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INTRODUCTION

Let G be a finite group, k an algebraically closed field whose characteristic p divides the order of G , and denote by A a symmetric interior G -algebra (that is a symmetric k -algebra A together with a homomorphism $\phi: G \rightarrow A^\times$). In [2], we introduce the notion of *Auslander-Reiten system* of G over A (cf. §1), as a “generalization” in terms of idempotents of the usual notion of almost split sequence of kG -modules, which then corresponds to the case where A is the algebra of k -endomorphisms of a kG -module. Furthermore we show in [2] that every primitive idempotent i of A^G such that $i \notin A_1^G$ is the right extremity of a unique Auslander-Reiten system, up to embedding of A into a symmetric interior G -algebra and to conjugacy by invertible G -fixed elements of that algebra (we will be talking abusively of “the Auslander-Reiten system ending with i ”); our construction of that system proceeds mainly like the construction of the almost split sequence terminating in a given kG -module M (that is we take the pullback of a projective cover of ΩM over a generator of the $\text{End}_{kG}(M)$ -socle of $\text{Ext}^1(M, \Omega^2 M)$).

Let k be the trivial kG -module and denote by $\mathcal{R}_k$ the almost split sequence terminating in k , and by $\mathcal{L}_k$ the corresponding Auslander-Reiten system of G . In [1], Auslander and Carlson show that the tensor product of any indecomposable kG -module M with the sequence $\mathcal{R}_k$ is either split or almost split up to an injective factor, and they give various criterions describing the second case (3.6., 4.7.). Using a different approach, we investigate a similar question for Auslander-Reiten systems : we give here sufficient conditions for certain pullbacks from the tensor product $i \otimes \mathcal{L}_k$ to be either split, or else equal up to a trivial system to the Auslander-Reiten system ending with i . In other words, denoting by u_G a representative of a generator of the socle of $\text{Ext}^1(k, \Omega^2 k)$ (see §1), we look for cases when the equivalence class of some tensor product with u_G lies in the socle of the bimodule which corresponds here to $\text{Ext}^1(M, \Omega^2 M)$. Since our sufficient conditions are not satisfied for all kG -modules, we obtain the related results of [1] (3.6, 4.7.) for very specific modules only. Yet our approach applies when i is a source idempotent of a block, and in that significant case (see [2], VII) we obtain an explicit generator of the socle. This last result, which was the starting point of this study, was suggested to me by Lluís Puig.

Section 1 presents our notations, the main definitions of [2] which are of use here, as well as two preliminary lemmas. We give our results in section 2.

§1 NOTATIONS AND PRELIMINARIES

We write $A^\times$ for the group of units of A and denote by a^x the element $\phi(x^{-1})a\phi(x)$ of A , where $a \in A$ and $x \in G$. The corresponding action of G makes A into a G -algebra. If H and K are two subgroups of G with $K \subset H$, we denote by A^H the algebra of H -fixed elements of A , and by $\text{Tr}_K^H: A^K \rightarrow A^H$ the relative trace map, defined by $\text{Tr}_K^H(a) = \sum a^x$, where x runs over a right transversal of H modulo K ; the image of this map is the two-sided ideal A_K^H of A^H . If P is any p -subgroup of G , we denote by $\text{Br}_P: A^P \rightarrow A(P)$ the Brauer morphism, that is the homomorphism corresponding to the quotient by the ideal $\sum_{Q \not\subset P} A_Q^P$ of A^P . Furthermore we set $\overline{A^H} = A^H/A_1^H$, and for any a in A^H we denote by $\overline{a}$ the image of a in $\overline{A^H}$. All modules and algebras are finite dimensional k -spaces, and the modules are left modules. We denote by A^o the opposite algebra of A , by $J(A)$ the Jacobson radical of A , and by ΠM (resp. ΩM) a projective cover (resp. a Heller translate) of the module M . Our tensor products are taken over k ; note that the tensor product of two (symmetric) interior G -algebras is again a (symmetric) interior G -algebra.

Suppose we are given three mutually orthogonal idempotents i, i^o and i' of A^G , together with two elements $d \in iA^G i^o$ and $d' \in i^o A^G i'$: we say that $\mathcal{S} = (i, i^o, i', d, d')$ is a *system* of G over A if we have $dd' = 0$ and if there exists (s, s') in $i^o A i \times i' A i^o$ satisfying the conditions :

$$i = ds, \quad i^o = sd + d's' \quad \text{and} \quad i' = s'd'.$$

In case $i^o \in A_1^G$, we call $\mathcal{S}$ a *Heller system*; we say $\mathcal{S}$ is *trivial* if $i = 0$, and *split* if $i \in dA^G$. Set $i^+ = i + i^o + i'$. The *commuting algebra* of the system $\mathcal{S}$ is by definition the interior G -subalgebra $A_{\mathcal{S}}$ of $i^+ A i^+$ whose elements commute with i, i^o, i', d and d' simultaneously. We call $\mathcal{S}$ an *Auslander-Reiten system* if it is a non trivial system, if the algebra $A_{\mathcal{S}}^G$ is local, and if for every symmetric interior G -algebra B and every embedding of interior G -algebras $f: A \rightarrow B$, we have $f(d)B^G = f(i)J(B^G)$ (by definition an embedding $f: A \rightarrow B$ is a homomorphism of interior G -algebras that it is one-to-one and satisfies $\text{Im } f = f(1)Bf(1)$). The idempotent i is then primitive in A^G (cf. [2]).

The following additional notations are fixed throughout this note. Considering projective covers of the kG -modules k and Ωk , we denote by E the interior G -algebra $\text{End}_k(k \oplus \Pi k \oplus \Omega k \oplus \Pi(\Omega k) \oplus \Omega^2 k)$ and write e, e^o, e', e'^o and e'' for the orthogonal idempotents of E corresponding to the projections on $k, \Pi k, \Omega k, \Pi(\Omega k)$ and $\Omega^2 k$ respectively. We consider Heller systems $\mathcal{H}_k = (e, e^o, e', h, h')$ and $\mathcal{H}_{\Omega k} = (e', e'^o, e'', m, m')$ of G over E (cf. [2]), and denote by u_G an element of $e'E^G e$ whose class $\overline{u_G}$ generates the 1-dimensional k -space $e'E^G e$ (cf. [2], III 3.).

On the other hand, we denote by P a Sylow p -subgroup of G , and we write $A(G)$ for the quotient of A^G by its two-sided ideal $\sum_{Q \not\subset P} A_Q^G = \text{Tr}_P^G(\ker \text{Br}_P)$ (if G is a p -group, then $P = G$ and we have $A(G) = \text{Br}_G(A^G)$, so the notation is consistent.) We begin with two lemmas which do not require A to be symmetric:

Lemma 1. *For any element a in $\sum_{Q \not\subset P} A_Q^G$, we have $a \otimes \overline{u_G} = 0$.*

Proof: Let Q be a proper subgroup of P . We have $u_G \in E_1^Q \cap E^G$ (cf. [2], IV 2.1.), so every $a' \in A^Q$ satisfies $\text{Tr}_Q^G(a') \otimes u_G \in (A \otimes E)_1^G$.

The converse of lemma 1 is true under certain conditions :

Lemma 2. *Suppose that $G = P$ and that the interior P -algebra A has a P -stable basis. Take u in E^P such that $\bar{u} \neq 0$ and let a be an element in A^P such that $\text{Br}_P(a) \neq 0$. Then $\bar{a} \otimes \bar{u} \neq 0$.*

Proof: Set $v = a \otimes u$ and let $\mathcal{B}$ be a P -stable basis of A . The condition $\text{Br}_P(a) \neq 0$ ensures the existence of b_0 in $\mathcal{B} \cap A^P$ such that a has a non zero b_0 -coordinate (cf. [4], 2.8.4.). We consider the projection of v onto $b_0 \otimes E$, in the decomposition $\bigoplus_{t \in \mathcal{B}} t \otimes E$ of $A \otimes E$; since u is not in E_1^P , we get $v \notin (A \otimes E)_1^P$.

§2

Fix an idempotent i in A^G , and set $B = iAi \otimes E$, $I = \sum_{Q \subseteq P} iA_i^Q i$. The tensor product with e defines an embedding of interior G -algebras from iAi to B , and B is symmetric. Denote by $\mathcal{H} = (j, j^\circ, j', c, c')$ the Heller system $i \otimes \mathcal{H}_k$ of G over B . We recall from [2] that the $(j'B^G j'; j'B^G j)$ -bimodule $j'B^G j$ has same socles as a left and as a right module, and that if i is primitive the socles have dimension 1 (cf. [2], III); furthermore in this case, if u is any element in $j'B^G j$ whose class $\bar{u}$ generates that socle, the Auslander-Reiten system ending with i is equal, up to a trivial system and to embedding, to the pullback of the Heller system $i \otimes \mathcal{H}_{\mathcal{H}k}$ over (j, u) (cf. [2], VI).

Lemma 1 shows in particular that the condition $i \in I$ (which in case i is primitive means that the Sylow subgroup P is not a defect group of i), implies that the tensor product with i , or with any idempotent of $iA^G i$, of the Auslander-Reiten system ending with e , is a split system. From now on we assume that the idempotent i does not belong to the ideal I .

Proposition 1. *For all a in $iA^G i$ whose class modulo I is in the $(iAi)(G)^{op}$ -socle of $(iAi)(G)$, the element $\bar{a} \otimes \bar{u}_G$ is in the socle of $j'B^G j$.*

Proof: Let a be such an element and set $u = a \otimes u_G$. Let $\hat{B}_{\mathcal{H}}$ denote the Heller algebra $B_{\mathcal{H}} \oplus j'Bj$ of $\mathcal{H}$ (cf. [2]). In [2], III 3. we show that every "symmetrising" form τ on B determines a central form $\tau_{\mathcal{H}, G}$ on $\hat{B}_{\mathcal{H}}^G$, which annihilates $(\hat{B}_{\mathcal{H}})_1^G$ and induces a symmetrising form on $\hat{B}_{\mathcal{H}}^G$; moreover the socle of $j'B^G j$ coincides with the orthogonal of the radical of $\hat{B}_{\mathcal{H}}^G$. Thus it is sufficient to prove that $\tau_{\mathcal{H}, G}(u \cdot J(\hat{B}_{\mathcal{H}}^G)) = 0$. Since we have $u \cdot J(\hat{B}_{\mathcal{H}}^G) = uJ((iAi \otimes eEe)^G)$ and $eEe = (eEe)^G \simeq k$, all we need to show is that the restriction of the form $\tau_{\mathcal{H}, G}$ to the space $(aJ(iA^G i)) \otimes u_G$ is zero. But the hypothesis yields $aJ(iA^G i) \in I$, so we conclude by lemma 1.

Corollary. *Suppose we have $J(iA^G i) = I$. Then for all a in $iA^G i$ the element $\bar{a} \otimes \bar{u}_G$ lies in the socle of $j'B^G j$.*

Proof: In this case the radical of $(iAi)(G)$ is $\{0\}$.

Let us assume next that $G = P$ and that i is primitive. Thus the algebra $(iAi)(P)$ is local, and it has a right socle of dimension 1. In certain cases we obtain a generator of the socle of $j'BPj$:

Proposition 2. Suppose that the P -algebra iAi has a P -stable basis. Let a be an element of $iA^P i$ whose image under Br_P generates the $(iAi)(P)^{op}$ -socle of $(iAi)(P)$. The socle of $\overline{j'B^P j}$ is the k -space generated by $\overline{a \otimes u_P}$.

Proof: Lemma 2 shows that $\overline{a \otimes u_P} \neq 0$. The result then follows from proposition 1.

Remark: (Application to kG -modules) Let us consider the special case where A is the algebra of k -endomorphisms of an indecomposable kG -module M with vertex P , and where $i = \text{id}_M$. If M is simple, or if $G = P$ and M is an endo-permutation kP -module, then $(iAi)(G) \simeq k$ (cf. [5], 5.8.), so the corollary applies: for $a = i$, our statement says that if M is simple, the tensor product of the almost split sequence $\mathcal{R}_k$ with M is either split, or almost split up to a projective direct summand. On the other hand if M is an endo-permutation kP -module, proposition 2 shows that this same tensor product is, up to a projective direct summand, the almost split sequence terminating in M . These are two special cases of the results of Auslander and Carlson on the tensor product of the sequence $\mathcal{R}_k$ with a kG -module, cf. [1], 3.6., 4.7. (indeed if M is an endo-permutation module, we have $p \nmid \dim M$, because every P -stable basis of $iAi = \text{End}_k(M)$ contains a unique fixed point, cf. [4], 2.8.4.).

Application to the source algebra of a block. Let b be a primitive idempotent of the center ZkG of kG , and set $A = kGb$. Assume b has a non trivial defect group, say D , and let i be a D -source of b , that is a primitive idempotent of A^D such that $b \in \text{Tr}_D^G(A^D i A^D)$. The hypotheses of proposition 2 are satisfied for the source algebra iAi and $P = D$. Let $SZ(D)$ denote the element $\sum_{x \in Z(D)} x$ of kD . We obtain an explicit generator of the socle of $\overline{j'B^D j}$:

Theorem. Set $a = SZ(D) \cdot i$. The socle of $\overline{j'B^D j}$ is the k -vector space generated by $\overline{a \otimes u_D}$.

Proof: Viewing the isomorphism of interior $Z(D)$ -algebras $(iAi)(D) \simeq kZ(D)$ (cf. [4], 14.5.) as an identification, the Brauer morphism $Br_D: iA^D i \rightarrow (iAi)(D)$ maps a to the element $SZ(D)$ of $kZ(D)$. Thus the $kZ(D)^{op}$ -module generated by $Br_D(a)$ is trivial, that is isomorphic to k and equal to the $kZ(D)^{op}$ -socle of $kZ(D)$. The conclusion now follows from proposition 2.

REFERENCES

- [1] M. Auslander and J.F. Carlson, *Almost-split sequences and group rings*, J. Alg. **103** (1986), 122–140.
- [2] O. Garotta, *Suites presque scindées d'algèbres intérieures et algèbres intérieures des suites presque scindées*, thesis, University Paris 7, 1988.
- [3] L. Puig, *Local fusions in block source algebras*, J. Alg. **104** (1986), 358–369.
- [4] L. Puig, *Pointed groups and construction of modules*, J. Alg. **116** (1988), 7–129.
- [5] L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.

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