Astérisque

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Astérisque, tome 218 (1993), p. 95-109
<http://www.numdam.org/item?id=AST_1993__218__95_0>

A FEW REMARKS ON THE LIFTING PROBLEM

by Luca Chiantini and Ciro Ciliberto*

0. Introduction

Let X be a reduced, non-degenerate variety of dimension n in $\mathbf{P}^r$, the projective space of dimension r over an algebraically closed field k of characteristic zero. If W is an irreducible variety of dimension n+m and degree s containing X, then for a general point t in the grassmannian Grass(h,r) of the h-planes in $\mathbf{P}^r$, with $h+n \geq r$, the corresponding h-plane L_t intersects X along a subvariety $X_t = X \cap L_t$ lying on the irreducible variety $W_t = W \cap L_t$ of dimension h+n+m-r and degree s.

Conversely, assume we have the following situation:

(0.1) Let X be a reduced, non-degenerate variety of dimension n in $\mathbf{P}^r$, let B be a smooth scheme and f: B→Grass(h,r) a dominant smooth morphism, $h+n \geq r$. For any $t \in B$ we let L_t be the h-plane corresponding to the point $f(t) \in$ Grass(h,r). Let $\mathcal{W}$ in $B \times \mathbf{P}^r$ be a family of projective varieties flat over B. For $t \in B$ we let W_t be the fibre of $\mathcal{W}$ over t. We suppose that the general fibre W_t of $\mathcal{W}$ is irreducible of dimension h+n+m-r and degree s, and that for $t \in B$ one has $L_t \supseteq W_t \supseteq X_t = X \cap L_t$.

In such a situation it is not true in general that there is a variety W of dimension n+m and degree s containing X and such that $W_t = W \cap L_t$ for $t \in B$: e.g. a general plane section of an irreducible curve of degree five in $\mathbf{P}^3$ lies on a conic, whereas there are such quintic curves lying in no quadrics.

The lifting problem consists in looking for suitable conditions on the variety X and the family $\mathcal{W}$ ensuring the existence of the variety W such that $W_t = W \cap L_t$ for $t \in B$.

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* Both authors have been supported by MURST and CNR of Italy. The research has been performed in the framework of Europroj's project "Hyperplane sections".

The problem has been first considered for the case of curves in $\mathbf{P}^3$, i.e. n=1, r=3, by Laudal [5], who gave a solution later refined by Gruson and Peskine [3]. Gruson-Peskine's result asserts that if X is a reduced, irreducible curve of degree d in $\mathbf{P}^3$, whose general plane section lies on some curve Γ of degree s, and if $d>s^2+1$, then X lies on a surface of degree s whose general plane section is Γ. Curves arising from sections of a null-correlation bundle show that the bound $d>s^2+1$ is sharp (see [3], [12]). More results on the lifting problem, especially for curves in $\mathbf{P}^3$, have been found by Strano with a purely algebraic approach relating the lifting problem to the syzigies of the resolution of the ideal of X_t (see [11], [9] and [6]).

Inspired by Gruson-Peskine's result mentioned above, we will restrict ourselves to the search of a function D(s,h,r,n,m) such that, if (0.1) holds, the lifting problem has a positive answer for d>D(s,h,r,n,m). And one could be so optimistic to try to find an optimal such function, i.e. a function D(s,h,r,n,m) with the above properties and such that there are counterexamples to the lifting problem for d≤D(s,h,r,n,m), e.g. in Gruson-Peskine's case (h=2, r=3, n=m=1) the optimal function is $D(s)=s^2+1$. The question, if one puts in this form, makes sense only if dim W_t=dim X_t+1, i.e. only if m=1 (see however § 3). Consider in fact the following:

Example: Let V be a smooth projection of the Veronese surface in $\mathbf{P}^4$, which is known to be not contained in any quadric 3-fold. Let X be an irreducible curve cut out on V by a hypersurface of degree d>3. By the theorem of Bezout X does not lie on any quadric 3-fold in $\mathbf{P}^4$, whereas its general hyperplane section is contained on a quartic rational curve, hence it does lie on a quadric surface in $\mathbf{P}^3$.

Hence in the present paper we will mainly restrict our attention to the case m=1, and we will determine a function D(s,h,r,n) such that if X has dimension n and degree d>D(s,h,r,n), and if there is a family W as in (0.1) with m=1, then there is a variety W of dimension n+1 such that $W_t=W\cap L_t$ for $t\in U$. The proof makes use of the differential-geometric concepts of foci and of focal locus for families of projective varieties, a classical notion firstly systematised by C. Segre [10] for families of linear subspaces and recently extended in [1] to any family of projective varieties. Similar ideas are already present in implicit form in [3]. We collect in § 1 all basic facts about foci and focal loci of a family which we need in the sequel. In § 2 we show that, if the lifting problem for X and the family W as in (0.1) with m=1 has a negative answer, then the points of X either lie in the focal locus of W or X_t lies in the singular locus of W_t for t a general point in B. Then by estimating the degrees of these loci, we prove the following:

Theorem (0.2).- Let X be a reduced, non-degenerate, projective subvariety of dimension n and degree d in $\mathbf{P}^r$ and let us suppose there is a family W as in (0.1) with m=1. If

$$d>D(s,h,r,n):=(r+h-3)s+k(k-1)(r-n-1)+2ek-2$$

where $s-1=k(r-n-1)+e$, $0\leq e<r-n-1$, then the image W of W in $\mathbf{P}^r$ is a variety of dimension n+1 and degree s, containing X and such that $W_t=W\cap L_t$ for $t\in B$.

Our function D(s,h,r,n) is not optimal in general. Slight improvements can be obtained in some cases with a more detailed analysis in the same vein of our proof below: for example the case of codimension two, n=r-2, has been recently carefully investigated by E. Mezzetti [7], whose result fully generalizes Gruson-Peskine's theorem to the case $r\leq 5$. She also makes a nice conjecture on the optimal function D(s,r-1,r,r-2). However we point out that, although in general not optimal, our function D(s,h,r,n) is asymptotically optimal. Indeed for instance in the case of curves we have that $D(s,r):=D(s,r-1,r,1)=[s^2/(r-2)]+o(s)$ and we find in § 3 curves X in $\mathbf{P}^r$ of degree $d=d(s)>>0$ with $d<D(s,r)$ but with $D(s,r)=d(s)+o(s)$, for which the lifting fails. These curves, as well as the curves in $\mathbf{P}^3$ achieving Gruson-Peskine's bound, are obtained as sections of suitable rank two vector bundles on certain rational normal scrolls. At the end of § 3 we will also briefly discuss an extension of theorem (0.2) to the case m>1.

In conclusion we want to mention that our approach via the focal loci has unexpected close relationships with Strano's algebraic approach mentioned above. We do not exploit this in the present paper, but we hope to come back on this subject in the future.

1. Generalities on foci.

In this section we let:

B be a non singular scheme of dimension b

W inside $B\times\mathbf{P}^r$ be a family, flat over B, of irreducible projective varieties of dimension w

V be a desingularization of W

After having shrinked B we may assume that V is flat over B, with smooth and irreducible fibres. Indeed, we may assume that for $t\in B$, the fibre V_t of $\mathsf{V}\to B$ over t is a desingularization of the corresponding fibre W_t of $\mathsf{W}\to B$.

The natural morphism u: $\mathsf{V}\to B\times\mathbf{P}^r$ yields the map of sheaves du: $T_{\mathsf{V}}\to u^*T_{B\times\mathbf{P}^r}$ which is generically injective, and therefore injective, since T_{V} is locally free. The cokernel of du is, by definition, the normal sheaf N_u to the map u, thus we have the exact sequence

(1.1) $$0\to T_{\mathsf{V}}\to u^*T_{B\times\mathbf{P}^r}\to N_u\to 0$$

and we notice that, in general, N_u is not necessarily torsion free.

We let p: $B\times \mathbf{P}^r \to B$ and q: $B\times \mathbf{P}^r \to \mathbf{P}^r$ be the projections. Then we have another natural map dq: $u^*T_{B\times \mathbf{P}^r} \to u^*q^*T_{\mathbf{P}^r}$ which is surjective. The kernel of dq is a locally free sheaf T(q) of rank b on V and we have the exact sequence

$$(1.2) \qquad 0 \to T(q) \to u^*T_{B\times \mathbf{P}^r} \to u^*q^*T_{\mathbf{P}^r} \to 0$$

The above sequences (1.1) and (1.2) fit into the commutative exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & L & \to & T(q) & \xrightarrow{\lambda} & N_u & \\
 & & \downarrow & & \downarrow & \| & \\
0 \to & T_{\mathsf{V}} & \to & u^*T_{B\times \mathbf{P}^r} & \longrightarrow & N_u & \to 0 \\
 & & \partial\downarrow & & \downarrow & & \\
 & & u^*q^*T_{\mathbf{P}^r} & = & u^*q^*T_{\mathbf{P}^r} & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

where ∂ is the differential of the map q∘u, λ is the <u>characteristic map</u> for the family V and L is the kernel of λ.

Since we are in characteristic 0, q is smooth at the general point of W. So if we set $w_0=\dim q(\mathsf{W})$, then we have

$$\mathrm{rk}\ \partial=w_0,\ \mathrm{rk}\ L=\mathrm{rk}\ T_{\mathsf{V}} -w_0=b+w-w_0,\ \mathrm{rk}\ \lambda=w_0-w$$

where of course $w_0-w=\dim q(\mathsf{W})-w\geq 0$.

Next we consider the restriction of λ to a general fibre of $\mathsf{V}\to B$. Take $t\in B$ and let V_t be the corresponding fibre of $\mathsf{V}\to B$. Let U be an affine open neighborhood of t in B over which T_B trivializes. Then over $p^{-1}(U)$ the map dq: $T_{B\times \mathbf{P}^r}\to q^*T_{\mathbf{P}^r}$ has a trivial kernel. Accordingly T(q) also trivializes over $V=u^{-1}p^{-1}(U)$, hence we have an isomorphism

$$(1.3) \qquad T(q)_{|V} \cong \mathcal{O}_V{}^b$$

Now we denote by N_t the normal sheaf to the induced map $u_t=q\circ u_{|V_t} : V_t\to \mathbf{P}^r$, and we prove the following basic:

<u>Proposition</u> (1.4).- One has $N_{u\,|\,V_t} \cong N_t$.

<u>Proof</u>. Consider the following commutative exact diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & T_{V_t} & \to & u^*q^*T_{\mathbf{Pr}\,|\,V_t} & \to & N_t & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & T_{V\,|\,V_t} & \to & u^*T_{B\times\mathbf{Pr}\,|\,V_t} & \to & N_{u\,|\,V_t} & \to & 0 \\
 & & \alpha\downarrow & & \downarrow & & & & \\
 & & N_{V_t,V} & \xrightarrow{\beta} & u^*N_{\mathbf{Pr},B\times\mathbf{Pr}\,|\,V_t} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since $V \to B$ is smooth, we have $N_{V_t,V} \cong T_{B,t} \otimes \mathcal{O}_{V_t}$. Similarly one has $u^*N_{\mathbf{Pr},B\times\mathbf{Pr}\,|\,V_t} \cong T_{B,t} \otimes \mathcal{O}_{V_t}$. A straightforward local computation shows now that β is in fact an isomorphism. The assertion follows then by the diagram. q.e.d.

In view of the isomorphism (1.3) and by proposition (1.4), we may interpret the restriction λ_t of the characteristic map λ to a fibre V_t as a map

$$\lambda_t\colon \mathcal{O}_{V_t}{}^b \to N_t$$

We notice that on a suitable dense open subset A_t of V_t the kernel of λ_t coincides with $L_t = L_{|V_t}$. Hence at a general point $p \in V_t$ we have

$$\mathrm{rk}\ \lambda_t = w_o - w$$

Futhermore if $p \in V$ is a general point, then we have

$$\dim\ ((q\circ u)^{-1}((q\circ u)(p))) = b + w - w_o$$

and the map

$$T_{V,p} \to u^*T_{B\times\mathbf{Pr},p}$$

is injective.

Now we are in position to give the following:

Definition (1.5).- A point $p \in V$ is called:

i) a focus, or a focal point, if the map

$$\lambda_p\colon T(q)\otimes k(p) \to N_u \otimes k(p)$$

has rank $r < w_o - w$;

ii) a fundamental point if the fibre $(q\circ u)^{-1}((q\circ u)(p))$ has dimension $\delta > b + w - w_o$;

iii) a cuspidal point, if the map

$$T_{V,p} \to u^*T_{B\times\mathbf{Pr},p}$$

is not injective.

The focal (resp. fundamental, cuspidal) locus is the set of all focal (resp. fundamental, cuspidal) points of V. V_t is a focal (resp. fundamental, cuspidal) fibre if it is contained in the focal (resp. fundamental, cuspidal) locus.

<u>Remark</u> (1.6).- i) The cuspidal locus is the set of all points $p\in V$ such that $Tor^1(N_u,k(p))\neq 0$, hence it is the torsion locus of N_u, thus it is Zariski closed. Notice that if p is a cuspidal point, then $p'=u(p)$ is a singular point of W, otherwise N_u would be locally free at p. Accordingly p' is singular in the fibre of $W\rightarrow B$ in which it sits.
ii) The focal locus is closed off the cuspidal locus. Indeed it is then defined as the set of points where the map

$$\Lambda^\rho \lambda\colon \Lambda^\rho T(q)\rightarrow\Lambda^\rho N_u, \quad (\rho=\mathrm{rk}\ \lambda)$$

drops rank.

<u>Proposition</u> (1.7).- A fundamental point is either a focal or a cuspidal point.
<u>Proof</u>. Consider the commutative exact diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & T(q)_p & \xrightarrow{\lambda_p} & N_{u,p} \\
 & & \downarrow & & \| \\
T_{V,p} & \rightarrow & u^*T_{B\times\mathbf{P}^r,p} & \longrightarrow & N_{u,p}\rightarrow 0 \\
\partial_p\downarrow & & \downarrow & & \\
u^*q^*T_{\mathbf{P}^r,p} & = & u^*q^*T_{\mathbf{P}^r,p} & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

and set $D_p=\ker\partial_p$. By assumption, since p is a fundamental point, we have dim $D_p>b+w-w_0$. If p is not cuspidal, then the map

$$T_{V,p}\rightarrow u^*T_{B\times\mathbf{P}^r,p}$$

is injective, hence D_p is nothing but the kernel of λ_p. q.e.d.

In the next two examples the reader will find an easy application of the above definitions and propositions and the description of a situation which shows that in general the behaviour of the focal and cuspidal loci can be rather tricky.

<u>Example</u> (1.8).- The classical <u>trisecant lemma</u> [5] says that a general chord of a reduced, non-degenerate space curve C is not a trisecant. An easy proof of this fact follows by proposition (1.7).

Let C' be the regular locus of C and let Δ be the diagonal in $C'\times C'$. We set $B=C'\times C'-\Delta$, and cosider the incidence correspondence

$$V=\{(x,y,z)\in B\times\mathbf{P}^3\colon z\in \text{line joining } x \text{ and } y\}$$

V is a smooth familiy of lines defined over B, and we use for it the notation introduced above. Since C is non-degenerate, we have that $q\colon V\rightarrow\mathbf{P}^3$ is dominant. Hence we have

$$b=2 \quad , \quad w=1 \quad , \quad w_0=3$$

thus L has rank 0, i.e. L=0. Let $t \in B$ be a general point. On the corresponding line V_t we have the map

$$\lambda_t: \mathcal{O}_{V_t}{}^2 \to N_t \cong {}_{V_t}(1)^2$$

which is given by a 2×2 matrix Λ_t of linear forms. Since the general fibre V_t is not focal because rk $\lambda_t = w_0 - w = 2$, then the focal locus on V_t consists of at most two points, defined by the quadratic equation $\det(\Lambda_t)=0$, unless V_t is a focal fibre. On the other hand the points in $q^{-1}(q(V_t) \cap C')$ are clearly fundamental points of V and since V is smooth they are focal points. Hence $q(V_t)$ intersects C at two distinct points, where the intersection has to be transverse, since the tangent lines to C form a 1-dimensional system only.

Essentially the same argument can be applied more generally to control the dimension of the family of (n+2)-secant lines to a variety of dimension n in $\mathbf{P}^{n+2}$, thus proving a theorem of Z. Ran's [8], whose approach is based on differential geometric ideas which are very close to the ones we introduce in the present paper.

Note that if C is a smooth complete intersection of a quadric cone Q and of a smooth quadric, then the vertex of the cone gives rise to a fundamental point of V. So any line V_t such that $q(V_t)$ is contained in Q has at least three focal points, thus is a focal fibre. A general point on such a line is a focal point which is not a fundamental point.

Example (1.9).- We sketch now an example which shows the existence of fundamental points which are not foci and an example of a focal locus which is not Zariski closed.

Let C be a smooth conic in $\mathbf{P}^2$. Let $p_1,\ldots,p_8$ be general points on C and $p_9,\ldots, p_{12}$ be four more general points in $\mathbf{P}^2$. Let us consider the rational map f: $\mathbf{P}^2 \to \mathbf{P}^2$ determined by the two-dimensional linear system of all quartics through the points $p_1,\ldots,p_{12}$. Consider now the smooth family p: $V \to \mathbf{P}^1$ given by the pencil of lines through $y=p_{12}$. The map f induces a map q: $V \to \mathbf{P}^2$ and accordingly a map $u=p \times q$: $V \to \mathbf{P}^1 \times \mathbf{P}^2$. Since f contracts C, every point x of C, regarded as a point of the line xy of V, is a fundamental point for q: $V \to \mathbf{P}^2$. Furthermore there are two points x, x' of C such that the lines xy, x'y are tangent to C. The points x, x', regarded as points of the lines xy, x'y, hence as points of V, are clearly cuspidal point with respect to q: $V \to \mathbf{P}^2$. In view of proposition (1.7), the general point of C is a focus, but x and x' are not foci. In fact with our usual notation, we have b=w=1, $w_0=2$ but the map

$$\lambda_x: T(q) \otimes k(x) \to N_u \otimes k(x)$$

is non-zero, since $N_u \otimes k(x)$ has dimension 2, acquiring a 1-dimensional torsion summand, and the image of λ_x is exactly the torsion summand. The same holds for x'.

2. Bounds for the degree of the focal locus.

In this section we prove the theorem (0.2) stated in the introduction, by giving a bound for the degree of the focal locus of a family W as in (0.1), with m=1. The bound will be derived from Castelnuovo's bound on the genus of projective curves.

Let V be a reduced, irreducible, non-degenerate, projective subvariety of degree s and dimension n in $\mathbf{P}^r$ and let π: $V'\to V$ be a desingularization of V. We denote by S(V) the reduced variety formed by the union of all codimension one irreducible components of Sing(V) and by s(V) its degree. We let H' be the pull-back by π of a general hyperplane section of V. Then H' is smooth and irreducible by Bertini's theorem. For all divisors D of V', we define the degree of D to be

$$\deg(D):=D\cdot H'^{n-1}$$

In particular we have deg(H')=s. Notice that the degree can be interpreted as a homomorphism of Pic(V') in $\mathbf{Z}$.

Notice that if Y is a reduced subvariety of V of pure dimension n-1 and D is an effective divisor on V' such that $\pi(D)\cup S(V)\supseteq Y$, then $\deg(Y)\leq\deg(D)+s(V)$. Indeed if Y' is the union of all components of Y not contained in S(V), then $\deg(Y')\leq\deg(\pi(D))=\deg(D)$.

Proposition (2.1).- Let V be as above. Then

$$s(V)\leq [k(k-1)/2]\cdot(r-n)+ke$$

where $s-1=k(r-n)+e$, $0\leq e<r-n$. Moreover, if K' is the canonical class of V', one has

$$\deg(K')\leq(1-n)s-2s(V)+k(k-1)(r-n)+2ek-2$$

Proof. A curve C' which is the pull-back via π of a general curve section C of V is smooth and irreducible. Indeed $\pi_{|C'}$: $C'\to C$ is the normalization morphism for C. We denote by g (resp. g') the arithmetic genus of C (resp. C'). Of course every point of $C\cap S(V)$ is singular for C, hence

$$0\leq g'\leq g-s(V)$$

Since C is a non-degenerate curve in $\mathbf{P}^{r-n+1}$, Castelnuovo's bound yields

$$g\leq[k(k-1)/2]\cdot(r-n)+ke$$

whence the estimate for s(V) easily follows. Furthermore the adjunction formula yields

$$2g'-2=K'\cdot C'+(n-1)H'\cdot C'=\deg(K')+(n-1)s$$

whence the estimate for deg(K') follows. q.e.d.

Remark (2.2).- The first part of proposition (2.1) can be extended as follows. Let V be a reduced variety of degree s and of pure dimension $n\geq1$ in $\mathbf{P}^r$ and let $S_i(V)$ be the Zariski closure of the locus of singular points of codimension i in V. Then

$$\deg(S_i(V))<s^{2i}$$

In fact consider a general $\mathbf{P}^{r-n-2}$ and project V from this $\mathbf{P}^{r-n-2}$ into a $\mathbf{P}^{n+1}$ as a hypersurface V'. Take a general polar of V' in $\mathbf{P}^{n+1}$. The cone from the original $\mathbf{P}^{r-n-2}$ over V" is a hypersurface of degree s-1 passing through all singular points of V. Now it is easy to see that the cones over V from the $\mathbf{P}^{r-n-2}$'s of $\mathbf{P}^r$ cut out V set theoretically and indeed scheme theoretically along the smooth points of V. This yields that the family of hypersurfaces like V" above has no base points on V except at the singular points. Hence if we take n-1 general such hypersurfaces, their intersection with V contains a one-dimensional component C passing through all isolated singularities of V and singular there. By Fulton's version of Bezout's theorem [F, pg. 223], we have

$$\deg(C) \leq s(s-1)^{n-1} < s^n$$

On the other hand the theorem clearly holds for n=i=1, whence the assertion.

We now go back to consider our original reduced, irreducible, non-degenerate variety X of dimension n in $\mathbf{P}^r$ with the family W as in (0.1), with m=1. For this family W we use the notation we introduced in § 2, e.g. V is a desingularization of W, etc. In particular we have the morphism q: $\mathsf{W} \to \mathbf{P}^r$ and we denote by W the Zariski closure of q(W), which is an irreducible subvariety of $\mathbf{P}^r$.

<u>Proposition</u> (2.3).- One has dim W≥n+1 and if dim W=n+1, then deg(W)=s.
<u>Proof</u>. For a general h-plane L_t corresponding to a general point $t \in B$, $W \cap L_t$ is irreducible and it contains W_t which has dimension h+n-r+1. Hence clearly dim W≥n+1 and if the equality holds, then $W \cap L_t = W_t$, whence the assertion. q.e.d.

Assume now dim W ≥ n+2. Consider then a general projection π of W onto $\mathbf{P}^{n+2}$. We denote by W ' the image of W via the map $p \times (\pi \circ q)$: $\mathsf{W} \to B \times \mathbf{P}^{n+2}$. We may assume, after perhaps having shrinked B, that:

i) π: $W \to \mathbf{P}^{n+2}$ is dominant;
ii) π maps X birationally onto its image;
iii) if $h \leq n+2$ then π restricts to an isomorphism to W_t for all $t \in B$;
iv) if $h > n+2$ then π restricts to a birational map to W_t for all $t \in B$; furthermore, since dim $W_t = h+n-r+1 \leq n$, then all components of $S(\pi(W_t))$ are birational projections of components of $S(W_t)$ and $s(W_t) = s(\pi(W_t))$, for all $t \in B$;
v) W '$\to$B is flat.

Then we may look at V as a desingularization of W ' and we denote by u' the obvious map $\mathsf{V} \to B \times \mathbf{P}^{n+2}$ and by q' the composition of u' with the projection onto the second factor.

<u>Proposition</u> (2.4).- Every point $x \in q'^{-1}(\pi(X))$ is a fundamental point.

Proof. Since q' is dominant, then a general fibre of q' has dimension b+h-r-1. Pick $x\in V$ such that $y=q(x)$ is a general point of X. The Shubert cycle G_y of h-planes of $\mathbf{P}^r$ containing y has codimension r-h in Grass(h,r). By the construction of $\mathcal{W}$ the projection on B of the fibre $q^{-1}(y)$ contains $f^{-1}(G_y)$. Hence dim $q^{-1}(q(x))\geq b+h-r$, and then for such a $x\in V$ one a fortiori has dim $q'^{-1}(q'(x))\geq b+h-r$, whence the assertion. q.e.d.

We are now in position to conclude the:

Proof of theorem (0.2). We keep the above notation. If dim W=n+1, we are done by proposition (2.3). Assume that dim W $\geq$n+2. Let t be a general point in B and let V_t be the corresponding fibre of $V\to B$ and let F_t be the focal locus of V_t in relation with the family $\mathcal{W}$ '. Since $\pi(X_t)$ has codimension one in $\pi(W_t)$, propositions (1.7) and (2.4) yield

$$q'(F_t)\cup S(\pi(W_t))\supseteq\pi(X_t)$$

so that

$$d=deg(X)=deg(\pi(X))=deg(\pi(X_t))\leq deg(q'(F_t))+s(\pi(W_t))$$

Look now at the characteristic map

$$\lambda_t: \mathcal{O}_{V_t}{}^b\to N_t$$

relative to the family $\mathcal{W}$ '. Since q' is dominant, λ_t is generically surjective. Hence, off the cuspidal locus, F_t is contained in some effective divisor D whose first Chern class is $c_1(N_t)$ defined by a non-zero section of $\mathcal{O}_{V_t}(c_1(N_t))$ given by $\Lambda^{r-h+1}\lambda_t$. One has

$$c_1(N_t)=K_t+(n+3)H_t$$

where K_t is the canonical class of V_t and H_t is the pull-back of a hyperplane of $\mathbf{P}^{n+2}$ via the map q'. Therefore

$$d\leq deg(K_t)+(n+3)s+s(\pi(W_t))$$

Then proposition (2.1) yields

$$d\leq(r-h+3)s+k(k-1)(r-n-1)+2ek-2=D(s,h,r,n)$$

a contradiction. q.e.d.

3. Comments, examples and extensions.

In this section we collect a few remarks and an example which shows that theorem (0.2) is asymptotically sharp. At the end of the section we briefly discuss an extension to the case m$\geq$2 of theorem (0.2).

Remark (3.1).- In the case of curves n=1, one has to take h=r-1, and our function D(s,h,r,n) becomes a function $D(s,r)=[s^2/(r-2)]+o(s)$. In particular for r=3 one has

$D(s,3)=s(s+1)$, and we thus recover Laudal's theorem [5] later extended by Gruson and Peskine [3] (see the Introduction).

Example (3.2).- Let M be a smooth threefold of degree r-2 in $\mathbf{P}^r$, $r\geq 5$. Such threefolds are described in [4]: M is a scroll in planes over a rational curve and Pic(M) is freely generated by the class F of a plane and by the hyperplane class H. The canonical class of M is

$$K_M = -3H+(r-4)F$$

In what follows we will need the:

Lemma (3.3).- $h^1(\mathcal{O}_M(aH+bF))=0$ for any $a\in \mathbf{Z}$ and for any $b\geq 0$.
Proof. It is well known that the assertion holds for b=0. We proceed by induction on b. The exact sequence

$$0\to\mathcal{O}_M(aH+bF)\to\mathcal{O}_M(aH+(b+1)F)\to\mathcal{O}_F(aH)\to 0$$

shows that $h^1(\mathcal{O}_M(aH+(b+1)F))=0$, since $h^1(\mathcal{O}_M(aH+bF))=0$ by induction and $h^1(\mathcal{O}_F(aH))=0$. q.e.d.

We will assume from now on that the class H-(r-4)F is effective on M, representing a smooth irreducible quadric surface Q inside M.

Let Y be a union of r-1 disjoint lines in M, each contained in a plane of M. We will assume Y to be general under the above conditions. We make the following:

Claim (3.4).- Let S be a surface in M containing Y then $\deg(S)\geq r-1$.
Proof of the claim. It goes by induction on r, the case r=5 being trivial. Assume $r\geq 6$ and let S be a surface of minimal degree containing Y. Then perform a projection of M from a point of one of the lines of Y to a scroll M' in $\mathbf{P}^{r-1}$. Then the remaining lines of Y are projected to a set Y' of r-2 general lines of M', contained in the projection S' of S. Then by induction $\deg(S)-1\geq\deg(S')\geq r-2$, whence $\deg(S)\geq r-1$. q.e.d.

Let ω_Y be the dualizing sheaf of Y and let $\mathcal{I}_Y$ the ideal sheaf of Y in M. Then

$$\mathcal{O}_Y \cong \mathcal{O}_Y(-2H-K_M-H-F) \cong \omega_Y(-H-F-K_M) \cong$$
$$\cong \mathcal{E}xt^2(\mathcal{O}_Y,\mathcal{O}_M(K_M))(-H-F-K_M) \cong \mathcal{E}xt^2(\mathcal{O}_Y,\mathcal{O}_M(-H-F)) \cong \mathcal{E}xt^1(\mathcal{I}_Y(H+F),\mathcal{O}_M)$$

The map

$$\mathrm{Ext}^1(\mathcal{I}_Y(H+F),\mathcal{O}_M)\to H^0(\mathcal{E}xt^1(\mathcal{I}_Y(H+F),\mathcal{O}_M)) \cong H^0(\mathcal{O}_Y) \cong k$$

is surjective, since its cokernel sits inside

$$H^2(\mathcal{H}om(\mathcal{I}_Y(H+F),\mathcal{O}_M)) \cong H^2(\mathcal{O}_M(-H-F)) = 0$$

So a constant in $k \cong H^0(\mathcal{O}_Y) \cong H^0(\mathcal{E}xt^1(\mathcal{I}_Y(H+F),\mathcal{O}_M))$ lifts to an extension

$$0\to\mathcal{O}_M\to E\to\mathcal{I}_Y(H+F)\to 0$$

with E locally free of rank two: indeed $Ext^1(E,\mathcal{O}_M)$ turns out to be zero since $Ext^1(\mathcal{O}_M,\mathcal{O}_M)$ is such and $\mathcal{O}_M \cong Hom(\mathcal{O}_M,\mathcal{O}_M) \to Ext^1(\mathcal{I}_Y(H+F),\mathcal{O}_M) \cong \mathcal{O}_Y$ is surjective by construction. Furthermore $c_1(E)=H+F$ and $c_2(E) = \deg(Y)=r-1$. Let X be a curve which is the 0-locus of some section of E(aH), for a>>0, which is clearly non-degenerate. We may also assume X to be smooth and irreducible. Its degree is

$$d=\deg(X)=H\cdot c_2(E(aH))=a^2(r-2)+(a+1)(r-1)$$

Let X_o be a general hyperplane section of X.

<u>Claim</u> (3.5).- X_o is contained in some curve of degree $s=(a+1)(r-2)$.
<u>Proof of the claim</u>. In fact X_o sits on the general hyperplane section M_o of M, and we have the exact sequence

$$0\to\mathcal{O}_{M_o}\to E_o\to\mathcal{I}_{Y_o}(H+F)\to 0$$

where Y_o is the general hyperplane section of Y and E_o is $E_{|M_o}$. Note that Y_o consists of r-1 points in $\mathbf{P}^{r-1}$, hence Y_o is degenerate, i.e. $h^0(\mathcal{I}_{Y_o}(H))\neq 0$. Hence $h^0(E_o(-F))\neq 0$ since $h^1(\mathcal{O}_{M_o}(-F)))=0$. From the exact sequence

$$0\to\mathcal{O}_{M_o}\to E_o(aH)\to\mathcal{I}_{X_o}((2a+1)H+F)\to 0$$

we have $h^0(\mathcal{I}_{X_o}((a+1)H))\neq 0$ proving the claim. q.e.d.

We remark now that $h^0(E)\neq 0$ yields $h^0(\mathcal{I}_X((a+1)H+F))\neq 0$, hence X lies on surfaces of degree $(a+1)(r-2)+1$. On the other hand we make the following:

<u>Claim</u> (3.6).- If $r\geq 7$ then X is not contained on any surface of degree $\sigma\leq(a+1)(r-2)$.
<u>Proof of the claim</u>. We argue by contradiction. Let S be such a surface and assume it has minimal degree, so that it is reduced and irreducible. By the theorem of Bezout S has to lie on M since a>>0, hence $h^0(\mathcal{I}_X(S))\neq 0$.

Notice that $S-F-(a+1)H$ has negative degree, hence $h^0(\mathcal{O}_M(S-F-(a+1)H))=0$. Thus if $h^0(E(S-F-(a+1)H))\neq 0$, then $h^0(\mathcal{I}_Y(S-aH))\neq 0$ contradicting the claim (3.4), since $\deg(S-aH)=\sigma-a(r-2)\leq r-2$. Hence we have $h^0(E(S-F-(a+1)H))=0$ which yields $h^1(\mathcal{O}_M(S-F-(2a+1)H))\neq 0$. Let $S=\alpha H+\beta F$ in Pic(M) hence $S-F-(2a+1)H=(\alpha-2a-1)H+(\beta-1)F$. By lemma (3.3) we must have $\beta\leq 0$. Then by the Kodaira vanishing theorem we must have $\alpha\geq 2a+1$.

Remark now that

$$S\cdot Q\cdot H=(\alpha H^2+\beta F\cdot H)\cdot(H-(r-4)F)=2\alpha+\beta$$

Since S and Q are irreducible and distinct, it is clear that $2\alpha+\beta\geq 0$. But then, since

$$\deg(S)=\alpha(r-2)+\beta\leq(a+1)(r-2)$$

we should have

$$(2a+1)(r-4)\leq\alpha(r-4)\leq(a+1)(r-2)$$

i.e. $r\leq 6$, a contradiction. q.e.d.

Finally we notice that we have

$$s-1=a(r-2)+(r-3)$$

hence with the notation of theorem (0.2), we have k=a, e=r-3. Hence

$$D(s,r)=(r-2)(a^2+5a+4)-2(a+1)$$

and therefore

(3.7) $$D(s,r)-d=(r-2)(5a+4)-(a+1)(r+1)=o(a)=o(s)$$

If $r\geq 7$ then, according to theorem (0,2), we have D(s,r)>d, but (3.7) shows that indeed the optimal function differs from our D(s,r) by a function $\delta(s,r)=o(s)$.

Remark (3.8).- In the proof of theorem (0.2) an important role is played by the hypothesis that the general fibre W_t of $\mathcal{W}$ is irreducible. Sometimes this assumption can be replaced by the assumption that X itself is irreducible. For example if n+m=r-1, i.e. W_t is a hypersurface in L_t and s is the minimal degree of such a hypersurface containing X_t, then W_t is clearly irreducible if $n\geq 2$, and the same happens by monodromy if n=1.

Remark (3.9).- Let us go back to the proof of theorem (0.2). One of the main points there is the fact that the focal locus F_t is contained in some effective divisor D whose first Chern class is $c_1(N_t)$. This follows from the consideration of the map of generically maximal rank

$$\lambda_t: \mathcal{O}_{V_t}{}^b \rightarrow N_t$$

where $b=\dim B\geq \dim \mathrm{Grass}(h,r)=(h+1)(r-h)>\mathrm{rk}\, N_t=r-h+1$. Let us then consider the map

$$\Lambda^{r-h+1}\lambda_t: \Lambda^{r-h+1}\mathcal{O}_{V_t}{}^b \rightarrow \det(N_t)$$

whose image gives rise to a linear system δ of divisors in the linear system $|\mathcal{O}_{V_t}(\det(N_t))|$, the so called focal linear system introduced in [7]. If dim δ is sufficiently large, then one has better estimates for the degree of the focal locus, thus improving the estimate for the function D(s,h,r,n). This idea, exploited in [7], is very useful in the case n=r-2. However it does not seem equally useful in the case of varieties of high codimension, in particular for curves.

Remark (3.10).- Let we weaken the hypotheses in (0.1) in the following way: f: $B\rightarrow \mathrm{Grass}(h,r)$ is no more necessarily dominant, but the union of the h-planes parametrized by the points of f(B) is dense in $\mathbf{P}^r$ and $b\geq r-h+1$. Then the map λ_t is still generically surjective and the proof of theorem (0.2) still goes through, except that proposition (2.3) could fail to hold.

For instance let us take for X the disjoint union of three lines on a smooth quadric surface W in $\mathbf{P}^3$ and let us take for B the set of all tangent planes to W. For all $t\in B$ the corresponding plane L_t meets X at three points on a line W_t and these lines form a two-dimensional flat family $\mathcal{W}$ verifying the assumptions of (0.1), modified as above. Of course proposition (2.3) does not hold for such a family, inasmuch as for a general point $t\in B$, the corresponding plane L_t is such that $W\cap L_t$ is reducible.

In order to let theorem (0.2) still work if f: $B\rightarrow Grass(h,r)$ is not dominant we must therefore make the following assumptions:

i) the union of the h-planes parametrized by the points of f(B) is dense in $\mathbf{P}^r$ and $b\geq r-h+1$;

ii) for a general point $t\in B$, the corresponding plane L_t is such that $W\cap L_t$ is irreducible, where as usual W is the Zariski closure of q(W).

Condition ii) is rather unpleasant. However it is automatically verified if, for instance, f(B) is dense in some Schubert cycle.

In conclusion we want to briefly point out the following extension of theorem (0.2) to the case $m\geq 2$:

Proposition (3.11).- Let X be a reduced, irreducible, non-degenerate variety of dimension $n\geq 2$ in $\mathbf{P}^r$, and suppose we have a situation like in (0.1) with $h+n>r$. Suppose that $d>(2rs)^{2^m}$. Then there is a variety Y containing X, with dim $Y=n+m-i$ and $\deg(Y)\leq(2rs)^{2^i}$ such that for $t\in B$ general, one has $L_t\supseteq W_t\supseteq Y_t\supseteq X_t$.

Proof. We proceed by induction on m. The case m=1 follows by theorem (0.2). Let $m\geq 2$. Now we use the notation introduced in § 2. As in proposition (2.3) we see that dim $W\geq n+m$ and if the eqauality holds then deg(W)=s. So we may assume dim $W\geq n+m+1$ and we make a general projection π to $\mathbf{P}^{n+m+1}$. The statement of proposition (2.4) still holds. Hence as in the proof of theorem (0.1) we have

$$q'(F_t)\cup Sing(\pi(W_t))\supseteq\pi(X_t)$$

Since dim $X\geq 2$ and therefore X_t is irreducible, we have either $Sing(\pi(W_t))\supseteq\pi(X_t)$ or $q'(F_t)\supseteq\pi(X_t)$. Now we claim that in either case $\pi(X_t)$ is contained in some irreducible subvariety of $\pi(W_t)$ of codimension one and of "low" degree. In fact in the first case one can prove, with an argument already used in remark (2.2), that $(Sing(\pi(W_t))$ is certainly contained in some hypersurface of degree s-1 not containing $\pi(W_t)$. In the latter case we notice that the map λ_t: $\mathcal{O}_{V_t}{}^b\rightarrow N_t$ relative to the family W ' is generically of maximal rank. Hence we can consider the focal linear system inside $|\mathcal{O}_{V_t}(\det(N_t))|=|K_{V_t}+(n+m+2)H_{V_t}|$ (see remark (3.9)), and by proposition (2.1) we have

$$\deg(K_{V_t}+(n+m+2)H_{V_t})\leq 2rs^2$$

Now, after may be a base change, we have a new family of varieties verifying (0.1) with m replaced by m-1. Futrthermore since

$$(2rs^2)^{2^{m-1}}<(2rs)^{2^m}<d$$

by induction we have that there is a variety Y containing X, with dim $Y=n+m-1-i$ and

$$\deg(Y)\leq[2r(2rs^2)]^{2^i}=(2rs)^{2^{i+1}}$$

such that for $t \in B$ general, one has $L_t \supseteq W_t \supseteq Y_t \supseteq X_t$. This proves our assertion. q.e.d.

It is useless to say that the hypothesis $d > (2rs)^{2^m}$ is very rough and could be refined as well as the bound for the degree of Y. It is also possible that the hypothesis dim $X \geq 2$, which we introduced for technical reasons, could be dropped.

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