Astérisque

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Astérisque, tome 210 (1992), p. 341-353
<http://www.numdam.org/item?id=AST_1992__210__341_0>

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Long Range Scattering and the Stark Effect

Denis A.W. White

1 Introduction.

In this Article we discuss long range quantum mechanical scattering in the presence of a constant electric field. The electric field is assumed to be of unit strength in the $\mathbf{e}_1 = (1, 0, \ldots, 0)$ direction of n-dimensional space, $\mathbf{R}^n$. The corresponding Hamiltonian for a quantum particle of unit mass is $H_0 = -(1/2)\Delta - x_1$, with $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$. ($H_0$ is essentially self adjoint (as an operator on $L^2(\mathbf{R}^n)$) on the Schwartz space of rapidly decreasing smooth functions.) A second Hamiltonian $H = H_0 + V$ is regarded as a perturbation of H_0 by a potential V. The potential $V = V_S + V_L$ consists of a "short range" term V_S and a "long range" term V_L. More precisely,

SR Hypothesis. *V_S is a symmetric operator, $V_S(H_0 + i)^{-1}$ is a compact operator and*

$$\int_1^\infty \|F(x_1 > r^2) V_S (H_0 + i)^{-1}\| \, dr < \infty$$

where $F(\cdot)$ is multiplication by the characteristic function of the indicated set.

LR Hypothesis. *$V_L(x)$ is real valued on $\mathbf{R}^n$, infinitely differentiable and for some $\epsilon > 0$ and for every multi-index α*

$$|D^\alpha V_L(x)| < C_\alpha \langle x_1 \rangle^{-|\alpha|/2 - \epsilon}$$
$$|D^\alpha V_L(x)| < o(1) \quad \textit{as } |x| \to \infty.$$

Here $\langle x_1 \rangle^2 = 1 + x_1^2$ and $D = -i\nabla$.

Example. If V_S is multiplication by a real valued function

$$V_S(x) = \{\chi(x_1)(1 + x_1^2)^{-\sigma/2} + \chi(-x_1)(1 + x_1^2)^{1/2}\} \tilde{V}_S(x)$$

where $\sigma > 1/2$ and $\tilde{V}_S = o(1)$ as $|x| \to \infty$ and $\tilde{V}_S$ is bounded and measurable and where

$$\chi(x_1) = \begin{cases} 1 & \text{if } x_1 > 1 \\ 0 & \text{if } x_1 < -1 \end{cases} \tag{1.1}$$

then V_S verifies the above short range hypothesis. (See Yajima [16]; local singularities may also be allowed.) The long range assumption is satisfied if, for some $0 \le \alpha, \beta \le 1/2$ and some real b_1 and b_2,

$$V_L(x) = \langle x\rangle^{-\epsilon}\cos(b_1|x_1|^{\alpha})\cos(b_2|x|^{\beta}).$$

In general these assumptions assure that $V(H_0+i)^{-1}$ is compact so that H is self adjoint on the domain of H_0. (Perry's book [14] is a good general reference.)

Introduce now the wave operators. Dollard's [3] *modified* wave operators W_D^- and W_D^+ are defined by

$$W_D^{\pm} = \text{s-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}e^{-iX_D(t)} \tag{1.2}$$

where "s-lim" indicates that the limit is taken in the strong operator topology. The "modifier" $e^{-iX_D(t)}$ was first introduced by J.D. Dollard [3] in the case of no electric field ($H_0 = -\Delta/2$) to extend the usual scattering theory which was based on the *Møller* wave operators,

$$W^{\pm} = \text{s-}\lim_{t\to\mp\infty} e^{itH}e^{-itH_0}, \tag{1.3}$$

to the case $V = V_L$ was the Coulomb potential ($V_L(x) = C/|x|$, for C a constant). An alternative choice of wave operators, are the *two Hilbert space* wave operators

$$W^{\pm}(J^{\pm}) = \text{s-}\lim_{t\to\mp\infty} e^{itH}J^{\pm}e^{-itH_0} \tag{1.4}$$

where $J^{\pm}$ are bounded operators conveniently chosen (as in §2 below.) The application of these operators to study long range scattering is due to Isozaki-Kitada [8] (who called $J^{\pm}$ "time independent modifiers") and Kitada-Yajima [12] who considered the case of no electric field. The two Hilbert space wave operators have certain technical advantages over the modified wave operators but the latter are the historical vehicle for studying long range scattering and are important for proving the non-existence of $W^{\pm}$; see Theorem 3.1 below. Each of the wave operators (for example W_D^+) is said to be (strongly asymptotically) *complete* if its range is the *subspace* $L^2(\mathbf{R}^n)_c$ *of continuity* of H. ($L^2(\mathbf{R}^n)_c$ is the orthogonal complement of all the eigenvectors of H.) Each wave operator (W_D^+, to be specific) is said to *intertwine* H and H_0 if

$$e^{-itH}W_D^+ = W_D^+e^{-itH_0}.$$

To state our results we must introduce the "modifiers." For the two Hilbert space wave operators we choose [8]

$$J^{\pm}u(x) = \int e^{ix\cdot\xi + i\theta^{\pm}(x,\xi)}\hat{u}(\xi)d\!\!{}^-\xi \tag{1.5}$$

where $\hat{u}$ denotes the Fourier transform of u and $\theta^{\pm}$ are smooth real valued functions to be specified in §2 below. ($J^{\pm}$ are not unique.) Here and below integrals are understood to be over all of $\mathbf{R}^n$ unless otherwise indicated and $d\!\!\!^{-}\xi = (2\pi)^{-n/2}\,d\xi$.

Theorem 1.1 *Hypotheses LR and SR imply that the two Hilbert space wave operators $W^{\pm}(J^{\pm})$ exist and are complete and are isometries that intertwine H and H_0. Moreover H has no singularly continuous spectrum and its eigenvalues are discrete and of finite multiplicity.*

Dollard's time dependent modifier can be defined as follows: Let $X_D(t)$ be Fourier equivalent to multiplication by a real valued function, X where

$$X(\xi_2,\dots,\xi_n,t) = \int_0^{\pm t} V_L(\tau Y(\xi_2,\dots,\xi_n,\tau) + (\tau^2/2)\mathbf{e}_1)\,d\tau \tag{1.6}$$

for $\pm t > 0$ and where Y is some smooth function of $n-1$ momentum variables plus time (t) taking values in $\mathbf{R}^n$ such that the first component $Y_1(\xi_2,\dots,\xi_n,t) \equiv 0$ and

$$\begin{aligned} |D^{\beta}_{\xi_\perp}(Y(\xi_2,\dots,\xi_n,t) - \xi_\perp)| &= O(|t|^{-\epsilon}); \\ |\frac{d}{dt}Y(\xi_2,\dots,\xi_n,t)| &= O(|t|^{-1-\epsilon}) \end{aligned}$$

for all multi-indices β, locally uniformly in $\xi_\perp = (0,\xi_2,\dots,\xi_n)$. (In particular in the one-dimensional case $Y \equiv 0$. In §3, Y is explicitly constructed.) Thus $X_D(t) = X(D_2,\dots,D_n,t)$.

Theorem 1.2 *Assume Hypotheses LR and SR. Then the modified wave operators $W_D^{\pm}$ exist and are complete and are isometries which intertwine H and H_0. Moreover the Møller wave operators $W^{\pm}$ exist if and only if $e^{iX(\xi_2,\dots,\xi_n,t)}$ converges in measure as $t \to \pm\infty$ on every compact subset of $\mathbf{R}^n$. Whenever $W^{\pm}$ exist, they are complete.*

Example. This continues the preceding example. Suppose for simplicity that b_1 and b_2 are nonzero and $\alpha \neq \beta$. Then the Møller wave operators (1.3) exist if and only if $\max\{\alpha,\beta\} + \epsilon > 1/2$ by Theorem 1.2. Ozawa [13] and Jensen-Ozawa [9] have already established a non-existence results for the Møller wave operators for a related class of potentials but by different methods.

Remark. In the case $n = 1$ the modifier depends only on time so that $e^{iX_D(t)} = e^{iX(t)}$ commutes with all operators. In particular, for any $u \in L^2(\mathbf{R}^n)$

$$|e^{-itH_0 - iX(t)}u(x)|^2 = |e^{-itH_0}u(x)|^2$$

which says that the position probability density of any free state is the same whether one uses the modified evolution or the usual free evolution. The same is true for the momentum probability density or any other observable in place of position or momentum. Therefore although the Møller wave operators $W^{\pm}$ do not exist the modified and free evolutions are indistinguishable by any quantum mechanical observable. It is therefore not surprising that in classical mechanics the usual wave operators exist as was observed by Jensen and Ozawa [9]. In general, for $n > 1$ the modifier is nontrivial. If however one further assumes

$$D^{\alpha}V_L(x) = O((1+|x|)^{-\alpha-\epsilon} \quad \text{for } |\alpha| \leq 1 \tag{1.7}$$

($\epsilon > 0$) then again one can replace $X(\xi, t)$ by a different modifier depending only on time (see Theorem 3.1 below) and which therefore cannot be observed. This last result is due to G.M. Graf [6] who assumed simply (1.7). Thus he requires less smoothness but more decay than here. He remarks that from the perspective of the Heisenberg picture of quantum mechanics there is no difference between quantum and classical mechanics in this setting. Graf uses Mourre's method.

In the remaining two Sections we outline the construction of $\theta^{\pm}$ for the proof of Theorem 1.1 (in §2). In §3 the proof of completeness in Theorem 1.2 is given; the remaining conclusions of Theorem 1.2 are standard and their proofs are only outlined.

2 Completeness of $W_D^{\pm}$.

In this Section we outline the construction of the operators $J^{\pm}$ of (1.5) or, more precisely, the phase terms $\theta^{\pm}$ as required for the proof of Theorem 1.1. In the process we indicate some key steps of the proof of Theorem 1.1 but our primary goal is to establish the properties of $\theta^{\pm}$ required for the proof of Theorem 1.2 in §3. A detailed proof of Theorem 1.1 is given in [15].

The construction of $\theta^{\pm}$ is as follows. It suffices to consider θ^{+}; the construction of θ^{-} is similar and in fact $\theta^{-}(x,\xi) = -\theta^{+}(x,-\xi)$. Choose $\chi_1 \in C^{\infty}(\mathbf{R})$ so that

$$\chi_1(x_1) = \begin{cases} 1 & \text{if } x_1 > 3 \\ 0 & \text{if } x_1 < 1 \end{cases} \tag{2.1}$$

The proof of Theorem 1.1 is based on the Enss method [4] in a two Hilbert space setting. One begins therefore with Cook's argument and so the key is to prove that the operator norm of $(d/dt)e^{itH}J^{+}e^{-itH_0}\chi_1(D_1)$ is an integrable function of $t > 1$, where $D_1 = -i\partial/\partial x_1$ so that $\chi_1(D_1)$ maps onto "outgoing states." The free evolution on outgoing states $e^{-itH_0}\chi_1(D_1)$ can be estimated as in the short range case [14] so that the crucial estimate to be established is: for arbitrary compact real interval I there is some integer $N \geq 0$, so that

$$\int_1^\infty \|E(I)(HJ^\pm - J^\pm H_0)\chi_1(\pm D_1/r)\chi_1(x_1/r^2)(H_0+i)^{-N}\|dr \ < \ \infty. \quad (2.2)$$

where E denotes the spectral measure of H. We consider this estimate in the case $V_S = 0$; the general case requires an auxiliary argument. To verify (2.2) we compute, for $\hat{u} \in C_0^\infty(\mathbf{R}^n)$,

$$[(H_0 + V_L)J^+ - J^+H_0]u\,(x) = \int e^{ix\cdot\xi + i\theta^+(x,\xi)}p^+(x,\xi)\hat{u}(\xi)d\xi \quad \text{where}$$

$$\begin{aligned} p^+(x,\xi) \ &= \ \xi\cdot\nabla_x\theta^+(x,\xi) \ + \ \frac{\partial}{\partial\xi_1}\theta^+(x,\xi) \ + \ \tfrac{1}{2}|\nabla_x\theta^+(x,\xi)|^2 \\ &\quad - \ \tfrac{i}{2}\Delta_x\theta^+(x,\xi) \ + \ V_L(x). \end{aligned} \quad (2.3)$$

Intuitively θ^+ should be chosen so that p^+ is roughly short range. More precisely (2.2) is verified if

$$D_x^\alpha D_\xi^\beta p^+(x,\xi) = O(\langle x_1\rangle^{-1/2-\epsilon}) \text{ for } x_1 > 0 \text{ and } \xi_1 > 0. \quad (2.4)$$

One tries to construct θ^+ as a solution of the equation $p^+(x,\xi) = 0$ but in fact it suffices to ignore the term $\frac{i}{2}\Delta_x\theta^+(x,\xi)$ in (2.3) intuitively because the second order derivatives of θ^+ should be better behaved than the lower order derivatives simply because V_L has this property. This leads to us to solving the transport equations,

$$\xi\cdot\nabla_x\theta_k + \partial\theta_k/\partial\xi_1 + b_k = 0 \quad (2.5)$$

where $b_0(x) = V_L(x)$ and for $k \geq 1$

$$b_k(x,\xi) = \frac{1}{2}\left(\left|\sum_{0\leq j\leq k-1}\nabla_x\theta_j(x,\xi)\right|^2 - \left|\sum_{0\leq j\leq k-2}\nabla_x\theta_j(x,\xi)\right|^2\right).$$

The transport equations are first order linear and there are many solutions but the solutions of interest are those that vanish as rapidly as possible as $x_1 \to \infty$. To enhance this decay we in fact settle for a solution of the transport equations with b_k replaced by $\tilde{b}_k$ where $\tilde{b}_k(x,\xi) = \chi(x_1)\chi(\xi_1)b_k(x,\xi)$. An appropriate solution is

$$\theta_k(x,\xi) = \int_0^\infty \tilde{b}_k(x + t\xi + (t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1) - \tilde{b}_k(t\xi_\perp + (t^2/2)\mathrm{e}_1, \xi_\perp + t\mathrm{e}_1)\,dt$$

where $\xi_\perp = (0, \xi_2, \ldots, \xi_n)$. (The second term in the above integrand is needed to assure that the integral exists.) One finds that

$$D_x^\alpha D_\xi^\beta\theta_k(x,\xi) = O(\langle x_1\rangle^{(1-|\alpha|)/2-(k+1)\epsilon} \text{ for } |\alpha| > 1.$$

and for $x_1 > 0$ and $\xi_1 > 0$ and all β. The improved decay for increased k is due to the squaring in b_k.) Then (2.4) is verified provided k is chosen so that $(k+2)\epsilon > 1/2$. This completes the construction of θ^+.

We pause now to record the properties of $\theta^{\pm}$, needed in §3 for the proof of Theorem 1.2. There we will need estimates on $\theta^{\pm}(x + (t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1)$ which are locally uniform in ξ (that is ξ is restricted to a compact set) as $t \to \infty$. Not surprisingly $\theta^{\pm}$ is larger in the direction opposite to the electric field, $x_1 < 0$.

For $\epsilon > 0$ as in Hypothesis LR, choose $\hat{\epsilon}$, $0 < \hat{\epsilon} < min\{1, \epsilon\}$. Then, for $\pm t > 1$

$$D_\xi^\beta \theta^{\pm}(x + \tfrac{t^2}{2}\mathrm{e}_1, \xi + t\mathrm{e}_1) = \begin{cases} O((|x| + |t|)(\langle x_1\rangle^{(|\beta|-1-\hat{\epsilon})/2} + t^{-\hat{\epsilon}/2})) & \text{if } x_1 < -\frac{t^2}{4}; \\ O((|x| + |t|)t^{-\epsilon}) & \text{if } x_1 > -\frac{t^2}{4}; \end{cases} \tag{2.6}$$

$$|D_\xi^\beta \frac{\partial}{\partial \xi_1} \theta^{\pm}(x + (t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1)| = \begin{cases} O(\langle x_1\rangle^{-\hat{\epsilon}/2}) & \text{if } x_1 < -t^2/4; \\ O(t^{-\epsilon}) & \text{if } x_1 > -t^2/4, \end{cases} \tag{2.7}$$

locally uniformly in ξ and for all multi-indices β. If $x_1 > -t^2/4$ then

$$|D_x^\alpha D_\xi^\beta \frac{\partial}{\partial x_1} \theta^{\pm}(x + (t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1)| = O(|t|^{-1-|\alpha|-\epsilon}); \tag{2.8}$$

$$|D_x^\alpha D_\xi^\beta \frac{\partial}{\partial \xi_1} \theta^{\pm}(x + \tfrac{t^2}{2}\mathrm{e}_1, \xi + t\mathrm{e}_1)| + |D_x^\alpha D_\xi^\beta \frac{\partial}{\partial x_j} \theta^{\pm}(x + \tfrac{t^2}{2}\mathrm{e}_1, \xi + t\mathrm{e}_1)| = O(|t|^{-|\alpha|-\epsilon}), \tag{2.9}$$

again locally uniformly in ξ and for each j, $1 \le j \le n$ and all α and β and $\pm t > 1$. These estimates follow from the construction of $\theta^{\pm}$. For the estimates (2.6; 2.7) observe that

$$\int_t^\infty \tau^k \langle y + \tau^2/2\rangle^{-(k+1+\epsilon)/2}\, d\tau = \begin{cases} O(\langle y\rangle^{(k-\hat{\epsilon})/2}) & \text{if } y < -t^2/4; \\ O(t^{-\epsilon}) & \text{if } y > -t^2/4, \end{cases} \tag{2.10}$$

for each nonnegative integer k. The same reasoning shows that, for $\pm t > 1$

$$D_\xi^\beta p^{\pm}(x + (t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1) = \begin{cases} O(\langle x_1\rangle^{-\hat{\epsilon}/2}) & \text{if } x_1 < -t^2/4; \\ O(|t|^{-1-\epsilon}) & \text{if } x_1 > -t^2/4. \end{cases} \tag{2.11}$$

3 Proof of Theorem 1.2.

In this Section we outline the proof of Theorem 1.2 emphasizing the proof of completeness of the modified wave operators $W_D^{\pm}$. Let us begin by noting that the conclusion about the non-existence of the Møller wave operators is a direct consequence of a result formulated by Hörmander [7, Theorem 3.1]:

Theorem 3.1 *Assume that the limits (1.2) exist with $X_D(t) = X(D,t)$ as well as the the corresponding limits $\tilde{W}_D^{\pm}$ when $X(D,t)$ is replaced by $\tilde{X}(D,t)$. Then $W_D^{\pm}$ has the same range as $\tilde{W}_D^{\pm}$ (same sign) if and only if $\exp i(X(\xi,t) - \tilde{X}(\xi,t))$ converges in measure as $t \to \pm\infty$ to functions $F^{\pm}$. In this case $\tilde{W}_D^{\pm} = W_D^{\pm} F^{\pm}(D)$.*

Applying this result with $\tilde{X} = 0$, we derive non-existence of the Møller wave operators from the completeness of the modified wave operators. (The ranges of $W_D^{\pm}$ are, in general, contained in $L^2(\mathbf{R}^n)_c$; see [14, p. 48].)

The existence of the modified wave operators can be established by an argument very similar to that given by Hörmander [7] for the case of no electric field. The reason similar arguments apply is the Avron-Herbst formula [1]:

$$e^{-itH_0} = e^{-it^3/6} e^{itx_1} e^{-iD_1 t^2/2} e^{-i(-\Delta)t/2}. \tag{3.1}$$

Therefore, up to an inconsequential phase factor $e^{-it^3/6}$, e^{-itH_0} is the evolution $e^{-i(-\Delta)t/2}$, free of the electric field, followed by a translation $e^{-iD_1t^2/2}$ of $t^2/2$ units in the $\mathbf{e}_1$ direction of configuration space followed by a translation e^{itx_1} of t units in the $\mathbf{e}_1$ direction in momentum space. With this formula, existence follows by the argument of [7] based on stationary phase and constructing a solution of a Hamilton-Jacobi equation.

The proof of the intertwining principle is well known (see Hörmander [7, p. 75], for example).

With these brief remarks about the other conclusions of Theorem 1.2 we proceed to the proof of completeness. This proof is entirely independent of the existence proof because, as we shall show, the wave operators $W_D^{\pm}$ exist at least on some subspace of $L^2(\mathbf{R}^n)$ and both have range the subspace $L^2(\mathbf{R}^n)_c$ of continuity of H. It is not necessary that the modifier be the same in both proofs, by Theorem 3.1.

To establish completeness it suffices by Theorem 1.1 to show that $W_D^{\pm}$ has the same range as $W^{\pm}(J^{\pm})$. To do so, we introduce the auxiliary operators

$$\tilde{W}_D^{\pm} \equiv \text{s-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0} e^{i\tilde{X}(D,t)} u$$

where $\tilde{X}(\xi,t)$ is some function chosen suitably for a stationary phase argument. Significantly $\tilde{X}$ may depend on all n of the ξ-variables whereas X of Theorem 1.1 depends only on (the last) $n-1$ variables. We show that $\tilde{W}_D^{\pm}$ has the same range as do $W^{\pm}(J^{\pm})$; this is the bulk of the work. We further show, with the help of Theorem 3.1, that $W_D^{\pm} = \tilde{W}_D^{\pm} F^{\pm}$ for some unitary operators $F^{\pm}$. This will establish the completeness of $W_D^{\pm}$. The intermediary operators $\tilde{W}_D^{\pm}$ are a convenience and not of independent interest because they do not intertwine H and H_0.

The operators $\tilde{W}_D^{\pm}$ and $W^{\pm}(J^{\pm})$ have the same range if, for every $u \in L^2(\mathbf{R}^n)$ there exists $v \in L^2(\mathbf{R}^n)$ so that

$$e^{itH} J^{\pm} e^{-itH_0} u - e^{itH} e^{-itH_0} e^{-i\tilde{X}(D,t)} v$$

converges to 0 as $t \to \pm\infty$ or, equivalently, if the operators

$$\Omega^{\pm} \equiv \operatorname{s-lim}_{t \to \pm\infty} e^{i\tilde{X}(D,t)} e^{itH_0} J^{\pm} e^{-itH_0} \tag{3.2}$$

exist. We therefore prove the Proposition below.

Proposition 3.2 *Assuming the hypotheses of Theorem 1.1 the operators $\Omega^{\pm}$ of (3.2) exist on all of $L^2(\mathbf{R}^n)$ when $\tilde{X}$ is defined by*

$$\tilde{X}(\xi, t) = \int_0^t V_L(\tau \tilde{Y}(\xi, \tau) + \tau^2/2\mathbf{e}_1)\, d\tau \tag{3.3}$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbf{R}^n$ and $\tilde{Y}$ is a smooth, real valued function such that

$$|D_\xi^\beta(\tilde{Y}(\xi,t) - \xi)| = O(|t|^{-\epsilon}) \tag{3.4}$$

$$|\frac{d\tilde{Y}}{dt}(\xi,t)| + |D_\xi^\beta \frac{\partial}{\partial \xi_1}(\tilde{Y}(\xi,t) - \xi)| = O(|t|^{-1-\epsilon}) \tag{3.5}$$

for all β, locally uniformly in ξ. In particular the operators $\tilde{W}_D^{\pm}$ of (1.2) are complete.

Before proving this Proposition, let us see how it implies completeness in Theorem 1.2. Define Y there componentwise: $Y_1 = 0$ and for $2 \le j \le n$

$$Y_j(\xi_2, \ldots, \xi_n, t) = \tilde{Y}_j(0, \xi_2, \ldots, \xi_n, t).$$

Completeness will follow from Theorem 3.1 if

$$e^{i\tilde{X}(\xi,t) - iX(\xi_\perp,t)}$$

converges locally in measure (or locally uniformly) as $t \to \pm\infty$. This follows from the mean value theorem and the estimates for $\tilde{Y}$. The only troublesome term is

$$\int_0^t \int_0^1 \tau \frac{\partial}{\partial x_1} V_L(\tau(s\tilde{Y}(\xi,\tau) + (1-s)Y(\xi_\perp,\tau)) + \frac{\tau^2}{2}\mathbf{e}_1)\, ds\, \tilde{Y}_1(\xi,\tau)\, d\tau. \tag{3.6}$$

Its convergence can be checked by integration by parts in the τ variable.

Proof of Proposition 3.2. We consider only the case of Ω^+ ($t > 0$). To prove the strong convergence in (3.2), it suffices to prove convergence on a subset of $L^2(\mathbf{R}^n)$ whose linear span is dense. This subset will consist of all u so that $\hat{u}$ is in $C_0^\infty(\mathbf{R}^n)$ and supported in a ball of radius $\eta/2$ centered at $\xi^0 \in \mathbf{R}^n$ where ξ^0 is arbitrary and $\eta > 0$ will be specified below. Apply Cook's method (differentiate and integrate in (3.2)): $\Omega^+ u$ exists if the L^2-norm of

$$X_t(D,t)e^{itH_0}J^+e^{-itH_0}u + e^{itH_0}[H_0J^+ - J^+H_0]e^{-itH_0}u$$

is an integrable function of t on an interval $[t_0, \infty)$, for some $t_0 > 1$. We add and subtract $e^{itH_0}V_L J^+ e^{-itH_0}u$ and apply the Avron-Herbst formula (3.1). It suffices to show that the L^2-norms $\|A(\cdot,t)\|$ and $\|C(\cdot,t)\|$ are integrable functions of $t > t_0$ where A and C are defined by

$$\begin{aligned} A(x,t) &= [X_t(D,t) - V_L(x + (t^2/2)\mathrm{e}_1)]B(x,t) \quad \text{where} \\ B(x,t) &= \int e^{ix\cdot\xi - it|\xi|^2/2 + i\theta^+(x+(t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1)}\hat{u}(\xi)d\xi; \\ C(x,t) &= e^{iD_1t^2/2}e^{-itx_1}[(H_0 + V_L)J^+ - J^+H_0]e^{-itH_0}u(x) \\ &= \int e^{ix\cdot\xi - it|\xi|^2/2 + i\theta^+(x+(t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1)}p^+(x + (t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1)\hat{u}(\xi)d\xi \end{aligned}$$

where p^+ was defined by (2.3).

We start by estimating $\|A(\cdot,t)\|$. As is typical in stationary phase arguments we estimate first the integral $B(x,t)$ far from the critical point of the phase function

$$\phi(\xi,x,t) = x\cdot\xi - t|\xi|^2/2 + \theta^+(x + (t^2/2)\mathrm{e}_1, \xi + t\mathrm{e}_1).$$

Choose therefore $\chi_0 \in C_0^\infty(\mathbf{R}^n)$ so that

$$\chi_0(x) = \begin{cases} 1 & \text{if } |x| < \eta \\ 0 & \text{if } |x| > 2\eta \end{cases} \tag{3.7}$$

Then

$$(1 - \chi_0\left(\frac{x - \xi^0 t}{1+t}\right))B(x,t) = O((1 + |x| + t)^{-N}). \tag{3.8}$$

because $|\nabla_\xi \phi| > c(1+|x|+t)$, for some $c > 0$, on the relevant region; see Fedoryuk [5] or Hörmander [7, Lemma A.1]. (The proof is essentially integration by parts.)

Therefore to check the integrability of $\|A(\cdot,t)\|$ in $t > t_0$ it suffices to check that of the $L^2(d\zeta)$-norm of

$$\int \chi_0\left(\frac{x - \xi^0 t}{1+t}\right)\int e^{-ix\cdot\zeta + i\phi(\xi,x,t)}[(\tilde{X}_t(\zeta,t) - V_L(x + t^2/2\mathrm{e}_1)]\hat{u}(\xi)d\xi dx. \tag{3.9}$$

It is again possible to estimate the $L^2(d\zeta)$ norm of the expression (3.9) away from the critical point of the phase function $-x\cdot\zeta + \phi(\xi,x,t)$, regarded as a function of x. Therefore multiply by $1-\chi_0(\zeta-\xi^0)$ in (3.9). Then there is no critical point for the phase: Indeed $|\zeta-\nabla_x\phi(\xi,x,t)|>c(1+|\zeta|)$ for some $c>0$, in the relevant region. Since

$$|D_x^\alpha\nabla_x\phi(\xi,x,t)|+|D_x^\alpha\{\chi_0\left(\frac{x-\xi^0t}{1+t}\right)[\tilde{X}_t(t,\zeta)-V_L(x+\frac{t^2}{2})]\}|=O(t^{-|\alpha|-\epsilon}),$$

it follows that the expression (3.9) times $1-\chi_0(\zeta-\xi^0)$ is $O((1+|\zeta|)^{-N}t^{-N-\epsilon})$ for any integer N and so its $L^2(d\zeta)$ is integrable in $t>t_0$. (See the references after equation (3.8).)

It remains to estimate the expression (3.9) times $\chi_0(\zeta-\xi^0)$. This is not quite the "usual" stationary phase estimate near the critical point because $X_t(\zeta,t)$ depends on ζ, not ξ. To remedy this we expand $X_t(\zeta,t)$ in a Taylor series, not around ξ but around the critical point for the phase, $\zeta=\nabla_x\phi(\xi,x,t)$. The expression (3.9) times $\chi_0(\zeta-\xi^0)$ is, for some positive integer k

$$\chi_0(\zeta-\xi^0)\hat{A}_0(\zeta,t)+\chi_0(\zeta-\xi^0)\sum_{1\le|\alpha|\le k}\frac{i^{|\alpha|}}{\alpha!}A_\alpha(\zeta,t)+\sum_{|\alpha|=k+1}\frac{i^{k+1}}{\alpha!}R_\alpha(\zeta,t)$$

where

$$\begin{aligned}
A_0(x,t) &= \chi_0\left(\frac{x-\xi^0t}{1+t}\right)\times \\
&\quad \int e^{i\phi(\xi,x,t)}[(\tilde{X}_t(\nabla_x\phi(\xi,x,t),t)-V_L(x+(t^2/2)\mathrm{e}_1)]\hat{u}(\xi)d\xi; \\
A_\alpha(\zeta,t) &= \int\chi_0\left(\frac{x-\xi^0t}{1+t}\right)\int e^{-i\zeta\cdot x+i\phi(\xi,x,t)}(\zeta-\nabla_x\phi(\xi,x,t))^\alpha\times \\
&\quad (D_\xi^\alpha\tilde{X}_t)(\nabla_x\phi(\xi,x,t),t)\hat{u}(\xi)d\xi dx; \\
R_\alpha(\zeta,t) &= \sum_{|\alpha|=k+1}\frac{1}{\alpha!}\chi_0(\zeta-\xi^0)\int\chi_0\left(\frac{x-\xi^0t}{1+t}\right)\int e^{-i\zeta\cdot x+i\phi(\xi,x,t)}\times \\
&\quad (\zeta-\nabla_x\phi(\xi,x,t))^\alpha\times \\
&\quad \int_0^1(1-s)^k(D_\xi^\alpha\tilde{X}_t)(s\zeta+(1-s)\nabla_x\phi(\xi,x,t),t)\,ds\,\hat{u}(\xi)d\xi dx
\end{aligned}$$

and $\hat{A}_0$ is the Fourier transform of $A_0(\cdot,t)$. We shall show that the $L^2(d\zeta)$ norms of each term is an integable function of $t>t_0$. (The factor $\chi_0(\zeta-\xi^0)$ only plays a role in the consideration of R_α.)

The $A_0(x,t)$ term is the most interesting because the choice of $\tilde{X}$ is critical here. First we change variables, $x=ty$: The $L^2(dx)$ norm of $A_0(x,t)$ equals the $L^2(dy)$ norm of $t^{n/2}A_0(ty,t)$. Optimally $\tilde{X}$ will satisfy

$$\tilde{X}_t((\nabla_x\phi)(\xi,ty,t),t)=V_L(ty+t^2/2\mathrm{e}_1)$$

at the critical point of the phase, that is where $\nabla_\xi\phi(\xi, ty, t) = 0$. In order to specify $\tilde{X}$ we begin by defining $y(\xi, t)$ by the equation for the critical point:

$$y(\xi,t) - \xi + t^{-1}\nabla_\xi\theta(ty(\xi,t) + t^2/2\mathrm{e}_1, \xi + t\mathrm{e}_1) = 0. \tag{3.10}$$

The implicit function theorem guarantees $y(\xi, t)$ is well defined for $(\xi, t) \in U_1 \times [t_1, \infty)$ where U_1 is any bounded open set and $t_1 > 1$ is suitably large; y is smooth and

$$\begin{aligned} |D_\xi^\beta(y(\xi,t) - \xi)| &= O(t^{-\epsilon}); \\ |\frac{dy}{dt}(\xi,t)| + |D_\xi^\beta \frac{\partial}{\partial \xi_1}(y(\xi,t) - \xi)| &= O(t^{-1-\epsilon}), \end{aligned}$$

for all β, by estimates (2.8, 2.9). Since U_1 is arbitrary, it is possible to extend $y(\xi, t)$ smoothly, by a partition of unity argument, to all of $\mathbf{R}^n \times [0, \infty)$ so that whenever ξ is restricted to a compact set the above bounds are valid and (3.10) holds for t large enough and $y(\xi, t) = \xi$ for small t.

The definition of $\tilde{X}$ further requires defining $\Xi(\cdot, t)$ to be the inverse of the mapping $\xi \mapsto (\nabla_x\phi)(\xi, ty(\xi, t), t)$. Provided ξ is restricted to a bounded open set and t is large then $\Xi(\cdot, t)$ indeed exists and

$$\begin{aligned} |D_\zeta^\beta(\Xi(\zeta,t) - \zeta)| &= O(t^{-\epsilon}) \\ |\frac{d\Xi}{dt}(\xi,t)| + |D_\zeta^\beta \frac{\partial}{\partial \zeta_1}(\Xi(\zeta,t) - \zeta)| &= O(t^{-1-\epsilon}) \end{aligned}$$

by (2.7). Extend Ξ to $\mathbf{R}^n \times [0, \infty)$ as was done with y and so that $\Xi(\zeta, t) = \zeta$ for small t.

Define the modifier $\tilde{X}$ as

$$\tilde{X}(\xi,t) = \int_0^t V_L(\tau\tilde{Y}(\xi,\tau) + (\tau^2/2)\mathrm{e}_1)\, d\tau \quad \text{where} \quad \tilde{Y}(\xi,t) = y(\Xi(\xi,t),t). \tag{3.11}$$

The estimates (3.4, 3.5) for $\tilde{Y}$ follow directly from the comparable estimates for y and Ξ.

We may now estimate the $L^2(dy)$ norm of $t^{n/2}A_0(ty, t)$ by a well known stationary phase argument [7, Lemma A.4]. Since the phase function in A_0 has a non-degenerate critical point, Hörmander's Lemma A.4 [7] applies and gives an expansion for A_0 at that critical point. Our choice of $\tilde{X}$ assures that the first term of that expansion is 0 and the remaining terms times $t^{n/2}$ have $L^2(dy)$-norms which are integrable functions of $t > t_0$.

The same type of argument applies to A_α, $|\alpha| > 0$ but first it is necessary to integrate by parts in the x variables several times. Each time the factor $e^{-i\zeta\cdot x + i\phi(\xi,x,t)}(\zeta - \nabla_x\phi(\xi, x, t))$ is integrated and the process is repeated until the symbol no longer contains the variable ζ (which is at most $|\alpha|$ times).

Then the outer integral over x is simply a Fourier transform so that we may estimate the L^2-norm of the inverse Fourier transform of $A_\alpha(\cdot,t)$. For example, the inverse Fourier transform of $A_\alpha(\cdot,t)$ in the special case $|\alpha|=1$, say $\alpha=\mathbf{e}_j$ for some $j, 1\le j\le n$, is

$$-i\int e^{i\phi(\xi,x,t)}\frac{\partial}{\partial x_j}\{\chi\left(\frac{x-\xi^0 t}{1+t}\right)(D_\xi^{\mathbf{e}_j}\tilde{X}_t)(\nabla_x\phi(\xi,x,t),t)\}\hat{u}(\xi)d\xi$$

We now argue as for A_0. We change variables $x=ty$ and apply Hörmander's Lemma [7, Lemma A.4]. Since, by (2.9), the x derivatives of θ and hence ϕ decay rapidly in t on the support of the above integrand, Hörmander's Lemma implies that $\|A_\alpha(\cdot,t)\|$ is an integrable function of $t>t_0$.

Next we estimate R_α when $|\alpha|=k+1$ and k is large. As above we integrate by parts repeatedly in x until all factors of $(\zeta-\nabla_x\phi(\xi,x,t))$ have been integrated (or differentiated) out. Here however the integral over x is not simply a Fourier transform but again the integrand will decay rapidly in t and in fact if k is large enough the integral may be estimated directly: $\|R_\alpha(\cdot.t)\|$ is an integrable function of $t>t_0$; there is no need for Hörmander's Lemma here.

The proof that $\|C(\cdot,t)\|$ is an integrable function of t follows arguments already given. The initial argument estimating B far from the critical point applies to C as it did to B and so it suffices to consider the $L^2(dx)$-norm of $\chi((x-\xi^0t)/(1+t))C(x,t)$. (See (2.11).) Changing variables $x=ty$ it suffices to show that the $L^2(dy)$-norm of $t^{n/2}\chi((y-\xi^0)/(1+t^{-1}))C(ty,t)$ is an integrable function of t. This follows again from Hörmander's Lemma A.4 [7] and the estimate (2.11). This proves the Proposition. □

References

[1] J.E. Avron and I.W. Herbst, *Spectral and scattering theory for Schrödinger operators related to the Stark effect,* Commun. Math. Phys. **52**(1977), 239-254.

[2] A. P. Calderón and R. Vaillancourt, *A class of bounded pseudo-differential operators,* Proc. Nat. Acad. Sci. U.S.A. **69**(1972), 1185-1187.

[3] J.D. Dollard, *Asymptotic convergence and the Coulomb interaction,* J. Math. Phys. **5**(1964), 729-738.

[4] V. Enss, *Asymptotic completeness for quantum mechanical potential scattering, I. Short range potentials,* Commun. Math. Phys. **61**(1978), 285-291.

[5] M.V. Fedoryuk, *The stationary phase method and pseudodifferential operators,* Russian Math. Surveys **26**(1971), 65-115.

[6] G.M. Graf, *A remark on long-range Stark scattering,* Helv. Phys. Acta **64**(1991). 1167-1174.

[7] L. Hörmander, *The existence of wave operators in scattering theory,* Math Z. **146**(1976), 69-91.

[8] H. Isozaki and H. Kitada, *Modified wave operators with time-independent modifiers,* J. Fac. Sci. Univ. Tokyo Sect. IA **32**(1985), 77-104.

[9] A. Jensen and T. Ozawa, *Classical and quantum scattering for Stark Hamiltonians with slowly decaying potentials,* Ann. Inst. Henri Poincaré, Phys. Théor., **54**(1991), 229-243.

[10] A. Jensen and K. Yajima, *On the long range scattering for Stark Hamiltonians,* J. reine angew. Math. **420**(1991), 179-193.

[11] H. Kitada, *A relation between the modified wave operators $W_J^{\pm}$ and $W_D^{\pm}$,* Sci. Pap. Coll. Arts Sci. Univ. Tokyo **36**(1987), 91-105.

[12] H. Kitada and K. Yajima, *A scattering theory for time-independent long-range potentials,* Duke Math. J. **49**(1982), 341-376.

[13] T. Ozawa, *Non-existence of wave operators for Stark effect Hamiltonians,* Math Z. **207**(1991), 335-339.

[14] P.A. Perry, *Scattering theory by the Enss Method,* Mathematical Reports, No.1, Harwood, New York, 1983.

[15] D.A.W. White, *The Stark effect and long range scattering in two Hilbert spaces,* Indiana Univ. Math. J. **39**(1990), 517-546.

[16] K. Yajima, *Spectral and scattering theory for Schrödinger operators with Stark-effect*, J. Fac. Sci. Univ. Tokyo, Sec IA, **26**(1979), 377-390.

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