_{loc}, aside from leading to inequality (K) or $(\mathrm{K})_{loc}$, is to be open in the L^∞ topology. Thus, we generalize the results in **[19]** and **[35, 37, 36]**, namely that (K) holds for perturbations of the identity or of constant elliptic matrices A_0, to the case where A_0 belongs to S or to S_{loc}. It is even possible to estimate how small the size of the perturbation may be: we do it in Appendix C when $A_0 = \mathrm{Id}$ and essentially find the same result as Journé **[45]**.

We also obtain Tb type results for a subclass of S which is characterized by a differential structure condition on its elements. For example, several well understood classes of matrices in homogenization theory, for which the homogenized matrix can be computed to some extent (see **[44]**) (though we do not require periodicity in our case) enter this class - we do not understand satisfactorily this coincidence. An example is the class of elliptic matrices depending on one variable, which are used to model stratified media.

The class S_{loc} contains matrices enjoying very mild smoothness properties. To this end, we introduce the notion of absolutely bounded mean oscillation ($ABMO$), which is the analog, in the sense of BMO, of absolute continuity. It encompasses regularity conditions such as uniform continuity, vanishing mean oscillation, or local uniform differentiability in some Sobolev space with positive regularity index. In

Appendix A, we give a few characterizations of this notion, including its Littlewood-Paley description, various examples and counterexamples. A surprising result is then that the map $A \to L^{1/2}$ (see question 2 in the Introduction) is analytic at any $A_0 \in ABMO$ for the BMO topology.

Let us stress the fact that no exhaustive description of S or S_{loc} is available at the present time. In particular, whether real and symmetric elliptic matrices belong to S is an open question.

This led us to revisit the selfadjoint case. Indeed, when $L^* = L$, the inequality $(\mathrm{K})_{loc}$ is true, hence the Carleson estimates described in Chapter 2, Theorems 24 (in the real case) and 25 (in the general complex case) must hold. We find nevertheless interesting to give a direct proof of these estimates. Though it is much longer than the abstract one, it has two advantages: it is constructive and it sheds light on the connection with the moments of Nash.

3.2. The class S and the inequality (K)

If $F = (F_1, \dots, F_n)$ is some $\mathbb{C}^n$-valued function defined on a domain of $\mathbb{R}^n$, denote by ∇F the matrix whose column vectors are $\nabla F_1, \dots, \nabla F_n$.

In the sequel φ is a function in $C_0^\infty(\mathbb{R}^n)$ with support in the ball $B(0,1)$ and $\int \varphi = 1$. As in Chapter 2, it is associated to an approximation of the identity $(P_t)_{t>0}$ and it is understood that P_t acts componentwise on vector-valued functions. Choosing φ non negative makes P_t a positivity preserving contraction on L^∞, but it is not necessary. We use the notation of Chapter 2.

Before we introduce the class S, it is worthwhile giving a heuristic (and incorrect) argument ignoring constants, localizations, etc. As before we set $\theta_t = e^{-t^2L} t \operatorname{div} A$.

Let Q be a cube and $f \in H^1(\mathbb{R}^n)$ with support in Q and $\|\nabla f\|_2^2 \sim |Q|$. Starting from Lemma 29 of Chapter 2, we have

$$\begin{aligned}\int_0^{\ell(Q)} \int_Q |(\theta_t 1)(x) P_t(\nabla f)(x)|^2 \frac{dxdt}{t} \\ &\leq \int_0^{\ell(Q)} \int_Q |(e^{-t^2L} t(\operatorname{div} A\nabla f)(x)|^2 \frac{dxdt}{t} + c|Q| \\ &\leq \int_0^{\ell(Q)} \int_Q |t(\operatorname{div} A\nabla f)(x)|^2 \frac{dxdt}{t} + c|Q|.\end{aligned}$$

Thus, if one also has $\|\operatorname{div} A\nabla f\|_2^2 \sim |Q|/\ell(Q)^2$ and, in the spirit of **[69]**, a control of the Carleson norm of $\theta_t 1$ by expressions of the form

$$\int_0^{\ell(Q)} \int_Q |(\theta_t 1)(x) P_t(\nabla f)(x)|^2 \frac{dxdt}{t}$$

then one can conclude that this Carleson norm is finite. This is the reason for the next definition.

Definition 1. — Let $A \in \mathcal{A}$. We say that $A \in S$ when there exist constants $C > 0$, $\alpha > 0$, $\mu > 1$, and, for each cube $Q \in \mathcal{Q}$, a function $F_Q \colon \mu Q \to \mathbb{C}^n$ fulfilling the following requirements.

(i) $F_Q \in H^1(\mu Q)$ and $\int_{\mu Q} |\nabla F_Q|^2 \leq C|Q|$.

(ii) $\operatorname{div}(A\nabla F_Q) \in L^2(\mu Q)$ and

$$\int_{\mu Q} |\operatorname{div}(A\nabla F_Q)|^2 \leq C\frac{|Q|}{\ell(Q)^2}.$$

Here, the divergence is taken in the sense of distributions on μQ.

(iii) For every family $(\gamma_t)_{t>0}$ in $L^\infty(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{C}^n)$, one has

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\gamma_t(x)P_t(\nabla F_Q)(x)|^2 \frac{dxdt}{t} \geq \alpha|\gamma_t|_c^2 - C\|\gamma_t\|_\infty^2,$$

where it is understood that $\nabla F_Q = 0$ outside μQ.

Here, μQ is the cube with the same center as Q, dilated by a factor μ.

Definition 2. — We say that $A \in S_{loc}$ when the above conditions hold for cubes $Q \in \mathcal{Q}_0$, where $\mathcal{Q}_0$ denotes the class of cubes in $\mathcal{Q}$ with $\ell(Q) \leq 1$.

Remark. — Note that if the family (F_Q) satisfies the above conditions, so does $(F_Q + c_Q)$ where c_Q are constants. This means that, in addition, we may assume that F_Q have mean value 0.

Remark. — The following conditions are equivalent to the ones in Definition 1.

There exist constants $C > 0$, $\alpha > 0$, $p > 2$, and, for each cube $Q \in \mathcal{Q}$, a function $F_Q \colon Q \to \mathbb{C}^n$ fulfilling the following requirements.

(i)' $F_Q \in W^{1,p}(Q)$ and $\left(\frac{1}{|Q|}\int_Q |\nabla F_Q|^p\right)^{1/p} \leq C$.

(ii)' $\operatorname{div}(A\nabla F_Q) \in L^2(Q)$ and

$$\int_Q |\operatorname{div}(A\nabla F_Q)|^2 \leq C\frac{|Q|}{\ell(Q)^2}.$$

(iii)' For every family $(\gamma_t)_{t>0}$ in $L^\infty(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{C}^n)$, one has

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\gamma_t(x)P_t(\nabla F_Q)(x)|^2 \frac{dxdt}{t} \geq \alpha|\gamma_t|_c^2 - C\|\gamma_t\|_\infty^2,$$

where it is understood that $\nabla F_Q = 0$ outside of Q.

That (i-iii) imply (i-iii)' follows from Meyers estimate (17) of Chapter 1. The converse is straightforward.

The main results of this section are the following ones.

Theorem 3. — *The inequality (K) (resp. $(K)_{loc}$) holds if $A \in S$ (resp. S_{loc}).*

Theorem 4. — *The class S (resp. S_{loc}) is open in $L^\infty(\mathbb{R}^n, M_n(\mathbb{C}))$.*

As a consequence, we have

Corollary 5. — *For any $A_0 \in S$, there exists $\varepsilon(A_0,n) > 0$ such that (K) holds for $L = -\operatorname{div}(A\nabla)$ whenever $\|A - A_0\|_\infty < \varepsilon(A_0,n)$.*

By taking $F_Q(x) = x$, one sees that any constant elliptic matrix A_0 belongs to S, so that Corollary 5 extends results in **[19]** and **[35, 37, 36]**.

Proof of Theorem 3. — Choose once for all an integer $k \geq (n-2)/4$ and set

$$\theta_t^{(k)} = e^{-t^{4k+2}L_k} t^{2k+1} \Delta^k \operatorname{div} A$$

and

$$(\theta_t^{(k)} 1)(x) = e^{-t^{4k+2}L_k} t^{2k+1} (\Delta^k \operatorname{div} A)(x),$$

the latter being a $\mathbb{C}^n$-valued function. Applying Theorem 25 of Chapter 2, it is enough to show that $(\theta_t^{(k)}1)(x)$ is a Carleson function. Observe that it is uniformly bounded. From (iii) in the definition of the class S, we are reduced to proving that there is a constant C such that for all cubes $Q \in \mathcal{Q}$,

$$a_1(Q) \leq C|Q|,$$

where

$$a_1(Q) = \int_0^{\ell(Q)} \int_Q |(\theta_t^{(k)}1)(x)(P_t \nabla F_Q)(x)|^2 \frac{dxdt}{t}.$$

Fix $Q \in \mathcal{Q}$. Recall that ∇F_Q is only defined on μQ, where $\mu > 1$, so that a localization argument is necessary. Let χ_Q be a $C_0^\infty(\mathbb{R}^n)$ function supported in $(\frac{1+2\mu}{3})Q$, with $\chi_Q = 1$ on $(\frac{2+\mu}{3})Q$ and $\|\nabla \chi_Q\|_\infty \leq c(n,\mu)\ell(Q)^{-1}$.

For $x \in Q$ and $t \leq \frac{\mu-1}{3}\ell(Q)$, we have

$$(P_t \nabla F_Q)(x) = (P_t \nabla(\chi_Q F_Q))(x).$$

Hence,

$$\begin{aligned} a_1(Q) = & \int_0^{\frac{\mu-1}{3}\ell(Q)} \int_Q |(\theta_t^{(k)}1)(x)(P_t \nabla(\chi_Q F_Q))(x)|^2 \frac{dxdt}{t} \\ & + \int_{\frac{\mu-1}{3}\ell(Q)}^{\ell(Q)} \int_Q |(\theta_t^{(k)}1)(x)(P_t \nabla F_Q)(x)|^2 \frac{dxdt}{t}. \end{aligned} \tag{1}$$

Using the uniform L^2-boundedness of P_t, one has

$$\int_{\frac{\mu-1}{3}\ell(Q)}^{\ell(Q)} \int_Q |(\theta_t^{(k)}1)(x)(P_t \nabla F_Q)(x)|^2 \frac{dxdt}{t} \leq c\|\theta_t^{(k)}1\|_\infty^2 \ln(\frac{3}{\mu-1}) \int |\nabla F_Q|^2.$$

Applying Lemma 29 of Chapter 2 with $f = \chi_Q F_Q$ to the other term in the right hand side of (1), we obtain

$$a_1(Q) \leq 2a_2(Q) + c\int |\nabla F_Q|^2, \tag{2}$$

where

$$a_2(Q) = \int_0^{\ell(Q)} \int_Q |(\theta_t^{(k)} \nabla(\chi_Q F_Q))(x)|^2 \frac{dxdt}{t}.$$

To control $a_2(Q)$, we compute $\theta_t^{(k)}\nabla(\chi_Q F_Q)$. For $x \in Q$, we have

$$(\theta_t^{(k)} \nabla(\chi_Q F_Q))(x) = (\theta_t^{(k)}((\nabla\chi_Q)F_Q))(x) \\ + e^{-t^{4k+2}L_k} t^{2k+1} \Delta^k \operatorname{div}(A\chi_Q \nabla F_Q)(x). \tag{3}$$

For the first term in the right hand side of (3), we use the following variant of Lemma 15 in Chapter 2.

Lemma 6. — *Let $\mu > \lambda > 1$, $Q \in \mathcal{Q}$ and $f \in L^2(\mathbb{R}^n; \mathbb{C}^n)$ supported in $\mu Q \setminus \lambda Q$. Then*

$$\int_0^{\ell(Q)} \int_Q |(\theta_t^{(k)} f)(x)|^2 \frac{dxdt}{t} \leq C \int |f|^2, \tag{4}$$

where C depends only on n, δ, λ, μ.

To prove this lemma write

$$(\theta_t^{(k)} f)(x) = \int \theta_t^{(k)}(x,y) f(y)\, dy$$

and observe that when $x \in Q$ and $y \in \operatorname{Supp} f$ then $|x-y| \sim \ell(Q)$. Hence, by the Cauchy-Schwarz inequality and estimate (76) in Chapter 1 (where the roles of x and y are reversed and $t^{1/2m}$ is changed to t),

$$|(\theta_t^{(k)} f)(x)|^2 \leq \int_{|x-y|\sim\ell(Q)} |\theta_t^{(k)}(x,y)|^2\, dy \int |f(y)|^2\, dy \\ \leq C \frac{1}{t^n} w\left(\frac{\ell(Q)}{t}\right) \int |f(y)|^2\, dy,$$

where $w(u) = u^{2\varepsilon} \exp\{-au^{(4k+2)/(4k+1)}\}$ with some $a, \varepsilon > 0$. A straightforward computation then gives us (4) and the proof of Lemma 6 is finished.

Applying this lemma with $f = (\nabla\chi_Q)F_Q$, we have

$$\int_0^{\ell(Q)} \int_Q |(\theta_t^{(k)}((\nabla\chi_Q)F_Q))(x)|^2 \frac{dxdt}{t} \leq C \int |\nabla\chi_Q(y)|^2 |F_Q(y)|^2\, dy. \tag{5}$$

For the other term in (3), observe that the operators $e^{-t^{4k+2}L_k} t^{2k} \Delta^k$ are uniformly bounded on $L^2(\mathbb{R}^n)$. Hence,

$$\int_0^{\ell(Q)} \int_Q |(\theta_t^{(k)}((\nabla F_Q)\chi_Q))(x)|^2 \frac{dxdt}{t} \leq C \int_0^{\ell(Q)} \int t^2 |\operatorname{div}(A\chi_Q\nabla F_Q)(x)|^2 \frac{dxdt}{t} \\ \leq C\ell(Q)^2 \int |\operatorname{div}(A\chi_Q \nabla F_Q)(x)|^2 dx. \tag{6}$$

Combining (2), (3), (5) and (6), we obtain

$$a_1(Q) \leq C \int |\nabla F_Q|^2 + C \int |\nabla(\chi_Q)|^2 |F_Q|^2 + C\ell(Q)^2 \int |\operatorname{div}(A\chi_Q \nabla F_Q)|^2.$$

Now, we use the hypotheses on F_Q and χ_Q together with the remark that F_Q can be chosen to have mean value 0. On the one hand, (i) and Poincaré inequality imply that

$$\int |\nabla F_Q|^2 + \int |\nabla(\chi_Q)|^2 |F_Q|^2 \leq C|Q|.$$

On the other hand, since $\operatorname{div}(A\chi_Q\nabla F_Q) = \chi_Q \operatorname{div}(A\nabla F_Q) + A\nabla\chi_Q \cdot \nabla F_Q$, we deduce from (i) and (ii) that $\operatorname{div}(A\chi_Q\nabla F_Q) \in L^2(\mathbb{R}^n)$ with

$$\int |\operatorname{div}(A\chi_Q\nabla F_Q)|^2 \leq \frac{C|Q|}{\ell(Q)^2}.$$

Hence, we have proved that

$$a_1(Q) \leq C|Q|,$$

where the constant C does not depend on Q. This concludes the argument. □

Remark. — A slight twist in the last part of the proof shows that condition (ii) in the definition of S can be weakened to the following:

(ii)' For some $0 < s \leq 1$, that does not depend on Q, $\operatorname{div}(A\nabla F_Q) \in H^{s-1}(Q)$ with

$$\|\operatorname{div}(A\nabla F_Q)\|_{H^{s-1}(Q)} \leq C\frac{|Q|^{1/2}}{\ell(Q)^s}.$$

Proof of Theorem 4. — Let $A \in S$ and $M \in L^\infty(\mathbb{R}^n, M_n(\mathbb{C}))$. We have to show that $A + M \in S$ provided $\|M\|_\infty$ is small enough.

Let $Q \in \mathcal{Q}$ and F_Q satisfying the conditions required by Definition 1 for A. If $\|M\|_\infty$ is small, then Lax-Milgram lemma tells us that $-\operatorname{div}(A+M)\nabla$ is an isomorphism from $H_0^1(\mu Q)$ onto $H^{-1}(\mu Q)$, hence there is a unique vector-valued function $H_Q \in H_0^1(\mu Q; \mathbb{C}^n)$ such that

$$-\operatorname{div}((A+M)\nabla H_Q) = \operatorname{div}(M\nabla F_Q),$$

and $\int_{\mu Q} |\nabla H_Q|^2 \leq c(n,\delta)\|M\|_\infty^2 \int_{\mu Q} |\nabla F_Q|^2$. Setting $G_Q = F_Q + H_Q$, we thus have $\int_{\mu Q} |\nabla G_Q|^2 \leq c|Q|$ and

$$-\operatorname{div}(A+M)\nabla G_Q = -\operatorname{div} A\nabla F_Q \in L^2(\mathbb{R}^n),$$

so that G_Q satisfies (i) and (ii) with respect to $A + M$.

It remains to show that (iii) also holds: this forces a further size condition on M. Extend ∇H_Q and ∇G_Q to be 0 outside of μQ. Carleson inequality implies

$$\int_0^{\ell(Q)} \int_Q |\gamma_t(x)(P_t\nabla H_Q)(x)|^2 \frac{dx\,dt}{t} \leq C|\gamma_t|_c^2 \int |\nabla H_Q|^2 \leq C|\gamma_t|_c^2 \|M\|_\infty^2 |Q|.$$

Hence by definition of G_Q,

$$\frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\gamma_t(x)(P_t\nabla G_Q)(x)|^2\frac{dxdt}{t}$$
$$\geq \frac{1}{2|Q|}\int_0^{\ell(Q)}\int_Q |\gamma_t(x)(P_t\nabla F_Q)(x)|^2\frac{dxdt}{t} - C|\gamma_t|_c^2\|M\|_\infty^2$$
$$\geq \left(\frac{\alpha}{2} - C\|M\|_\infty^2\right)|\gamma_t|_c^2 - C\|\gamma_t\|_\infty^2.$$

We obtain (iii) by imposing $\|M\|_\infty^2 < \alpha/(4C)$ and this ends the proof of Theorem 4. □

Remark. — We see in this perturbation argument the role of the arbitrariness of the functions γ_t in Definition 1.

3.3. Applications

3.3.1. Structure conditions: *Tb*-type results. — The *Tb*-Theorem provides boundedness for operators annihilating bounded invertible and accretive functions. See **[61]** and **[27]**. In the setting of square roots, there is again a dichotomy between one and several dimensions.

In one dimension, we have taken advantage of the cancellation condition $\theta_t a^{-1} = 0$ (see Chapter 2, Section 2.2.2). However, a closer look shows that a plays a more subtle role than just being bounded invertible and accretive: its inverse is also the derivative of a Lipschitz function. We make use of this observation by claiming that a is in the class S. Indeed, the function $f(x) = \int_0^x a^{-1}(y)dy$ has a bounded and accretive derivative and $D^*aDf = 0$. Thus, for each cube $Q \in \mathcal{Q}$, the function $f_Q = f$ on $2Q$ satisfies (i) and (ii) of Definition 1, while (iii) comes from the inequality $\operatorname{Re} P_t(Df)(x) \geq \alpha > 0$ a.e. for some $\alpha > 0$ provided P_t is positivity preserving, which we may impose. This, therefore, gives another proof of the one dimensional square root problem on applying Theorem 3.

In higher dimensions, we also have $\theta_t A^{-1} = 0$, but A^{-1} need not be the gradient of some Lipschitz (or even less regular) function. A *Tb*-type result in our context must, therefore, include a structure condition of a differential nature. We come to this now.

The following definition is taken from **[27]**.

Definition 7. — A locally integrable matrix-valued function M is said to be pseudo-accretive when there exist $\alpha > 0$ and a compactly supported approximation of the identity $(P_t)_{t>0}$ such that $(P_tM)(x)$ is almost everywhere invertible for every $t > 0$ with $\|(P_tM)^{-1}\|_\infty \leq 1/\alpha$.

Example 1. — Accretivity implies pseudo-accretivity.

Example 2. — A triangular matrix-valued function whose diagonal entries are accretive functions or even pseudo-accretive in the sense of G. David [**25**], is pseudo-accretive.

Definition 8. — A bounded and accretive matrix M is said to satisfy the structure condition (SC) when there exists a Lipschitz function $F\colon \mathbb{R}^n \to \mathbb{C}^n$ such that (i) ∇F is pseudo-accretive, and (ii) $\operatorname{div}(A\nabla F) = 0$ in the sense of distributions.

Remarks

1. If $A^{-1} = \nabla F$ for such an F, then A satisfies (SC). See below for an example.
2. We could clearly make this condition more local by replacing F by a family (F_Q) of functions defined on cubes, and what follows would remain true. We shall leave the details to the reader.

Theorem 9 **(Tb-type result).** — *If A satisfies (SC) then (K) holds for the square root of $L = -\operatorname{div}(A\nabla)$.*

Proof. — Just observe that matrices satisfying the structure condition form a subclass of S, the argument being the same as the one presented for the one dimensional case. Thus, Theorem 9 follows from Theorem 3. □

Let us describe some examples of matrices satisfying the structure condition. The first two are commonly used in homogenization theory [**44**].

In the solenoidal case, *i.e.*, if $\operatorname{div} A = 0$, the choice $F(x) = x$ shows that A satisfies (SC).

The irrotational case is when $A^{-1} = \nabla F$ for some Lipschitz F. For example, take A diagonal of the form

$$A(x) = \begin{pmatrix} a_1(x_1) & 0 & \dots & 0 \\ 0 & a_2(x_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_n(x_n) \end{pmatrix}$$

where the functions a_j are bounded and accretive on $\mathbb{R}$.

Remark. — That (K) holds for such an example may also be seen using the one dimensional result and functional calculi of commuting operators (personal communication of A. McIntosh and E. Franks).

A further example of a matrix satisfying (SC) is

$$A(x) = \begin{pmatrix} k(x) & 0 & \dots & 0 \\ 0 & k(x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & k(x) \end{pmatrix}$$

where $k(x) = a_1(x_1)a_2(x_2)\dots a_n(x_n)$. It is not of either of the above types.

The next example arises in the study of stratified media.

Proposition 10. — *If A is a bounded and accretive matrix on $\mathbb{R}^n$ that depends only on one coordinate variable, then A satisfies (SC).*

Proof. — Assume that A depends on, say, the first variable and write $A(x_1)$ for simplicity. We look for F as in Definition 8 starting from the ansatz

$$F_1(x) = f_1(x_1) \quad \text{and} \quad F_j(x) = x_j + f_j(x_1), \quad 2 \le j \le n.$$

Calling a_{jk} the entries of A, then a_{11} must be a bounded and accretive function. Elementary calculations show that the choice

$$f_1' = \frac{1}{a_{11}} \quad \text{and} \quad f_j' = -\frac{a_{1j}}{a_{11}}$$

gives us $\operatorname{div}(A\nabla F) = 0$. It is easy to see that ∇F is upper triangular so that it is pseudo-accretive (see example 2). □

Combining Theorems 3, 4 and this proposition, we obtain the following result.

Proposition 11. — *For every matrix $A_0 \in \mathcal{A}(\delta)$ that depends on one coordinate variable only, there exists $\varepsilon > 0$ such that (K) and (K*) hold for $L = -\operatorname{div}(A\nabla)$ when $\|A - A_0\|_\infty < \varepsilon$.*

Here (K*) is the inequality corresponding to (K) for L^*. We obtain it in the conclusion because the assumptions are stable under taking adjoints.

Let us study three amusing examples in dimension two.

Example 3. — It is easy to show that a matrix-valued function $B(x)$ in $\mathbb{R}^2$ satisfies $B = \nabla F$ and $B^* = \nabla G$ for some Lipschitz functions F and G if and only if

$$B(x) = Hu(x) + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

where $b \in \mathbb{R}$, u is a $C^{1,1}$ complex-valued function on $\mathbb{R}^2$, and Hu is its Hessian. As a consequence, if u is strictly convex with positive definite Hessian and $b = 0$, then $A_0 = (Hu)^{-1}$ satisfies (SC) and, therefore, $L = -\operatorname{div}(A\nabla)$ will satisfy (K) and (K*) for any A with $\|A - (Hu)^{-1}\|_\infty$ small enough.

Example 4. — Similarly, a bounded matrix-valued function satisfies $\operatorname{div} A = 0$ and $\operatorname{div} A^* = 0$ on $\mathbb{R}^2$ if and only if

$$A(x) = \begin{pmatrix} \dfrac{\partial^2 v}{\partial x_2 \partial x_2} & -\dfrac{\partial^2 v}{\partial x_1 \partial x_2} \\ -\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & \dfrac{\partial^2 v}{\partial x_1 \partial x_1} \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

where $b \in \mathbb{R}$ and v is a $C^{1,1}$ complex-valued function on $\mathbb{R}^2$. Hence, we see that such matrices with $b = 0$ and v strictly convex with positive definite Hessian satisfy (SC): just take $F(x) = x$.

Example 5. — Assume that $A \in \mathcal{A}$ and that $\det A$ is constant on $\mathbb{R}^2$. Then a straightforward computation shows that

$$A^{-1} = \nabla F \quad \text{if and only if} \quad \operatorname{div}({}^tA) = 0.$$

(This explains, in part, the similarity between the preceding examples.) Hence, for elements of $\mathcal{A}$ with $\operatorname{div} A = 0$ and $\det A$ constant and their L^∞ perturbations, (K) and (K*) hold simultaneously for $L = -\operatorname{div}(A\nabla)$.

3.3.2. Functions of absolutely bounded mean oscillations. — Let us begin by introducing a new function space. We say that two cubes do not overlap when their intersection is a null set for the Lebesgue measure.

Definition 12. — A locally square integrable (scalar- or vector-valued) function f is of absolutely bounded mean oscillation whenever $f \in BMO$ and, for each $\varepsilon > 0$, there exists $\eta > 0$ such that for all $Q \in \mathcal{Q}_0$ and any finite family Q_i, $i \in I$, of non overlapping cubes in $\mathcal{Q}_0$ with $Q_i \subset Q$ and $\ell(Q_i) \le \eta \ell(Q)$, we have

$$\sum_{i\in I} \int_{Q_i} |f - m_{Q_i} f|^2 \le \varepsilon |Q|. \tag{7}$$

Here, $m_{Q_i} f$ is the mean value of f over Q_i. The space of such functions is denoted by $ABMO$.

Note that one could make this definition scale invariant by replacing the class of cubes $\mathcal{Q}_0$ by $\mathcal{Q}$. Doing this imposes some behavior at infinity on f and this is not desirable for the application we have in mind.

Proposition 13. — *If $A \in \mathcal{A} \cap ABMO$ then $A \in \mathcal{S}_{loc}$.*

From this and Theorems 3 and 4, we obtain the following result (where $(\mathrm{K}^*)_{loc}$ is the inequality corresponding to $(\mathrm{K})_{loc}$ for L^*).

Proposition 14. — *If $A_0 \in \mathcal{A} \cap ABMO$ then, for some $\varepsilon > 0$, $(K)_{loc}$ and $(K^*)_{loc}$ are valid for any $L = -\operatorname{div}(A\nabla)$ provided $\|A - A_0\|_\infty < \varepsilon$.*

See the next section for a stronger result (Theorem 18).

Before going into the proof of Proposition 13, we give a few characterizations of $ABMO$ functions and some examples, postponing most proofs to Appendix A.

Proposition 15. — *For $f \in BMO$, the following assertions are equivalent.*

(i) $f \in ABMO$.

(ii) *For all $\varepsilon > 0$, there exists $\eta > 0$ such that for all cubes $Q \in \mathcal{Q}_0$,*

$$\int_Q |f - P_{\eta\ell(Q)}(f)|^2 \leq \varepsilon|Q|, \tag{8}$$

where $(P_t)_{t>0}$ is an approximation of the identity as usual.

(iii) *For all $\varepsilon > 0$, there exist $c > 0$ and, for each cube $Q \in \mathcal{Q}_0$, two functions $b_Q \in L^2(Q)$, $h_Q \in H^1(Q)$ with*

$$f(x) - m_Q f = b_Q(x) + h_Q(x), \quad x \in Q,$$

$$\int_Q |b_Q|^2 \leq \varepsilon|Q| \quad \text{and} \quad \int_Q |\nabla h_Q|^2 \leq \frac{c|Q|}{\ell(Q)^2}.$$

Example 1. — The spaces BUC and vmo are contained in $ABMO$.

Indeed, since $BUC \subset vmo$ (see Chapter 1), it suffices to prove the inclusion $vmo \subset ABMO$. If $f \in vmo$ and $\varepsilon > 0$ then $\int_Q |f - m_Q f|^2 \leq \varepsilon|Q|$ for $\ell(Q)$ small enough. Hence (iii) in Proposition 15 holds with $h_Q = 0$.

Example 2. — If $f \in L^\infty(\mathbb{R}^n)$ is such that, for some $c > 0$ and $s \in (0,1)$, $f\mathbf{1}_Q \in H^s(\mathbb{R}^n)$ with

$$\|f\mathbf{1}_Q\|_{H^s(\mathbb{R}^n)} \leq C\frac{|Q|^{1/2}}{\ell(Q)^s}, \quad \forall Q \in \mathcal{Q}_0$$

then $f \in ABMO$.

To see this, use the classical observation that, given any $R > 0$, any function $g \in H^s(\mathbb{R}^n)$ can be written as $g = b + h$ where $b \in L^2(\mathbb{R}^n)$ with $\|b\|_2 \leq CR^{-s}$ and $h \in H^1(\mathbb{R}^n)$ with $\|\nabla h\|_2 \leq CR^{1-s}$, and where C is proportional to $\|g\|_{H^s(\mathbb{R}^n)}$. Given $\varepsilon > 0$, apply this to $g = f\mathbf{1}_Q$ with $R = (\varepsilon\ell(Q))^{-1}$ to obtain (iii) in Proposition 15 and the proof is finished.

For $s \in (0, 1/2)$, charateristic functions of cubes belong to $H^s(\mathbb{R}^n)$, hence pointwise multipliers of $H^s(\mathbb{R}^n)$ satisfy the above condition. Examples are characteristic functions of Lipschitz domains (see [**76**]). The square root problem with coefficients being pointwise H^s-multipliers was first solved by McIntosh [**57**].

The characteristic function of a cube is not in vmo. Hence, the inclusion $vmo \subset ABMO$ is strict.

Example 3. — An integral regularity condition. If $f \in BMO$ is such that

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|}\int_{|h|\leq\ell(Q)}\int_Q \frac{|f(x+h) - f(x)|^2}{|h|^n}\,dxdh < \infty,$$

then $f \in ABMO$. The proof is done in Appendix A.

Example 4. — Any continuous function away from the origin that is homogeneous of degree 0 belongs to $ABMO$.

Example 5. — Characteristic functions of domains whose boundaries possess a mild regularity are in $ABMO$. Here is a precise condition.

Assume that Ω is an open set such that there exists a non decreasing function $d\colon [0,1] \to \mathbb{R}$ verifying

(a) $\lim_{s\to 0} d(s) = 0$,
(b) for any cube $Q \in \mathcal{Q}_0$,

$$\min \left\{ \begin{array}{l} |\{ x \in \Omega \cap Q\,;\, d(x, {}^c\Omega) \le s\ell(Q) \}|, \\ |\{ x \in {}^c\Omega \cap Q\,;\, d(x, \Omega) \le s\ell(Q) \}| \end{array} \right\} \le d(s)|Q|,$$

where $|E|$ denotes the Lebesgue measure of E.

Then $\mathbf{1}_\Omega \in ABMO$.

Examples of such domains are Lipschitz domains for which $d(s) \sim s$ and domains with fractal boundaries such as snowflakes for which $d(s) = O(s^\alpha)$ for some $\alpha > 0$.

The argument is as follows. Let $f = \mathbf{1}_\Omega$, $Q \in \mathcal{Q}_0$, and $(Q_i)_{i\in I}$ be a family of non overlapping subcubes of Q as in the definition of $ABMO$. Set

$$c = \frac{1}{|Q|} \sum_{i\in I} \int_{Q_i} |f - m_{Q_i} f|^2.$$

A direct computation gives

$$c = \sum_{i\in I} \frac{|Q_i \cap \Omega|}{|Q_i|} \frac{|Q_i \cap {}^c\Omega|}{|Q_i|} \frac{|Q_i|}{|Q|}.$$

If $\ell(Q_i) \le \dfrac{\eta \ell(Q)}{\sqrt{n}}$, using that the cubes Q_i do not overlap, we have

$$\begin{aligned} c &\le \sum_{Q_i \cap {}^c\Omega \ne \emptyset} \frac{|Q_i \cap \Omega|}{|Q_i|} \frac{|Q_i|}{|Q|} \\ &\le \frac{|\{ x \in \Omega \cap Q\,;\, d(x, {}^c\Omega) \le \eta\ell(Q) \}|}{|Q|}, \end{aligned}$$

and symmetrically

$$c \le \frac{|\{ x \in {}^c\Omega \cap Q\,;\, d(x, \Omega) \le \eta\ell(Q) \}|}{|Q|}.$$

Hence $c \le d(\eta)$ by (b) and the conclusion that $\mathbf{1}_\Omega \in ABMO$ follows from (a).

Example 6. — The characteristic function of the set $\cup_{n\ge 1}(\frac{1}{(2n+1)^\alpha}, \frac{1}{(2n)^\alpha})$, where $\alpha > 0$, does not belong to $ABMO$. This fact is related to oscillations, since this function alternates the values 0 and 1. A similar example of a bounded function that does not belong to $ABMO$ is $\sin(\frac{1}{x^{1/\alpha}})$ for any $\alpha > 0$. The verifications of these assertions are in Appendix A.

Proof of Proposition 13. — Let A belong to $\mathcal{A} \cap ABMO$. We have to verify the required properties in the definition of S_{loc}.

Let $Q \in \mathcal{Q}_0$ and $\varepsilon > 0$. By (ii) in Proposition 15, there exists $\eta > 0$ which does not depend on Q such that

$$\int_{2Q} |A - P_{\eta\ell(Q)}(A)|^2 \leq \varepsilon |Q|. \tag{9}$$

Observe that, with $F(x) = x$, by letting the divergence fall on P_t, we have

$$\operatorname{div}(P_{\eta\ell(Q)}(A)\nabla F) = \operatorname{div}(P_{\eta\ell(Q)}(A)) \in L^2(2Q)$$

and

$$\int_{2Q} |\operatorname{div}(P_{\eta\ell(Q)}(A)\nabla F)|^2 \leq \frac{C|Q|}{\ell(Q)^2},$$

where C depends only on η, n, and $\|A\|_\infty$. We now construct F_Q by solving

$$\begin{cases} -\operatorname{div}(A\nabla F_Q) = -\operatorname{div}(P_{\eta\ell(Q)}(A)\nabla F), & \text{on } 2\text{Q}, \\ F_Q - F \in H_0^1(2Q). \end{cases}$$

Clearly (ii) in Definition 1 is satisfied by F_Q and (i) comes from ellipticity. To see that (iii) holds, it suffices to control

$$I = \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\gamma_t(x)P_t(\nabla F - \nabla F_Q)(x)|^2 \frac{dxdt}{t}$$

from above by $C|\gamma_t|_c^2\varepsilon$ and to choose $\varepsilon > 0$ appropriately (recall that ∇F and ∇F_Q are 0 outside of $2Q$). Again, we have by Carleson inequality,

$$I \leq C|\gamma_t|_c^2 \frac{1}{|Q|}\int_{2Q} |\nabla F - \nabla F_Q|^2 \leq C|\gamma_t|_c^2 \frac{1}{|Q|}\int_{2Q} |(P_{\eta\ell(Q)}(A) - A)\nabla F|^2$$

where the last inequality comes from the Lax-Milgram lemma which guarantees the existence of F_Q. Using (9) ends the proof. □

3.3.3. Perturbations in the BMO topology. — As we saw, S and S_{loc} are open in the L^∞ topology. Some subclasses are also open in the BMO topology. This shows that the space of holomorphy of the map $A \to L^{1/2}$ can be taken as BMO when it is restricted to these subclasses. Here is the first one.

Proposition 16. — *Let $A_0 \in \mathcal{A}$ be such that $\operatorname{div} A_0$ is locally square integrable with the estimates*

$$\int_Q |\operatorname{div} A_0|^2 \leq \frac{C|Q|}{\ell(Q)^2} \tag{10}$$

for all $Q \in \mathcal{Q}$. Then the class S is a neighborhood of A_0 in the BMO topology.

Proof. — Let A_0 satisfy (10). It clearly belongs to S. Let $A \in \mathcal{A}$ such that $\|A - A_0\|_{BMO} < \varepsilon$. We must show that if $\varepsilon > 0$ is small enough, then $A \in S$.

Fix a cube $Q \in \mathcal{Q}$ and define F_Q on $2Q$ by solving $\operatorname{div}(A\nabla F_Q) = \operatorname{div} A_0$ on $2Q$ and $F_Q(x) - x \in H_0^1(2Q)$. Clearly, (ii) in Definition 1 is verified. Set $F(x) = x$. By construction, we have on $2Q$,

$$\operatorname{div}(A\nabla(F_Q - F)) = \operatorname{div}(A_0 - A) = \operatorname{div}(A_0 - A - m_{2Q}(A_0 - A)).$$

Hence, by ellipticity of A,

$$\int_{2Q} |\nabla F_Q - \nabla F|^2 \leq C \int_{2Q} |A_0 - A - m_{2Q}(A_0 - A)|^2 \leq C2^n \|A_0 - A\|_{BMO}^2 |Q|.$$

In particular, the condition (i) in Definition 1 holds. Condition (iii) is proved as for Proposition 13. This ends the argument. □

Remark. — Again, one can relax the requirement on $\operatorname{div} A_0$ and only assume that it belongs to $H^{s-1}(Q)$ for some $s \in (0, 1]$ with the appropriate norm estimate.

The second subclass is $ABMO$.

Proposition 17. — *Let $A_0 \in \mathcal{A} \cap ABMO$. Then the class S_{loc} is a neighborhood of A_0 in the BMO topology.*

Proof. — The argument is a variant of that of Propositions 13 and 16. Let us sketch the main lines. Let $A \in \mathcal{A}$. We must show that if $\|A - A_0\|_{BMO}$ is small enough, then $A \in S_{loc}$.

Fix a cube $Q \in \mathcal{Q}$ and define F_Q on $2Q$ by solving $\operatorname{div}(A\nabla F_Q) = \operatorname{div} P_t(A_0)$ on $2Q$ and $F_Q(x) - x \in H_0^1(2Q)$. Here, $t = \eta\ell(Q)$ and η will be chosen later.

Again, (i) and (ii) are easily checked.

As in the proof of Proposition 13, it remains to show that $\int_{2Q} |\nabla F - \nabla F_Q|^2$ can be controlled by $\varepsilon|Q|$ for some $\varepsilon > 0$ that is not too large. To this end, observe that, if $F(x) = x$, we have

$$\operatorname{div}(A\nabla(F_Q - F)) = \operatorname{div}(P_t(A_0) - A_0) - \operatorname{div}(A - A_0 - m_{2Q}(A - A_0))$$

on $2Q$. Hence,

$$\begin{aligned}\int_{2Q} |\nabla F - \nabla F_Q|^2 &\leq c \int_{2Q} |P_t(A_0) - A_0|^2 + c \int_{2Q} |A - A_0 - m_{2Q}(A - A_0)|^2 \\ &\leq c \int_{2Q} |P_t(A_0) - A_0|^2 + c2^n \|A - A_0\|_{BMO}^2 |Q|.\end{aligned}$$

For all $\varepsilon > 0$, the first integral in the right hand side is controlled by $\varepsilon|Q|/2$ upon choosing η small by (ii) in Proposition 15. The conclusion follows readily. □

We thus obtain the following improvement of Corollary 5 and Proposition 14.

Theorem 18

(i) *Let $A \in \mathcal{A}(\delta)$. There exists $\varepsilon = \varepsilon(n, \delta) > 0$ such that (K) and (K*) hold for $L = -\operatorname{div}(A\nabla)$ when $\|A\|_{BMO} \leq \varepsilon$.*

(ii) *Let $A_0 \in \mathcal{A}(\delta) \cap ABMO$. There exists $\varepsilon = \varepsilon(n, \delta) > 0$, such that $(K)_{loc}$ and $(K^*)_{loc}$ hold for any $L = -\operatorname{div}(A\nabla)$ with $\|A - A_0\|_{BMO} < \varepsilon$.*

3.3.4. Open questions relative to the class S. — We list three questions.

Question 1. — *Is S stable under bilipichitz change of variables?*

As we have observed in Preliminaries, the Kato square root problem is stable under a bilipischitz change of variables. It is not clear that our class S has the same property. The difficulty lies in (iii) of Definition 1. This is an indication that the class S may be enlarged so that Theorems 3 and 4 still hold.

Question 2. — *Is S stable under taking the adjoint?*

This is likely not to be the case. The subclass (SC) studied in Section 3.3.1 is clearly not.

Question 3. — *Does S contains the class of real uniformly positive definite bounded matrix-valued functions?*

We know from Preliminaries that the map $A \to (-\operatorname{div}(A\nabla))^{1/2}$ is analytic on S. A positive answer to this question is therefore of importance toward perturbation and regularity results.

3.4. The real symmetric case

In this section, we assume that $A(x)$ is a real positive definite $n \times n$ matrix with $0 < \delta I \leq A(x) \leq \delta^{-1} I$ a.e..

We have already observed that (K) holds for a purely abstract reason. Since $L = -\operatorname{div}(A\nabla)$ has the property (G), Theorem 24 of Chapter 2 is applicable. Therefore, $\theta_t 1$ must be a Carleson function, where we have set $\theta_t = e^{-t^2 L}\, t \operatorname{div} A$. Our goal here is to obtain directly the Carleson measure estimate. As mentioned above, we do not know whether $A \in S$, so we proceed using integration by parts. We shall see in the course of the proof that this method gives a Carleson measure estimate on the first order moments of Nash.

Remark. — The assumption that $A(x)$ be real symmetric can be replaced by $A(x)$ selfadjoint. Theorem 25 is used in that case and the strategy developed below should be adapted.

Let e_j denotes the jth vector in the canonical basis of $\mathbb{C}^n$. We have to estimate $\theta_t(e_j)$, $1 \le j \le n$. On a formal level,

$$\theta_t(e_j) = e^{-t^2L}\, t\,\mathrm{div}\,(Ae_j) = e^{-t^2L}\, t\,\mathrm{div}\,(A\nabla x_j) = \frac{1}{2}\frac{d e^{-t^2L}(x_j)}{dt}, \tag{11}$$

where x_j denotes the jth coordinate function. As this function plays no particular role, we set, for any Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{C}$,

$$\gamma_t(x) = \int K_t(x,y)\,(f(y) - f(x))\,dy, \tag{12}$$

where $K_t(x,y)$ is the kernel of e^{-tL}. We omit the dependence on f to keep the notation simple. Let us just note that when $f(x) = x_j$, this integral is a first order moment for the heat kernel, as introduced by Nash **[66]**. With this notation, we have

$$\theta_t(e_j) = t\dot\gamma_{t^2} \quad \text{with } f(x) = x_j, \tag{13}$$

where $\dot g$ stands for $\dfrac{dg}{dt}$.

To justify this equality, recall that for $f \in \mathcal{D}(L)$

$$\theta_t \nabla f = -e^{-t^2L} tLf = \frac{1}{2}\frac{d e^{-t^2L} f}{dt}, \tag{14}$$

hence,

$$\int \theta_t(x,y)\nabla f(y)\,dy = \frac{1}{2}\int \frac{\partial K_{t^2}(x,y)}{\partial t} f(y)\,dy, \quad a.e.. \tag{15}$$

By density of $\mathcal{D}(L)$ in $H^1(\mathbb{R}^n)$ and since θ_t is L^2-bounded, the latter equality extends to $H^1(\mathbb{R}^n)$.

Now, given a Lipschitz function f, it can be approximated by a sequence of compactly supported Lipschitz functions f_k in the sense that $\lim f_k(x) = f(x)$ and $\lim \nabla f_k(x) = \nabla f(x)$ as $k \to \infty$ for almost all x, with $\|\nabla f_k\|_\infty \le C\|\nabla f\|_\infty$ uniformly. Since (15) holds for each f_k, it remains to pass to the limit as $k \to \infty$ with the dominated convergence theorem using the estimates on $\theta_t(x,y)$ and $\dfrac{\partial K_{t^2}(x,y)}{\partial t}$ proved in Chapter 1.

Finally, since $e^{-tL}1 = 1$, we have $\gamma_t = e^{-tL}f - f$ so that $\dfrac{1}{2}\dfrac{d e^{-t^2L} f}{dt} = t\dot\gamma_{t^2}$ as desired.

We have to show, therefore, that $t\dot\gamma_{t^2}$ is a Carleson function. This property does not depend on a specific choice of f and we do it for f arbitrary. Also, it is easier to use the parabolic homogeneity by changing t^2 to t.

Theorem 19. — *With the notation above, $t^{1/2}\dot\gamma_t$ is a parabolic Carleson function, and*

$$\sup_{Q\in\mathcal{Q}} \left\{ \frac{1}{|Q|}\int_Q \int_0^{\ell(Q)^2} |t^{1/2}\dot\gamma_t(x)|^2 \frac{dxdt}{t} \right\} \le c\|\nabla f\|_\infty^2. \tag{16}$$

Before proving this, let us derive a consequence for the Nash moments.

Corollary 20. — *The first order moments $t^{-1/2}\gamma_t$ are parabolic Carleson functions and*

$$\sup_{Q\in\mathcal{Q}}\left\{\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)^2}|\gamma_t(x)|^2\frac{dxdt}{t^2}\right\}\le c\|\nabla f\|_\infty^2. \tag{17}$$

Notice that this is stronger than

$$\|\gamma_t\|_\infty\le ct^{1/2}\|\nabla f\|_\infty,\quad t>0, \tag{18}$$

which is a direct consequence of (G). This corollary sheds light on the cancellations contained in these moments, thus improving (18) to (17). Observe, for example, that these moments vanish whenever $f(x)=x_j$ and $A(x)$ is a constant matrix, since the heat kernel is an even convolution kernel in that case.

Also, (17) is an estimate for the parabolic Cauchy problem

$$\frac{du_t}{dt}+Lu_t=-\operatorname{div}(A\nabla f)\quad\text{on }\mathbb{R}^n\times(0,\infty)$$

with $u_0=0$ on $\mathbb{R}^n$ of which γ_t is a weak solution.

Remark. — Instead of using the heat semigroup, we may use the resolvent to define θ_t. Then, we obtain analogous statements about Carleson measure estimates. They may also be reformulated in terms of estimates for boundary value elliptic problems.

Let us first derive Corollary 20. By the Hardy inequality

$$\int_0^a|u(t)|^2\frac{dt}{t^2}\le 4\int_0^a|\dot u(t)|^2\,dt$$

applied to $u(t)=\gamma_t(x)$, we have

$$\int_0^{\ell^2}|\gamma_t(x)|^2\frac{dt}{t^2}\le 4\int_0^{\ell^2}|t^{1/2}\dot\gamma_t(x)|^2\frac{dt}{t}.$$

Integrating over Q of sidelength $\ell(Q)=\ell$, we obtain (17) from (16) and Corollary 20 is proved.

Remark. — It is also true that (16) is controlled by (17) up to an error term. We shall not use this.

In the proof of Theorem 19 we use the following lemma which holds under more general assumption than $A(x)$ being real symmetric. Its proof is postponed until the end of this section.

Lemma 21. — *If $L=-\operatorname{div}(A\nabla)\in\mathcal{E}$ has the property (G) and if f is Lipschitz, then*

(i) $\|\gamma_t\|_\infty\le ct^{1/2}\|\nabla f\|_\infty$,
(ii) $\|\dot\gamma_t\|_\infty\le ct^{-1/2}\|\nabla f\|_\infty$,

(iii) $\sup_{0<t\le \ell(Q)^2} \int_{2Q} |\nabla\gamma_t|^2 \le c|Q|\|\nabla f\|_\infty^2$,

(iv) $\int_{2Q} |\nabla\dot\gamma_t|^2 \le ct^{-2}|Q|\|\nabla f\|_\infty^2$ *when* $0 < t \le \ell(Q)^2$.

Proof of Theorem 19. — The inequality (16) holds provided we can show that there exists a constant C such that for all cube Q and all $\epsilon > 0$, $I_\varepsilon(Q) \le C|Q|\|\nabla f\|_\infty^2$, where

$$I_\varepsilon(Q) = \int_Q\int_{\varepsilon^2}^{\ell(Q)^2} |\dot\gamma_t(x)|^2\,dxdt.$$

By homogeneity, we may assume that $\|\nabla f\|_\infty = 1$. Let us also make the further assumption that f has compact support, which makes $f \in H^1(\mathbb{R}^n)$. Then we have

$$\int \dot\gamma_t\psi\,dx = -\int A(\nabla\gamma_t + \nabla f)\cdot\nabla\psi\,dx, \quad t>0, \tag{19}$$

for all $\psi \in H^1(\mathbb{R}^n)$ with compact support. Indeed, since $\dot\gamma_t = \dfrac{d\,e^{-tL}f}{dt}$, by definition of the semigroup, one has

$$\int \dot\gamma_t\psi\,dx = -\int A\nabla(e^{-tL}f)\cdot\nabla\psi\,dx = -\int A(\nabla\gamma_t + \nabla f)\cdot\nabla\psi\,dx.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \ge 0$, such that $\varphi = 1$ on Q and $\varphi = 0$ on $(2Q)^c$ with $\|\varphi\|_\infty \le 1$ and $\|\nabla\varphi\|_\infty \le 10\ell(Q)^{-1}$. By (19) and Fubini theorem, we have

$$\begin{aligned} I_\varepsilon(Q) &\le \operatorname{Re}\int\int_{\varepsilon^2}^{\ell(Q)^2} \dot\gamma_t\overline{\dot\gamma_t\varphi}\,dtdx \\ &= -\operatorname{Re}\int\int_{\varepsilon^2}^{\ell(Q)^2} A(\nabla\gamma_t + \nabla f)\cdot\overline{\nabla(\dot\gamma_t\varphi)}\,dxdt. \end{aligned}$$

Note that the use of $\dot\gamma_t\varphi$ as a test function is justified by (ii) and (iv) in Lemma 21. The last integrand gives us three terms to which Fubini theorem applies.

First, we have

$$\operatorname{Re} A\nabla\gamma_t\cdot\overline{\nabla\dot\gamma_t}\varphi = \frac{1}{2}\frac{\partial A\nabla\gamma_t\cdot\overline{\nabla\gamma_t}\,\varphi}{\partial t}$$

since $A^* = A$. Integrating this equality with respect to dt and then to dx, we obtain a bound $C|Q|$ by (iii) in Lemma 21.

The second term is

$$\operatorname{Re} A\nabla\gamma_t\cdot\nabla\varphi\,\overline{\dot\gamma_t}.$$

Integrating with respect to x, using Cauchy-Schwarz inequality and Lemma 21, we obtain a bound

$$ct^{-1/2}\|\nabla\varphi\|_\infty|Q| \le ct^{-1/2}\ell(Q)^{-1}|Q|.$$

Integration with respect to dt yields the desired estimate.

The last term is

$$\operatorname{Re} A\nabla f\cdot\overline{\nabla(\dot\gamma_t\varphi)}=\frac{\partial\operatorname{Re} A\nabla f\cdot\overline{\nabla(\gamma_t\varphi)}}{\partial t}.$$

Thus, integration with respect to dt and then with respect to dx gives us again a bound $c|Q|$ by use of Lemma 21.

It remains to remove the assumption that f has compact support. Pick a sequence of uniformly Lipschitz functions f_k with compact support that converges in all points to f as k tends to ∞. By the dominated convergence theorem

$$\dot\gamma_t(x)=\int\frac{\partial K_t(x,y)}{\partial t}(f(y)-f(x))\,dy=\lim_{k\to\infty}\dot\gamma_{t,k}(x),$$

where $\dot\gamma_{t,k}(x)=\displaystyle\int\frac{\partial K_t(x,y)}{\partial t}(f_k(y)-f_k(x))\,dy$. Thus, Fatou lemma gives us

$$I_\varepsilon(Q)\le\liminf_{k\to\infty}\int_Q\int_{\varepsilon^2}^{\ell(Q)^2}|\dot\gamma_{t,k}(x)|^2\,dxdt\le C\sup_{k>0}\|\nabla f_k\|_\infty|Q|,$$

and we are finished. □

Proof of Lemma 21. — Assume without loss of generality that $\|\nabla f\|_\infty=1$. First, (i) and (ii) follow directly from the Gaussian decay of $K_t(x,y)$ or its time derivative, and use of the mean value inequality for f.

To prove (iii) introduce a function φ with compact support on $6Q$ such that $\|\varphi\|_\infty=1$ and $\varphi=1$ on $4Q$. Set

$$\mu_t(x)=\int K_t(x,y)\,(f(y)-f(x))(1-\varphi(y))\,dy,$$

so that

$$\gamma_t(x)=[f,e^{-tL}](\varphi)(x)+\mu_t(x),$$

where $[f,e^{-tL}]$ is the commutator between the multiplication by f and the semigroup. We have

$$\nabla_x\mu_t(x)=\int\nabla_xK_t(x,y)\,(f(y)-f(x))(1-\varphi(y))\,dy\\ -\nabla f(x)\int K_t(x,y)\,(1-\varphi(y))\,dy.$$

Observing that $|x-y|\ge2\ell(Q)$ for $x\in2Q$ and y in the support of $1-\varphi$, we have

$$|\nabla\mu_t(x)|\le\int_{|x-y|\ge2\ell(Q)}|\nabla_xK_t(x,y)|\,|y-x|\,dy\\ +\int_{|x-y|\ge2\ell(Q)}|K_t(x,y)|\,dy.$$

Now, use the condition (G) on $K_t(x,y)$, the estimates in Theorem 7 of Chapter 1 on $\nabla_xK_t(x,y)$, and the fact that $t\le\ell(Q)^2$, to obtain (by adapting the argument to estimate $U_t f_2$ in Lemma 15 of Chapter 2) that

$$\int_{2Q} |\nabla \mu_t(x)|^2 \, dx \le C|Q|.$$

It remains to control the commutator. This is a general fact.

***Lemma* 22.** — *For any $L \in \mathcal{E}(\delta)$ and any Lipschitz function f, the operator $\nabla[f, e^{-tL}]$ is continuous on $L^2(\mathbb{R}^n)$ with norm bounded uniformly by $c\|\nabla f\|_\infty$, where $c = c(n, \delta)$.*

Accepting the truth of this operator bound, we have

$$\int_{2Q} |\nabla[f, e^{-tL}](\varphi)|^2 \le c\|\varphi\|_2^2 \le c|Q|.$$

which ends the proof of (iii).

To finish the proof of Lemma 21, it remains to establish (iv). Thanks to the results in Chapter 1 (Lemma 20 and the remark at the end of Section 1.4.7) one could have replaced in the proof of (iii) positive times t by complex times z in some appropriate sector. Doing this allows us to use complex function theory and (iv) follows easily from Cauchy estimates. The details are left to the reader. □

Proof of Lemma 22. — It suffices to establish a similar bound for the resolvent, namely

$$\|\nabla[f, (L+\lambda)^{-1}]\varphi\|_2 \le c|\lambda|^{-1}\|\nabla f\|_\infty \|\varphi\|_2 \tag{20}$$

for $-\lambda$ in any strict subsector of the resolvent set. Indeed, integrating this in the Cauchy formula

$$e^{-tL} = \frac{1}{2\pi i}\int_\gamma e^{t\lambda}(L+\lambda)^{-1}\, d\lambda,$$

used in Chapter 1 (formula (61)) yields the result.

Now, an explicit computation gives us

$$\nabla[f, (L+\lambda)^{-1}] = \nabla(L+\lambda)^{-1}(b\cdot\nabla + \operatorname{div}\tilde{b})(L+\lambda)^{-1},$$

where $b, \tilde{b}$ are bounded $\mathbb{C}^n$-valued functions with $\|b\|_\infty + \|\tilde{b}\|_\infty \le c\|\nabla f\|_\infty$. The inequality (20) follows on applying the estimates (4) in Proposition 1 of Chapter 1. □

3.5. A variation on regularity conditions

We consider the class of matrices $A \in \mathcal{A}$ such that $(A - P_t A)(x)$ is a local Carleson function (see (41) in Chapter 2), $(P_t)_{t>0}$ being a usual approximation to the identity. As we shall see in Appendix A, such functions belong to $ABMO$ in view of the next result.

***Proposition* 23.** — *Let $f \in BMO$. Then $(f - P_t f)(x)$ is a local Carleson function if, and only if, f satisfies the integral regularity condition in Example 3. In particular, any of these conditions implies that $f \in ABMO$.*

Hence, we could invoke Proposition 14 to show that $(\mathrm{K})_{loc}$ holds for $L = -\operatorname{div}(A\nabla)$. Here is a direct argument only using quadratic functionals.

Proof. — We have to show that

$$\int_0^1\int_{\mathbb{R}^n} |\theta_t \nabla f(x)|^2 \,\frac{dxdt}{t} \le c\|\nabla f\|_2^2,$$

where $\theta_t = e^{-t^2L} t \operatorname{div} A$.

Again, write $\theta_t\nabla = \theta_t P_t \nabla + \theta_t(1-P_t)\nabla$ and the last term is taken care of as in the proof of Theorem 21 in Chapter 2. Now,

$$\theta_t P_t \nabla f = e^{-t^2L} t \operatorname{div}\big[(P_tA)(P_t\nabla f)\big] + e^{-t^2L} t \operatorname{div}\big[(A-P_tA)(P_t\nabla f)\big].$$

For the last term, by the uniform L^2-boundedness of $e^{-t^2L}t \operatorname{div}$ and Carleson inequality, we have

$$\int_0^1\int_{\mathbb{R}^n} \big|e^{-t^2L} t \operatorname{div}\big[(A-P_tA)(P_t\nabla f)\big]\big|^2 \,\frac{dxdt}{t}$$
$$\le C\int_0^1\int_{\mathbb{R}^n} |(A-P_tA)(P_t\nabla f)|^2 \,\frac{dxdt}{t} \le C|A-P_tA|_c^2\|\nabla f\|_2^2.$$

For the first term, we have

$$\int_{\mathbb{R}^n} \big|e^{-t^2L} t \operatorname{div}\big[(P_tA)(P_t\nabla f)\big]\big|^2 \,dx \le \int_{\mathbb{R}^n} \big|t \operatorname{div}\big[(P_tA)(P_t\nabla f)\big]\big|^2 \,dx.$$

By differentiating, we find that

$$t \operatorname{div}\big[(P_tA)(P_tH)\big] = (P_t\, a_{k\ell})\ (Q_t^k\, H_\ell) + (Q_t^k\, a_{k\ell})\ (P_t H_\ell),$$

where $H = \nabla f$, $a_{k\ell}$ are the entries of A and $Q_t^k = t\dfrac{\partial}{\partial x_k} P_t$. It remains to integrate with respect to dt/t. For the term involving $(P_t\, a_{k\ell})\ (Q_t^k H_\ell)$, we use that $\|P_t\, a_{k\ell}\|_\infty \le \|A\|_\infty$ and the fact that $(Q_t^k)_{t>0}$ is a bounded family, which can be checked using (2) in Chapter 2. For the term involving $(Q_t^k a_{k\ell})\ (P_t H_\ell)$ we use again Carleson inequality invoking the fact that $Q_t^k\, a_{k\ell}$ is a Carleson function when $a_{k\ell}$ is bounded (Lemma 7 of Chapter 2). □

Remark. — For example pointwise multipliers of $H^s(\mathbb{R}^n)$ for some $s > 0$ have the above regularity property. This furnishes another simple proof of a result by McIntosh **[57]**.

CHAPTER 4

SQUARE ROOTS OF DIFFERENTIAL OPERATORS, SINGULAR INTEGRALS AND L^p THEORY

4.1. Introduction

Unless explicit mention all the results presented in this chapter are in dimension larger than two and L denotes an operator of the form $-\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$, as defined in Chapter 1. The number δ in this condition is referred to as the ellipticity constant of A.

In the previous chapters, we have studied the validity of

$$\|L^{1/2}f\|_2 \leq c\|\nabla f\|_2, \quad f \in \mathcal{D}(L), \tag{K}$$

under various hypotheses on A. Recall that the conjunction of both (K) and (K*) —the similar inequality for L^*— implies that the domain of $L^{1/2}$ is the Sobolev space $H^1(\mathbb{R}^n)$.

Our purpose is to investigate the validity of the corresponding L^p estimates when $1 < p < \infty$, and of the endpoint estimates when $p = 1$ or $p = \infty$. By L^p estimates, we mean the *a priori* inequalities

$$\|L^{1/2}f\|_p \leq c_p\|\nabla f\|_p, \tag{1}$$

$$\|\nabla f\|_p \leq c'_p\|L^{1/2}f\|_p. \tag{2}$$

By Proposition 3 of Preliminaries, (K*) is equivalent to (2) for $p = 2$.

Toward this aim, we study the link between $L^{1/2}$ and Calderón-Zygmund operators. In spirit of classical Calderón-Zygmund theory, we start from (1) and (2) when $p = 2$ and look for extensions to other values of p.

The motivating example for doing so is the Laplace operator. When $L = -\Delta$, (2) is equivalent to the L^p boundedness of the Riesz transforms:

$$R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}, \qquad j = 1, \ldots, n,$$

which are historical examples of Calderón-Zygmund operators in higher dimensions.

Hence, a natural question is whether the Riesz transforms associated to L, namely the operators $\frac{\partial}{\partial x_j}L^{-1/2}$, are Calderón-Zygmund operators.

In dimension 1, the authors have shown in [**14**] that $a(x)\frac{d}{dx}L^{-1/2}$, where $L = -\frac{d}{dx}\big(a(x)\frac{d}{dx}\big)$, is a Calderón-Zygmund operator.
Using homogenization theory, Alexopoulos obtained that the Riesz transforms associated to L are Calderón-Zygmund operators when A has real-valued Hölder continuous coefficients that are periodic with common period [**2**].

But the proofs cannot generalize: an example shows that the L^p estimate (2) may fail for $p > 2$ in $\mathbb{R}^2$ and, hence, the Riesz transforms associated to L are not Calderón-Zygmund operators in general. This is due to C. Kenig. In particular, this example can be adapted to show the necessity of assuming some regularity condition for Alexopoulos' result to hold.

A partial result concerning (2) has been obtained by David and Journé in [**26**]: using their $T1$-Theorem and multilinear estimates, they show that (2) holds when $1 < p \leq 2$ and $\|A - I\|_\infty$ is small (see also [**22**] for an exposition).

One of the main results of this chapter is that, when L has the Gaussian property (G) defined in Chapter 1, the L^p estimates but the one excluded by Kenig's example are a consequence of the L^2 estimates.

Theorem 1. — *Assume that L has the Gaussian property (G) and that (K) and (K*) hold. Then, the following a priori inequalities hold for $f \in \mathcal{D}(L^{1/2}) = H^1(\mathbb{R}^n)$:*

$$\begin{aligned} \|L^{1/2}f\|_p &\leq c_p\|\nabla f\|_p, \quad 1 < p < \infty, \\ \|\nabla f\|_p &\leq c'_p\|L^{1/2}f\|_p, \quad 1 < p < 2+\varepsilon, \\ \|L^{1/2}f\|_{BMO} &\leq c\|A\nabla f\|_{BMO}, \\ \|L^{1/2}f\|_{\mathcal{H}^1(\mathbb{R}^n)} &\sim \|\nabla f\|_{\mathcal{H}^1(\mathbb{R}^n)}. \end{aligned}$$

Here c_p, c'_p, c do not depend on f, and $\varepsilon > 0$ depends only on L. Moreover, these estimates are optimal, in the sense that, for every $\varepsilon > 0$, there exists an operator L with the required properties such that (2) fails when $p \geq 2+\varepsilon$.

In this statement, $\mathcal{H}^1(\mathbb{R}^n)$ is the atomic Hardy space, BMO being its dual (see [**74**]). Note also that no regularity assumption is made on A.

Typical situations where Theorem 1 is valid are:

(i) A is real symmetric,
(ii) $\|A\|_{BMO} < \chi$ where $\chi > 0$ depends on n and the ellipticity constant of A,
(iii) A depends on one variable.

Parts (i) and (ii) follow by combining results of Chapter 1 and Chapter 3. Part (iii) will be considered in Section 4.5.

Note that part (ii) covers the case of L^∞ perturbations of constant (complex) elliptic matrices, thus extending the L^p results in **[26]**.

As explained in the general introduction, L^p estimates for $L^{1/2}$ can be reformulated in terms of elliptic boundary value problems. Inequality (2) implies a solvability result for a Neumann problem with data in L^p. Similarly, inequalities (1) and (2) imply a solvability result for a regularity problem with data in an L^p Sobolev space. Thus, part (i) should be compared with the main results in **[51]**.

A positive solution to the conjecture stated in Section 0.5 of Preliminaries would imply the validity of Theorem 1 for perturbations of real symmetric L.

As observed, the Riesz transforms associated to L are not Calderón-Zygmund operators; yet their kernels satisfy a Hörmander condition, and this yields parts of the L^p estimates in Theorem 1. Additional estimates are obtained via a non-standard factorization of $L^{1/2}$ which makes Calderón-Zygmund theory fully available. More precisely, we have

Theorem 2. — *Assume that L has the property (G) and that (K) holds. Then, there is a Calderón-Zygmund operator U such that*

$$L^{1/2}f = U\nabla f, \quad f \in H^1(\mathbb{R}^n).$$

There are extensions of these results in various directions.

As was mentioned before, all of this applies when A has coefficients depending on one variable. But more is true, namely that the missing L^p estimates are valid in this situation. This generalizes the one dimensional results obtained in **[14]**.

Going back to the general situation, the replacement of (K) and (K*) by the corresponding local inequalities gives us local L^p estimates with the same ranges of p's as in Theorem 1. An application of this combined with resolvent estimates of **[3]** enables us to prove

Theorem 3. — *Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$, where A has vmo coefficients. Assume also that the coefficients are real-valued if $n \geq 3$. Then, for $p \in (1,\infty)$, $L^{1/2}$ extends to a bounded operator from $W^{1,p}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with the estimates*

$$\begin{aligned} \|L^{1/2}f\|_p &\leq c_p(\|\nabla f\|_p + \|f\|_p), \\ \|\nabla f\|_p &\leq c'_p(\|L^{1/2}f\|_p + \|f\|_p). \end{aligned}$$

4.2. Standard factorization of $L^{1/2}$ and Riesz transforms associated to L

4.2.1. Generalities. — We start from

$$L^{1/2}f = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2 L} L f \, dt, \quad f \in \mathcal{D}(L). \tag{3}$$

Factoring out ∇f in Lf gives us

$$L^{1/2}f = T\nabla f, \quad f \in \mathcal{D}(L), \tag{4}$$

where

$$T = \int_0^\infty \theta_t \, \frac{dt}{t} \tag{5}$$

and $\theta_t \colon L^2(\mathbb{R}^n;\mathbb{C}^n) \to L^2(\mathbb{R}^n)$ is given by

$$\theta_t = -\frac{2}{\sqrt{\pi}} e^{-t^2L}\, t \operatorname{div} A. \tag{6}$$

Here, A stands for multiplication by $A(x)$ on $L^2(\mathbb{R}^n;\mathbb{C}^n)$ (whose elements are arranged as column vectors). More explicitely,

$$\theta_t F = -\frac{2}{\sqrt{\pi}} \sum_{k,\ell=1}^{n} e^{-t^2L}\, t \frac{\partial}{\partial x_k}(a_{k\ell}F_\ell).$$

Note that T^* is related to the Riesz transforms associated to L^* by the formal relation

$$T^* = A^*\nabla L^{*-1/2}$$

(see Section 4.2.2).

Our first task is to give a precise meaning to the factorization (4) of $L^{1/2}$.

Consider the subspace

$$N = \{\ F \in L^2(\mathbb{R}^n;\mathbb{C}^n)\ ;\ \operatorname{div} AF = 0\ \},$$

where the divergence is taken in the sense of distributions (actually, $\operatorname{div} AF$ is a distribution in $H^{-1}(\mathbb{R}^n)$).

***Lemma 4* (Hodge decomposition).** — *N is closed in $L^2(\mathbb{R}^n;\mathbb{C}^n)$, and the sum*

$$L^2(\mathbb{R}^n;\mathbb{C}^n) = \nabla \dot{H}^1(\mathbb{R}^n) + N,$$

is a topological direct sum.

Here, $\dot{H}^1(\mathbb{R}^n)$ is the space of distributions (modulo constants) having a square-integrable gradient, equipped with the inner product $\int_{\mathbb{R}^n} \nabla f \cdot \overline{\nabla g}$.

Proof. — Let $F \in L^2(\mathbb{R}^n;\mathbb{C}^n)$. By a classical representation theorem (see [**48**]), there is a unique $f \in \dot{H}^1(\mathbb{R}^n)$ such that

$$\forall\, g \in \dot{H}^1(\mathbb{R}^n) \qquad \int_{\mathbb{R}^n} AF \cdot \overline{\nabla g} = \int_{\mathbb{R}^n} A\nabla f \cdot \overline{\nabla g}.$$

Indeed, the right-hand side defines a coercive sesquilinear form on $\dot{H}^1(\mathbb{R}^n)$, while the left-hand side defines a conjugate linear functional. Moreover we have,

$$\|\nabla f\|_2 \le \frac{1}{\delta}\|A\|_\infty \|F\|_2 \le \frac{1}{\delta^2}\|F\|_2,$$

where δ is the ellipticity constant of the matrix $A(x)$.

In particular, we have

$$\int_{\mathbb{R}^n} A(\nabla f - F)\cdot\nabla\varphi = 0$$

for every test function φ, which means that $G = F - \nabla f$ belongs to N. We have obtained the required decomposition of F and the accompanying norm estimates. Its uniqueness follows by density of $C_0^\infty(\mathbb{R}^n)$ in $\dot{H}^1(\mathbb{R}^n)$. □

We now define the space $E = \nabla\mathcal{D}(L) + N$ as the subspace of those $F \in L^2(\mathbb{R}^n;\mathbb{C}^n)$ such that $F = \nabla f + G$, $f \in \mathcal{D}(L)$, $G \in N$. Set

$$\|F\|_E = \|f\|_2 + \|Lf\|_2 + \|G\|_2.$$

Thanks to the preceding lemma, $\|\ \|_E$ is a norm on E. Since the embeddings

$$\mathcal{D}(L) \hookrightarrow H^1(\mathbb{R}^n) \hookrightarrow \dot{H}^1(\mathbb{R}^n)$$

are dense, E is a dense subspace of $L^2(\mathbb{R}^n;\mathbb{C}^n)$. We shall use E as a space of test functions. It is not a classical space, and does not contain $C_0^\infty(\mathbb{R}^n)$ in general.

Lemma 5. — *If $F \in E$, the integral $\int_0^\infty \theta_t F\,\frac{dt}{t}$ converges normally in $L^2(\mathbb{R}^n;\mathbb{C}^n)$.*

Proof. — Let $F \in E$ and write $F = \nabla f + G$, where $f \in \mathcal{D}(L)$ and $G \in N$. Observe that

$$\theta_t G = -\frac{2}{\sqrt{\pi}}e^{-t^2L}\,t\operatorname{div} AG = 0,$$

(recall from Preliminaries that e^{-t^2L} extends to a bounded operator from $H^{-1}(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$) so that

$$\theta_t F = \theta_t\nabla f = \frac{2}{\sqrt{\pi}}e^{-t^2L}\,tLf.$$

Therefore, as in Section 0.1 of Preliminaries we have

$$\int_0^\infty \|\theta_t F\|_2\,\frac{dt}{t} \le c\int_0^\infty \min(t\|Lf\|_2, t^{-1}\|f\|_2)\,\frac{dt}{t} = c(\|Lf\|_2\|f\|_2)^{1/2}.$$

□

We can define the operator T on E by

$$TF = \int_0^\infty \theta_t F\,\frac{dt}{t},\quad F \in E,$$

and by construction

$$L^{1/2}f = T\nabla f,\quad f \in \mathcal{D}(L). \tag{7}$$

Now that T is defined, we immediately forget its construction and, following a classical procedure, we consider the truncated operators

$$T_\varepsilon = \int_\varepsilon^{1/\varepsilon} \theta_t\,\frac{dt}{t}, \tag{8}$$

where $0 < \varepsilon \leq 1$. They are continuous from $L^2(\mathbb{R}^n;\mathbb{C}^n)$ into $L^2(\mathbb{R}^n)$ with norm bounded by $O(|\ln\varepsilon|)$.

Proposition 6. — *The following assertions are equivalent.*

(i) $\|L^{1/2}f\|_2 \leq c\|\nabla f\|_2, \quad f \in \mathcal{D}(L)$.
(ii) *T extends boundedly from $L^2(\mathbb{R}^n;\mathbb{C}^n)$ to $L^2(\mathbb{R}^n)$.*
(iii) $\sup_{\varepsilon>0}|T_\varepsilon|_{2,2} < \infty$.

Moreover, if the assertions above hold, T is the strong limit of the operators T_ε as ε tends to 0 and the equality (7) holds on all of $H^1(\mathbb{R}^n)$.

Thanks to this proposition, one can do various calculations with the operators T_ε instead of T. Provided all the estimates hold uniformly in ε, they extend to T.

Proof. — Recall that L is ω-accretive for some $0 < \omega < \pi/2$ (see Sections 0.1 and 0.2 of Preliminaries). Let $\omega < \mu < \pi/2$. For $z \in \Gamma_\mu$ define

$$\psi_\varepsilon(z) = \frac{2}{\sqrt{\pi}}\int_\varepsilon^{1/\varepsilon} e^{-t^2z}tz^{1/2}\,\frac{dt}{t}.$$

Then we have

$$\sup_{0<\varepsilon\leq 1}\ \sup_{z\in\Gamma_\mu}|\psi_\varepsilon(z)| \leq c_\mu = (\cos\mu)^{-1/2},$$

and

$$\lim_{\varepsilon\to 0}\psi_\varepsilon(z) = 1$$

uniformly on compact subsets of Γ_μ. Hence by H^∞ functional calculus, if $f \in \mathcal{D}(L)$,

$$\|\psi_\varepsilon(L)L^{1/2}f\|_2 \leq cc_\mu\|L^{1/2}f\|_2, \tag{9}$$

and

$$\lim_{\varepsilon\to 0}\|\psi_\varepsilon(L)L^{1/2}f - L^{1/2}f\|_2 = 0.$$

Finally notice that

$$\psi_\varepsilon(L)L^{1/2} = T_\varepsilon\nabla \qquad \text{on } \mathcal{D}(L).$$

Let us now prove that (i) implies (iii). Take $F \in E$ and write $F = \nabla f + G$, where $f \in \mathcal{D}(L)$ and $G \in N$, so that

$$T_\varepsilon F = \psi_\varepsilon(L)L^{1/2}f.$$

By (9) and (i), we obtain

$$\|T_\varepsilon F\|_2 \leq cc_\mu\|L^{1/2}f\|_2 \leq c\|\nabla f\|_2 \leq c\|F\|_2.$$

We conclude on using the density of E.

To prove that (iii) implies (ii) observe that, by Lemma 5,

$$\lim_{\varepsilon\to 0}\|TF - T_\varepsilon F\|_2 = 0,$$

whenever $F \in E$. Hence, the hypothesis (iii) implies

$$\|TF\|_2 \leq c\|F\|_2.$$

Finally, that (ii) implies (i) is a consequence of (7). □

4.2.2. Negative results on Riesz transforms. — Recall that when A is real symmetric, L is selfadjoint. Therefore, (K) and (K*) hold so that, by Proposition 6, T extends continuously from $L^2(\mathbb{R}^n;\mathbb{C}^n)$ to $L^2(\mathbb{R}^n)$. The operator T is not in general a Calderón-Zygmund operator, because of the following result.

***Theorem 7* (Kenig).** — *For every $\alpha > 0$ there is a real symmetric $L = -\operatorname{div}(A\nabla)$ in $L^2(\mathbb{R}^2)$ such that the inequality*

$$||\nabla f||_p \le C||L^{1/2} f||_p \tag{10}$$

fails when $p \ge 2+\alpha$.

This means that the Riesz transforms associated to L are not bounded on L^p if p is large.

Corollary 8. — *Let L be as in Theorem 7. Then the operator T defined by (5) is not bounded on L^p if $p \le 2-\alpha(1+\alpha)^{-1}$. In particular, it is not a Calderón-Zygmund operator.*

Let us first prove the corollary. Since T is bounded on L^2, so is $\widetilde{T} = A^{-1}T^*$. We claim that for $f \in \mathcal{D}(L)$

$$\widetilde{T}L^{1/2}f = \nabla f. \tag{11}$$

This claim and Theorem 7 imply that $\widetilde{T}$ is not bounded on L^p if $p \ge 2+\alpha$. Hence, T cannot be bounded on $L^{p'}$ if $p' \le 2-\alpha(1+\alpha)^{-1}$.

To prove (11), let $f \in \mathcal{D}(L)$, and set

$$u_\varepsilon = \frac{2}{\sqrt{\pi}}\int_\varepsilon^{1/\varepsilon} e^{-t^2L}tL^{1/2}f\,\frac{dt}{t} = \psi_\varepsilon(L)f.$$

Arguing as in the proof of Proposition 6, we have that

$$\begin{aligned}\lim_{\varepsilon\to 0}||u_\varepsilon - f||_2 &= 0\\ \lim_{\varepsilon\to 0}||L^{1/2}u_\varepsilon - L^{1/2}f||_2 &= 0.\end{aligned}$$

Also, since L is selfadjoint,

$$||\nabla u_\varepsilon - \nabla f||_2 \le C||L^{1/2}u_\varepsilon - L^{1/2}f||_2,$$

hence ∇u_ε converges to ∇f in L^2.

Next, let $\widetilde{T}_\varepsilon = A^{-1}T_\varepsilon^*$. An easy calculation yields $\widetilde{T}_\varepsilon L^{1/2}f = \nabla u_\varepsilon$. As it follows from Proposition 6 that $\widetilde{T}_\varepsilon$ converges weakly to $\widetilde{T}$ when ε tends to 0, we obtain (11) by taking weak limits.

Now, we turn to the proof of Theorem 7. The starting point is Meyers example (see [38]). Let

$$A(x) = I + \frac{\beta(\beta+2)}{|x|^2}\begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix},$$

with $x = (x_1, x_2) \in \mathbb{R}^2$ and $\beta > -1$. The matrix $A(x)$ is definite positive with

$$\min(1, (1+\beta)^2)I \le A(x) \le \max(1, (1+\beta)^2)I,$$

for all $x \neq 0$, so that $L = -\operatorname{div}(A\nabla)$ is selfadjoint and that (K) and (K*) hold. It is an elementary computation to check that $u_0(x) = r^{1+\beta}\cos\theta$, where (r, θ) are polar coordinates of x, is a classical solution of L in $\mathbb{R}^2 \setminus \{0\}$ and a weak solution in $\mathbb{R}^2$. Moreover $|\nabla u_0| \sim r^\beta$ near 0 so that $\nabla u_0 \notin L^p_{loc}$ if $\beta p \le -2$.

Fix $\alpha > 0$ and choose $\beta \in (-1, -2(2+\alpha)^{-1}]$. If $B = B(0,1)$ is the unit ball, pick $\varphi \in C_0^\infty(B)$ with $\varphi = 1$ on $B(0, 1/2)$, and set $u = u_0\varphi$.

By construction, $\nabla u \notin L^p(\mathbb{R}^2)$ when $p \ge 2+\alpha$. We next show that $L^{1/2}u \in L^p$ for every $p > 2$ (in addition to being in L^2), which proves the theorem. This is done by writing

$$L^{1/2}u = L^{-1/2}Lu, \tag{12}$$

and using simple estimates on $f = Lu$ and $L^{-1/2}$.

We first compute f and find out that

$$f = 2A\nabla\varphi \cdot \nabla u_0 + u_0 L\varphi,$$

which implies that f is C^∞ with support contained in the annulus defined by $1/2 \le |x| \le 1$. Hence, $u \in \mathcal{D}(L)$ and (12) is justified. As a consequence, we have

$$L^{1/2}u = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2L}f\,dt. \tag{13}$$

By Aronson estimates (Theorem 4 of Chapter 1), there exist $c > 0$ and $\gamma > 0$ such that

$$S_t(x,y) \le \frac{c}{t^2}\exp\left\{-\gamma\frac{|x-y|^2}{t^2}\right\},$$

where $S_t(x,y)$ denotes the kernel of e^{-t^2L}. Using this estimate in (13), together with the support condition on f, one finds that

$$|L^{1/2}u(x)| \le \frac{C}{1+|x|},$$

which gives $L^p(\mathbb{R}^2)$-integrability if $p > 2$.

Remarks

1. The decay of $L^{1/2}u(x)$ found here is not optimal. Indeed, observing that $\int f = 0$, one can use the Hölder estimate on $S_t(x,y)$ to find that

$$|L^{1/2}u(x)| \le \frac{C}{1+|x|^s}$$

for some $s > 1$.

2. It is clear from the proof that the failure of L^p estimates for ∇u is a local property near 0. Let B coincide with A on the cube $(-2,2)^2$ and be periodic with period 4 in each direction. Then $L_B = -\operatorname{div}(B\nabla)$ is selfadjoint. Since u_0 is also a weak solution of L_B on $(-2,2)^2$, the same argument applies. This proves our assertion in the introduction concerning the result in [**2**].

4.2.3. Positive results on Riesz transforms. — Our aim in this section is to describe a positive counterpart to Corollary 8. It will be obtained assuming that L has the Gaussian property (G) of Chapter 1. For the convenience of the reader, let us recall what it means and extract a few consequences proved in Chapter 1.

Set $S_t = e^{-t^2L}$ and denote by $S_t(x,y)$ its distributional kernel (this means that $S_t(x,y) = K_{t^2}(x,y)$ where the notation $K_t(x,y)$ is used in Chapter 1). In this section, we assume

$$\text{(G)}\quad \begin{cases} \exists\, c,\eta,\alpha > 0 \quad \forall\, t > 0 \quad \forall\, x,y,h \in \mathbb{R}^n \quad 2|h| \leq t + |x-y| \\ |S_t(x,y)| \leq \dfrac{c}{t^n} \exp\left\{-\dfrac{\alpha|x-y|^2}{t^2}\right\}, & \text{(G1)} \\ |S_t(x,y) - S_t(x+h,y)| \leq \dfrac{c}{t^n}\left(\dfrac{|h|}{t+|x-y|}\right)^{\eta} \exp\left\{-\dfrac{\alpha|x-y|^2}{t^2}\right\}, & \text{(G2)} \\ |S_t(x,y+h) - S_t(x,y)| \leq \dfrac{c}{t^n}\left(\dfrac{|h|}{t+|x-y|}\right)^{\eta} \exp\left\{-\dfrac{\alpha|x-y|^2}{t^2}\right\}. & \text{(G3)} \end{cases}$$

Under such estimates, we know from Proposition 25 of Chapter 1 that

$$S_t 1 = 1, \quad t > 0. \tag{14}$$

Denoting by $\theta_t(x,y)$ the kernel of the operator θ_t defined by (6), we have

$$\theta_t(x,y) = \frac{2}{\sqrt{\pi}} A(y)^T\, t\nabla_y S_t(x,y).$$

It follows from Theorem 7 of Chapter 1 that there are constants $c,\beta,\mu > 0$ depending only on the constants in (G), the dimension n, and the ellipticity constant of A, such that

$$\forall\, t > 0 \quad \forall r > 0 \quad \forall\, x \in \mathbb{R}^n \quad \forall\, h \in \mathbb{R}^n \text{ with } 2|h| \leq r + t$$

$$\int_{r\leq|x-y|\leq 2r} |\theta_t(x,y)|\, dy \leq c\left(\frac{r}{t}\right)^{n-1} \exp\left\{-\frac{\beta r^2}{t^2}\right\} \tag{15}$$

$$\left(\int_{r\leq|x-y|\leq 2r} |\theta_t(x+h,y) - \theta_t(x,y)|^2\, dy\right)^{1/2} \leq \frac{c}{r^{n/2}}\left(\frac{r}{t}\right)^{n-1}\left(\frac{|h|}{t+r}\right)^{\mu} \exp\left\{-\frac{\beta r^2}{t^2}\right\}. \tag{16}$$

Moreover, θ_t cancels each column vector of A^{-1}, viewed as a vector-valued function, which we write

$$\theta_t(A^{-1}) = 0, \quad t > 0. \tag{17}$$

It is meaningful, since (see Proposition 24 of Chapter 1),

$$\int_{\mathbb{R}^n} |\theta_t(x,y)|\, dy \leq C$$

uniformly over $x \in \mathbb{R}^n$ (and $t > 0$), which implies that θ_t extends boundedly from $L^\infty(\mathbb{R}^n;\mathbb{C}^n)$ into $L^\infty(\mathbb{R}^n)$.

Inequality (16) is a property of regularity with respect to the x variable of the kernel of θ_t, and we are going to show that the operator T has a similar property. Without knowledge about L^2 boundedness of T, its kernel is not defined in the sense of distributions as T only operates on E, a space which may not contain smooth test functions, unless the matrix A itself is smooth. For the time being, it is enough to deal with the kernels of the truncated operators.

Lemma 9. — *Let $T_\varepsilon(x,y)$ be the distributional kernel of T_ε. Its restriction off the diagonal is a locally integrable function, and we have*

$$\exists\, c > 0 \quad \forall\, \varepsilon \in (0,1] \quad \forall\, r > 0 \quad \forall\, x \in \mathbb{R}^n \quad \forall\, h \in \mathbb{R}^n \text{ with } 2|h| \leq r$$

$$\left(\int_{r\leq|x-y|\leq 2r} |T_\varepsilon(x+h,y) - T_\varepsilon(x,y)|^2\, dy\right)^{1/2} \leq \frac{c}{r^{n/2}}\left(\frac{|h|}{r}\right)^\mu. \tag{18}$$

Here, μ is the same exponent as in (16).

Proof. — Since

$$T_\varepsilon(x,y) = \int_\varepsilon^{1/\varepsilon} \theta_t(x,y)\, \frac{dt}{t},$$

we obtain from (15) that, if $r > 0$ and $x \in \mathbb{R}^n$, then

$$\begin{aligned}\int_{r\leq|x-y|\leq 2r} |T_\varepsilon(x,y)|\, dy \quad &\leq \quad c\int_\varepsilon^{1/\varepsilon} \left(\frac{r}{t}\right)^{n-1} \exp\left\{-\frac{\beta r^2}{t^2}\right\} \frac{dt}{t} \\ &\leq \quad c\Gamma((n-1)/2)\beta^{-(n-1)/2},\end{aligned}$$

which proves local integrability. The inequality (18) is obtained similarly using (16) and Minkowski integral inequality. This finishes the proof. □

A first consequence of Lemma 9 is that, by Cauchy-Schwarz inequality, we have

$$\int_{r\leq|x-y|\leq 2r} |T_\varepsilon(x+h,y) - T_\varepsilon(x,y)|\, dy \leq c\left(\frac{|h|}{r}\right)^\mu. \tag{19}$$

Integrating this expression over r in $[2|h|,\infty)$ with respect to the measure dr/r yields

$$\int_{2|h|\leq|x-y|} |T_\varepsilon(x+h,y) - T_\varepsilon(x,y)|\, dy \leq c, \tag{20}$$

which is the standard Hörmander condition.

Assuming, furthermore, the (uniform in ε) boundedness on L^2 of T_ε, it is classical that (20) implies its (uniform in ε) boundedness on L^p, $2 \leq p < \infty$, and from L^∞ to BMO (see [**63**] or [**74**]).

In addition, the stronger inequality (18) combined with the cancellation property (17) enables us to prove that the operators $T_\varepsilon A$ are uniformly bounded from $BMO(\mathbb{R}^n; \mathbb{C}^n)$ to $BMO(\mathbb{R}^n)$.

Before entering into details, let us summarize these results and write down our first L^p estimates.

Proposition 10. — *Assume that L has the property (G) and that (K) and (K*) hold. Then, we have the a priori inequalities for $f \in \mathcal{D}(L^{1/2}) = H^1(\mathbb{R}^n)$:*

$$||L^{1/2}f||_p \leq c_p||\nabla f||_p, \qquad 2 \leq p < \infty, \tag{21}$$

and

$$||\nabla f||_p \leq c_p||L^{1/2}f||_p, \qquad 1 < p \leq 2, \tag{22}$$

with the endpoint estimates

$$||L^{1/2}f||_{BMO} \leq c||A\nabla f||_{BMO} \tag{23}$$

and

$$||\nabla f||_{\mathcal{H}^1(\mathbb{R}^n)} \leq c||L^{1/2}f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{24}$$

Remarks

1. (K*) is not used when proving (21) and (23) and, for a dual reason, (K) is not used when proving (22) and (24).
2. A proof of (21) and (22) that does not make use of the hypotheses (G2-3) (regularity in x and y) has been recently found by X.T. Duong and A. McIntosh [**32**]. In this case, the Hörmander condition is replaced by a weaker condition which still guarantees weak type 1-1. But the estimates (23) and (24) are unclear under this weaker condition. See also the related work [**24**] in the context of Riesz transforms for the Laplace-Beltrami operator on a Riemannian variety.

Proof. — For $1 \leq j \leq n$ and $\varepsilon \in (0, 1]$, set

$$W_{j,\varepsilon} = \int_\varepsilon^{1/\varepsilon} e^{-t^2L} t \frac{\partial}{\partial x_j} \frac{dt}{t}$$

so that for $F \in L^2(\mathbb{R}^n; \mathbb{C}^n)$,

$$T_\varepsilon F = -\frac{2}{\sqrt{\pi}} \sum_{j=1}^n W_{j,\varepsilon}(AF)_j,$$

where G_j denotes the jth component of a $\mathbb{C}^n$-valued function G.

If the inequality (K) holds, then $W_{j,\varepsilon}$ are uniformly bounded on $L^2(\mathbb{R}^n)$ by Proposition 6, and by (19), on L^p, $2 \leq p < \infty$, and from L^∞ to BMO. Moreover we have

$W_{j,\varepsilon}(1) = 0$ in BMO. The boundedness on BMO of $W_{j,\varepsilon}$ then follows from the next lemma (compare with Lemma 2.7 in [**27**]).

Lemma 11. — *Let K be a bounded operator on $L^2(\mathbb{R}^n)$. Assume that its distributional kernel $K(x,y)$ is locally square integrable off the diagonal and that, for some $c_0, \mu > 0$,*

$$\left(\int_{r\leq|x-y|\leq 2r} |K(x+h,y) - K(x,y)|^2\,dy\right)^{1/2} \leq \frac{c_0}{r^{n/2}}\left(\frac{|h|}{r}\right)^{\mu} \tag{25}$$

for all $r > 0$, $x, h \in \mathbb{R}^n$, $2|h| \leq r$. Assume that $K(1) = 0$ in BMO. Then K maps continuously BMO into itself, with norm bounded by $c(c_0 + |K|_{2,2})$.

Let us finish the proof of Proposition 10 before proving this lemma. So far, we have obtained (23), and (21) follows by interpolation with (K). For the remaining estimates, we proceed by duality. If T_{A^*} denotes the operator associated to L^* and A^* by (5) and (6), then $A\nabla = (T_{A^*})^* L^{1/2}$. Hence, as soon as (K*) holds, we have (22) and (24). □

To prove Lemma 11, it suffices to establish

$$|\langle Kf, a\rangle| \leq c\|f\|_{BMO} \tag{26}$$

for all atoms $a \in \mathcal{H}^1(\mathbb{R}^n)$ and all BMO functions f. By stantard approximation arguments, we may take f bounded (as long as we do not use its L^∞ norm quantitatively).

Let a be an atom, that is, a is bounded and compactly supported in a ball B, with $|B|\|a\|_\infty \leq 1$ and $\int_B a = 0$. Let $f \in L^\infty(\mathbb{R}^n)$ and set $f_{3B} = \dfrac{1}{|3B|}\displaystyle\int_{3B} f$, where $\lambda B = B(x_0, \lambda r)$ if $B = B(x_0, r)$. We have

$$\langle Kf, a\rangle = \langle K(f - f_{3B}), a\rangle$$

since $K(1) = 0$. Proceeding as usual, split $f - f_{3B}$ as $f_1 + f_2$ where $f_1 = f - f_{3B}$ on $3B$ and 0 elsewhere. The contribution of f_1 is controlled by the L^2 boundedness of K:

$$\begin{aligned}
|\langle Kf_1, a\rangle| &\leq |K|_{2,2}\|f_1\|_2\|a\|_2 \\
&\leq 3^{n/2}|K|_{2,2}\left(\frac{1}{|3B|}\int_{3B}|f_1|^2\right)^{1/2} \\
&\leq 3^{n/2}|K|_{2,2}\|f\|_{BMO}.
\end{aligned}$$

To estimate the contribution of f_2, we first use the oscillation of a to obtain

$$\langle Kf_2, a\rangle = \int_{x\in B}\int_{y\notin 3B} f_2(y)[K(x,y) - K(x_0,y)]\overline{a(x)}\,dydx,$$

and, if we set $C_k = \{y \in \mathbb{R}^n; 2^k r \le |x_0 - y| \le 2^{k+1} r\}$, by (25) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle Kf_2, a\rangle| &\le \sum_{k=1}^{\infty} \int_{x\in B}\int_{y\in C_k} |f_2(y)||K(x,y)-K(x_0,y)||a(x)|\,dydx\\ &\le c_0 \sum_{k=1}^{\infty}\left(\int_{C_k}|f_2|^2\right)^{1/2}(2^k r)^{-n/2}2^{-k\mu}\|a\|_1.\end{aligned}$$

Since $C_k \subset 2^{k+1}B$, the well known inequality

$$\frac{1}{|2^k B|}\int_{2^k B}|f - f_{3B}|^2 \le c(k+1)^2\|f\|_{BMO}^2$$

yields

$$|\langle Kf_2, a\rangle| \le cc_0 \sum_{k=1}^{\infty}(k+1)2^{-k\mu}\ \|f\|_{BMO}.$$

This concludes the proof of Lemma 11, and that of Proposition 10.

Remark. — The conclusion of Lemma 11 remains valid if the L^2 average in (25) is replaced by an L^p average for some $p > 1$.

4.3. Non standard factorization of $L^{1/2}$ by a Calderón-Zygmund operator

To derive more L^p estimates, we take advantage of the following observation. In the factorization

$$L^{1/2} = T\nabla,$$

only the action of T on the space $\nabla\mathcal{D}(L)$ is relevant (see Lemma 4), which leaves room for modifying T without affecting the product $T\nabla$. In other words, the above factorization is not unique: our point is that among all possible factorizations, there is a remarkable one. The next statement is a more precise version of Theorem 2.

Theorem 12. — *Assume that L has the property (G). Then, there is an operator U with Calderón-Zygmund kernel such that*

$$L^{1/2}f = U\nabla f, \quad f \in \mathcal{D}(L). \tag{27}$$

Furthermore, there is an equivalence between the following assertions.

(i) $\|L^{1/2}f\|_2 \le c\|\nabla f\|_2, \quad f \in \mathcal{D}(L)$.

(ii) *U has a bounded extension from $L^2(\mathbb{R}^n;\mathbb{C}^n)$ to $L^2(\mathbb{R}^n)$ (this extension being a Calderón-Zygmund operator).*

(iii) *$U(e_j) \in BMO(\mathbb{R}^n)$ for $1 \le j \le n$.*

Moreover, when the above assertions hold, (27) extends to all of $H^1(\mathbb{R}^n)$.

Here, e_j is the jth vector in the canonical basis for $\mathbb{C}^n$.

The construction of U is done by two different means, both requiring smoothing techniques.

Take a function $\varphi \in C_0^\infty(\mathbb{R}^n)$, supported in the unit ball and such that $\int \varphi = 1$ and $\int x_j\varphi = 0$ for $j = 1, \ldots, n$. If $t > 0$, set $\varphi_t(x) = t^{-n}\varphi(x/t)$, and $P_t f = f * \varphi_t$. As in the previous chapters, the action of P_t on vector-valued functions is meant to be componentwise.

Denote also by $R\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n; \mathbb{C}^n)$ the array of Riesz transforms (see the introduction of this chapter) given by

$$Rf = (R_1 f, \ldots, R_n f)^T.$$

Its adjoint $R^*\colon L^2(\mathbb{R}^n; \mathbb{C}^n) \to L^2(\mathbb{R}^n)$ is given by

$$R^*F = R_1^* F_1 + \cdots + R_n^* F_n.$$

We have that $RR^*\nabla = \nabla$ and by duality $\operatorname{div} RR^* = \operatorname{div}$ in the sense of distributions on $\mathbb{R}^n$. Moreover, RR^* is the orthogonal projector on $\nabla\dot{H}^1(\mathbb{R}^n)$.

The first definition of U is that of a linear operator from $C_0^\infty(\mathbb{R}^n)$ into its dual by

$$U = V + W, \tag{28}$$

where

$$V = \int_0^\infty \theta_t P_t \,\frac{dt}{t}, \tag{29}$$

and

$$W = \int_0^\infty \theta_t (I - P_t) RR^* \,\frac{dt}{t}. \tag{30}$$

We set $V_t = \theta_t P_t$ and $W_t = \theta_t(I - P_t)RR^*$. The key result is the following lemma.

Lemma 13. — *The operators V_t and W_t are bounded from $L^2(\mathbb{R}^n; \mathbb{C}^n)$ to $L^2(\mathbb{R}^n)$ uniformly for $t > 0$; their kernels $V_t(x, y)$ and $W_t(x, y)$ are continuous functions, and there exist constants $c, \gamma, \mu > 0$ depending only on the constants in (G), the dimension n, the function φ and the ellipticity constant of A, such that for all $t > 0$, $x, y, h \in \mathbb{R}^n$, with $|h| \le t$,*

$$\begin{aligned} |V_t(x,y)| &\le \frac{c}{t^n} \exp\left\{-\frac{\gamma|x-y|^2}{t^2}\right\}, \\ |W_t(x,y)| &\le \frac{c}{t^n}\left(1 + \frac{|x-y|}{t}\right)^{-n-1}, \end{aligned} \tag{31}$$

$$\begin{aligned} |V_t(x+h,y) - V_t(x,y)| &\le \frac{c}{t^n}\left(\frac{|h|}{t}\right)^\mu \exp\left\{-\frac{\gamma|x-y|^2}{t^2}\right\}, \\ |W_t(x+h,y) - W_t(x,y)| &\le \frac{c}{t^n}\left(\frac{|h|}{t}\right)^\mu \left(1 + \frac{|x-y|}{t}\right)^{-n-1}, \end{aligned} \tag{32}$$

$$\begin{aligned} |\nabla_y V_t(x,y)| &\leq \frac{c}{t^{n+1}} \exp\left\{-\frac{\gamma|x-y|^2}{t^2}\right\}, \\ |\nabla_y W_t(x,y)| &\leq \frac{c}{t^{n+1}} \left(1+\frac{|x-y|}{t}\right)^{-n-2}. \end{aligned} \tag{33}$$

Proof. — The L^2 boundedness of V_t and W_t is obvious.

The estimates for the kernel of V_t are easy. We have

$$V_t(x,y) = \int \theta_t(x,z)\varphi_t(z-y)\, dz.$$

If $|x-y| \leq 4t$, we have

$$|V_t(x,y)| \leq t^{-n}\|\varphi\|_\infty \int |\theta_t(x,z)|\, dz \leq ct^{-n}.$$

If $|x-y| \geq 4t$ then $3|x-y|/4 \leq |x-z| \leq 5|x-y|/4$ on the support of the integral. Using (15) gives us

$$|V_t(x,y)| \leq ct^{-n}\|\varphi\|_\infty \exp\left\{-\frac{9\beta|x-y|^2}{16t^2}\right\}.$$

Hence, (31) for $V_t(x,y)$ follows. The Hölder estimate (32) is dealt with similarly by applying (16) instead of (15). The y-gradient estimate is obtained as (31), replacing φ_t by $\nabla\varphi_t = t^{-1}(\nabla\varphi)_t$.

To derive the estimates on W_t, we recall the following identity (see Lemma 20 of Chapter 2).

Lemma 14. — $\theta_t \nabla = -\dfrac{1}{\sqrt{\pi}}\dfrac{dS_t}{dt}$, *where* $S_t = e^{-t^2L}$.

This means that, while $\theta_t(x,y)$ has no regularity with respect to y, its y-divergence, the kernel of $-t\theta_t\nabla$, satisfies (G) (see Lemma 19 of Chapter 1).

Applying Lemma 14, and using the fact that P_t and R commute, we obtain

$$W_t = \theta_t(I-P_t)RR^* = \theta_t\nabla(-\Delta)^{-1/2}(I-P_t)R^* = -\frac{1}{\sqrt{\pi}}\, t\,\frac{dS_t}{dt}\, Q_t, \tag{34}$$

where

$$Q_t = (-t^2\Delta)^{-1/2}(1-P_t)R^*.$$

This is a bounded convolution operator bounded from $L^2(\mathbb{R}^n;\mathbb{C}^n)$ to $L^2(\mathbb{R}^n)$ given by $Q_tF = \sum_j (\psi_j)_t * F_j$ where $(\psi_j)_t(x) = \dfrac{1}{t^n}\psi_j(x/t)$ and ψ_j is defined by its Fourier transform

$$\widehat{\psi_j}(\xi) = \frac{-i\xi_j(1-\widehat{\varphi}(\xi))}{|\xi|^2}.$$

Using the moment conditions on φ, standard Littlewood-Paley analysis gives us

$$|\psi_j(x)| \leq \frac{c}{|x|^{n-1}(1+|x|)^2} \tag{35}$$

and

$$|\nabla\psi_j(x)| \leq \frac{c}{|x|^n(1+|x|)^2}. \tag{36}$$

By (34), one can represent $W_t(x,y)$ as

$$W_t(x,y) = -\frac{t}{\sqrt{\pi}}\int \frac{\partial S_t(x,z)}{\partial t}\psi_t(z-y)\,dz.$$

Using the estimates (G1) and (G2) for $t\partial S_t(x,z)/\partial t$ and the size estimate (35) just obtained for ψ_j, one easily checks (31) and (32) for $W_t(x,y)$. We skip details. The y-gradient estimate is a little more subtle.

If $h \in \mathbb{R}^n$, we have

$$W_t(x,y+h) - W_t(x,y) = \\ -\frac{t}{\sqrt{\pi}}\int\left[\frac{\partial S_t(x,z)}{\partial t} - \frac{\partial S_t(x,y)}{\partial t}\right]\left[\psi_t(z-y-h) - \psi_t(z-y)\right]dz$$

since ψ is integrable and $\int\psi = 0$. Using (G1) and (G2) for $t\partial S_t(x,y)/\partial t$ and (36), the Lebesgue dominated convergence theorem shows that $\nabla_y W_t(x,y)$ exists everywhere and

$$\nabla_y W_t(x,y) = \frac{t}{\sqrt{\pi}}\int\left[\frac{\partial S_t(x,z)}{\partial t} - \frac{\partial S_t(x,y)}{\partial t}\right]\nabla(\psi_t)(z-y)\,dz,$$

the integral being absolutely convergent. Hence, $|\nabla_y W_t(x,y)|$ is dominated by

$$\frac{c}{t}\int_{2|z-y|\leq t+|x-y|}\frac{1}{t^n}\left(\frac{|z-y|}{t+|x-y|}\right)^{\eta}\frac{e^{-\alpha|x-y|^2/t^2}}{|z-y|^n(1+|z-y|/t)^2}\,dz \\ +\frac{c}{t}\int_{2|z-y|\geq t+|x-y|}\frac{e^{-\alpha|x-z|^2/t^2}+e^{-\alpha|x-y|^2/t^2}}{t^n|z-y|^n(|z-y|/t)^2}\,dz \\ \leq \frac{c}{t^{n+1}}\left(1+\frac{|x-y|}{t}\right)^{-\eta}e^{-\alpha|x-y|^2/t^2} + \frac{c}{t^{n+1}}\left(1+\frac{|x-y|}{t}\right)^{-n-2}.$$

This ends the proof of Lemma 13. □

Lemma 15. — *We have the following cancellation properties.*

$$V_t^*(1) = 0 \tag{37}$$

$$W_t^*(1) = 0 \textit{ and } W_t(e_j) = 0, \quad j = 1,\dots,n. \tag{38}$$

The first two equalities follow from (14), which yields $\partial S_t^*(1)/\partial t = \theta_t^*(1) = 0$, while the last one is a consequence of $Q_t(e_j) = \int(\psi_j)_t = 0$.

By standard results on singular integrals **[26]**, **[63]**, one deduces from Lemma 13 and Lemma 15 that V and W are defined as continuous operators from $C_0^\infty(\mathbb{R}^n;\mathbb{C}^n)$ into $(C_0^\infty(\mathbb{R}^n))'$ by the absolutely convergent integral

$$\langle VF, g\rangle = \int_0^\infty \langle V_t F, g\rangle\,\frac{dt}{t}$$

and the similar equation for W. Summarizing, we have obtained

Lemma 16

(i) *V is a weakly bounded singular integral operator, with $V^*(1) = 0$.*

(ii) *W extends to a Calderón-Zygmund operator, with $W^*(1) = W(e_j) = 0$, for $1 \le j \le n$.*

Hence, we deduce from the $T1$-Theorem **[26]** the

Corollary 17. — *The operator $U = V + W$ extends to a bounded operator from $L^2(\mathbb{R}^n;\mathbb{C}^n)$ to $L^2(\mathbb{R}^n)$ if and only if $U(e_j) \in BMO$ for all $1 \le j \le n$.*

In order to explain the relation between $L^{1/2}$ and U let us now give the second definition of U. To this end, we use the following lemma.

Lemma 18. — *Let $U_t \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $t > 0$, be bounded operators whose distributional kernels $U_t(x,y)$ are measurable functions that satisfy*

$$|U_t(x,y)| \le \frac{1}{t^n}\left(1 + \frac{|x-y|}{t}\right)^{-n-1} \tag{39}$$

and

$$\int U_t(x,y)\,dx = 0. \tag{40}$$

Then

$$\int_0^\infty |\langle U_t f, g\rangle| \,\frac{dt}{t} \le c(n)\|f\|_2(|g|_{1/2} + \|g\|_1).$$

Here, $|g|_{1/2} = \sup_{x\neq y} \dfrac{|g(x)-g(y)|}{|x-y|^{1/2}}$.

Proof. — First, using (39), one has

$$\begin{aligned} |\langle U_t f, g\rangle| &= \left|\iint U_t(x,y) f(y)\overline{g(x)}\,dydx\right| \\ &\le \int \left(\int |U_t(x,y)|^2\,dy\right)^{1/2} \|f\|_2 |g(x)|\,dx \\ &\le ct^{-n/2}\|f\|_2\|g\|_1. \end{aligned}$$

Next, using (40), one has $U_t^*(g)(x) = \int \overline{U_t(y,x)}(g(y) - g(x))\,dy$, which implies

$$|U_t^*(g)(x)| \le \int t^{-n}\left(1 + \frac{|x-y|}{t}\right)^{-n-1} |g(y) - g(x)|\,dy.$$

Hence, by (39), $\|U_t^* g\|_\infty \le ct^{1/2}|g|_{1/2}$. Since $\|U_t^* g\|_1 \le c\|g\|_1$, one finds that

$$\|U_t^* g\|_2 \le ct^{1/4}(|g|_{1/2}\|g\|_1)^{1/2} \le 2ct^{1/4}(|g|_{1/2} + \|g\|_1)$$

and, therefore,

$$|\langle U_t f, g\rangle| = |\langle f, U_t^* g\rangle| \le ct^{1/4}\|f\|_2(|g|_{1/2} + \|g\|_1).$$

The conclusion follows readily. □

Applying this lemma, we see that for $F \in L^2(\mathbb{R}^n;\mathbb{C}^n)$, the linear functional

$$g \to \int_0^\infty \langle V_tF + W_tF, \overline{g}\rangle \frac{dt}{t}$$

is continuous on $C_0^\infty(\mathbb{R}^n)$ and defines a distribution which we call $\widetilde{U}F$. The application $\widetilde{U}\colon F \to \widetilde{U}F$ is linear and continuous from $L^2(\mathbb{R}^n;\mathbb{C}^n)$ to $(C_0^\infty(\mathbb{R}^n))'$.

It is clear from their constructions that U and $\widetilde{U}$ agree on $C_0^\infty(\mathbb{R}^n;\mathbb{C}^n)$. Thus, if Corollary 17 applies the extension of U must be $\widetilde{U}$. So it is no harm not to distinguish them from now on.

With this in hand, one can establish (27). Let $f \in \mathcal{D}(L)$. If $t > 0$, using that $RR^*\nabla = \nabla$, it is clear that $(V_t+W_t)\nabla f = \theta_t\nabla f$, hence for all $\varepsilon > 0$ and $g \in C_0^\infty(\mathbb{R}^n)$,

$$\int_\varepsilon^{1/\varepsilon} \langle V_t\nabla f + W_t\nabla f, g\rangle \frac{dt}{t} = \langle T_\varepsilon \nabla f, g\rangle.$$

By Lemma 5 and the definition of U, passing to the limit as ε tends to 0 proves (27).

Let us turn to the proof of the equivalence between (i), (ii) and (iii) in Theorem 12.

The equivalence between (ii) and (iii) follows from Corollary 17. That (ii) implies (i) is obvious. We are left with the implication (i) $\Rightarrow$ (iii).

If (i) holds, then T is L^2-bounded. Its kernel is well defined in the sense of distributions; it is the limit in $(C_0^\infty(\mathbb{R}^{2n}))'$ of $T_\varepsilon(x,y)$ and, therefore, satisfies the Hörmander condition (20). Hence, T maps L^∞ to BMO. Since $U(e_j) = T(e_j)$, we have obtained $U(e_j) \in BMO$, which proves (iii).

It remains to check that (27) extends to $H^1(\mathbb{R}^n)$ when assertion (i) holds. This assertion implies that the domain of $L^{1/2}$ contains $H^1(\mathbb{R}^n)$. So, for $f \in H^1(\mathbb{R}^n)$, the equality $T_\varepsilon\nabla f = \psi_\varepsilon(L)L^{1/2}f$ used in Proposition 6 makes sense. Letting ε tend to 0, the left hand side converges to $T\nabla f$ in $L^2(\mathbb{R}^n)$ by Proposition 6 while the right hand side converges to $L^{1/2}f$. This proves that (27) holds on $H^1(\mathbb{R}^n)$ and the proof of Theorem 12 is finished.

Remark. — A consequence of Lemma 18 and of (27) is that, whenever L has the Gaussian property (G), $L^{1/2}$ extends to a continuous linear operator from $H^1(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$.

4.4. Further L^p estimates and invertibility of $L^{1/2}$

We are now in a position to add new L^p estimates to those given in Proposition 10.

Proposition 19. — *Assume that L has the property (G) and that (K) holds. Then, we have the a priori inequalities for $f \in H^1(\mathbb{R}^n)$:*

$$||L^{1/2}f||_p \leq c_p||\nabla f||_p, \qquad 1 < p < \infty, \tag{41}$$

and

$$||L^{1/2}f||_{\mathcal{H}^1(\mathbb{R}^n)} \leq c||\nabla f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{42}$$

Proof. — By Theorem 12, the assumptions imply that U is L^2 bounded. By Calderón-Zygmund theory, it is bounded on L^p for $1 < p < \infty$. Furthermore, since $U^*(1) = 0$ it is bounded on $\mathcal{H}^1(\mathbb{R}^n)$. The conclusion of Proposition 19 is now immediate from equality (27) on $H^1(\mathbb{R}^n)$. □

Remark. — The operator W is always L^2 bounded. Indeed, by (34) and Cauchy-Schwarz inequality, we have

$$\int_0^\infty |\langle W_t f, g\rangle| \frac{dt}{t} \leq \frac{1}{\sqrt{\pi}} \left(\int_0^\infty ||Q_t f||_2^2 \frac{dt}{t}\right)^{1/2} \left(\int_0^\infty ||t\frac{dS_t^*}{dt}g||_2^2 \frac{dt}{t}\right)^{1/2}.$$

Thus, we have to control quadratic functionals as in Chapter 2. But the one given by Q_t is bounded thanks to (2) of Chapter 2 and the other one is bounded using the maximal accretivity of L^* and (4) of Preliminaries. Of course, without further assumption and, in particular, the property (G), W may not be a Calderón-Zygmund operator.

At this point, almost all estimates stated in Theorem 1 have been obtained. The remaining ones are in the next result.

Proposition 20. — *Assume that L has the property (G) and that (K) and (K*) hold. Then, there exists $\varepsilon > 0$ such that for $|1/2 - 1/p| < \varepsilon$,*

$$||\nabla f||_p \leq C||L^{1/2}f||_p, \quad f \in H^1(\mathbb{R}^n). \tag{43}$$

We denote by $\dot{W}^{1,p}(\mathbb{R}^n)$ the homogeneous Sobolev space of the distributions f defined modulo constants by $||\nabla f||_p < \infty$. If $p = 2$, $\dot{W}^{1,2}(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n)$.

Proof. — Since (K) and (K*) hold, we have $||\nabla f||_2 \sim ||L^{1/2}f||_2$ and the domain of $L^{1/2}$ is $H^1(\mathbb{R}^n)$ (Proposition 3 of Preliminaries). By the previous results, we have $L^{1/2} = U\nabla$ on $H^1(\mathbb{R}^n)$ and U is a Calderón-Zygmund operator. Thus, $L^{1/2}$ extends to a bounded operator from $\dot{W}^{1,p}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. The inequality (43) then follows on applying Sneiberg's result (Lemma 23 of Chapter 1). □

We now study invertibility properties of $L^{1/2}$.

Theorem 21. — *Assume that L has the property (G) and that (K) and (K*) hold. Then, $L^{1/2}$ extends to a continuous operator from $\dot{W}^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$, which is invertible for $1 < p < 2 + \varepsilon$ for some $\varepsilon > 0$.*

Proof. — We have already seen that $L^{1/2}$ extends to a continuous operator from $\dot{W}^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Furthermore, combining Proposition 10 and Proposition 20, we have for some $\varepsilon > 0$,

$$\|\nabla f\|_p \leq C\|L^{1/2}f\|_p, \quad f \in \dot{W}^{1,p}(\mathbb{R}^n) \cap \dot{H}^1(\mathbb{R}^n), \quad 1 < p < 2+\varepsilon.$$

Thus, the extension of $L^{1/2}$ is one-one from $\dot{W}^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for this range of p's.

To see that it is onto, it now suffices to show that it has dense range. Let $g \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Since $g \in L^2(\mathbb{R}^n)$, we have $g = L^{1/2}L^{-1/2}g$ by functional calculus. By (22) and (43) we know that $\|\nabla L^{-1/2}g\|_p \leq c\|g\|_p < \infty$, and this shows that g is in $L^{1/2}(\dot{W}^{1,p}(\mathbb{R}^n))$. □

Remarks

1. Consider the space of tempered distributions modulo constants f such that $\nabla f \in \mathcal{H}^1(\mathbb{R}^n)$, equipped with the semi-norm $\|\nabla f\|_{\mathcal{H}^1(\mathbb{R}^n)}$. This space coincides with the Triebel-Lizorkin space $\dot{F}_1^{1,2}(\mathbb{R}^n)$ **[79]**. Since test functions are dense in $\dot{F}_1^{1,2}(\mathbb{R}^n)$, the same argument as above shows that $L^{1/2}$ extends to a bounded and invertible operator from $\dot{F}_1^{1,2}(\mathbb{R}^n)$ onto $\mathcal{H}^1(\mathbb{R}^n)$.
2. Call $(L^{1/2})_p$ the extension of $L^{1/2}$ on $W^{1,p}(\mathbb{R}^n)$ for $1 < p < 2+\varepsilon$. There is another way of defining this extension. Assume that L has the property (G). Then, for $1 < p < \infty$, the semigroup e^{-tL} extends to a C^0-semigroup on $L^p(\mathbb{R}^n)$ and its infinitesimal generator, which is denoted by $-L_p$, is an unbounded operator of type ω on $L^p(\mathbb{R}^n)$ **[77]**. Thus one can define its square root on $L^p(\mathbb{R}^n)$, which we denote by $(L_p)^{1/2}$ with domain D_p **[1]**. It is easy to show that D_p is the range of the bounded extension on $L^p(\mathbb{R}^n)$ of $(I+L)^{-1/2}$. Hence, Theorem 21 implies that $D_p = W^{1,p}(\mathbb{R}^n)$ for $1 < p < 2+\varepsilon$ and that $(L_p)^{1/2} = (L^{1/2})_p$. The situation for p large is not settled: neither D_p is known nor can one define appropriately an extension of $L^{1/2}$. See Section 4.7.4 for further information.

4.5. Coefficients depending on one variable

If $L = -\frac{d}{dx}\big(a(x)\frac{d}{dx}\big)$, then $a(x)\frac{d}{dx}L^{-1/2}$ is a Calderón-Zygmund operator **[14]**. In particular $\|df/dx\|_p \leq c_p\|L^{1/2}f\|_p$ for all $p \in (1,\infty)$. This L^p inequality, false in general for large p's in higher dimensions, is valid in specific situations such as the following one.

Theorem 22. — *Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ on $\mathbb{R}^n$ and assume that A depends only on one of the coordinate variables. Then, for $1 < p < \infty$,*

$$\|\nabla f\|_p \sim \|L^{1/2}f\|_p, \tag{44}$$

and $L^{1/2}$ extends to an isomorphism from $\dot{W}^{1,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof. — There is no loss of generality to assume that the entries $a_{k\ell}$ of A are functions of x_1 where $(x_1, x_2, \dots, x_n)$ are rectangular coordinates in $\mathbb{R}^n$. Introduce the vector fields

$$X_1 = a_{11}(x_1)\,\frac{\partial}{\partial x_1} + \cdots + a_{1n}(x_1)\,\frac{\partial}{\partial x_n}$$

and, for $2 \le k \le n$, $X_k = \partial/\partial x_k$, and set $X = (X_1, \dots, X_n)$. Note that $Xf = M\nabla f$ where, by the ellipticity of A, M is bounded with bounded inverse. Thus, there exists $c > 0$ such that

$$c^{-1}|Xf| \le |\nabla f| \le c|Xf|, \quad a.e. \tag{45}$$

for all f for which ∇f exists almost everywhere.

The result will follow from

Proposition 23. — *Under the assumptions of Theorem 22, the operators $X_k L^{-1/2}$, $1 \le k \le n$, are Calderón-Zygmund operators.*

Indeed, Calderón-Zygmund theory and (45) imply, for $1 < p < \infty$, that $\|\nabla f\|_p \le c\|Xf\|_p \le cc_p\|L^{1/2}f\|_p$. The reverse inequality is already known from Theorem 1. Eventually, the extension of $L^{1/2}$ to an isomorphism can be done as in the proof of Theorem 21. □

Proof of Proposition 23. — It is interesting to note that among the three operators $\nabla L^{-1/2}$, $A\nabla L^{-1/2}$ and $M\nabla L^{-1/2}$, only the last one is a Calderón-Zygmund operator. This is due to some natural algebraic relation on the heat kernel and justify the introduction of the vector fields X_k.

Set $T_k = X_k L^{-1/2}$. We have

$$T_k = \frac{2}{\sqrt{\pi}} \int_0^\infty t X_k e^{-t^2 L}\,\frac{dt}{t}.$$

The key point is that we have good kernel estimates.

Lemma 24. — *Under the assumptions of Theorem 22, for all $1 \le k \le n$, the operators $tX_k e^{-t^2L}$ satisfy (G1-3).*

This lemma is proved in Appendix B.

It classically implies that the off-diagonal restriction of the kernel of T_k is a Calderón-Zygmund kernel **[26]**. Thus, it remains to show that T_k is bounded on $L^2(\mathbb{R}^n)$. We give two proofs of this fact.

The first one makes use of Proposition 11 of Chapter 3, which tells us that $\|\nabla f\|_2 \sim \|L^{1/2}f\|_2$. The L^2 boundedness of T_k then follows from (45).

In the second proof we establish the L^2 boundedness of T_k as follows. Assume first that $2 \le k \le n$. Since $X_k = \partial/\partial x_k$ and $e^{-t^2L}1 = 1$, it is easy to see that $T_k 1 = T_k^* 1 = 0$. Hence, T_k is bounded by invoking the $T1$-Theorem.

Next, for the same reasons we have $T_1 1 = 0$. Observe that $b = \overline{a_{11}^{-1}}$ is a bounded and accretive function on $\mathbb{R}^n$ and, by a simple calculation, that

$$T_1^*(b) = \sum_{k=2}^{n} T_k^*(\overline{a_{1k}}b).$$

Now, this sum is in BMO because we already know that for $k \geq 2$, T_k^* extends boundedly from L^∞ to BMO. Hence $T_1^*(b) \in BMO$ and the boundedness of T_1 follows by invoking the Tb-Theorem **[27]**. □

4.6. Local L^p estimates

In this section, we are interested in the local inequalities corresponding to (1) and (2). Namely, for $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$, we consider the *a priori* inequalities

$$||L^{1/2}f||_p \leq c_p(||\nabla f||_p + ||f||_p), \tag{46}$$

$$||\nabla f||_p \leq c'_p(||L^{1/2}f||_p + ||f||_p). \tag{47}$$

The strategy to obtain such inequalities is basically the same as before with some modifications that need to be explained.

Let us begin with a simple observation.

Lemma 25. — *Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ and assume that L has the Gaussian property (G). Then, for all $p \in (1,\infty)$ there is a constant $C = C(p,\delta) > 0$ such that*

$$C^{-1}||(L+1)^{1/2}f||_p \leq ||L^{1/2}f||_p + ||f||_p \leq C||(L+1)^{1/2}f||_p \tag{48}$$

for all $f \in \mathcal{D}(L^{1/2})$.

Remark. — The assumption $(\mathrm{G})_{loc}$ would not suffice to obtain both inequalities.

Proof. — This lemma relies on the following result. The bounded holomorphic functional calculus for L on $L^2(\mathbb{R}^n)$ extends to $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$. This means that for all $\pi > \mu > \omega$ (see Preliminaries), there exists a constant $c = c(\mu,p) > 0$ such that for all $\varphi \in H^\infty(\Gamma_\mu)$

$$||\varphi(L)f||_p \leq c||f||_p, \quad f \in L^p \cap L^2(\mathbb{R}^n). \tag{49}$$

This fact can be proved as in Theorem 6.3 of **[11]** by showing that $\varphi(L)$ is a Calderón-Zygmund operator. An alternative approach is to invoke a general result of Duong and Robinson **[33]**.

Applying twice (49) to $\varphi(z) = (z^{1/2}+1)(z+1)^{-1/2}$ and to its inverse readily yields (48) and the proof of the lemma is finished. □

Set $D = (\nabla, \mathrm{Id})$. Since $||Df||_p$ is equivalent to $||\nabla f||_p + ||f||_p$, when this lemma applies, (46) and (47) become respectively equivalent to

$$||(L+1)^{1/2} f||_p \leq C_p ||Df||_p, \tag{50}$$

$$||Df||_p \leq C_p ||(L+1)^{1/2} f||_p. \tag{51}$$

Note that when $p = 2$, $(\mathrm{K})_{loc}$ is precisely (50) and $(\mathrm{K}^*)_{loc}$ is equivalent to (51) by duality.

Now recall from Chapter 2 that we can factor $L + 1$ as $D^* \widetilde{A} D$ where $\widetilde{A}$ is the multiplication by the matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. It is then just a matter of adapting *mutatis mutandis* the methods developed in Sections 4.2, 4.3 and 4.4 of this chapter to the new situation. For this we need to assume (G), $(\mathrm{K})_{loc}$ and $(\mathrm{K}^*)_{loc}$ and we obtain that (50) holds for all $p \in (1, \infty)$ and (51) for $p \in (1, 2 + \varepsilon)$ for some $\varepsilon > 0$.

Next, let us assume that, in addition to the preceding hypotheses, for a given $q \in (1, \infty)$, $(1 + L)^{-1}$ extends to a bounded operator from $W^{-1,q}(\mathbb{R}^n)$ into $W^{1,q}(\mathbb{R}^n)$ with the estimate

$$||(1+L)^{-1} f||_{W^{1,q}} \leq C_q ||f||_{W^{-1,q}}, \quad f \in L^2(\mathbb{R}^n) \cap W^{-1,q}(\mathbb{R}^n). \tag{52}$$

We claim that

$$||Df||_q \leq C ||(L+1)^{1/2} f||_q, \quad f \in H^1(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n).$$

Indeed, let $f \in L^2(\mathbb{R}^n) \cap W^{-1,q'}(\mathbb{R}^n)$, where q' is the dual exponent of q. Since $(L^* + 1)^{-1/2} f = (L^* + 1)^{1/2} (L^* + 1)^{-1} f$, by (50) for L^* with $p = q'$, one has that

$$||(L^* + 1)^{-1/2} f||_{q'} \leq c ||D(L^* + 1)^{-1} f||_{q'} \leq c C_q ||f||_{W^{-1,q'}}$$

using (52) and duality. Hence $(L + 1)^{-1/2}$ extends continuously from $L^q(\mathbb{R}^n)$ into $W^{1,q}(\mathbb{R}^n)$. Finally, since the domain of $(L + 1)^{1/2}$ is $H^1(\mathbb{R}^n)$, we obtain the desired inequality.

Summarizing the above discussion, we have obtained the following result.

Theorem 26. — *Assume $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ has the property (G) and that $(K)_{loc}$ and $(K^*)_{loc}$ hold. Then there is $\varepsilon > 0$ such that*

$$||L^{1/2} f||_p \quad \leq c_p (||\nabla f||_p + ||f||_p), \quad 1 < p < \infty, \tag{53}$$

$$||\nabla f||_p \quad \leq c'_p (||L^{1/2} f||_p + ||f||_p), \quad 1 < p < 2 + \varepsilon. \tag{54}$$

If, in addition, for a given $q \in (1, \infty)$, $(1 + L)^{-1}$ extends to a bounded operator from $W^{-1,q}(\mathbb{R}^n)$ into $W^{1,q}(\mathbb{R}^n)$, then (54) holds for this value of q.

Remark. — By combining these estimates appropriately, one sees that the converse of the last statement is true if, in addition, (54) also holds for L^* with $p = q'$.

Let us finish with a statement that encompasses Theorem 3.

Corollary 27. — *Let $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$, where A has vmo coefficients. Assume also that the coefficients are real-valued if $n \geq 3$. Then, for $p \in (1,\infty)$, $L^{1/2}$ (resp. $(L+1)^{1/2}$) extends to a bounded (resp. bounded and invertible) operator from $W^{1,p}(\mathbb{R}^n)$ into (resp. onto) $L^p(\mathbb{R}^n)$ with the estimates*

$$||L^{1/2}f||_p + ||f||_p \sim ||\nabla f||_p + ||f||_p.$$

Remark. — The hypotheses of this statement are selfadjoint.

Proof. — The hypothesis that A be real if $n \geq 3$ ensures the validity of the Gaussian estimates on L by the results of Chapter 1. Next, $(\mathrm{K})_{loc}$ and its adjoint follow from Chapter 3. For the resolvent estimate, we quote the following result from **[3]**.

Lemma 28. — *Assume $A \in \mathcal{A}(\delta)$ has vmo coefficients. Then, for all $p \in (1,\infty)$*

$$||(1 - \operatorname{div}(A\nabla))^{-1}f||_{W^{1,p}} \leq c_p ||f||_{W^{-1,p}}.$$

The proof of the corollary is complete provided the invertibility of $(L+1)^{1/2}$ is established, which can be done as in the proof of Theorem 21 and is left to the reader. □

Remark. — Theorem 26 is known under the additional hypothesis that A has Hölder continuous entries. In this case, further L^p-Sobolev results can be obtained. See **[11]** for statements and proofs.

4.7. Miscellaneous results

4.7.1. Perturbation theory. — We have shown in Section 0.5 of Preliminaries a strong continuity result for the map $A \to (-\operatorname{div}(A\nabla))^{1/2}$ defined on real symmetric elliptic matrices.

The real variable representation of square roots given by Theorem 2 and the perturbation results for heat kernels (Theorem 6, (ii), of Chapter 1) have the following consequence for this map.

Proposition 29. — *Let $f \in H^1(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then, the map $A \to \langle L^{1/2}f, \varphi\rangle$ is analytic on $\mathcal{G}$.*

Recall that $\mathcal{G}$ is the class defined in Section 1.6 of Chapter 1 of all complex $A \in \mathcal{A}$ for which $L = -\operatorname{div}(A\nabla)$ has the Gaussian property (G) and that $\mathcal{G}$ is open in $\mathcal{A}$. It follows from Theorem 4 of Chapter 1 that this result applies to real symmetric elliptic matrices. Of course, this is a weak analyticity result as $L^{1/2}$ is considered here as an element of the space of bounded linear applications from $H^1(\mathbb{R}^n)$ into the space of distributions on $\mathbb{R}^n$ (see the remark after Lemma 18). One can recover from it the strong continuity result mentioned above but it gives no hints so far to whether the conjecture formulated in Section 0.5 of Preliminaries holds.

The proof of this proposition is similar to that of Theorem 8 of Preliminaries. The only difference is in the topology put on square roots. We leave the details to the reader.

4.7.2. Extensions to higher order operators. — If

$$L = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha\beta}\partial^\beta)$$

is an elliptic homogeneous differential operators of order $2m$ with bounded coefficients, satisfying the Gårding inequality

$$\operatorname{Re}\langle Lf, f\rangle \geq \delta \|\nabla^m f\|_2^2$$

and for which (K) and (K*) hold (see Preliminaries), the same questions about the L^p counterparts may be formulated. It turns out that an entirely similar strategy that uses the estimates of Section 1.7 in Chapter 1 gives results similar to Theorems 1 and 2. More precisely, the statement of these results generalize to such L, where A becomes the multiplication with $(a_{\alpha\beta})$ and where $H^m(\mathbb{R}^n)$ and ∇^m replace respectively $H^1(\mathbb{R}^n)$ and ∇.

If the Gårding inequality occurs with a factor $-C\|f\|_2^2$ with a non-negative C in the right hand side then the local analogs of these results are valid: they correspond to the ones established in Section 4.6 for second order operators.

One can also develop the L^p theory of square roots for elliptic systems along the same lines.

4.7.3. L^p theory and bilipschitz changes of variables. — Let us go back to second order operators. The L^p theory also can be transferred under a bilipschitz change of variables. From the discussion in Section 0.8 of Preliminaries, it suffices to extend our results to square roots of mL, where $L = -\operatorname{div}(A\nabla)$ and m is a bounded non-negative function with bounded inverse (in fact, all what follows generalizes to the case where m is a bounded and accretive function on $\mathbb{R}^n$).

We claim that the L^p estimates of Theorem 1 and Theorem 2 hold for $(mL)^{1/2}$ in lieu of $L^{1/2}$. Let us give a hint, leaving details to the reader. We assume that (G) holds for L and that $\|(mL)^{1/2}f\|_2 \sim \|\nabla f\|_2$.

First, starting from (5) of Preliminaries we have

$$(mL)^{1/2}f = \frac{2}{\pi}\int_0^\infty (1+t^2mL)^{-1} mLf\, dt = \int_0^\infty \theta_t^{(m)}(\nabla f)\,\frac{dt}{t},$$

where $\theta_t^{(m)} = -\dfrac{2}{\pi}(m^{-1}+t^2L)^{-1} t \operatorname{div} A$.

As observed in a remark of Section 1.5.1 in Chapter 1, $\theta_t^{(m)}$ has the kernel estimates that make the methods of Section 4.3 applicable. We obtain that

$$(mL)^{1/2} = U\nabla, \tag{55}$$

where U is a Calderón-Zygmund operator, which gives part of the L^p-inequalities. For the converse inequalities, the starting point is

$$\nabla(mL)^{-1/2}f = \frac{2}{\pi}\int_0^\infty \nabla(m^{-1}+t^2L)^{-1}(m^{-1}f)\,\frac{dt}{t} \tag{56}$$

and L^p-estimates for $1 < p \leq 2$ follow from L^2-inequalities and Hörmander's condition on the kernel of $\nabla(mL)^{-1/2}m$. The improvement from 2 to $2+\varepsilon$ is obtained as in Proposition 20.

For the limiting cases $p = 1$ and $p = \infty$, we need to understand the cancellations. Observe that

$$(m^{-1}+t^2L^*)^{-1}(m^{-1}) = 1, \quad t > 0,$$

hence $U^*(m^{-1}) = 0$. This can be used to show that U maps $\mathcal{H}^1(\mathbb{R}^n)$ into $m\mathcal{H}^1(\mathbb{R}^n)$, the image of $\mathcal{H}^1(\mathbb{R}^n)$ under the operator of multiplication by m, and thus

$$\|(mL)^{1/2}f\|_{m\mathcal{H}^1(\mathbb{R}^n)} \leq c\|\nabla f\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

The converse inequality also holds using (56).

Finally, using the same method as the one in Section 4.2 one has

$$\|(mL)^{1/2}f\|_{BMO(\mathbb{R}^n)} \leq c\|A\nabla f\|_{BMO(\mathbb{R}^n)}.$$

4.7.4. Estimates on Morrey spaces and application. — We go back to the situation where $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$.

Let us define the homogeneous Morrey space $\dot{M}^\gamma$, $0 \leq \gamma \leq n$, as the space of $f \in L^2_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{M}^\gamma} := \sup\left(\rho^{-\gamma}\int_{B_\rho}|f|^2\right)^{1/2} < \infty, \tag{57}$$

where the supremum is taken over all Euclidean balls B_ρ of radius $\rho > 0$ (compare with Chapter 1).

It should be noticed that for $p > 2$, Hölder inequality yields the embedding

$$L^p(\mathbb{R}^n) \subset \dot{M}^\gamma(\mathbb{R}^n) \quad \text{if and only if} \quad \gamma = n\left(1-\frac{2}{p}\right).$$

While L^p estimates for square roots fails when $p > 2$, we do have substitute estimates on Morrey spaces.

First, a result of M. Taylor asserts that Calderón-Zygmund operators are bounded on homogeneous Morrey spaces **[78]**. Therefore, we have the following result as a corollary of Theorem 2.

Proposition 30. — *Assume that L has the property (G) and that (K) holds. Then, for all* $0 < \gamma < n$,

$$\|L^{1/2}f\|_{\dot{M}^\gamma} \leq c\|\nabla f\|_{\dot{M}^\gamma}.$$

For the converse, we have

Proposition 31. — *Assume that L has the property (G) and that (K*) holds or, equivalently,* $||\nabla L^{-1/2} f||_2 \leq c||f||_2$. *Then for some* $\gamma_0 \in (n-2, n]$ *and all* $0 < \gamma < \gamma_0$, *we have*

$$||\nabla L^{-1/2} f||_{\dot{M}^\gamma} \leq c||f||_{\dot{M}^\gamma}.$$

A consequence is a better understanding of the L^p domain of $L^{1/2}$ (see Section 4.4).

Corollary 32. — *Under the assumptions of Proposition 31, there exists a real* $p_0 > n$ *such that if* $g \in L^p(\mathbb{R}^n)$ *and* $L^{1/2} g \in L^p(\mathbb{R}^n)$ *with* $n < p < p_0$, *then* $g \in \dot{C}^\eta(\mathbb{R}^n)$ *for* $\eta = 1 - n/p$.

Proof. — Define p_0 by $\gamma_0 = n(1-2/p_0)$. If $n < p < p_0$, we obtain that

$$||g||_{\dot{C}^\eta} \leq c||\nabla g||_{\dot{M}^\gamma} \leq c||L^{1/2} g||_{\dot{M}^\gamma} \leq c||L^{1/2} g||_p,$$

where $\gamma = n(1-2/p)$ and $\eta = (\gamma + 2 - n)/2$. This proves the corollary. □

To prove Proposition 31 we need the following lemma.

Lemma 33. — *If L has the property (G), then the heat kernel* $K_t(x, y)$ *satisfies the following estimate: there exists* $\gamma_0 \in (n-2, n]$, *such that for all* $0 < \gamma < \gamma_0$, $\nabla_x K_t(x, y) \in \dot{M}^\gamma(\mathbb{R}^n)$ *and there are non negative constants* $c, \alpha > 0$ *such that*

$$\int_{B_\rho(x_0)} |t\nabla_x K_{t^2}(x, y)|^2 \, dx \leq \frac{c}{t^n}\left(\frac{\rho}{t}\right)^\gamma e^{-\alpha|x_0 - y|^2/t^2},$$

for all $x_0, y \in \mathbb{R}^n$ *and* $\rho, t > 0$ *provided* $2\rho \leq |x_0 - y|$

Proof. — In the course of the argument done in Chapter 1 that to show (D) implies (G), we have seen that, if k is large enough and if λ belongs to an appropriate sector, $\nabla(\lambda + L)^{-k}$ is bounded from $L^1(\mathbb{R}^n)$ into $\dot{M}^\gamma(\mathbb{R}^n)$ for all $\gamma \in (0, \gamma_0)$ for some $\gamma_0 > n-2$. Integrating this in the Cauchy formula (61) of Chapter 1 gives the same boundedness result for ∇e^{-tL}. Taking into account homogeneity with respect to $t > 0$, this is equivalent to an estimate of the form

$$\int_{B_\rho(x_0)} |t\nabla_x K_{t^2}(x, y)|^2 \, dx \leq \frac{c}{t^n}\left(\frac{\rho}{t}\right)^\gamma.$$

(Note the exponent $\gamma > n-2$: compare with (13) of Chapter 1.) The Gaussian decay now comes by interpolating with (13) in Chapter 1 which changes γ to an arbitrary smaller value. This proves the lemma. □

Proof of Proposition 31. — Fix $0 < \gamma < \gamma' < \gamma_0$, $f \in \dot{M}^\gamma$ and a ball $B_\rho(x_0)$ centered at x_0. Write

$$f = f_0 + f_1 + f_2 + \cdots,$$

where $f_0 = f$ on $B_0 = B_{2\rho}(x_0)$ and $f_0 = 0$ elsewhere, and if $j \geq 1$, $f_j = f$ on $B_j = B_{2^{j+1}\rho}(x_0) \setminus B_{2^j\rho}(x_0)$ and $f_j = 0$ elsewhere. The assumption on f implies $\|f_j\|_2^2 \leq c(\rho 2^j)^\gamma$. To prove (57), it suffices to establish that for all $j \geq 0$,

$$\left(\int_{B_\rho(x_0)} |\nabla L^{-1/2} f_j|^2\right)^{1/2} \leq c\rho^{\gamma/2} 2^{-j\beta} \tag{58}$$

for some $\beta > 0$ independent of j.

For $j = 0$, this follows from the L^2-boundedness of $\nabla L^{-1/2}$.

Suppose that $j \geq 1$. Since $\nabla L^{-1/2} = c\int_0^\infty t\nabla e^{-t^2L}\, dt/t$, from Minkowski integral inequality and Lemma 33 with γ', we deduce that

$$\begin{aligned}\left(\int_{B_\rho(x_0)} |\nabla L^{-1/2} f_j|^2\right)^{1/2} & \\ &\leq c\int_0^\infty \int_{B_j} \left(\int_{B_\rho(x_0)} |t\nabla_x K_{t^2}(x,y)|^2\, dx\right)^{1/2} |f_j(y)|\, dy \frac{dt}{t} \\ &\leq c\int_0^\infty \int_{B_j} \frac{c}{t^{n/2}}\left(\frac{\rho}{t}\right)^{\gamma'/2} e^{-\alpha|x_0-y|^2/2t^2} |f_j(y)|\, dy \frac{dt}{t}.\end{aligned}$$

Next, for $y \in B_j$, $|x_0 - y| \sim \rho 2^j$, hence integrating in t and then using Hölder's inequality, we obtain a bound

$$c(\rho 2^j)^{-n/2} 2^{-j\gamma'/2} \int_{B_j} |f_j(y)|\, dy \leq c 2^{-j\gamma'/2} (\rho 2^j)^{\gamma/2},$$

which proves (58) and the proof is finished. □

APPENDIX A

THE SPACE ABMO

This space $ABMO$ has been defined in Chapter 3. We give the proofs of results announced there.

A.1. Proof of Proposition 15 of Chapter 3

That (ii) implies (iii) is evident.

Let us prove that (iii) implies (i). Let $f \in BMO$. Fix $\varepsilon > 0$ and $Q \in \mathcal{Q}_0$. Take b_Q and h_Q as in (iii). Now, let Q_i, $i \in I$, be a finite family of nonoverlapping subcubes of Q with $\ell(Q_i) \leq \eta\ell(Q)$. By Poincaré inequality

$$\int_{Q_i} |h_Q - m_{Q_i} h_Q|^2 \leq c\ell(Q_i)^2 \int_{Q_i} |\nabla h_Q|^2 \leq c\eta^2 \ell(Q)^2 \int_{Q_i} |\nabla h_Q|^2.$$

On the other hand,

$$\int_{Q_i} |b_Q - m_{Q_i} b_Q|^2 \leq 2 \int_{Q_i} |b_Q|^2,$$

hence

$$\sum_{i \in I} \int_{Q_i} |f - m_{Q_i} f|^2 \leq 2c\eta^2 \ell(Q)^2 \int_Q |\nabla h_Q|^2 + 4 \int_Q |b_Q|^2 \leq (2c\eta^2 + 4\varepsilon)|Q|$$

by definition of b_Q and h_Q, and (i) follows readily.

We now assume that $f \in ABMO$ and prove (ii). Fix a cube $Q \in \mathcal{Q}_0$ and normalize f by imposing $m_Q f = 0$. By scaling there is no loss of generality to assume that $Q = [0,1)^n$. Consider the dyadic cubes Q_{jk}, $j \geq 0$ and $k \in \mathbb{Z}^n$, and $(S_j f)(x) = m_{Q_{jk}} f$ whenever $x \in Q_{jk} \subset Q$ and $S_j f = 0$ elsewhere. By assumption, for any $\varepsilon > 0$, there exists j which is chosen once for all such that

$$\int_Q |f - S_j f|^2 = \sum_{Q_{jk} \subset Q} \int_{Q_{jk}} |f - m_{Q_{jk}} f|^2 \leq \varepsilon. \tag{1}$$

We show that for some $t_0 > 0$ depending only on ε and f, $\int_Q |f - P_t f|^2 \le 50\varepsilon$ if $t < t_0$.

To this end, write

$$f - P_t f = (I - P_t)(f - S_j f) + (I - P_t)(S_j f).$$

If χ_{jk} is the characteristic function of the cube Q_{jk}, the function $(I - P_t)\chi_{jk}$ is bounded uniformly with respect to t and j and has support contained in the set of $x \in \mathbb{R}^n$ such that $d(x, Q_{jk}) \le t$ (recall that P_t has a kernel supported in $|x - y| \le t$). Hence

$$\int |(I - P_t)\chi_{jk}|^2 \le ct2^{-j(n-1)} = ct2^j |Q_{jk}|.$$

If t is small the functions $(I - P_t)\chi_{jk}$ have supports with finite overlap. Hence,

$$\begin{aligned}\int |(I - P_t)(S_j f)|^2 &\le c \sum_k \int |(I - P_t)(m_{Q_{jk}} f \chi_{jk})|^2 \\ &\le ct2^j \sum_k |m_{Q_{jk}} f|^2 |Q_{jk}| \\ &\le ct2^j \sum_k \int_{Q_{jk}} |f|^2 \\ &\le ct2^j \int_Q |f|^2.\end{aligned}$$

Next, with χ being the charateristic function of Q, write

$$(I - P_t)(f - S_j f) = f - S_j f - P_t(\chi(f - S_j f)) - P_t((1 - \chi)f).$$

In the last term we have used the fact $S_j f = 0$ outside of Q. Thus, by (1), the L^2-boundedness of P_t and localization, we have

$$\int_Q |(I - P_t)(f - S_j f)|^2 \le 6\varepsilon + 3 \int |P_t((1 - \chi)f)|^2 \le 6\varepsilon + 3 \int_{(1+t)Q \setminus Q} |f|^2.$$

At this point, it is useful to state the following result.

Lemma 1. — *Let $f \in BMO$, $Q \in \mathcal{Q}$ and $0 < t < 1$. Then,*

$$\int_{dist(x,\partial Q) \le t\ell(Q)} |f(x) - m_Q f|^2 \, dx \le c\, t\, (|\ln t| + 1)|Q| \|f\|_{BMO}^2,$$

where c depends only on dimension.

Applying this result and collecting all the estimates, we have

$$\int_Q |f - P_t f|^2 \le 12\varepsilon + ct2^j \|f\|_{BMO}^2 + c(|\ln t| + 1)t\|f\|_{BMO}^2,$$

and choosing t small enough gives us (ii).

Proof of Lemma 1. — Decompose the set $\operatorname{dist}(x,\partial Q) \le t\ell(Q)$ as a non-overlapping union of $O(t^{1-n})$ cubes Q_i with sidelength $t\ell(Q)$ and adjacent to Q. Classical BMO inequalities yields for each i,

$$\int_{Q_i} |f(x) - m_Q f|^2 \le c(|\ln t| + 1)|Q_i| \|f\|_{BMO}^2,$$

and, since $|Q_i| = t^n|Q|$, we conclude by summing over all i's. □

A.2. Littlewood-Paley characterization of ABMO

We take $Q_t = -t\dfrac{dP_t}{dt}$.

Proposition 2. — *Let $f \in BMO$. The following statements are equivalent.*

(i) $f \in ABMO$.

(iv) *For any $\varepsilon > 0$, there is $\eta > 0$ such that for all $Q \in \mathcal{Q}_0$,*

$$\frac{1}{|Q|}\int_0^{\eta\ell(Q)}\int_Q |Q_t f(x)|^2 \frac{dxdt}{t} \le \varepsilon.$$

(v) *There exists a non-increasing weight $w\colon [0,1) \to [0,\infty)$ such that*

$$\lim_{t\to 0} w(t) = \infty$$

and

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |Q_t f(x)|^2 w\Big(\frac{t}{\ell(Q)}\Big)\frac{dxdt}{t} < \infty.$$

Remarks

1. In **[15]**, we took (v) as a definition. We gave no proofs there.
2. As usual, these characterizations are independent of the choice of Q_t.

Proof. — That (v) implies (iv) is evident. Conversely, assume that (iv) holds. To each $j \ge 0$, apply (iv) with $\varepsilon = 2^{-j}$ and call (η_j) the sequence thus obtained. With no loss of generality, it can be assumed non-increasing and converging to 0. The desired weight in (v) is defined by $w(s) = 2^{j/2}$ when $\eta_{j+1} \le s < \eta_j$.

Next, we prove that (iii) of Proposition 15 in Chapter 3 implies (iv). Let $\varepsilon > 0$ and $Q \in \mathcal{Q}_0$. Construct b_{3Q} and h_{3Q} by (iii) (the fact that $3Q$ may not be in $\mathcal{Q}_0$ is not a major difficulty and we ignore it). By standard Littlewood-Paley analysis and localization, we have

$$\int_0^{\ell(Q)}\int_Q |Q_t h_{3Q}(x)|^2 \frac{1}{t^2}\frac{dxdt}{t} \le c\int_{3Q} |\nabla h_{3Q}|^2,$$

hence

$$\int_0^{\eta\ell(Q)}\int_Q |Q_t h_{3Q}(x)|^2 \frac{dxdt}{t} \le c\eta^2\ell(Q)^2\int_{3Q} |\nabla h_{3Q}|^2 \le c\eta^2|Q|$$

by definition of h_{3Q}. Next,

$$\int_0^{\ell(Q)}\int_Q |Q_t b_{3Q}(x)|^2 \frac{dxdt}{t} \leq c \int_{3Q} |b_{3Q}|^2 \leq c\varepsilon|Q|$$

and (iv) follows from $f = b_{3Q} + h_{3Q}$ on $3Q$.

Finally, we prove that (iv) implies (ii) with a modified approximation to the identity $(\tilde{P}_t)_{t>0}$ defined by

$$I - \tilde{P}_t = \int_0^t Q_s^2 \frac{ds}{s}.$$

Let $Q \in \mathcal{Q}_0$ and χ be the characteristic function of $2Q$. Observe that since Q_s has a kernel supported by $|x-y| \leq s$, we have

$$Q_s^2 f(x) = Q_s(\chi Q_s f)(x), \quad x \in Q, s \leq \ell(Q).$$

Hence, using standard Littlewood-Paley techniques and $\langle , \rangle$ as the inner product on $L^2(\mathbb{R}^n)$, if $t \leq \ell(Q)$, we have

$$\begin{aligned} \int_Q |f - \tilde{P}_t f|^2\, dx &\leq \int_{\mathbb{R}^n} \left| \int_0^t Q_s(\chi Q_s f)(x) \frac{ds}{s} \right|^2 dx \\ &\leq \int_0^t \int_0^t \langle Q_s^* Q_u(\chi Q_u f), \chi Q_s f\rangle \frac{ds}{s}\frac{du}{u} \\ &\leq c \int_0^t \int_0^t \inf\left(\frac{s}{u}, \frac{u}{s}\right) \|\chi Q_u f\|_2 \|\chi Q_s f\|_2 \frac{ds}{s}\frac{du}{u} \\ &\leq c \int_0^t \int_{\mathbb{R}^n} |\chi(x) Q_s f(x)|^2 \frac{dxds}{s}, \end{aligned}$$

where the last inequality follows from Schur lemma.

Now, let $\varepsilon > 0$ and $\eta > 0$ given by (iv). Splitting $2Q$ into 2^n cubes with sidelength $\ell(Q)$, we can apply the inequality in (iv) to each of these cubes. Choosing $t = \eta\ell(Q)$ in the preceding calculations, we obtain

$$\int_Q |f - \tilde{P}_t f|^2 \leq c\varepsilon 2^n |Q|$$

which is the desired inequality. Proposition 2 is proved. □

A.3. A subclass of ABMO

We consider Example 3 of *ABMO* functions and Proposition 23 of Chapter 3.

Proposition 3. — *Let $f \in BMO$. Then the following assertions are equivalent*

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|} \int_{|h|\le \ell(Q)} \int_Q \frac{|f(x+h)-f(x)|^2}{|h|^n}\, dxdh < \infty. \tag{a}$$

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |Q_t f(x)|^2 \ln\left(\frac{\ell(Q)}{t}\right) \frac{dxdt}{t} < \infty. \tag{b}$$

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |f(x)-P_t f(x)|^2 \frac{dxdt}{t} < \infty. \tag{c}$$

The condition (b) is clearly a special case of (v) in Proposition 2. So it follows from this lemma that any f satisfying (a) or (c) is in $ABMO$. Also, the equivalence between (a) and (c) gives us a proof of Proposition 23 in Chapter 3.

Proof. — We begin with some localisation arguments. Let $f \in BMO$ and set, for each cube Q, $f_Q(x) = f(x) - m_Q f$ when $x \in Q$ and $f_Q(x) = 0$ otherwise.

First, (a) is equivalent to

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|} \int_{|h|\le \ell(Q)} \int_{\mathbb{R}^n} \frac{|f_Q(x+h)-f_Q(x)|^2}{|h|^n}\, dxdh < \infty. \tag{a'}$$

Indeed, for each Q, the error terms are controlled by integrals of the form

$$\frac{c}{|Q|} \int_{|h|\le \ell(Q)} \int_{\mathrm{dist}(x,\partial Q)\le |h|} |f(x)-m_Q f|^2 \frac{dxdh}{|h|^n}$$

which, by Lemma 1, are bounded by

$$\frac{c}{|Q|} \int_{|h|\le \ell(Q)} \frac{|h|}{\ell(Q)} (\ln(\frac{\ell(Q)}{|h|})+1)|Q| \frac{dh}{|h|^n}\, \|f\|_{BMO}^2 = c\|f\|_{BMO}^2.$$

Next, (b) is equivalent to

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_{\mathbb{R}^n} |Q_t f_Q(x)|^2 \ln\left(\frac{\ell(Q)}{t}\right) \frac{dxdt}{t} < \infty. \tag{b'}$$

Using the the fact that the kernel of Q_t is supported in $|x-y| \le t$ and that Q_t is uniformly bounded on $L^2(\mathbb{R}^n)$, the error terms can be shown to be controlled by

$$\frac{c}{|Q|} \int_0^{\ell(Q)} \int_{\mathrm{dist}(x,\partial Q)\le t} |f(x)-m_Q f|^2 \ln\left(\frac{\ell(Q)}{t}\right) \frac{dxdt}{t},$$

and again, Lemma 1 gives a bound $c\|f\|_{BMO}^2$.

Finally, by similar arguments, (c) is equivalent to

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_{\mathbb{R}^n} |f_Q(x)-P_t f_Q(x)|^2 \frac{dxdt}{t} < \infty. \tag{c'}$$

It remains to prove that (a'), (b') and (c') are equivalent conditions. Since, by definition of BMO, we have

$$\sup_{Q\in\mathcal{Q}_0} \frac{1}{|Q|} \int_{\mathbb{R}^n} |f_Q(x)|^2 dx \le \|f\|_{BMO}^2,$$

this follows from the next result and rescaling applied to each f_Q.

Lemma 4. — *For $f \in L^2(\mathbb{R}^n)$, set*

$$q_1(f) = \int_{|h|\le 1}\int_{\mathbb{R}^n} \frac{|f(x+h)-f(x)|^2}{|h|^n}\,dxdh,$$

$$q_2(f) = \int_0^1\int_{\mathbb{R}^n} |Q_tf(x)|^2|\ln t|\frac{dxdt}{t},$$

$$q_3(f) = \int_0^1\int_{\mathbb{R}^n} |f(x)-P_tf(x)|^2\frac{dxdt}{t}.$$

Then, the quantities $q_i(f)+\|f\|_2^2$, $i=1,2,3$, are equivalent (in the sense of norms).

The proof of this lemma consists in observing that each of these quantities is, via Plancherel theorem, equivalent to

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 m(\xi)\,d\xi,$$

where $m(\xi)$ is a continuous functions satisfying $m(\xi)\sim 1$ near 0 and $m(\xi)\sim \ln|\xi|$ near ∞.

This concludes the proof of Proposition 3. □

A.4. Further examples

Example 4 in Section 3.3.2 of Chapter 3. — We consider a function f that is continuous and homogeneous of degree 0 on $\mathbb{R}^n\setminus\{0\}$.

To show that $f\in ABMO$, pick a cube $Q\in\mathcal{Q}_0$ and remark that we may use any of the conditions (i) to (v) on Q since the various implications are local (more precisely, the proofs show that any of the conditions from (i) to (v) valid on Q implies the other ones on subcubes of Q).

If $0\notin 3Q$, we use condition (ii). Observe that since f is homogeneous, we may assume by a linear change of variables that $\ell(Q)=1$. In such a case, elementary geometry gives us that $2Q\subset E=\{\,x\in\mathbb{R}^n; 1/2\le|x|\,\}$. Now, for $t\le\ell(Q)$ and $x\in Q$, using the support condition on the kernel of P_t, we have

$$|f(x)-(P_tf)(x)|\le w(t)=\sup\{\,|f(x)-f(y)|; x,y\in E \text{ and } |x-y|\le t\,\}.$$

Hence,

$$\int_Q |f(x)-P_tf(x)|^2\,dx\le cw(t)^2,$$

and since f is uniformly continuous on E, this tends to 0 with t.

If $0\in 3Q$, we use condition (iv). Again, the observation on homogeneity applies and we may assume that $\ell(Q)=1/4$. Then we have that Q is contained in the cube

$Q_0 = [-1/2, 1/2]^n$ and it suffices to estimate

$$\int_0^1 \int_{Q_0} |Q_t f(x)|^2 w(t) \frac{dxdt}{t}$$

for a convenient weight.

To this end, remark that since $f \in L^\infty$,

$$\int_0^1 \int_{Q_0} |Q_t f(x)|^2 \frac{dxdt}{t} \leq c$$

by Littlewood-Paley theory. To complete the proof, it suffices to apply the following lemma to the function $g(t) = \frac{1}{t}\int_{Q_0} |Q_t f(x)|^2 \, dx$.

Lemma 5. — *For any $g \in L^1([0,1])$, there exists a non-increasing weight $w\colon [0,1] \to [0,\infty)$ with $\lim_{t\to 0} w(t) = \infty$ and*

$$\int_0^1 |g(t)| w(t) \, dt \leq 2 \int_0^1 |g(t)| \, dt.$$

Proof. — Assume $\int_0^1 |g(t)| \, dt = 1$ and select a non-increasing sequence t_j by

$$\int_0^{t_j} |g(t)| \, dt = 4^{-j}.$$

If $\lim t_j > 0$, then g vanishes in a neighborhood of 0 and the existence of w is trivial. Otherwise, we define w by setting $w(t) = 2^j$ when $t_{j+1} < t \leq t_j$. Thus, $\lim_{t\to 0} w(t) = \infty$ and

$$\int_0^1 |g(t)| w(t) \, dt = \sum_{j=0}^{\infty} 2^j \int_{t_j+1}^{t_j} |g(t)| \, dt \leq \sum_{j=0}^{\infty} 2^j 4^{-j} = 2.$$

□

Example 6 in Section 3.3.2 of Chapter 3. — Let f be the characteristic function of the set E defined by

$$\bigcup_{n\geq 1} \left(\frac{1}{(2n+1)^\alpha}, \frac{1}{(2n)^\alpha} \right)$$

where $\alpha > 0$ is fixed. Let $\eta > 0$ be arbitrary. Choose $n_o \in \mathbb{N}$ with

$$\frac{2\alpha}{2n_o - 1} \leq \eta.$$

Choose

$$I = \left[0, \frac{1}{(2n_o - 1)^\alpha}\right] \quad \text{and} \quad I_n = \left[\frac{1}{(2n+1)^\alpha}, \frac{1}{(2n-1)^\alpha}\right], \quad n \geq n_o.$$

We have, if $n \leq n_o$,

$$\ell(I_n) \leq \frac{2\alpha}{(2n-1)^{\alpha+1}} \leq \eta \ell(I),$$

and it is easy to see that

$$\int_{I_n} |f - m_{I_n} f|^2 dx = c_n \ell(I_n),$$

where

$$\lim_{n\to\infty} c_n = 1.$$

Thus, if n_1 is chosen large enough, we have

$$\sum_{n=n_o}^{n_1} \int_{I_n} |f - m_{I_n} f|^2 dx \geq \frac{1}{2} \sum_{n=n_o}^{n_1} \ell(I_n) \geq \frac{1}{4} \ell(I).$$

This proves that $f \notin ABMO$.

For $g(x) = \sin(x^{-1/\alpha})$, we proceed analogously by choosing

$$I = \left[0, \frac{1}{(n_o\pi)^\alpha}\right] \quad \text{and} \quad I_n = \left[\frac{1}{((n+1)\pi)^\alpha}, \frac{1}{(n\pi)^\alpha}\right], \quad n \geq n_o.$$

Easy calculations show that $m_{I_n} g \sim 2(-1)^n/\pi$ as n increases indefinitely so that

$$\int_{I_n} |g - m_{I_n} g|^2 dx = c_n \ell(I_n),$$

with $c_n \sim \int_0^\pi |\sin x - 2/\pi|^2\, dx$. The conclusion follows as before.

APPENDIX B

COEFFICIENTS DEPENDING ON ONE VARIABLE

This appendix is concerned with the proof of Lemma 24 in Chapter 4, dealing with $L \in \mathcal{E}$ having coefficients depending only on the first coordinate variable. With the notation of this chapter this amounts to showing that the kernel $M_t(x,y)$ of tXe^{-t^2L} satisfies

$$|M_t(x,y)| \leq \frac{c}{t^n} \exp\left\{-\frac{\alpha|x-y|^2}{t^2}\right\} \tag{1}$$

$$|M_t(x,y) - M_t(x+h,y)| \leq \frac{c}{t^n}\left(\frac{|h|}{t+|x-y|}\right)^{\eta} \exp\left\{-\frac{\alpha|x-y|^2}{t^2}\right\} \tag{2}$$

$$|M_t(x,y+h) - M_t(x,y)| \leq \frac{c}{t^n}\left(\frac{|h|}{t+|x-y|}\right)^{\eta} \exp\left\{-\frac{\alpha|x-y|^2}{t^2}\right\} \tag{3}$$

whenever $2|h| \leq t + |x-y|$.

We assume throughout this appendix that $L = -\operatorname{div}(A\nabla)$ where A has entries $a_{k\ell}$ that are functions of x_1, $(x_1, x_2, \ldots, x_n)$ being the rectangular coordinates in $\mathbb{R}^n$ and $X = (X_1, \ldots, X_n)$ is the arrow of vector fields

$$X_1 = a_{11}(x_1)\frac{\partial}{\partial x_1} + \cdots + a_{1n}(x_1)\frac{\partial}{\partial x_n}$$

and, for $2 \leq k \leq n$, $X_k = \partial/\partial x_k$.

The proof is long so let us explain the strategy.

The first step is to show that L has the Gaussian property (G).

The second step is to obtain boundedness and regularity with respect to both variables for $M_t(x,y)$. For the boundedness and regularity with respect to x, we show that $x \to t\nabla_x X K_{t^2}(x,y) = \nabla_x M_t(x,y)$, the variables t, y being fixed, is in a Morrey space, which implies that $M_t(x,y)$ is bounded and Hölder continuous in x. The regularity in the y variable is obtained in a similar way.

The third step consists in obtaining the decay in (1-3) from the Gaussian decay of $K_t(x,y)$ and the Hölder regularity of $M_t(x,y)$ via an interpolation technique.

Remark. — Before going on into the proof, a further simplification can be obtained by noticing that the class of operators L considered in this argument is invariant under scaling and translations. Thus, it suffices to obtain estimates for $t = 1$ and $y = 0$ in most cases and the full estimates with the correct dependence in t and y follow by an appropriate linear change of variables. For this reason, we do not care in the proof about controlling constants.

Lemma 1. — *L has the Gaussian property (G).*

Proof. — By Theorem 10 of Chapter 1, L has the property (G) if L and L^* have the property (D). Since this class of operators is stable under taking adjoints, we restrict our attention to L.

Let us pick a cube $Q = I \times J$ in an open set Ω where I is an interval of $\mathbb{R}$ and J a cube of $\mathbb{R}^{n-1}$, and u a weak solution of L on Ω.

As the coefficients a_{kl} depend only on x_1, L and $\partial/\partial x_k$ commute for $2 \le k \le n$. Using only Cacciopoli inequality as in **[39]**, one has that $\partial u/\partial x_k \in H^1_{loc}(Q)$ and $L(\partial u/\partial x_k) = 0$ for $2 \le k \le n$. Note that only the ellipticity of A is used in this argument. By induction, one has that $\partial^\alpha u \in H^1_{loc}(Q)$ for all multiindices $\alpha = (0, \alpha_2, \ldots, \alpha_n)$. Using the inequality (Sobolev embeddings in $\mathbb{R}^{n-1}$)

$$\int_I |\partial^\alpha \varphi(s,y)|^2\, ds \le c_n \int_I \int_{\mathbb{R}^{n-1}} \sum_{|\beta| \le n-1, \beta_0 = 0} |\partial^{\alpha+\beta} \varphi(s,z)|^2\, dz ds$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and all multiindices α with first coordinate 0, and localization and density arguments, one obtains

$$\|\partial^\alpha u(\cdot, y)\|_{H^1_{loc}(I)} \le c(n, \delta, \alpha) \sum_{|\beta| \le n-1, \beta_0 = 0} \|\partial^{\alpha+\beta} u\|_{H^1_{loc}(Q)}.$$

Therefore $u \in C^\infty(J; H^1_{loc}(I))$ and since $H^1_{loc}(I) \subset C^{1/2}_{loc}(I)$, we have shown that u is locally Hölder continuous with exponent $1/2$.

Let us show that u satisfies (18) in the definition of (D) with $\mu = 1/2$. Let $\Omega = B_1$ be a ball of radius 1 and assume that $u \in H^1(B_1)$ and $Lu = 0$. Let $0 < \rho \le 1$. If $\rho > 1/3$, the inequality is obvious, so we assume $\rho \le 1/3$. From Cacciopoli inequality and what we just established,

$$\int_{B_\rho} |\nabla u|^2 \le \frac{c}{\rho^2} \int_{B_{2\rho}} |u - c_\rho|^2 \le c\rho^{n-1},$$

where c_ρ is the mean of u on $B_{2\rho}$. Note that c depends on n and ellipticity and on u. The correct dependence in u follows from the uniform boundedness principle as follows.

Let E be the space of all weak solutions $u \in H^1(B_1)$ of $Lu = 0$. Then $E/\mathbb{C}$ equipped with the norm $\|u\| = (\int_{B_1} |\nabla u|^2)^{1/2}$ is a Banach space (it is easy to show that the space of all weak solutions is closed in $H^1(B_1)$). Define $T_\rho \colon E \to L^2(\mathbb{R}^n)$ by $T_\rho u(x) = \rho^{-(n-1)/2} (\nabla u)(x)$ for almost all $x \in B_\rho$ and $T_\rho u(x) = 0$ otherwise. We have shown that $\sup_{0<\rho\leq 1/3}\|T_\rho u\|_2 < +\infty$ for each $u \in E$. Thus $\sup_{0<\rho\leq 1/3}\|T_\rho\| < +\infty$. Writing out this inequality means precisely that

$$\int_{B_\rho} |\nabla u|^2 \leq C\rho^{n-1}\int_{B_1} |\nabla u|^2, \tag{4}$$

where the constant C depends only on n and δ.

On balls B_R with arbitrary radius, a linear change of variables (which does not affect the ellipticity constant) brings us back to the preceding case. □

Remark. — A further investigation on the equation $Lu = 0$ shows that, in fact, u is locally Lipschitz so that $n-1$ in (4) becomes n. We do not need such a refinement here. However, see the next remark for an alternate way of seeing this.

Lemma 2. — *If $K_t(x,y)$ denotes the heat kernel of L then for all $t > 0$ and $y \in \mathbb{R}^n$, $x \mapsto \nabla X K_t(x,y)$ belongs to the Morrey spaces M^γ for all $\gamma < n-1$.*

Proof. — We start from the equation

$$\operatorname{div}_x(A(x_1)\nabla_x K_t(x,y)) = \frac{dK_t(x,y)}{dt}. \tag{5}$$

Fix $2 \leq k \leq n$. Differentiating, we obtain

$$\operatorname{div}_x\left(A(x_1)\nabla_x \frac{\partial K_t(x,y)}{\partial x_k}\right) = \operatorname{div}\left(e_k \frac{dK_t(x,y)}{dt}\right) \tag{6}$$

where e_k is the kth canonical basis vector in $\mathbb{R}^n$. It follows from Lemma 19 of Chapter 1 and Lemma 1 that $x \mapsto dK_t(x,y)/dt$ is bounded, hence it belongs to all Morrey spaces M^β where $\beta \leq n$. Using Lemma 12 of Chapter 1, (6) implies have that $x \mapsto \nabla_x(\partial K_t(x,y)/\partial x_k)$ belongs to M^γ for all $\gamma < n-1$. Since $X_k = \partial/\partial x_k$ we have obtained that $\nabla X_k K_t(x,y)$ belongs to M^γ for $\gamma < n-1$. It remains to study $\nabla X_1 K_t(x,y)$.

If $\ell \geq 2$, since $\partial/\partial x_\ell$ and X_1 commute, we have that

$$\frac{\partial(X_1 K_t(x,y))}{\partial x_\ell} = Ae_1 \cdot \nabla \frac{\partial K_t(x,y)}{\partial x_\ell}$$

belongs to M^γ for $\gamma < n-1$ by what we have just proved.

Next, the function $x \mapsto \partial(X_1 K_t(x,y))/\partial x_1$ occurs as one term in equation (5). More precisely, we have

$$\frac{\partial X_1 K_t(x,y)}{\partial x_1} = \frac{dK_t(x,y)}{dt} - \sum_{k\geq 2,\ell\geq 1} \frac{\partial}{\partial x_k}\left(a_{k\ell}(x_1)\frac{\partial K_t(x,y)}{\partial x_\ell}\right).$$

Commuting $\partial/\partial x_k$ with the other operators in the sum, we see that the function $x \mapsto \partial X_1 K_t(x,y)/\partial x_1$ belongs to M^γ for $\gamma < n-1$. □

Lemma 3. — *$M_t(x,y)$ is bounded and Hölder continuous in both variables (x,y) for some exponent in $(0,1)$.*

Proof. — We have seen that $x \mapsto \nabla X K_{t^2}(x,y)$ belongs to M^γ for $\gamma < n-1$. By the Morrey embeddings (Lemma 11 of Chapter 1) and since $M_t(x,y) = tXK_{t^2}(x,y)$, this implies that $x \mapsto M_t(x,y)$ is bounded and Hölder continuous with exponent $\eta < 1/2$.

It remains to obtain the Hölder regularity in the y variable for $M_t(x,y)$. The argument of Lemma 2 can easily be adapted starting from $K_t(x,y+h) - K_t(x,y)$ instead of $K_t(x,y)$ in (5). This shows the boundedness of $x \mapsto M_t(x,y+h) - M_t(x,y)$. Its supremum depends on the Hölder bound of the heat kernel, which gives the usual growth $|h|^\eta$. □

Remark. — One can improve the conclusion of Lemma 2 by a bootstrap. Indeed, the proof gives that $x \mapsto K_t(x,y)$ is Lipschitz continuous. Thus, the term in the right hand side of (6) is bounded. Starting again the argument shows that $x \mapsto \nabla X K_t(x,y)$ belongs to the Morrey spaces M^γ for all $\gamma < n$. Hence $M_t(x,y)$ is Hölder continuous in both variables (x,y) for any exponent in $(0,1)$.

Lemma 4. — *Let $f\colon \mathbb{R}^n \to \mathbb{C}$ and fix $1 \le k \le n$. Assume that $\partial f/\partial x_k$ exists and $\partial f/\partial x_k \in C^\eta(\mathbb{R}^n)$ for some $\eta \in (0,1)$. Assume, furthermore, that $|f(x)| \le w(|x|)$ where w is a non-increasing weight on $[0,\infty)$. Then*

$$|\frac{\partial f}{\partial x_k}(x)| \le c(n,\eta) w(|x|)^{\eta/(\eta+1)}. \tag{7}$$

Proof. — We begin with Taylor formula

$$f(x+he_k) = f(x) + h\frac{\partial f}{\partial x_k}(x) + O(|h|^{1+\eta})$$

for all $x \in \mathbb{R}^n$, $h \in \mathbb{R}$ and $O(|h|^{1+\eta})$ is uniform in x. Choosing h with the sign of x_k we have $|x + he_k| \ge |x|$, and since w is non-increasing we have $|f(x+he_k)| \le w(|x|)$. Thus, for all h with $hx_k \ge 0$,

$$\left|\frac{\partial f}{\partial x_k}(x)\right| \le \frac{2w(|x|)}{|h|} + C|h|^\eta.$$

Optimizing over all possible h gives us (7). □

Lemma 5. — *$M_t(x,y)$ satisfies (1).*

Proof. — The preceding lemma applies to $f(x) = K_t(x+y,y)$ when $2 \le k \le n$ and $t=1$ with $w(x) = e^{-\alpha|x|^2}$ and gives a Gaussian decay for $\partial f/\partial x_k = X_k f$. For the remaining term we proceed as follows by adapting Lemma I.4 in **[14]**.

By the fundamental theorem of calculus, for all $h \in \mathbb{R}$ with $hx_1 \ge 0$,

$$f(x+he_1) = f(x) + h\int_0^1 \frac{\partial f}{\partial x_1}(x+the_1)\,dt. \tag{8}$$

Using the explicit form of X_1, one has

$$\frac{\partial f}{\partial x_1} = \frac{1}{a_{11}}X_1 f - \frac{a_{12}}{a_{11}}\frac{\partial f}{\partial x_2} - \cdots - \frac{a_{1n}}{a_{11}}\frac{\partial f}{\partial x_n}. \tag{9}$$

Now, we insert (9) into (8) and replace $X_1 f(x + the_1)$ by $X_1 f(x)$ since

$$X_1 f(x + the_1) = X_1 f(x) + O(|h|^\eta)$$

uniformly in x, t. Invoking the Gaussian bound on f and (7) for $2 \leq k \leq n$, we obtain

$$\left| \int_0^1 \frac{1}{a_{11}(x_1 + th)} \, dt \right| |X_1 f(x)| \leq \frac{2w(|x|)}{|h|} + C|h|^\eta + c(n, \eta) w(|x|)^{\eta/(\eta+1)}.$$

Since $\|a_{11}\|_\infty \leq 1/\delta$ and $\operatorname{Re} a_{11} \geq \delta$ almost everywhere, the integral involving this function is bounded below by δ^3. Thus, optimizing over h gives us $|X_1 f(x)| \leq \delta^{-3} c(n, \eta) w(|x|)^{\eta/(\eta+1)}$ as desired.

□

So far we have obtained (1), and (2-3) without the decay. But Remark 2 after Definition 3 of Chapter 1 applies and gives us the desired decay. This completes the proof of Lemma 24 in Chapter 4.

APPENDIX C

IMPROVED CONSTANTS

Journé studied in **[45]** how far a perturbation of the identity matrix one can take to control the multilinear expansions involved in the square root problem.

Although we do not use multilinear expressions, we can ask whether our methods improve Journé's constants.

In fact, we more or less find the same results, and this indicates that these constants may have a geometric meaning.

In this appendix, we need to care about algebra on matrices and their norms. Recall that the space $M_n(\mathbb{C})$ is equipped with the operator norm induced by the hermitian structure on $\mathbb{C}^n$.

Let us begin by recalling a basic fact.

Lemma 1. — *The following assertions are equivalent.*
(i) $A \in \mathcal{A}$.
(ii) *There exist* $\lambda \in \mathbb{R}$ *and* $M \in L^\infty(\mathbb{R}^n, M_n(\mathbb{C}))$ *with* $\|M\|_\infty < 1$ *such that* $A(x) = \lambda(I - M(x))$ *a.e..*
(iii) *There exist* $\lambda \in \mathbb{R}$ *and* $M \in L^\infty(\mathbb{R}^n, M_n(\mathbb{C}))$ *with* $\|M\|_\infty < 1$ *such that* $A^{-1}(x) = \lambda(I - M(x))$ *a.e..*

Define

$$\kappa_0 = \sup\{\, \|I - A\|_\infty \,;\, (K) \text{ holds for } -\operatorname{div}(A\nabla)\,\}$$

and

$$\kappa_1 = \sup\{\, \|I - A^{-1}\|_\infty \,;\, (K) \text{ holds for } -\operatorname{div}(A\nabla)\,\},$$

and ask for the values of κ_0 and κ_1. By Lemma 1, the square root problem will be solved for all $-\operatorname{div}(A\nabla)$, $A \in \mathcal{A}$, if one can show that $\kappa_0 = 1$ or $\kappa_1 = 1$. This is the case when $n = 1$.

Theorem 2. — *If* $n \geq 2$, *then*

$$\frac{1}{1 + 2\sqrt{n}} \leq \kappa_0 \tag{1}$$

and

$$\frac{1}{\sqrt{1+4n}} \leq \kappa_1. \tag{2}$$

Journé obtained essentially the same lower bounds for κ_0 and κ_1, and we give proofs based on our approach. He also observed that an abstract nonsense argument shows that these numbers must be equal but our proofs do not give equal lower bounds.

Before we proceed, let us introduce, following Journé, the family of dyadic averages defined by

$$S_t f(x) = |Q|^{-1} \int_Q f,$$

where $|Q|$ is the dyadic cube of $\mathbb{R}^n$ which contains x and with sidelength satisfying $t < \ell(Q) \leq 2t$. It is a classical result that the associated maximal operator, S^*, is bounded on $L^2(\mathbb{R}^n)$ with norm equal to 2. Also, the geometrical properties of dyadic cubes make the constant in Carleson's inequality equal to 1:

$$\int_{\mathbb{R}^n} \int_0^\infty |S_t f(x)|^2 \, d\mu(x,t) \leq |d\mu|_c^2 \int_{\mathbb{R}^n} |S^* f(x)|^2 \, dx, \quad f \in L^2(\mathbb{R}^n).$$

Thus, we have

$$\int_{\mathbb{R}^n} \int_0^\infty |b_t(x) S_t f(x)|^2 \frac{dxdt}{t} \leq 4 \, |b_t|_c^2 \int_{\mathbb{R}^n} |f(x)|^2 \, dx, \tag{3}$$

for scalar functions f and b_t and it extends to $\mathbb{C}^n$-valued functions where the product $b_t(x) S_t f(x)$ becomes a scalar product.

The other interesting point concerning the family of operators $(S_t)_{t>0}$ is that, given any approximation to the identity $(P_t)_{t>0}$ as in Chapter 2, then $(P_t - S_t)_{t>0}$ is a bounded family with

$$|(P_t - S_t)|_S \leq c(n, \varphi). \tag{4}$$

The proof of this inequality is merely sketched in **[45]** and we propose another proof at the end of this appendix.

Proof of Theorem 2. — Let us begin the proof of (2). It consists in optimizing the constant c_1 in Theorem 21 of Chapter 2 in a specific situation.

We consider $V_t \colon L^2(\mathbb{R}^n, \mathbb{C}^p) \to L^2(\mathbb{R}^n)$ such that $(V_t)_{t>0}$ is a bounded family satisfying the size and partial regularity assumptions of the above mentioned theorem. Furthermore, we assume that

$$V_t = W_t \Pi A, \tag{5}$$

where A is a bounded and accretive operator on $L^2(\mathbb{R}^n, \mathbb{C}^p)$, which, by the operator-valued version of Lemma 1, is normalized by

$$A^{-1} = I - M$$

with $||M|| < 1$, $||M||$ denoting the operator norm on $L^2(\mathbb{R}^n, \mathbb{C}^p)$. The operator Π is assumed to be an orthogonal projection in $L^2(\mathbb{R}^n, \mathbb{C}^p)$ and W_t is L^2-bounded but we never use its norm in a quantitative way.

It is easy to check that (5) with the conditions on Π and A implies the structural condition with $X = \Pi(\Pi A \Pi)^{-1}\Pi A$ (see Lemma 23 of Chapter 2).

Lemma 3. — *For some constant C we have, for all $F \in L^\infty(\mathbb{R}^n, \mathbb{C}^p)$,*

$$|V_tF|_c \leq \frac{2|V_t1|_c}{\sqrt{1-||M||^2}}||F||_\infty + C||F||_\infty. \tag{6}$$

Remark. — Recall that V_t1 is the $\mathbb{C}^n$-valued function $(V_t\, e_1, \ldots, V_t\, e_n)$, while V_tF is scalar-valued. This remark will be of importance in the proof.

Proof. — We follow the main steps of the argument of Theorem 21 in Chapter 2.

The three important inequalities in this proof are:

$$|V_tF|_c \leq (|V_t|_{\mathcal{S}} + C)||F||_\infty,$$

$$|V_t|_{\mathcal{S}} \leq \Upsilon_0|V_t\Pi|_{\mathcal{S}},$$

and

$$|V_t\Pi|_{\mathcal{S}} \leq \Upsilon_1|V_t1|_c + C.$$

Combining these inequalities gives us

$$|V_tF|_c \leq \Upsilon_0\Upsilon_1|V_t1|_c||F||_\infty + C||F||_\infty, \tag{7}$$

and our task is to minimize $\Upsilon_0\Upsilon_1$.

Lemma 3 thus follows from the claims that

$$\Upsilon_0 \leq \frac{1}{\sqrt{1-||M||^2}} \tag{8}$$

and that

$$\Upsilon_1 \leq 2. \tag{9}$$

To estimate Υ_0, use $A = I + AM$, the structural condition (5) on V_t and the fact that $\Pi(1-\Pi) = 0$ (since Π is a projection) to obtain

$$V_t(1-\Pi) = W_t\Pi A(1-\Pi) = W_t\Pi AM(I-\Pi) = V_tM(I-\Pi).$$

Thus, for $F \in L^2(\mathbb{R}^n; \mathbb{C}^p)$, we have

$$V_tF = V_t\Pi F + V_t(1-\Pi)F = V_t\Pi(\Pi F) + V_tM(1-\Pi)F$$

and taking quadratic norms, we get

$$\left(\int_0^\infty ||V_tF||_2^2\, \frac{dt}{t}\right)^{1/2} \leq |V_t\Pi|_{\mathcal{S}}||\Pi F||_2 + |V_t|_{\mathcal{S}}||M||||(1-\Pi)F||_2.$$

Now, Π is an orthogonal projection, hence $||F||_2^2 = ||\Pi F||_2^2 + ||(1-\Pi)F||_2^2$. Optimizing the above inequality over all F gives us

$$|V_t|_S \leq \sqrt{|V_t\Pi|_S^2 + |V_t|_S^2||M||^2},$$

and (8) is proved.

To control Υ_1, begin with the inequality

$$|V_t\Pi|_S \leq |(V_t 1)\, P_t|_S + C$$

which follows from Chapter 2. Now,

$$|(V_t1)\,P_t|_S \leq |(V_t1)\,S_t|_S + |(V_t1)\,(P_t - S_t)|_S \leq 2|V_t(1)|_c + ||V_t1||_\infty |P_t - S_t|_S$$

by (3) applied to vector-valued functions. Applying (4) finishes the proof of (9) and that of Lemma 3. □

As shown in Section 2.5 of Chapter 2, this lemma applies to

$$V_t = \theta_t^{(k)} = e^{-t^{4k+2}L_k} t^{2k+1} \Delta^k \operatorname{div} A$$

when k is large enough. The equality $A = I + AM$ yields

$$(\theta_t^{(k)} e_j)(x) = (\theta_t^{(k)}\, M_j)(x),$$

where M_j denotes the jth column vector of M. Hence, for all cubes Q,

$$\frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |(\theta_t^{(k)} e_j)(x)|^2 \frac{dxdt}{t} \leq \frac{4|\theta_t^{(k)}1|_c^2||M_j||_\infty^2}{1-||M||_\infty^2} + C,$$

for some $C > 0$. Now, sum over j and take the supremum over all cubes to obtain

$$|\theta_t^{(k)}1|_c^2 \leq \frac{4n|\theta_t^{(k)}1|_c^2||M||_\infty^2}{1-||M||_\infty^2} + C,$$

and (2) follows readily.

Remark. — A reexamination of the argument shows that the condition

$$\frac{4||\ ||M(x)||_{HS}^2\ ||_\infty}{1-||M||_\infty^2} < 1$$

implies $|\theta_t^{(k)}1|_c^2 < \infty$. Since the Hilbert-Schmidt norm on $M_n(\mathbb{C})$ satisfies

$$||M||_{HS} \leq \sqrt{n}||M||,$$

this is a stronger result. We leave details to the reader.

We continue the proof of Theorem 2 and establish (1). To this end, let us introduce an estimator which was defined in **[15]**. For each cube Q and $1 \leq j \leq n$, define $f_{j,Q}$ as the unique solution in $H_0^1(Q)$ of

$$\int_Q A\nabla f_{j,Q}\cdot\nabla\varphi = \int_Q Ae_j\cdot\nabla\varphi, \quad \forall\varphi \in H_0^1(Q).$$

In other words,

$$\operatorname{div} A\nabla f_{j,Q} = \operatorname{div} Ae_j \quad \text{in } \mathcal{D}'(Q).$$

Set

$$\gamma(L) = \inf_{Q\in\mathcal{Q}} \sup_{1\le j\le n} \left(\frac{1}{|Q|}\int_Q |\nabla f_{j,Q}|^2\right)^{1/2}.$$

Writing $A = I - M$, we have

$$\int_Q (I-M)\nabla f_{j,Q}\cdot\overline{\nabla f_{j,Q}} = -\int_Q Me_j\cdot\overline{\nabla f_{j,Q}}, \tag{10}$$

thus

$$\frac{1}{|Q|}\int_Q |\nabla f_{j,Q}|^2 \le \frac{\|Me_j\|_\infty^2}{(1-\|M\|_\infty)^2}$$

whence

$$\gamma(L) \le \frac{\|M\|_\infty}{1-\|M\|_\infty}. \tag{11}$$

Next, by a localization argument analogous to the one made in Chapter 3, one can show that, for all cube Q and $\mu > 1$,

$$(\theta_t^{(k)}e_j)(x) = (\theta_t^{(k)}\nabla f_{j,\mu Q})(x) + b_t(x), \quad x\in Q,$$

where $\int_0^{\ell(Q)}\int_Q |b_t(x)|^2 dxdt/t \le c|Q|$, the constant c being independent of Q. Now,

$$\int_0^{\ell(Q)}\int_Q |(\theta_t^{(k)}\nabla f_{j,\mu Q})(x)|^2\frac{dxdt}{t} \le |\theta_t^{(k)}\Pi|_S^2 \int_{\mu Q} |\nabla f_{j,\mu Q}|^2,$$

where Π is the projection on gradient vectors (note that we extended $\nabla f_{j,\mu Q}$ by 0 outside μQ). A straightforward computation yields, therefore, that

$$|\theta_t^{(k)}e_j|_c^2 \le 4\mu^n|\theta_t^{(k)}1|_c^2\gamma(L)^2 + C.$$

Summing over all j and using (11) shows that $|\theta_t^{(k)}1|_c < \infty$ provided

$$\frac{2\mu^{n/2}\|M\|_\infty}{1-\|M\|_\infty} < 1.$$

Since this holds for any $\mu > 1$, this proves (1).

Remark. — The right hand side of (10) can be written as

$$-\int_Q (M - c_Q)e_j\cdot\overline{\nabla f_{j,Q}}$$

where c_Q is any constant. Thus, we also have

$$\gamma(L) \le \frac{\|M\|_{BMO}}{1-\|M\|_\infty},$$

with the definition of the BMO using cubes instead of balls. Finishing the proof as before and noticing that M and A agree in BMO we obtain that

$$\|A\|_{BMO} \le \frac{1 - \|A - I\|_\infty}{2\sqrt{n}}$$

implies (K) for $-\operatorname{div}(A\nabla)$ (note that A is normalized by $\|A - I\|_\infty < 1$). This refines the estimate obtained in Theorem 18 of Chapter 3.

Proof of (4). — By Theorem 11 of Chapter 2, it is enough to show that $(S_t - P_t)_{t>0}$ is a weakly regular family. The size condition on the kernel is easily verified since both S_t and P_t have bounded kernels supported in a band $|x - y| \le ct$. The regularity estimate

$$\|U_t(-\Delta)^{s/2} f\|_2 \le c_0 t^{-s} \|f\|_2$$

is true for $U_t = P_t$ and $s > 0$ by Plancherel theorem. For $U_t = S_t$, it holds for $0 < s < 1/2$. Indeed, for $2^{-j-1} \le t < 2^{-j}$ and $g \in L^2(\mathbb{R}^n)$, we have

$$\langle S_t(-\Delta)^{s/2} f, g\rangle = \sum_{k \in \mathbb{Z}^n} \langle (-\Delta)^{s/2} f, 2^{nj} \chi(2^j x - k)\rangle m_{Q_{jk}}(\bar{g}),$$

where Q_{jk} are dyadic cubes of sidelength 2^{-j} and χ is the characteristic function of the cube $[0, 1)^n$. Hence

$$\langle S_t(-\Delta)^{s/2} f, g\rangle = \langle f, \sum_{k \in \mathbb{Z}^n} c_{jk} \chi_s(2^j x - k)\rangle,$$

with $c_{jk} = m_{Q_{jk}}(g) 2^{j(n+s)}$ and $\chi_s = (-\Delta)^{s/2}\chi$. Using Plancherel theorem and a classical periodization argument (see **[63]**) we have that

$$\int_{\mathbb{R}^n} \left| \sum_{k \in \mathbb{Z}^n} c_{jk}\ \chi_s(2^j x - k) \right|^2 dx \le c(n, s) 2^{-nj} \sum_{k \in \mathbb{Z}^n} |c_{jk}|^2,$$

with

$$c(n, s) = \sup_{\xi \in \mathbb{R}^n} \sum_{\ell \in \mathbb{Z}^n} |\widehat{\chi}_s(\xi + 2\ell\pi)|^2.$$

Since

$$\widehat{\chi}_s(\xi) = |\xi|^s \prod_{j=1}^n \frac{1 - e^{-i\xi_j}}{i\xi_j}$$

one has that $c(n, s) < \infty$ when $s < 1/2$. In this case, we deduce that

$$\begin{aligned} |\langle S_t(-\Delta)^{s/2} f, g\rangle| &\le c\|f\|_2 2^{js} \left(2^{nj} \sum_{k \in \mathbb{Z}^n} |m_{Q_{jk}}(g)|^2 \right)^{1/2} \\ &\le c t^{-s} \|f\|_2 \|g\|_2. \end{aligned}$$

□

APPENDIX D

REDUCTION OF DIMENSION PRINCIPLE

We have seen that for matrices depending only on one variable, the square root of the associated accretive operator behaves as in the one dimensional situation. In fact, there is a principle that governs this fact and that we call the reduction of dimension principle for square roots. The situation is as follows.

We are given $A \in \mathcal{A}(\delta)$ in $\mathbb{R}^n$ and $L = -\operatorname{div}(A\nabla)$ on $L^2(\mathbb{R}^n)$. We assume that A depends only on the m first variables $x_1, \ldots, x_m$, $m < n$. We shall put $x = (y, z) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ with $y = (x_1, \ldots, x_m)$.

Now, extract from A the matrix consisting in the first m rows and m columns of A and call it B. It is obvious that $B \in \mathcal{A}(\delta)$ in $\mathbb{R}^m$. We set $M = -\operatorname{div}(B\nabla)$ the associated maximal accretive operator on $L^2(\mathbb{R}^m)$.

Theorem 1. — *With the assumptions above, then*

$$\int_{\mathbb{R}^m} |M^{1/2} f(y)|^2\, dy \le c \int_{\mathbb{R}^m} |\nabla f(y)|^2\, dy, \quad f \in \mathcal{D}(M), \tag{1}$$

is equivalent to

$$\int_{\mathbb{R}^n} |L^{1/2} f(x)|^2\, dx \le c \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx, \quad f \in \mathcal{D}(L). \tag{2}$$

The interesting implication is from (1) to (2). For example, when A is real, it suffices that B be symmetric for (2) to be valid. There are inhomogeneous variants of this implication. We leave to the reader the care of checking them. If $B \in ABMO(\mathbb{R}^m)$ this gives $(\mathrm{K})_{loc}$ for L.

Proof. — Let us first assume that both L and M have the Gaussian property. Set

$$\theta_t^A = (I + t^2 L)^{-1} t \operatorname{div}_x A,$$

and

$$\theta_t^B = (I + t^2 M)^{-1} t \operatorname{div}_y B,$$

where we indicate the variables to ease the exposition.

We first prove that (1) implies (2). By Theorem 24, (iii), of Chapter 2, we have to show that $\theta_t^A e_j$ is a Carleson function in $\mathbb{R}^n \times \mathbb{R}^+$.

Since A depends only on y, we have for $1 \le j \le n$,

$$\operatorname{div}_x(Ae_j) = \frac{\partial a_{1j}}{\partial x_1} + \cdots + \frac{\partial a_{mj}}{\partial x_m} = \operatorname{div}_y(BF_j),$$

where a_{kj} are the coefficients of A and where $F_j(y) \in L^\infty(\mathbb{R}^m; \mathbb{C}^m)$ and, $B(y)$ being invertible, is completely determined by the equation

$$B(y)F_j(y) = ((a_{1j}(y), \ldots, a_{mj}(y))^T.$$

Note in particular that, when $1 \le j \le m$, we have $F_j(y) = e_j$, identified with the jth basis vector in $\mathbb{C}^m$.

At this point, we quote the following result whose proof is presented later.

Proposition 2. — *Let $n \ge 1$ and $L = -\operatorname{div}(A\nabla) \in \mathcal{E}(\delta)$ and assumed to satisfy the property (G). Fix $F \in L^\infty(\mathbb{R}^n; \mathbb{C}^n)$ and $t > 0$. Then $(1+t^2L)^{-1}\, t \operatorname{div} F$ is the unique function $u \in L^\infty(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n)$ such that*

$$\int u(x)\phi(x)\,dx + t^2 \int A(x)\nabla u(x) \cdot \nabla\phi(x)\,dx = -t \int F(x) \cdot \nabla\phi(x)\,dx \tag{3}$$

for all $\phi \in C^1_0(\mathbb{R}^n)$.

For t fixed, set $u_j(x) = (\theta_t^A e_j)(x)$ and $v_j(y) = (\theta_t^B F_j)(y)$. The equations that characterize u_j in $L^\infty \cap H^1_{loc}(\mathbb{R}^n)$ and v_j in $L^\infty \cap H^1_{loc}(\mathbb{R}^m)$ enable us to show by an easy but lengthy calculation using separation of variables that $u_j = v_j \otimes 1$, that is u_j depends only on y. Hence, given a cube Q in $\mathbb{R}^n$ and Q' its projection on $\mathbb{R}^m$, we have

$$\int_Q\int_0^{\ell(Q)} |(\theta_t^A e_j)(x)|^2 \frac{dxdt}{t} = \ell(Q)^{n-m} \int_{Q'}\int_0^{\ell(Q')} |(\theta_t^B F_j)(y)|^2 \frac{dydt}{t}.$$

Using our hypothesis on M and applying successively Theorem 24 and Theorem 21 of Chapter 2 we obtain that the latter integral is bounded by $c\ell(Q')^m = c\ell(Q)^m$, where c does not depend on Q'. This proves that (1) implies (2) under the assumption that both L and M have the Gaussian property.

The converse implication is dealt with using similar calculations; we skip details.

The assumption above is removed by raising the order of L and M to a large enough order using the familiar technique by now so as to obtain Gaussian estimates. Then the statement corresponding to Proposition 2 is valid in this case and the rest of the argument is similar. The proof is complete. □

Remarks

1. It is not clear at all how to work out an argument of separation of variables starting directly from (1). Thus, the reduction to the Carleson functions $\theta_t^A e_j$ seems crucial.

2. As the argument will show, the only estimate in the property (G) that is used in Proposition 2 is the Gaussian decay, not the Hölder bounds.

The rest of this appendix is devoted to proving Proposition 2. It is well-known that for a data $F \in L^2$, the weak solution of (3) coincide with $(1+t^2L)^{-1}\,t\,\mathrm{div}F$. This result is an extension to the case of L^∞ data. The main difficulty is to show that $(1+t^2L)^{-1}\,t\,\mathrm{div}F$ belongs to H^1_{loc}.

We begin with some further local estimates on the semigroup and the resolvent, complementing those obtained in Chapter 1.

Proposition 3. — *Assume $n \geq 1$. Let L in $\mathcal{E}(\delta)$ satisfy the pointwise estimate (7) in the property (G). Then $e^{-tL}\,\mathrm{div}$ and $(1+t^2L)^{-1}\,\mathrm{div}$ extend to bounded operators from $L^\infty(\mathbb{R}^n;\mathbb{C}^n)$ into $L^\infty \cap H^1_{loc}(\mathbb{R}^n)$ with, for all $t>0$ and all compact $K \subset \mathbb{R}^n$, the estimates*

$$\|e^{-t^2L}\,\mathrm{div}\,F\|_\infty + \|(1+t^2L)^{-1}\,\mathrm{div}\,F\|_\infty \leq ct^{-1}\|F\|_\infty \tag{4}$$

$$\int_K |\nabla e^{-t^2L}\,\mathrm{div}\,F|^2 + \int_K |\nabla(1+t^2L)^{-1}\,\mathrm{div}\,F|^2 \leq c_K t^{-4-n}\|F\|_\infty^2. \tag{5}$$

Proof. — For $n=1$ we refer the reader to **[14]** and **[11]**: the pointwise estimates obtained in these works imply (4) and (5). For the rest of the argument, we assume that $n \geq 2$.

The uniform L^∞ boundedness for $e^{-t^2L}t\,\mathrm{div}$ and $(1+t^2L)^{-1}t\,\mathrm{div}$ follows easily from the results in Section 1.5.2 of Chapter 1.

Let us next prove the part of (5) involving the semigroup. By scale invariance, it suffices to establish the inequality when $t^2 = 2$.

By Proposition 24 in Chapter 1, ∇e^{-L^*} is bounded from $L^1(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Hence, $e^{-L}\,\mathrm{div}$ is bounded from $L^2(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, and the similar result holds with L replaced by L^*. Thus,

$$\|e^{-L}\,\mathrm{div}\,F\|_\infty + \|e^{-L^*}\,\mathrm{div}\,F\|_\infty \leq c\|F\|_2. \tag{6}$$

Let $T_1(x,y)$ and $T_2(x,y)$ be the distribution-kernels of $e^{-L}\,\mathrm{div}$ and $e^{-L^*}\,\mathrm{div}$ respectively. Using a duality argument, the gradient estimates of Theorem 7 in Chapter 1 become

$$\int_{r\leq|x-y|\leq 2r} |T_i(x,y)|^2\,dy \leq ce^{-ar^2}, \quad i=1,2, \quad x\in\mathbb{R}^n, r>0, \tag{7}$$

for some $c,a>0$.

Cover $\mathbb{R}^n$ with balls $B(x_k,1/2)$, $k\in\mathbb{Z}^n$, of radii $1/2$ such that $|x_k-x_\ell| \sim |k-\ell|$ uniformly. Using a partition of unity subordinated to this covering, write $F=\sum F_k$, with $\mathrm{Supp}\,F_k \subset B(x_k,1)$ and $\|F_k\|_\infty \leq C\|F\|_\infty$, and $G=\sum G_\ell$ with $\mathrm{Supp}\,G_\ell \subset B(x_\ell,1)$ and $\|G_\ell\|_2 \leq C\|G\|_2$. We claim that

$$\left|\int_{\mathbb{R}^n} e^{-L}\,\mathrm{div}\,F_k\,\overline{e^{-L^*}\,\mathrm{div}\,G_\ell}\right| \leq ce^{-a|x_k-x_\ell|^2}\|F_k\|_2\|G_\ell\|_2. \tag{8}$$

(Here and in similar inequalities, we do not follow the precise values of the exponent a, which, as for other constants, may change from line to line.)

Admitting this inequality, if $G \in L^2(\mathbb{R}^n; \mathbb{C}^n)$ has support in K then the sum $G = \sum G_\ell$ has a finite number of terms and we have

$$\left| \int_{\mathbb{R}^n} \nabla e^{-2L} \operatorname{div} F \cdot \overline{G} \right| = \left| \sum_{k,\ell} \int_{\mathbb{R}^n} e^{-L} \operatorname{div} F_k \, \overline{e^{-L^*} \operatorname{div} G_\ell} \right|$$

$$\leq \sum_{k,\ell} c e^{-a|x_k - x_\ell|^2} \|F\|_\infty \|G\|_2 \leq c\|F\|_\infty \|G\|_2,$$

where we used $|x_k - x_\ell| \sim |k - \ell|$ to bound the series.

To prove the claim, set $d = |x_k - x_\ell|$ and distinguish two cases. If $d \leq 4$, there is nothing to prove by the L^2 boundedness of $e^{-L} \operatorname{div}$ and $e^{-L^*} \operatorname{div}$. If $d \geq 4$, split the domain of integration in three parts: $|x - x_k| \leq d/2$, $|x - x_\ell| \leq d/2$ or else. This gives us three terms: I, II and III.

If $|x - x_k| \leq d/2$, then using support considerations and (7), we have

$$|e^{-L^*} \operatorname{div} G_\ell(x)| \leq \int_{d/4 \leq |x-y| \leq 7d/4} |T_2(x,y)||G_\ell(y)| \, dy$$

$$\leq \left(\int_{d/4 \leq |x-y| \leq 7d/4} |T_2(x,y)|^2 \, dy \right)^{1/2} \|G_\ell\|_2$$

$$\leq c e^{-ad^2} \|G_\ell\|_2.$$

Also, $|e^{-L} \operatorname{div} F_k(x)| \leq c\|F_k\|_2$ by (6). Thus,

$$|I| \leq c d^n e^{-ad^2} \|F_k\|_2 \|G_\ell\|_2,$$

which is on the correct order of magnitude.

The second integral on $|x - x_\ell| \leq d/2$ is handled similarly.

Next, assume $|x - x_k| \geq d/2$ and $|x - x_\ell| \geq d/2$. Then we obtain as before

$$|e^{-L} \operatorname{div} F_k(x)| \leq c e^{-ad^2} \|F_k\|_2, \quad |x - x_k| \geq d/2.$$

while

$$|e^{-L^*} \operatorname{div} G_\ell(x)| \leq c e^{-a4^j d^2} \|G_\ell\|_2,$$

if $2^{j-1} d \leq |x - x_\ell| \leq 2^j d$ and $j \in \mathbb{N}$. By splitting the domain of integration of III on the rings $2^{j-1} d \leq |x - x_\ell| \leq 2^j d$, $j \geq 0$, one sees that

$$|III| \leq c e^{-ad^2} \sum_{j \geq 0} (2^j d)^n e^{-a4^j d^2} \|F_k\|_2 \|G_\ell\|_2$$

and the series is bounded above by a numerical constant that is independent of d.

We have proved one part of (5) and we now turn to the part involving the resolvent. Again, by scaling we can assume without loss of generality that $t = 1$. If we apply Laplace formula and use only what we just proved, we obtain a divergent integral.

To get around this difficulty, we use the following off diagonal local estimate, which follows from (7) by a similar covering argument to the one above and scaling.

Lemma 4. — *If $n \geq 2$ and $L \in \mathcal{E}(\delta)$ satisfies the pointwise Gaussian estimate (7) in (G), then we have*

$$\left| \int_{\mathbb{R}^n} \nabla e^{-tL} \operatorname{div} F \cdot \overline{G} \right| \leq \frac{c}{t}\, e^{-ad^2/t}\, \|F\|_2 \|G\|_2, \tag{9}$$

whenever $F, G \in L^2(\mathbb{R}^n; \mathbb{C}^n)$ with $d = \operatorname{dist}(\operatorname{Supp} F, \operatorname{Supp} G) \geq 4t^{1/2}$.

Continuing the argument, let $F \in L^\infty(\mathbb{R}^n; \mathbb{C}^n)$ and K be a compact. For simplicity assume that K is the unit ball. Write $F = F_0 + F_1 + F_2 + \cdots$ where $F_0(x) = F(x)$ if $|x| \leq 2$ and $F_0 = 0$ elsewhere, $F_1(x) = F(x)$ if $2 < |x| \leq 4$ and $F_1 = 0$ elsewhere, etc. We claim that $\sum \|\nabla(1+L)^{-1} \operatorname{div} F_j\|_{L^2(K)} \leq C\|F\|_\infty$, which implies $\|\nabla(1+L)^{-1} \operatorname{div} F\|_{L^2(K)} \leq C\|F\|_\infty$.

First,

$$\int_K |\nabla(1+L)^{-1} \operatorname{div} F_0|^2 \leq C\|F_0\|_2^2 \leq C2^n \|F\|_\infty^2$$

by (4) in Proposition 1 of Chapter 1. Now, for $j \geq 1$, Minkowski integral inequality and (9) imply

$$\begin{aligned}\left(\int_K |\nabla(1+L)^{-1} \operatorname{div} F_j|^2\right)^{1/2} &\leq \int_0^\infty \left(\int_K |\nabla e^{-sL} \operatorname{div} F_j|^2\right)^{1/2} e^{-s}\, ds\\ &\leq \int_0^\infty \frac{c}{s}\, e^{-a4^j/s} e^{-s}\, ds\ \|F_j\|_2\\ &\leq ce^{-a2^j} \|F_j\|_2\\ &\leq ce^{-a2^j} 2^{jn/2} \|F\|_\infty,\end{aligned}$$

for some $a > 0$ (break the integral at $s = 2^j$). The claim follows readily and the proof of Proposition 3 is finished. □

Proof of Proposition 2. — Without loss of generality, assume $t = 1$ as it plays no role and let $u = (1+L)^{-1} \operatorname{div} F$. We have just seen that $u \in L^\infty \cap H^1_{loc}(\mathbb{R}^n)$. Observe also that we have the kernel representation

$$u(x) = \int_{\mathbb{R}^n} T(x,y) F(y)\, dy \quad a.e., \tag{10}$$

where $T(x, y)$, the kernel of $(1+L)^{-1}\operatorname{div}$, satisfies

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |T(x,y)|\, dy \leq c.$$

We now show that u verifies (3) by an approximation argument.

Consider a sequence of compactly supported and uniformly bounded smooth functions F_n that converges to F almost everywhere. Define $u_n = (1+L)^{-1} \operatorname{div} F_n$. By

(10) and dominated convergence, we see that the sequence (u_n) is bounded and converges to u a.e.. Also, since (∇u_n) is bounded in L^2_{loc}, up to passing to a subsequence, (∇u_n) converges weakly to ∇u in L^2_{loc}. Finally, since $\operatorname{div} F_n \in L^2$, $u_n \in \mathcal{D}(L)$ so that the equality (3) holds for u_n, F_n in place of u, F respectively. It remains to let n tend to ∞ and (3) is proved.

To prove uniqueness, it suffices to establish that any $v \in L^\infty \cap H^1_{loc}(\mathbb{R}^n)$ satisfying

$$\int v(x)\phi(x)\,dx + \int A(x)\nabla v(x)\cdot\nabla\phi(x)\,dx = 0$$

for all $\phi \in C^1_0(\mathbb{R}^n)$ must be 0. We do it via the classical localization argument.

Let $\Phi \in C^1_0(\mathbb{R}^n)$. Then $w = v\Phi \in L^\infty \cap H^1(\mathbb{R}^n)$ and w satisfies

$$\int w(x)\phi(x)\,dx + \int A(x)\nabla w(x)\cdot\nabla\phi(x)\,dx = \int v(x)A(x)\nabla\Phi(x)\cdot\nabla\phi(x)\,dx$$

for all $\phi \in C^1_0(\mathbb{R}^n)$. Thus $(1+L)(w) = -\operatorname{div}(A(\nabla\Phi)\,v)$, where this equality lies in $H^{-1}(\mathbb{R}^n)$. Since $(1+L)^{-1}$ is bounded from $H^{-1}(\mathbb{R}^n)$ onto $H^1(\mathbb{R}^n)$, we have $w = -(1+L)^{-1}\operatorname{div}(A(\nabla\Phi)v)$. Now, $A(\nabla\Phi)v$ is bounded, thus

$$v(x)\Phi(x) = -\int T(x,y)A(y)(\nabla\Phi(y))\,v(y)\,dy \quad a.e..$$

Next, apply this representation with $\Phi_n(x) = \Phi(x/n)$ where $\Phi(0) = 1$ and let n tend to ∞. Since $(\nabla\Phi_n)$ converges to 0 and since $(\|\nabla\Phi_n\|_\infty)$ is bounded,

$$\int T(x,y)A(y)(\nabla\Phi_n(y))\,v(y)\,dy \longrightarrow 0$$

by the dominated convergence theorem and (10). On the other hand, $v(x)\Phi_n(x)$ converges to $v(x)$. Therefore, $v = 0$. □

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