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TWO RESULTS IN NUMBER THEORY

S. Srinivasan

*In memory of my friend
Mr. Ellis Martin Richstone*

§1. Introduction. Let $N \geq 1$ be an integer and, for a reduced fraction a/N , let $K(a/N)$ denote the largest partial quotient in the continued fraction expansion of a/N . Set $K_0(N) = \min K(a/N)$ for $1 \leq a \leq N$, $(a, N) = 1$. Let, for a sequence B of integers, $B(x)$ denote the number of b in B , but not exceeding x . With this notation, a conjecture of S.K. Zaremba can be stated as $K_0(N) \leq 5$, for all N (cf., p.76 of [4]). There has also been some numerical evidence for this conjecture (cf., p.989 of [2]).

Now considering for given integer $\ell \geq 2$, the sequence A_ℓ consisting of N with $K_0(N) \leq \ell$, it has been shown in [3] that

$$A_\ell(x) > \frac{1}{\sqrt{2\ell}} x^{\frac{1}{2}(1-\ell^{-2})}, \text{ for } x \geq 1. \quad (1)$$

In this context, we observe the following proposition, which also leads to a qualitative result of type (1). For a given collection of a.p.s (i.e., arithmetic progressions) of (positive) integers, let $\alpha(m)$ denote the number of a.p.s containing m (but not as the smallest) and, $\delta(m)$ denote the number of a.p.s with common difference m .

Proposition 1. *Let $k(\geq 2)$ be an integer, β and β' some positive constants. Suppose A is the union of a collection of a.p.s, each consisting of k integers, satisfying (i) $\alpha(m) \leq \beta\delta(m)$ for all m and (ii) the smallest member of each a.p. is $\leq \beta'$ times its common difference. Then, for any b such that $\alpha(b) > 0$ and for all sufficiently large x , we have*

$$A(x) \geq \frac{\sqrt{\beta}}{k-1} \left(\frac{x}{b}\right)^\theta, \text{ for some } \theta = \theta(\beta, \beta', k) > 0, \quad (2)$$

provided $(k-1)$ exceeds β . (An expression for θ is given in (4) below.)

Our second observation gives the following

Proposition 2. *Any solution in integers of the equation*

$$(\xi^2 + \eta^2 - 1) = t(\xi u + \eta v)(\xi v - \eta u) \quad (3)$$

satisfies $tuv = 0$.

In § 3, we shall give an example from R.Tijdeman in connection with Proposition 1. And in § 4, we have some remarks about these results.

§2. Proofs. We start with the proof of Proposition 1. Consider

$$T(x) := \sum_{m \leq x} \alpha(m) \leq \beta \sum_{m \leq x} \delta(m).$$

Since every a.p. counted by the last sum, in view of the assumption (ii), consists of members not exceeding $(k - 1 + \beta')x$, we see that it occurs $(k - 1)$ times in $T((k - 1 + \beta')x)$ and so,

$$(k - 1)T(x) \leq \beta T((k - 1 + \beta')x).$$

On letting $x_r = (k - 1 + \beta')^r b$, we obtain

$$T(x_r) \geq \left(\frac{k - 1}{\beta}\right)^r, \quad r = 1, 2, 3, \dots$$

From here, after a short calculation, (2) is obtained on noting that $T(x) \leq (k - 1)(A(x))^2$. To see the last inequality observe that every a.p. counted in $\alpha(m)$ determines the pair (a, m) , a being its smallest member and, each such pair arises from at most $(k - 1)$ of the a.p.s. Also, we may take

$$\theta = \frac{1}{2} \left\{ \log \left(\frac{k - 1}{\beta} \right) / \log(k - 1 + \beta') \right\}. \quad (4)$$

Now we give a proof of Proposition 2. Let $tuv \neq 0$. First, observe that, after changing notations if necessary, we can assume that

$$(u, v) = 1, \quad \min(\xi, \eta, t, x, y) > 0, \quad (5)$$

where $x := \xi u + \eta v$, $y := \xi v - \eta u$. Now, since $(x, y) \nmid (\xi^2 + \eta^2)$, we can conclude, by (5), that $(x, y) = 1$. Then (3) can be rewritten as

$$(x^2 + y^2) = d(1 + txy), \quad d := u^2 + v^2 > 1.$$

Next we obtain from this

$$w \geq \varphi(x, y) := (x^2 + y^2 - wxy) = d > 1, \quad w := dt. \quad (6)$$

We have here

$$\varphi(x, y) = y^2 - xz = \varphi(y, z); \quad z := wy - x.$$

Thus starting with a solution (x, y) of (6), we obtain another solution (y, z) through $z = wy - x$. Now we observe that if $x \geq y > 1$ and $\gcd(x, y) = 1$, then $y > z > 0$, and obviously $\gcd(y, z) = 1$ so that, on iteration, we finally get a solution of (6) with $y = 1$, whereas (6) has no such solution and therefore $tuv = 0$, which proves Proposition 2.

§3. About β . The following example illuminates the significance of β occurring in Proposition 1: Let $k > 2$, and let r be a positive integer such that $r + 1, \dots, r + k - 2$ are all composite. (We can take $r < k$!) Consider the sequence A of positive integers composed of primes at most r . Take any element $m > 1$ of A . Then m is divisible by a prime p which is at most r . The numbers $p - 1, p, \dots, p + k - 2$ are all smaller than $r + k - 1$ and therefore composed of primes at most r . Thus m is the second element of the following a.p. of length k with entries from A : $(p - 1)m/p, m, (p + 1)m/p, \dots, (p + k - 2)m/p$. Now it is easily seen that

$$A(x) < 2(\log x)^t,$$

where t is the number of primes at most r . Hence t is a constant depending only on k .

This example satisfies all conditions of Proposition 1, except the last one. For, we have $\delta(m) = t$ and $\alpha(m) \leq t(k - 1)$ and further, $\alpha(m) = t(k - 1)$ for all m belonging to A and which are multiples of K , defined as the product of $(p + j)$, as p runs through all of the t primes not exceeding r , and j takes values $0, 1, \dots, (k - 2)$. So, $\beta = k - 1$.

§4. Some remarks. It can easily be seen that the conjecture of Zaremba (in §1) may also be stated as follows: Let $V_q := \begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}$ and Z_ℓ denote the semigroup of matrices generated by $V_1, \dots, V_\ell$. Then every $N \geq 1$ occurs as an entry of some element in Z_5 . It is this formulation involving substitutions like $x_{j+1} = qx_j + x_{j-1}$ which links Propositions 1 and 2 with the conjecture; in fact, in Proposition 1 this connection comes by considering members of an a.p., with least element a and common difference m , as values of $qm + a$ ($0 \leq q < k$), whereas in Proposition 2 this is more explicit through the substitution $z = wy - x$.

For obtaining a result of the form (1), from (2), we need only observe that A_ℓ can be considered as a sequence A of Proposition 1, with $k = \ell + 1$,

$\beta = 1 = \beta'$. (Here, with regard to $N \in A_\ell$, $N \geq 2$ and a/N with $K(a/N) \leq \ell$ we consider the a.p.s (of length $\ell+1$): (i) with smallest member b and common difference a , where $N = aj + b$ ($0 \leq b < a, 1 \leq j \leq \ell$), and (ii) with smallest member a and common difference N .)

Also, it is apparent from the proof that for an estimate like (2) it suffices to have (i), in Proposition 1, for all sufficiently large values of m .

Incidentally, it may be noted that the argument subsequent to (6) would lead to all solutions of the simultaneous congruences

$$\xi^2 \equiv 1 \pmod{\eta}, \quad \eta^2 \equiv 1 \pmod{\xi}. \quad (7)$$

This is because (7) implies that $\xi^2 + \eta^2 - 1 = w\xi\eta$ for some positive integer w and so we obtain (6), but instead with $d = 1$. Now starting with any solution (ξ, η) of (7) we can iterate the passage from (ξ, η) to $(\eta, w\eta - \xi)$ from the proof following (6) until, after only finitely many steps, we reach $(w, 1)$. It is obvious that $(w, 1)$ is a solution of (7) for every positive integer w . Hence we obtain a complete parametrization of the set of solutions of (7). Congruences of the type (7) were earlier considered in [1].

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