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CONTENTS

INTRODUCTION

1. The themes treated in this paper have their origin in the classical theory of special functions, namely the functions that arise as solutions of linear differential equations with rational or algebraic coefficients. The study of special functions certainly goes back to the work of Gauss and Kummer on the hypergeometric differential equation. Riemann, who followed them, had a more conceptual point of view that focussed attention on the singularities of the equation and attempted to determine their influence on the solutions. In particular the programme of studying the solutions of linear differential equations with meromorphic coefficients on a compact Riemann surface undoubtedly originates with Riemann.

If the singular points of the differential equation are all regular, the monodromy group contains all the essential information. Indeed, this was the point of view of Riemann who proceeded to calculate the monodromy group of the hypergeometric equation. Moreover it is characteristic of a regular singular point that, locally at that point, the formal and analytic theories coincide. The picture changes significantly at an irregular singular point. Let P be an irregular singular point, and let us write the differential equation as a first order linear system

$$(*) \qquad du/dz = A(z)u,$$

where z is a local coordinate at P, u is an $N \times 1$ column vector, and A is an $N \times N$ matrix of functions meromorphic at $z = 0$. One then finds that, typically, formal solutions are divergent and that the formal theory of $(*)$ is no longer adequate to obtain a full understanding of the local structure of $(*)$ and its solutions. Nevertheless the formal structure of $(*)$ is the foundation on which one can erect its complete study. This is due to the fact, first discovered by Poincaré, that any formal solution of $(*)$ is *asymptotic* to an analytic solution on a sector with vertex at P, provided only that the angle of the sector is small enough. This analytic solution is however *not unique* , and will in general change when we rotate the sector keeping P as well as the formal solution fixed; this is the *Stokes phenomenon* for the system $(*)$. The constant matrices by which a fundamental matrix of $(*)$ with a prescribed asymptotic behaviour changes as we vary the sector are called the *Stokes multipliers* . It is a fundamental theorem of the subject, due to Sibuya and Malgrange, that the formal data and the Stokes multipliers associated to $(*)$ will determine it upto meromorphic equivalence.

If one is interested in a local theory of linear meromorphic differential equations it is natural to proceed as follows. Let us say that two systems $(*)$ with matrices A and B are *meromorphically equivalent* if there is an invertible $N \times N$ matrix g of functions meromorphic at $z = 0$ such that

$$B = g[A] := g A g^{-1} + (dg/dz) g^{-1}$$

This definition reflects the fact that the substitution $v = g u$ takes the system $(*)$ into the system $(*)$ with B in place of A. It is important to note that if we replace the field of germs of functions meromorphic at $z = 0$ by its formal counterpart, the quotient field of the ring $\mathbf{C}[[z]]$ of formal power series over $\mathbf{C}$, we obtain a corresponding framework of formally meromorphic systems $(*)$ and their formal meromorphic equivalence classes. Similarly the notion of meromorphic equivalence of two *analytic families* is defined in the same fashion except that the matrix g is allowed to depend analytically on the parameters of the family.

Our concern in this paper is entirely with the local structure of linear meromorphic systems. In the classical language we can describe our aims as follows : (i) to classify the systems $(*)$ upto meromorphic equivalence (ii) to give the space of equivalence classes a natural structure as an analytic space so that analytic families of systems $(*)$ are classified upto meromorphic equivalence by analytic maps into this space. It turns out that these questions are reasonable when we consider families that are *isoformal* , i. e., when *all the formal invariants of the system* $(*)$ *are fixed* . We shall find that if we fix a formal model and consider the pairs consisting of a system $(*)$ and a formal isomorphism of it with the model, the Stokes multipliers may be viewed as the elements of the first cohomology of a certain sheaf (the Stokes sheaf) of groups and that this space classifies such pairs upto meromorphic equivalence; and further that this space, which is a complex affine space $\mathbf{C}^d$ in a natural manner, is the *moduli space* for the pairs considered above. For the corresponding problems involving the systems themselves the answers are essentially the same; one has to replace the affine space by a quotient of it by an algebraic group.

Our treatment of all these questions is in the framework of vector bundles and connections. This, or alternatively, the framework of differential modules which we also make use of rather frequently, is the natural language to use for studying problems of differential equations on compact Riemann surfaces, as well as problems in higher dimensions. It is our view that it is a reasonable language also in the local context studied in this paper. In any case it is entirely adequate for treating all the problems that arise, including questions of moduli.

2. We now give a brief description of the organization of the paper. There are three parts and an appendix. The parts are divided into chapters which are in turn subdivided into sections (§). References to items within the same part omit the part number.

Part **I** is an exposition of the basic theory of meromorphic connections and their Stokes phenomena. As mentioned a little earlier, the fundamental objects of study are germs of pairs (V, ∇), where V is a holomorphic vector bundle defined on a disk Δ in the complex plane $\mathbf{C}$ containing the origin and ∇ is a holomorphic connection on $\Delta \setminus (0)$ which is meromorphic at $z = 0$. If $\nabla_{d/dz}$ is the covariant derivative defined by the connection, then choosing a trivialization at $z = 0$ allows us to represent it as $d/dz - A(z)$ where A is a matrix of size $N \times N$ with entries that are meromorphic at $z = 0$; the horizontal sections are then the vector functions u such that $du/dz = A(z)u$. In Chapter 1 we introduce the definitions and concepts and discuss the formal aspects of the theory. To any pair (V, ∇) is associated a differential module over $\mathfrak{F} = \mathbf{C}[[z]][z^{-1}]$, its *formalization*, thus giving us a functor from the category of germs of pairs to the category of formal differential modules, namely, differential modules over $\mathfrak{F}$. The structure theory of formal differential modules is well known and goes back to Hukuhara, Turrittin, and Levelt ; we formulate it in categorical terms, essentially in the form given by Deligne. In the last section of this chapter we treat isoformal families of formal differential modules and prove a fundamental structure theorem for them. In Chapter 2 the asymptotic aspects of pairs (V, ∇) are treated in detail. The basic result is Theorem 2.2.1 which asserts that for any horizontal section σ of the formalization of (V, ∇) we can find horizontal sections s of (V, ∇) on sectors Γ with sufficiently small angles such that s is asymptotic to σ on Γ. Our proof follows rather closely the discussion of Wasow [W] (§§ 12-19), but is adapted to the setting of families in which the theorem is proved. In Chapter 3 the Stokes sheaf and the Stokes lines of a pair (V, ∇) are introduced, and a formula for the so called *irregularity* of the pair is proved; this is due to Deligne. The Stokes sheaf of (V, ∇) is a sheaf of groups defined on the unit circle S^1, and its stalk at $u \in S^1$ is the group of all germs of automorphisms g of (V, ∇) defined on sectors containing u that are flat, i. e., that satisfy the asymptotic condition $g \sim 1$ on these sectors. The development contained in the first three chapters is then used in Chapter 4 to prove the fundamental theorems of the subject, namely the theorems of Malgrange-Sibuya and Deligne. Let us fix a pair (V_0, ∇_0) and consider the set $\mathcal{M}(V_0, \nabla_0)$ of all isomorphism classes of $((V, \nabla), \xi)$ where ξ is an isomorphism of the formalization of (V, ∇) with that of (V_0, ∇_0) (we shall refer to $((V, \nabla), \xi)$ as a *marked pair*). The theorem of Malgrange-Sibuya (Theorem 4.5.1) gives a canonical isomorphism of $\mathcal{M}(V_0, \nabla_0)$ with the cohomology $H^1(S^1, St_0)$ where St_0 is the Stokes sheaf of (V_0, ∇_0). This is thus the precise formulation of the result that the Stokes multipliers and formal data determine the differential equations upto meromorphic equivalence. This is then used to prove the theorem of Deligne (Theorem 4.7.3) which gives a complete description of the category of germs of pairs. To any pair (V, ∇) we can associate the sheaf of sectorial horizontal sections $\mathcal{H}(V, \nabla)$ on S^1 on whose stalks a filtration can be defined via the asymptotic growths at $z = 0$ of the elements of the stalks. This gives a functor from the category of germs of pairs to the category of certain types of filtered local systems on S^1, and Deligne's theorem is the assertion that this functor is an equivalence of categories. The final chapter of this part treats a few examples that illustrate the various aspects of the theory. In particular we give a detailed discussion of the differential equations of Bessel and Whittaker from our point of view, describing their formal reduction, the associated Malgrange-Sibuya map, and the cohomology of the Stokes sheaf.

Part **II** is devoted to a detailed study of the Stokes sheaf and its cohomology. Chapter 1 of this part is a treatment of cohomology of groups that is more or less self-contained; in particular we focus attention on the concept of *twisting* which plays an important role later. In Chapters 2 and 3 we take up the study of the cohomology of the Stokes sheaf St of a meromorphic pair. The starting point is the fundamental fact that St is a sheaf of *unipotent affine algebraic groups* . Following a beautiful suggestion of Deligne we shall view this as a sheaf of *unipotent group schemes over* **C** defined on S^1 and so obtain a functor

$$R \longrightarrow St(R)$$

from the category of **C**-algebras to the category of sheaves of groups over S^1. It follows from this that the assignment

$$R \longrightarrow H^1(S^1, St(R))$$

is a functor from the category of **C**-algebras to the category of pointed sets. The fundamental theorem is then Theorem 3.4.1 which asserts that this functor is *representable by an affine space of dimension equal to the irregularity of the endomorphism bundle* . We follow Deligne in proving this theorem as a consequence of a rather general result on sheaves of unipotent group schemes (Theorem 2.4.1). This theorem deals with sheaves $\mathcal{U}$ of unipotent group schemes that admit a filtration of normal subsheaves such that the successive quotients are again sheaves of unipotent group schemes which are in addition *elementary* in a certain sense; we remark that the notion of an elementary sheaf of group schemes is to be understood in the context of the result that the Stokes sheaf of a pair whose formalization has only one canonical level is elementary. Theorem 2.4.1 asserts that the cohomology $H^1(S^1, \mathcal{U})$ of sheaves of group schemes $\mathcal{U}$ of the type considered is representable by affine space. Theorem 3.4.1 is then proved by simply verifying that the Stokes sheaf satisfies the conditions of Theorem 2.4.1.

Part **III** deals with the problem of moduli. In Chapter 1 we establish the basic result that the space $H^1(S^1, St(V_0, \nabla_0)) := H^1$ is the moduli space for the category of marked pairs $((V, \nabla), \xi)$. In Chapter 2 we consider unmarked pairs which are intuitively more natural and examine in what sense the quotient of the space H^1 by the automorphism group $G^\wedge(V_0, \nabla_0) := G^\wedge_0$ of the formalization of (V_0, ∇_0), is a moduli space for the category of germs of pairs themselves. Since $G^\wedge_0$ is an affine algebraic group and H^1 is an affine space we are in the context of algebraic group actions studied by Mumford [MF]. In particular, if $G^\wedge_0$ is *reductive* (this is the case generically), we can construct a *geometric quotient* in the neighbourhoods of points in H^1 that lie in orbits of maximal dimension that are closed (*stable*). We give examples of stable orbits and note that if the formalization of (V_0, ∇_0) has only one canonical level, then a pair (V, ∇) defines a stable point in H^1 as soon as its *Galois differential group is irreducible* . For many classical families it is the case that for generic values of the parameters the Galois differential group is irreducible (see [DM]).

The theory of meromorphic differential equations has a long history and a very large number of mathematicians have contributed to its themes and results. Even in the limited circle of ideas that are the focus of attention of this paper, the foregoing summary has done hardly any justice to the historical aspects of the subject. We have attempted to remedy this in an appendix that contains a brief historical survey of the main themes of this paper; for further information and greater perspective the reader should consult [Be] [J] [Maj] [Mal] and the references given there.

3. We would like to express our gratitude to a large number of our friends and colleagues in various institutions who willingly gave their time and advice and helped us understand many aspects of this theory. Above all we would like to thank Professor Deligne who generously gave us his ideas to work with and who helped us when we had difficulties in understanding them. In particular, his letters to Malgrange [De 2] and to one of us [De 3], in which he sketched the essential outlines of his way of viewing and proving the fundamental theorems of the subject, were indispensable for us during the preparation of this paper. We have followed his approach in our proof of the representability theorem for the cohomology of the Stokes sheaf, not only because it is more beautiful and more elegant than our original method worked out in [BV 4], but also because of the fact that it is the only way we know to prove this theorem in the ramified case ([BV 4] treats only the unramified case). We are very grateful to him for giving us permission to use his ideas and write up his results. We would like to thank

Professor Sibuya for the extensive discussions we had with him during his visits to UCLA in 1983 and 1988-89; Professors Malgrange and Ramis for the discussions at Strasbourg and Kyoto; Professors Levelt and van den Essen for the discussions at Nijmegen; and to Professors Balser, Duval, Lutz, Jurkat, Ramis, and Sibuya for their participation in an informal seminar at UCLA during October-April of 1986. Finally we would like to thank the authorities of the Nato Institute on deformation theory held in Il Ciocco, Italy, in the Summer of 1986, and the authorities of the Taniguchi Symposium held in Katata, Japan, in the Summer of 1987, for inviting us to participate in their conferences and present the results that form the essential content of this paper.

PART I : MEROMORPHIC CONNECTIONS AND THEIR STOKES PHENOMENA

1 MEROMORPHIC CONNECTIONS, DIFFERENTIAL EQUATIONS, AND DIFFERENTIAL MODULES

1.1. As we have mentioned in the introduction, the themes treated in this paper originate from very classical questions concerning systems of linear meromorphic differential equations. However to get a deeper understanding of these problems it is essential to study them on Riemann surfaces, and more generally, on complex manifolds of arbitrary dimension. Indeed, the idea that Riemann surfaces form a natural setting for problems of ordinary differential equations appears already in the works of Riemann (perhaps only implicitly; cf. the various articles, notes, and fragments in his Collected papers), Fuchs, Poincaré, Thomé, and many others. Unfortunately the classical language is not adequate for working in this more general context, and it becomes necessary to use the more modern point of view of vector bundles with connections, or equivalently, of differential modules. This section contains a brief discussion of these languages leading to a presentation of the formal theory of linear meromorphic differential equations from the categorical point of view. The categorical approach that we have decided to take has the advantage that it allows one to formulate all the relevant results in a form that is not only the most elegant and far-reaching but also the most suitable for use in global situations. As one of the best illustrations of this point of view we recommend to the reader Deligne's solution of the Riemann-Hilbert problem [De 1].

We start with the framework of vector bundles and connections which provides a coordinate free and geometric language for treating problems of linear differential equations in the complex domain. We assume that the reader is familiar with this language, but for the sake of completeness we shall begin with a brief review of its main features that we shall need (see [De 1]). *We shall al-*

ways be in the holomorphic category unless we indicate otherwise. To any complex manifold X one can associate the category of pairs (V, ∇) where V is a vector bundle on X and ∇ is a connection on V. This category is equipped with $\oplus$, $\otimes$, $*$, and Hom, according to the following definitions (see [De 1], p.8) :

$$\nabla_\xi(s_1 \oplus s_2) = \nabla_{1,\xi}(s_1) \oplus \nabla_{2,\xi}(s_2), \qquad \nabla = \nabla_1 \oplus \nabla_2$$

$$\nabla_\xi(s_1 \otimes s_2) = \nabla_{1,\xi}(s_1) \otimes s_2 + s_1 \otimes \nabla_{2,\xi}(s_2), \qquad \nabla = \nabla_1 \otimes \nabla_2$$

$$\nabla_\xi(h)(s_1) = \nabla_{2,\xi}(h(s_1)) - h(\nabla_{1,\xi}(s_1), \quad \nabla = \mathrm{Hom}\,(\nabla_1, \nabla_2)$$

$$(\nabla^*)_\xi(s^*)(s) = - s^*(\nabla_\xi(s)) + \xi((s^*(s))$$

Here ξ is an arbitrary vector field and ∇_ξ is the covariant derivative in the direction of ξ. A local section u of (V, ∇) is said to be *horizontal* if $\nabla_\xi u = 0$ for all ξ. It is clear that a map h $(V_1 \longrightarrow V_2)$ is a morphism from (V_1, ∇_1) to (V_2, ∇_2) if and only if h is a horizontal section for $\nabla = \mathrm{Hom}\,(\nabla_1, \nabla_2)$. This simple fact is however absolutely fundamental because it allows one to reduce questions about morphisms to questions about horizontal sections; we shall make frequent use of this principle in this context as well as in others. If we choose local coordinates x_μ on X and a local trivialization for V, the covariant derivatives ∇_μ corresponding to $\partial/\partial x_\mu = \partial_\mu$ may be written as $\partial_\mu - A_\mu$ where the A_μ, the so-called *connection matrices* , are $N \times N$ matrices of holomorphic functions of the x_ν . The connection ∇ is said to be *flat* or *integrable* if its curvature is zero, the condition for which in local coordinates is

$$\partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu] = 0 \qquad (1 \leq \mu,\nu \leq N) \quad .$$

These are the classical *Frobenius* conditions of *integrability* that are necessary and sufficient for the system of partial differential equations

$$\nabla_\mu u = \partial_\mu u - A_\mu u = 0 \quad (1 \leq \mu \leq N)$$

that describe the horizontal sections to have unique local solutions for arbitrary initial data. Thus, for flat connections, the sheaf of germs of local horizontal sections is a *local system* of rank N, the rank of the bundle. Here we use the term local system of rank N on X in its usual sense, namely, a sheaf of vector spaces defined on X which is locally isomorphic to the constant sheaf with coefficients in $\mathbf{C}^N$, $1 \leq N \leq \infty$ (cf. [De 1], p 3). We note that *the integrability condition is automatic in dimension* 1, i. e., when X is a Riemann surface. We shall be exclusively concerned with this case. The flat pairs (V, ∇) form a full subcategory closed under $\oplus$, $\otimes$, $*$, and Hom.

Let X be a Riemann surface. For $x \in X$, let $\mathcal{O}_x$ be the algebra of germs of analytic functions at x, $\mathcal{M}_x$, its quotient field of germs of functions meromorphic at x, and $\mathcal{O}_X$, $\mathcal{M}_X$, the corresponding sheaves on X. For a vector bundle V of rank N defined on X let $\mathcal{O}_x(V)$ be the $\mathcal{O}_x$–module of germs of holomorphic sections of V at x, $\mathcal{M}_x(V)$, the N-dimensional vector space over the field $\mathcal{M}_x$ of germs of meromorphic sections of V at x, and $\mathcal{O}_X(V)$, $\mathcal{M}_X(V)$, the corresponding sheaves. If W is a sufficiently small open neighborhood of x, any basis of $\mathcal{M}_x(V)$ defines a trivialization of the restriction of V to $W \setminus \{x\}$. We shall refer to such trivializations as (*meromorphic*) *trivializations at* x. If ∇ is a connection defined on the restriction of V to $W\setminus\{x\}$, we say that ∇ or the pair (V, ∇) is *meromorphic at* x if ∇_ξ leaves $\mathcal{M}_x(V)$ invariant for any local vector field holomorphic at x ; this is equivalent to the requirement that for some (any) local uniformisant z at x, and with respect to some (any) local trivialization of V at x, the covariant derivative $\nabla_{d/dz}$ has the form

$$\nabla_{d/dz} = d/dz - A(z), \qquad A \in \mathfrak{gl}(N, \mathcal{M}_x);$$

here $\mathfrak{gl}(N, \mathcal{M}_x)$ is the Lie algebra of $N \times N$ matrices over $\mathcal{M}_x$. By using trivializations it is easy to see the equivalence between the abstract language and the classical one of systems of differential equations. Thus, once we choose a trivializtion, the horizontal local sections may be identified with $N \times 1$ vectors u satisfying the system of ordinary differential equations

$$du/dz = Au.$$

If we consider another connection ∇' with $\nabla'_{d/dz} = d/dz - A'$, then the pairs (V, ∇) and (V, ∇') are locally isomorphic at $z = 0$ if and only if the matrices A and A' are related by

$$B = g[A] := g A g^{-1} + (dg/dz) g^{-1} \qquad (g \in GL(N, \mathcal{M}_x)).$$

If we think of A and A' as *connection matrices* , then g may be viewed as the *gauge transformation* that defines the bundle automorphism. If V_1 and V_2

are two bundles defined on a neighborhood of x, a bundle map from V_1 to V_2 is *meromorphic at* x if it is given by a matrix of meromorphic functions of z with respect to trivializations of the two bundles at x. As usual, two pairs (V, ∇) and (V', ∇') defined and meromorphic at x, are *equivalent* if they coincide on a neighborhood of x; and the equivalence classes are known as the *germs* . The local theory deals with the germs rather than the pairs themselves, but we shall generally not insist on this distinction. If we replace the bundles, connections, and the maps by their germs we obtain the category $\mathfrak{C}_x$ of germs of meromorphic pairs at x. If z is a local uniformisant at x we can identify $\mathfrak{C}_x$ with the category $\mathfrak{C}_0$ of germs of pairs at the origin of the complex plane $\mathbf{C}_z$ of the complex variable z. Let b be an integer ≥ 1, let $\mathbf{C}_\zeta$ be the plane of a complex variable ζ, and let f_b be the map $\zeta \longrightarrow z = \zeta^b$. Associated to $\mathbf{C}_\zeta$ we have the category $\mathfrak{C}_{0,\zeta}$, and $f_b{}^*$ defines a " pull-back functor " $f_b{}^*$ from $\mathfrak{C}_0$ to $\mathfrak{C}_{0,\zeta}$; if (V, ∇) represents a germ at $0 \in \mathbf{C}_z$ and we choose a local trivialization for it at 0 so that $\nabla_{d/dz} = d/dz - A(z)$, then, with respect to the pull-back trivialization for $(V^\sim, \nabla^\sim) = f_b{}^* (V, \nabla)$, we have

$$\nabla^\sim{}_{d/d\zeta} = d/d\zeta - A^\sim, \qquad A^\sim(\zeta) = b\zeta^{b-1} A(\zeta^b).$$

Let S be a discrete subset of X. If V is a vector bundle on X and ∇ is a connection defined on $X \setminus S$, ∇ (or the pair (V, ∇)) is said to be *meromorphic at* S if it is meromorphic at each point of S; the notion of a bundle map to be meromorphic at S is defined in an analogous manner. Then it is clear that we can associate to the pair X, S the category whose objects are pairs (V, ∇) meromorphic at S, and whose maps are the bundle maps that are meromorphic at S. The basic problem of the theory of linear meromorphic differential equations is that of understanding the structure of this category, and one of the essential steps in such a programme is the study of the local version of this problem, namely, the elucidation of the category $\mathfrak{C}_0$. This is one of the main concerns of this paper.

1.2. From the algebraic point of view the basic notion is that of a D-module that goes back to Manin [Ma]. Let R be a commutative ring with unit and let D be an R–module of derivations of R; a special case is when R is a differential ring, i. e., a commutative ring with unit equipped with a derivation ∂, with D =

$R\partial$. By a *D–module over* (R, D) *or a differential module over* R we mean an R–module M together with an R–linear map ∇ ($\xi \longrightarrow \nabla_\xi$) of D into the R–module of additive maps of M into itself such that

$$\nabla_\xi(a\,m) = a\,\nabla_\xi(m) + (\xi a)\,m \qquad (a \in R,\ m \in M,\ \xi \in D).$$

For given R and D the category of D–modules comes equipped with $\oplus$, $\otimes$, $*$, and Hom. If (M, ∇) is a differential module over R we refer to it as free, projective, etc., if M, as a module over R, is free, projective etc. The solutions m to the "system of differential equations"

$$\nabla_\xi\, m = 0 \qquad (m \in M,\ \xi \in D)$$

are known as the *horizontal* elements of M. As in the geometric situation, morphisms h $((M_1, \nabla_1) \longrightarrow (M_2, \nabla_2))$ are precisely the horizontal elements of Hom $(M_1, M_2,)$. *Unless it is otherwise stated explicitly, we shall suppose that all differential modules are finitely generated over their base rings.*

If X is a Riemann surface and $S \subset X$ is a discrete set, we may take R to be the ring of meromorphic functions on X which are holomorphic on $X \setminus S$ and D to be the R–module of derivaions of R defined by meromorphic vector fields on X that are holomorphic on $X \setminus S$. Let V be a vector bundle on X and ∇ a connection on $X \setminus S$ that is meromorphic at the points of S; if we take M to be the R–module of sections of V on $X \setminus S$ that are meromorphic at S, we obtain a D–module, and the assignment that takes (V, ∇) to this D–module is a functor. At the local level, if (V, ∇) is a pair meromorphic at x, we can take R to be the field $\mathcal{M}_x$ and D to be $\mathcal{D}_x = \mathcal{M}_x\, d/dz$ where z is a local uniformisant at x; then $M = \mathcal{M}_x(V)$ is a $\mathcal{D}_x$–module which depends only on the germ defined by (V, ∇) at x. The assignment that takes this germ to the $\mathcal{D}_x$–module M is a functor from $\mathcal{C}_x$ to the category of finite dimensional $\mathcal{D}_x$–modules over $\mathcal{M}_x$ which is an equivalence of categories. In view of this we shall often permit ourselves to interchange these categories. Let $\mathcal{O}^\wedge_x$ be the formal completion of $\mathcal{O}_x$ at x, $\mathcal{M}^\wedge_x$ its quotient field, and $\mathcal{D}^\wedge_x = \mathcal{M}_x^\wedge\, d/dz$; then we have the *formalization* functor which assigns to any $\mathcal{D}_x$– module M over $\mathcal{M}_x$ the $\mathcal{D}^\wedge_x$–module $M^\wedge$ defined by $M^\wedge = \mathcal{M}^\wedge_x \otimes_{\mathcal{M}_x} M$. If $M = \mathcal{M}_x(V)$, we shall view $M^\wedge$ as the formalization of the germ of the pair (V, ∇) at x. We denote the category of $\mathcal{D}^\wedge_x$–modules over $\mathcal{M}^\wedge_x$ by $\mathcal{C}^\wedge_x$, or $\mathcal{C}_0^\wedge$, when $X = \mathbf{C}_z$, and x is the origin. In order to be consistent with the notation of our earlier papers we shall often write, when $X = \mathbf{C}_z$ and $x = 0$, $\mathfrak{F}$ for $\mathcal{M}^\wedge_x$ and $\mathfrak{F}_{cgt}$ for $\mathcal{M}_x$; moreover, when working in the complex plane $\mathbf{C}_\zeta$ we shall denote the corresponding objects by $\mathfrak{F}_\zeta$ and $\mathfrak{F}_{\zeta,cgt}$ respectively. $\mathfrak{F}$ and $\mathfrak{F}_\zeta$ differential fields with d/dz and $d/d\zeta$ as their basic derivations.

Let $\mathfrak{F}^{cl}$ be the algebraic closure of $\mathfrak{F}$. One knows from the classical theorem of Puiseux (cf. [Se], Proposition 8, p. 76) that

$$\mathfrak{F}^{cl} = \cup_{b\geq 0}\ \mathfrak{F}_b, \qquad \mathfrak{F}_b = \mathfrak{F}(z^{1/b})\ ;$$

here $\mathfrak{F}_b$ is the Galois extension of $\mathfrak{F}$ obtained by adjoining a b^{th} root of z. For any $b \geq 1$, the " pull-back " imbedding $\mathfrak{F} \hookrightarrow \mathfrak{F}_\zeta$ defined by the substitution $z = \zeta^b$ extends to an isomorphism of $\mathfrak{F}_b$ with $\mathfrak{F}_\zeta$; the extensions are not unique, and correspond to the choice of a branch $z^{1/b}$ that is mapped onto ζ. The Galois group $Gal(\mathfrak{F}_b/\mathfrak{F})$ is μ_b, the group of b^{th} roots of unity, acting by σ, $z^{1/b} \longrightarrow \sigma z^{1/b}$ $(\sigma \in \mu_b)$. The full Galois group $Gal(\mathfrak{F}^{cl}/\mathfrak{F})$ is the topological group $\mu = \lim\ \mu_b$, the inverse limit of the μ_b; we shall identify μ with $\mathbf{Z}^\wedge$, the completion of $\mathbf{Z}$, the imbedding of $\mathbf{Z}$ in $\mathbf{Z}^\wedge$ corresponding to the identification of $m \in \mathbf{Z}$ with the element of μ which projects to $\exp(2i\pi m/b)$ in μ_b. The convergent subfield $\mathfrak{F}_{b,cgt} \subset \mathfrak{F}_b$ is defined in the obvious way as $\mathfrak{F}_{cgt}[z^{1/b}]$ and is seen to be identical with the preimage of $\mathfrak{F}_{\zeta,cgt}$ under some (any) isomorphism $\mathfrak{F}_b \cong \mathfrak{F}_\zeta$ that extends the pull-back imbedding. We put

$$\mathfrak{F}_{cgt}{}^{cl} = \cup_{b\geq 0}\ \mathfrak{F}_{b,cgt}\,.$$

It is known that $\mathfrak{F}_{cgt}{}^{cl}$ is algebraically closed. Indeed, one knows that $\mathfrak{F}_{cgt}$ is algebraically closed in $\mathfrak{F}$ (see [A], p. 48, Theorem 14), so that $\mathfrak{F}_{b,cgt}$ is algebraically closed in $\mathfrak{F}_b$ for all $b \geq 1$, and hence $\mathfrak{F}_{cgt}{}^{cl}$ is algebraically closed in $\mathfrak{F}^{cl}$, thus algebraically closed.

For any $f \in \mathfrak{F}^{cl}$, ord(f), the *order* of f, is defined as usual as follows:

$$f = c_{r/b}\, z^{r/b} + c_{(r+1)/b}\, z^{(r+1)/b} + \ldots, c_{r/b} \neq 0, \ \Rightarrow\ \operatorname{ord}(f) = r/b.$$

The topology defined by the absolute value

$$|f| = c^{-\operatorname{ord}(f)} \quad (c > 1)$$

is the *adic topology* . The continuous derivations of $\mathfrak{F}^{cl}$ (resp. $\mathfrak{F}_b$) are precisely the ones of the form f d/dz, f $\in \mathfrak{F}^{cl}$ (resp. $\mathfrak{F}_b$). They form a vector space over $\mathfrak{F}^{cl}$ (resp. $\mathfrak{F}_b$) whose dual is the space of differential forms ω over $\mathfrak{F}^{cl}$ (resp. $\mathfrak{F}_b$),

$$\omega = \omega^{\#}. dz, \qquad \omega^{\#} \in \mathfrak{F}^{cl} \text{ (resp. } \mathfrak{F}_b),$$

where dz is the form that takes the value 1 at d/dz. Under an isomorphism ι ($\mathfrak{F}_b \cong \mathfrak{F}_\zeta$) that is defined by the choice of a branch $z^{1/b}$ it is easy to see that $\omega = \omega^{\#}. dz$ goes over to $\omega_\zeta = \omega_\zeta^{\#}. d\zeta$ where

$$\omega_\zeta^{\#} = b\zeta^{b-1} \iota(\omega^{\#}).$$

These definitions and formulae are consistent with the corresponding ones in the geometric context involving holomorphic differential forms and their images under the pull-back maps f_b^*, f_b ($\mathbf{C}_\zeta \longrightarrow \mathbf{C}$) being the usual covering map.

To any differential module (M, ∇) over $\mathfrak{F}$ we can associate the " pull-back " module (M_ζ, ∇_ζ) as follows: $M_\zeta = \mathfrak{F}_\zeta \otimes_{\mathfrak{F}} M$, $\nabla_{\zeta, d/d\zeta} = b\zeta^{b-1} \nabla_{d/dz}$, and

$$\nabla_{\zeta, d/d\zeta} (u \otimes m) = (du/d\zeta) \otimes m + b\zeta^{b-1} u \otimes \nabla_{d/dz} m.$$

In particular, if $A(z)$ is the connection matrix of $\nabla_{d/dz}$ with respect to the basis (m_i) of M, then the connection matrix $A^\sim(\zeta)$ of ∇_ζ with respect to the basis $(1 \otimes m_i)$ of M_ζ is given by the formula

$$A^\sim(\zeta) = b\zeta^{b-1} A(\zeta^b).$$

More generally, let us fix a branch $z^{1/b}$ and hence an isomorphism $\mathfrak{F}_b \cong \mathfrak{F}_\zeta$ extending the pull-back imbedding. If $M_\zeta = \mathfrak{F}_\zeta \otimes_{\mathfrak{F}_b} M$, then these formulae define an isomorphism from the category of differential modules over $\mathfrak{F}_b$ to the category of differential modules over $\mathfrak{F}_\zeta$.

1.3 On a complex manifold, the basic object associated to a pair (V, ∇) is the local system of horizontal sections. In the local situation with which we are concerned here it is better to work with sections defined on sectorial domains. A *sector* (in $\mathbf{C}_z$) is a subset of $\mathbf{C}_z^\times$ of the form

$$\{ z = r e^{i\theta} : \alpha < \theta < \beta \} \qquad \varphi \leq \alpha < \beta \leq 2\pi + \varphi$$

Note that sectors are always *proper* subsets of $\mathbf{C}_z^\times$. The *angle* of the sector is then $\beta - \alpha$. If $W \subset S^1$ is an open arc, $\Gamma(W)$ is the sector of all points $z = r w$, with $r > 0$, and $w \in W$. A *sectorial domain* is a region of the form $\Gamma_\delta = \Gamma \cap \Delta_\delta$ where Γ is a sector, $\delta > 0$, and Δ_δ is the disc in the z-plane of radius δ and center 0. Given the germ of a pair (V, ∇) at $z = 0$, we associate to it the sheaf $\mathcal{H}(V, \nabla) = \mathcal{H}(V)$ of germs of *sectorial* horizontal sections. This is a sheaf defined on the unit circle S^1 in the z-plane; for any $u \in S^1$, its stalk $\mathcal{H}(V, \nabla)(u)$ at u is the space of germs of horizontal sections of (V, ∇) defined on sectorial domains $\Gamma_\delta = \Gamma \cap \Delta_\delta$ where Γ is a sector containing u. If $W \subset S^1$ is an open arc, $\mathcal{H}(V)(W)$ is the space of germs of horizontal sections of V defined on $\Gamma(W)_\delta$ for some $\delta > 0$. This is a local system of rank N = the rank of V, and the assignment

$$(V, \nabla) \longrightarrow \mathcal{H}(V, \nabla)$$

defines a covariant functor from $\mathcal{C}_0$ into the category of local systems on S^1 which is compatible with $\otimes$, $*$, and Hom. In general this will not be an equivalence of categories because the nature of the singularity at $z = 0$ is not encoded in this functor. However for *regular singularities* this functor does contain all the pertinent information. In this paragraph we shall give a brief review of the local theory of regular singular connections.

A pair (V, ∇) is said to be *Fuchsian* or *Regular Singular* if on any sector Γ with vertex at $z = 0$, any horizontal section s of (V, ∇) is of *moderate growth*, i. e., for some $N \geq 0$,

$$s(z) = O(|z|^{-N}) \qquad (z \longrightarrow 0 \text{ in } \Gamma).$$

Here the O refers to the components of the section in some (hence every) trivialization at $z = 0$. For the pair (V, ∇) the point $z = 0$ is then said to be a *regular singularity.*

THEOREM 1.3.1 *The functor* $(V, \nabla) \longrightarrow \mathcal{H}(V, \nabla)$ *is an equivalence of categories when restricted to the subcategory of Fuchsian pairs .*

PROOF Let us begin the proof by recalling ([Mi], pp. 51-52) that for verifying that a functor F from a category C_1 to a category C_2 is an equivalence of categories one must prove two things : (i) it is *fully faithful* ; this means that for any two objects A and B of C_1, the map Morph(A, B) $\longrightarrow$ Morph(F(A), F(B)) is bijective, and (ii) F is *essentially surjective* , that is, every object in C_2 is isomorphic to one of the form F(A) for some $A \in C_1$. To verify that the functor under consideration is fully faithful we use the compatibility of the functor with Hom to reduce it to the proof that the assignment taking a germ of a meromorphic horizontal section of (V, ∇) to the corresponding global section of $\mathcal{H}(V, \nabla)$ is *bijective* . It is clearly injective; if s is a global section of $\mathcal{H}(V, \nabla)$ it defines a horizontal section $s^{\times}$ of (V, ∇) on the *punctured disc* $\Delta_\delta^{\times} = \Delta_\delta \setminus (0)$. But $s^{\times}$ is now of moderate growth at $z = 0$ and so meromorphic by the theorem of Riemann on removable singularities. To complete the proof it remains to show that any local system on S^1 arises as a $\mathcal{H}(V, \nabla)$ upto isomorphism. Let $\mathcal{V}$ be a local system on S^1 and $U = \mathcal{V}(1)$. We then have a (monodromy) action of the fundamental group of S^1 with base point $z = 1$ on U (see [De 1], p3). Identifying the fundmental group with **Z** in the usual way we obtain an element $\gamma \in GL(U)$ for the action of $1 \in \mathbf{Z}$. It is now a question of constructing a pair (V, ∇) such that the monodromy action of 1 on the stalk of $\mathcal{H}(V, \nabla)$ at $z = 1$ is equal to γ. Select an endomorphism C of U such that $\gamma = \exp(2\pi i C)$. Let V_U be the trivial bundle $\mathbf{C} \times U$, $\nabla_{C,d/dz} = d/dz - z^{-1}C$. Then the horizontal sections of (V_U, ∇_C) are the multi-valued functions

$$z \longrightarrow \exp(\log z \, . \, C)\, u, \qquad u \in U,$$

and so the monodromy action of 1 is $\exp(2\pi i C) = \gamma$. So $\mathcal{H}(V_U)$ is isomorphic to $\mathcal{V}$. ◆

REMARK To ensure that the assignment

$$(*) \qquad (U, \gamma) \longrightarrow (V_U, \nabla_C)$$

constructed above is functorial it is necessary to choose C to depend functorially on γ. This can be done in many ways. We may, for instance, require that all eigenvalues λ of C satisfy $0 \le \mathrm{Re}(\lambda) < 1$; C is then said to be *reduced*

([BV 1]) and we call it a *reduced logarithm* of γ. With this choice (*) is functorial and inverts the functor $(V, \nabla) \longrightarrow \mathcal{H}(V, \nabla)$.

We shall now consider the corresponding formal category. Let K denote one of $\mathfrak{F}$, $\mathfrak{F}_{cgt}$, or $\mathfrak{F}^{cl}$, the algebraic closure of $\mathfrak{F}$, and let O be the corresponding integer ring, so that $O = \mathfrak{O} = \mathbf{C}[[z]]$ if $K = \mathfrak{F}$, $O = \mathfrak{O}_{cgt} = \mathbf{C}\{z\}$ when $K = \mathfrak{F}_{cgt}$, and $O = \mathfrak{O}^{cl} = \mathbf{C}[[z]] [z^{1/2}, z^{1/3},......]$ when $K = \mathfrak{F}^{cl}$. If M is a finite dimensional vector space over K and $E \subset M$, E is called an *O-lattice* if it is a free O-module of rank equal to the dimension of M. It is well known that a subset $E \subset M$ is an O-lattice if and only if it is a finitely generated O-module whose K-span is M. We now follow Manin [Ma] and define a differential module (M, ∇) over K to be *Fuchsian* if $\exists$ an O-lattice $L \subset M$ with $z\nabla_{d/dz} L \subset L$. Thus M is Fuchsian if and only if there is a basis of M with respect to which the connection matrix of ∇ has at most a simple pole; in the classical language, such a connection matrix is said to be of the *first kind*. It is not difficult to check that this is equivalent to requiring that for each $m \in M$, the smallest O-module in M containing m and stable under $z\nabla_{d/dz}$ is finitely generated over O (see [Ma]).

The basic results in the formal theory of Fuchsian modules are the following .

1. *Every such module is isomorphic to the pair* $(U(\mathfrak{F}), \nabla_C{}^\wedge)$; *here* C *is an endomorphism of a vector space* U *over* **C**, $U(\mathfrak{F}) = \mathfrak{F} \otimes_{\mathbf{C}} U$, $\nabla_{C,}{}^\wedge{}_{d/dz} = d/dz - z^{-1}C$, *so that* $(U(\mathfrak{F}), \nabla_C{}^\wedge)$ *is the formalization of* (V_U, ∇_C).

2. *If* (V, ∇) *is a meromorphic pair and* M *is the associated module over* $\mathfrak{F}_{cgt}$ *of germs of meromorphic sections , then* (V, ∇) *is Fuchsian if and only if* (M, ∇) *is Fuchsian in the sense of the definition above .*

3. *Formal meromorphic solutions are always convergent* : *if* (V, ∇) *is Fuchsian and* $(M^\wedge, \nabla^\wedge)$ *is the differential module over* $\mathfrak{F}$ *which is its formalization , then the natural inclusion map from the space of horizontal sections of* V *to* $H(M^\wedge)$, *the space of horizontal elements of* $M^\wedge$, *is a bijection . In conjunction with* 1. *this implies that formalization is an equivalence of categories .*

Let $\mathfrak{C}_{0,Fuchs}$ (resp. $\mathfrak{C}^\wedge{}_{0,Fuchs}$) be the category of germs of Fuchsian (resp. formal Fuchsian) pairs (resp. modules), and $\mathbf{Loc}(S^1)$ the category of local systems on S^1. Let us write $\mathcal{E}$ for the full subcategory of $\mathcal{C}_{0,\text{Fuchs}}$ of all pairs (V_U, ∇_C). We shall introduce the diagram

$$\begin{array}{ccc} \mathcal{C}_{0,\text{Fuchs}} & = & \mathcal{C}_{0,\text{Fuchs}} \\ \downarrow \text{ formalization} & & \downarrow \text{ sectorial horizontal sections} \\ \mathcal{C}^{\wedge}{}_{0,\text{Fuchs}} & \longrightarrow & \mathbf{Loc}\,(S^1) \end{array}$$

Since the vertical arrows are equivalences of categories, it follows from purely categorical arguments that there are functors representing the bottom arrow for which the above diagram is commutative in the sense of equivalence of categories, and that all such functors are mutually naturally equivalent. Furthermore, as the natural inclusion of $\mathcal{E}$ into $\mathcal{C}_{0,\text{Fuchs}}$ is an equivalence of categories, these are precisely the functors $F^{\wedge}$ from $\mathcal{C}^{\wedge}{}_{0,\text{Fuchs}}$ to $\mathbf{Loc}(S^1)$ with the following property : if F is the composition $F^{\wedge} \circ$ formalization and L is the functor on $\mathcal{E}$ that takes (V_U, ∇_C) to the corresponding local system of sectorial horizontal sections, then the functors F and L are naturally equivalent on $\mathcal{E}$:

$$(*) \qquad\qquad F \approx L$$

THEOREM 1.3.2 *The above diagram is commutative in the sense of equivalences of categories for all choices of functors* $(*)$ *and only for those* ; *all the arrows are equivalences* ; *and they are all compatible with* $\bigotimes$, $*$, *and Hom*.

We shall now complete this discussion by giving an *explicit* construction of a functor representing the bottom arrow in the above diagram. Let $\mathfrak{A}$ be the differential $\mathbf{C}$-algebra of germs of analytic functions defined on sectorial domains $\Gamma_\delta = \Gamma \cap \Delta_\delta$ (Γ a sector containing $z = 1$), and let Φ be the differential subalgebra of $\mathfrak{A}$ generated by $\mathfrak{F}_{cgt}$, z^λ ($\lambda \in \mathbf{Q}$). We shall identify $\mathfrak{F}_{cgt}{}^{cl}$ with Φ and write Φ_1 for the subalgebra generated by $\{\Phi,\ z^\lambda\ (\lambda \in \mathbf{C}), \log z\}$. Put $\Psi = \mathfrak{F}^{cl} \bigotimes_\Phi \Phi_1$. Clearly Φ, Φ_1, and Ψ are differential algebras over $\mathbf{C}$ which are integral domains. $\mathbf{Z}$ acts through analytic continuation around S^1 on Φ_1, on $\mathfrak{F}^{cl}$ by Galois, and so acts on Ψ. Since $\mathfrak{F}_{cgt}{}^{cl}$ is algebraically closed (cf. §1.2) and has characteristic 0, one knows that Ψ is a domain (see [Z S], p. 198). If M is any differential module over $\mathfrak{F}^{cl}$ we define

$$M(\Psi) = \Psi \bigotimes_{\mathfrak{F}^{cl}} M.$$

Observe that if M is defined over $\mathfrak{F}$ and $M^{cl} = \mathfrak{F}^{cl} \bigotimes_{\mathfrak{F}} M$, then the action of **Z** on Ψ described above, and on M^{cl} through the imbedding of **Z** in $\mathrm{Gal}(\mathfrak{F}^{cl}/\mathfrak{F})$, leads to an action on $M^{cl}(\Psi)$ by $\mathfrak{F}$–linear mappings that preserve the connection. Finally, for any differential module U we write $H(U)$ for its space of horizontal elements.

LEMMA 1.3.3 *Let* (M, ∇) *be a Fuchsian module of dimension* N *over* $\mathfrak{F}^{cl}$ *and let* $M(\Psi)$ *be defined as above. Then* $H(M(\Psi))$ *has dimension* N *over* **C** *and*

$$(M, \nabla) \longrightarrow H(M(\Psi))$$

is a functor with values in the category of finite dimensional vector spaces and compatible with $\bigotimes$, $*$, *and Hom*.

PROOF Only the dimension statement is not immediate. To prove this we may assume that $M = (\mathfrak{F}^{cl})^N$ and $\nabla_{d/dz} = d/dz - z^{-1}C$, C being a block diagonal matrix

$$C = \text{bl. diag.}\ (\mu_1 1 + N_1, \dots , \mu_k 1 + N_k) \quad (\mu_j \in \mathbf{C} \text{ and } N_j \text{ nilpotent})$$

The columns f_i $(1 \leq i \leq N)$ of the matrix $z^C = \exp(\log z.\ C)$ are in $\Phi_1{}^N \subset \Psi^N = M(\Psi)$. As **C** is the ring of constants of Ψ and $\det(z^C)$ is a unit of Ψ, the f_i are easily seen to form a **C**-basis of $H(M(\Psi)$. ◆

PROPOSITION 1.3.4 *Let* (M, ∇) *be a Fuchsian module over* $\mathfrak{F}$ *and let* (M^{cl}, ∇^{cl}) *be the corresponding module over* $\mathfrak{F}^{cl}$. *Then there is an action of* **Z** *on* $H(M^{cl}(\Psi))$, *and the assignment*

$$(M, \nabla) \longrightarrow H(M^{cl}(\Psi))$$

is an equivalence of categories satisfying $(*)$ *above*.

PROOF Here we are identifying (through the monodromy action) **Loc**(S^1) with the category of vector spaces over **C** with **Z**-actions. Now **Z** acts on $M^{cl}(\Psi)$, the action leaving the connection invariant. This leads to the action of **Z** on $H(M^{cl}(\Psi))$. We claim that $(M, \nabla) \longrightarrow H(M^{cl}(\Psi))$ is a fully faithful assignment. Since this functor is compatible with Hom, it comes down to proving that $H(M) = H(M^{cl}(\Psi))^{\mathbf{Z}}$. Take M and ∇ as in the proof of the lemma but with C reduced. The horizontal section $z^C u$ ($u \in \mathbf{C}^N$) is invariant under $\mathbf{Z}$ if and only if $\exp(2\pi i C)\, u = u$, or if and only if $Cu = 0$. But then $z^C u = u \in H(M)$. Since the functor assigns $(U, \exp(2i\pi C)$ to $(U(\mathfrak{F}), \nabla_C{}^\wedge)$, we are through. ◆

The above argument involved the consideration of maps between differential modules that are not linear but only semi linear. This can be done systematically by introducing the *extended* category of differential modules over $\mathfrak{F}^{cl}$ or $\mathfrak{F}_{cgt}{}^{cl}$ in which the morphisms are *extended*, i.e., are allowed to be σ–llinear, $\sigma \in \mathrm{Gal}(\mathfrak{F}^{cl}/\mathfrak{F})$ or $\mathrm{Gal}(\mathfrak{F}_{cgt}{}^{cl}/\mathfrak{F})$ while preserving the connections ; we recall that a map L between vector spaces over a field E is σ–linear, σ being an automorphism of E, if it is additive and satisfies $L(cu) = \sigma(c)\, L(u)$ for all vectors u and all $c \in E$. If M, M' are two differential modules over $\mathfrak{F}_{cgt}{}^{cl}$, a $(M \longrightarrow M')$ is a σ–linear morphism, and if we write

$$M^{cl} = \mathfrak{F}^{cl} \otimes_{\mathfrak{F}_{cgt}{}^{cl}} M, \qquad M'^{cl} = \mathfrak{F}^{cl} \otimes_{\mathfrak{F}_{cgt}{}^{cl}} M',$$

then there is a well defined σ–linear morphism $a^{cl}\,(M^{cl} \longrightarrow M'^{cl})$ such that

$$a^{cl}\,(f \otimes m) = (\sigma.f) \otimes a(m).$$

It follows from this that formalization $M \longrightarrow M^{cl}$ is a well defined functor in the context of the extended categories. We now have

PROPOSITION 1.3.5 *Formalization is an equivalence on the Fuchsian extended subcategories* .

PROOF It is only a question of verifying that it is fully faithful. Choose bases for M and M' so that $M = (\mathfrak{F}_{cgt}{}^{cl})^N$, $M' = (\mathfrak{F}_{cgt}{}^{cl})^{N'}$, $\nabla_{d/dz} = d/dz - z^{-1}C$ and $\nabla'_{d/dz} = d/dz - z^{-1}C'$ where C and C' are complex matrices. If now $\beta\,(M^{cl} \longrightarrow M'^{cl})$ is a σ–linear morphism, then there is a $\gamma\,(M^{cl} \longrightarrow M'^{cl})$ which is a morphism in the *usual sense* such that $\beta = \gamma \circ \sigma$. But by the result for the usual categories γ is represented by a convergent matrix and so the same is true of β. In other words, $\beta = b^{cl}$ for a uniquely determined σ–linear morphism $b\,(M \longrightarrow M')$. ◆

1.4. The theory of the category of not necessarily Fuchsian pairs is dominated by the fact that formalization is not an equivalence of categories. Furthermore the formal theory itself is much richer. We shall now give a brief review of it, referring the reader to [BV 1] [Be] and [J] for more detailed expositions. The fundamental results are due to Hukuhara [Hu], Turrittin [Tu]. and Levelt [Le] with important additions from the categorical point of view due to Deligne [De 2].

Let $\mathfrak{X}^{cl} = \mathfrak{X}(\mathfrak{F}^{cl})$ be the vector space of differential forms $\omega = \omega^{\#}.\,dz$ where $\omega^{\#} \in \mathfrak{F}^{cl}$ is a linear combination of the powers z^a, $a \in \mathbf{Q}$, $a < -1$, and let $\mathfrak{X}^b = \mathfrak{X}(\mathfrak{F}_b)$ be the subspace of these ω that are defined over $\mathfrak{F}_b$, namely, for which $\omega^{\#} \in \mathfrak{F}_b$. We select a finite nonempty subset $\Sigma \subset \mathfrak{X}^{cl}$ and a finite dimensional vector space U over **C** equipped with a grading by $\mathfrak{X}^{cl}$ such that the nonzero components of the grading correspond to the elements of Σ:

$$U = \bigoplus_{\omega} U_{\omega}, \qquad U_{\omega} \neq 0 \Leftrightarrow \omega \in \Sigma.$$

Let P_{ω} $(U \longrightarrow U_{\omega})$ be the associated projections. For each ω we choose an endomorphism C_{ω} of U_{ω} and define B as the endomorphism of $U(\mathfrak{F}^{cl}) = \mathfrak{F}^{cl} \otimes_{\mathbf{C}} U$ given by

$$B = \sum_{\omega \in \Sigma} \omega^{\#}.\, 1 \otimes P_{\omega} + z^{-1} \otimes C \qquad (C = \bigoplus_{\omega} C_{\omega}).$$

We refer to $U, \Sigma, (U_{\omega})_{\omega \in \Sigma}, (C_{\omega})_{\omega \in \Sigma}$ as *formal data* , and to them we associate the differential module $(M^{\wedge}_B, \nabla^{\wedge}_B)$ over $\mathfrak{F}^{cl}$, where $M^{\wedge}_B = U(\mathfrak{F}^{cl})$, $\nabla^{\wedge}_{B,d/dz} = d/dz - B$. Note that $(M^{\wedge}_B, \nabla^{\wedge}_B)$ is the formalization of (M_B, ∇_B) where $M_B = U(\mathfrak{F}_{cgt}{}^{cl}) = \mathfrak{F}_{cgt}{}^{cl} \otimes_{\mathbf{C}} U$, and $\nabla_{B,d/dz} = d/dz - B$. The module $(M^{\wedge}_B, \nabla^{\wedge}_B)$ is called a *canonical form* . The rational numbers $a < -1$ such that z^a occurs with a nonzero coefficient in some $\omega^{\#}$ $(\omega \in \Sigma)$ are the *canonical levels* , and the smallest of these is known as the *principal level* or the *Katz invariant* ; Σ itself is called the *spectrum* . If Σ consists only of 0, it is clear that $U = U_0$, $C \in \mathrm{End}(U)$, and $B = z^{-1} \otimes C$. If $b \geq 1$ is an integer such that all the $\omega^{\#} \in \mathfrak{F}_b$ $(\omega \in \Sigma)$, and the endomorphism C of U has the property that the real parts of all its eigenvalues are in $[0, 1/b)$, i. e., bC is reduced, we shall say that the canonical form (resp. C, the formal data) is b-*reduced* . Let us now choose a branch $z^{1/b}$ and let $\Sigma \subset \mathfrak{L}(\mathfrak{F}_b)$; and let L be the set of canonical levels so that $L = \{ r_1, \ldots, r_m \} \subset (1/b)\,\mathbf{Z}$ ($r_1 < \ldots < r_m < -1$). Clearly there are uniquely defined endomorphisms C, D_r ($r \in L$) of U such that

(a) C, D_r ($r \in L$) commute with each other

(b) $D_r \neq 0$ and is semisimple for all $r \in L$

(c) $B = \sum_{r \in L} z^r \otimes D_r + z^{-1} \otimes C$

It is easy to check that Σ is the set of forms $\omega = (\Sigma_{r\in L}\, c_r z^r).\, dz$ where $(c_r)_{r\in L}$ is in the joint spectrum of $(D_r)_{r\in L}$, $U = \bigoplus_\omega U_\omega$ is the spectral decomposition of U with respect to the D_r. This is the way canonical forms were defined by us in [BV 1]. The fundamental result of the Hukuhara- Turrittin- Levelt theory is now the following (see [BV 1], §§ 6-7).

PROPOSITION 1.4.1 *Any differential module over $\mathfrak{F}^{cl}$ is isomorphic to a canonical form defined by some formal data* U, Σ, $(U_\omega)_{\omega\in\Sigma}$, $(C_\omega)_{\omega\in\Sigma}$. *The spectrum Σ and the dimensions of the U_ω are uniquely determined by the isomorphism class of the module .*

If M is a differential module over $\mathfrak{F}$, the above result may be applied to the module $M^{cl} = \mathfrak{F}^{cl} \otimes_{\mathfrak{F}} M$, and the theory of the modules over $\mathfrak{F}$ may be worked out with some additional Galois descent arguments. We shall define the canonical levels and spectrum of M to be those of M^{cl}. M is said to be *unramified* if its canonical levels are all integers. More generally, we shall say that M is *unramified over* $\mathfrak{F}_b$ if all its canonical levels are in $(1/b)\mathbf{Z}$; the *ramification index* of M is defined as the smallest of such integers b. It is known that b is a divisor of the least common multiple of {1,2,... , N} where $N = \dim_{\mathfrak{F}}(M)$ ([BV 1], Proposition 7.6). From [BV 1] (§§6-7) we have

PROPOSITION 1.4.2 *Let* M *be a differential module over* $\mathfrak{F}_b$ *unramified over* $\mathfrak{F}_b$. *Then* M *is isomorphic to a canonical form determined by a unique (upto isomorphism) set of* b-*reduced data* U, Σ, $(U_\omega)_{\omega\in\Sigma}$, $(C_\omega)_{\omega\in\Sigma}$. *More precisely , the assignment that takes* b-*reduced formal data to the corresponding canonical forms is a functor ,with values in the category of differential modules over* $\mathfrak{F}_b$ *unramified over* $\mathfrak{F}_b$, *that is an equivalence of categories .*

Consider now a b-reduced canonical form defined by the formal data U, Σ, $(U_\omega)_{\omega\in\Sigma}$, $(C_\omega)_{\omega\in\Sigma}$ but with the additional property that Σ is stable under the Galois action of μ_b. A *descent structure* for this formal data is then any representation t ($\sigma \longrightarrow t(\sigma)$) of μ_b in U such that

(DES 1) $$t(\sigma)\, P_\omega\, t(\sigma)^{-1} = P_{\sigma\omega}\,,\quad t(\sigma)\, C_\omega\, t(\sigma)^{-1} = C_{\sigma\omega}\,.$$

Two descent structures t, t' are said to be *isomorphic* if there is a $\tau \in GL(U)$ such that

(DES 2) $$t'(\sigma) = \tau\, t(\sigma)\tau^{-1}\,,\quad \tau\, P_\omega\tau^{-1} = P_\omega\,,\quad \tau\, C_\omega\tau^{-1} = C_\omega\,.$$

If M is a differential module over $\mathfrak{F}$ unramified over $\mathfrak{F}_b$, Proposition 1.4.2 shows that there is an isomorphism h of $\mathfrak{F}_b\otimes_{\mathfrak{F}}M$ over $\mathfrak{F}_b$ to a b-reduced canonical form $(M^\wedge{}_B, \nabla^\wedge{}_B)$:

$$h : \mathfrak{F}_b\otimes_{\mathfrak{F}}M \cong (\mathfrak{F}_b\otimes_{\mathbf{C}}U, \nabla^\wedge{}_B).$$

For any $\sigma \in \mu_b$, $(\mathfrak{F}_b\otimes_{\mathbf{C}}U, \nabla^\wedge{}_{\sigma[B]})$ is also a b-reduced canonical form where

$$\sigma[B] = \Sigma_{\omega\in\Sigma}\ \sigma.\omega^{\#}.\, 1\otimes P_\omega + z^{-1}\otimes C\,,$$

and $\sigma\otimes 1$ transforms $(\mathfrak{F}_b\otimes_{\mathbf{C}}U, \nabla^\wedge{}_B)$ into $(\mathfrak{F}_b\otimes_{\mathbf{C}}U, \nabla^\wedge{}_{\sigma[B]})$. On the other hand, as $\mathfrak{F}_b\otimes_{\mathfrak{F}}M$ is invariant under $\sigma^{-1}\otimes 1$, $(\sigma\otimes 1)\, h\, (\sigma^{-1}\otimes 1)\, h^{-1}$ is an isomorphism of $(\mathfrak{F}_b\otimes_{\mathbf{C}}U, \nabla^\wedge{}_B)$ onto $(\mathfrak{F}_b\otimes_{\mathbf{C}}U, \nabla^\wedge{}_{\sigma[B]})$. Proposition 1.4.1 now shows that $\sigma.\Sigma = \Sigma$ for all σ and Proposition 1.4.2 shows that the isomorphism in question must be of the form $1\otimes t(\sigma)^{-1}$ for a unique $t(\sigma) \in GL(U)$ satisfying (DES 1). We thus have for all $\sigma \in \mu_b$,

(*) $$h\,(\sigma\otimes 1)\, h^{-1} = \sigma\otimes t(\sigma).$$

It is immediate from (*) that t is a representation of μ_b in U. In other words, t is a descent structure. If we choose another isomorphism h' instead of h, $h'h^{-1}$ is an automorphism of $(\mathfrak{F}_b\otimes_{\mathbf{C}}U, \nabla^\wedge{}_B)$, and so it follows from Proposition 1.4.2 that $h' = (1\otimes\tau)\, h$ for some $\tau \in GL(U)$ satisfying (DES 2). This means that t and t' are isomorphic. We have thus associated to M in a natural manner an isomorphism class of descent structures. At the same time, (*) shows that the Galois action $\sigma \longrightarrow \sigma\otimes t(\sigma)$ leaves $\nabla^\wedge{}_B$ invariant, and that the fixed point subspace $M^\wedge{}_t$ of $\mathfrak{F}_b\otimes_{\mathbf{C}}U$ for this action inherits the structure of a differential module over $\mathfrak{F}$, and finally that h is an isomorphism over $\mathfrak{F}$ of M with $M^{\wedge}_t$. If Σ is stable under μ_b and t is any descent structure for the formal data,exactly the same argument applies to show that $M^{\wedge}_t$ is a differential module over $\mathfrak{F}$ with t as the associated descent structure. We have thus obtained the central result of the theory of differential modules over $\mathfrak{F}$ (see [BV 1], §§ 6-7) :

PROPOSITION 1.4.3 *There is a canonical bijection from the set of isomorphism classes of differential modules over $\mathfrak{F}$ unramified over $\mathfrak{F}_b$ and isomorphism classes of* b-*reduced formal data equipped with descent structures .*

Actually, the entire discussion preceding this proposition could have been carried out starting with $\mathfrak{F}_{b,cgt}$ instead of $\mathfrak{F}_b$; the module determined by the subspace M_t of fixed points for the Galois action in $U(\mathfrak{F}_{b,cgt})$ would then be defined over $\mathfrak{F}_{cgt}$. It is obvious that $M_t = M_t^{\wedge} \cap (\mathfrak{F}_{b,cgt} \otimes_{\mathbf{C}} U)$ and that $M_t^{\wedge}$ is the formalization of M_t.

It is useful to have an explicit construction of the meromorphic pair in the z-plane whose module of sections is the $\mathfrak{F}_{cgt}$-module M_t described above. We go over to the plane $\mathbf{C}_\zeta$ of the complex variable $\zeta = z^{1/b}$ and consider the pair (V_ζ , ∇_ζ) where V_ζ is the trivial bundle $\mathbf{C}_\zeta \times U$ and ∇_ζ is determined by $\nabla_{\zeta,d/d\zeta} = d/d\zeta - B^{\sim}(\zeta)$, with

$$B^{\sim}(\zeta) = \Sigma_{\omega \in \Sigma} \; \omega_\zeta^{\#}. 1 \otimes P_\omega + \zeta^{-1} \otimes bC .$$

We seek a pair (V_1, ∇_1) with $\nabla_{1,d/dz} = d/dz - B_1(z)$ together with an isomorphism h of the pull back of (V_1, ∇_1) with (V_ζ , ∇_ζ). If $\varepsilon = \exp(2i\pi/b)$, the equation $h (\varepsilon \otimes 1) h^{-1} = \varepsilon \otimes t(\varepsilon)$ then becomes, on computing the effect of both sides on arbitrary sections of (V_ζ , ∇_ζ),

$$h(\zeta) h(\varepsilon\zeta)^{-1} = t(\varepsilon).$$

As $t(\varepsilon)^b = 1$, $t(\varepsilon)$ is diagonizable with eigenvalues that are b^{th} roots of unity, and so it follows from Hilbert's Theorem 90 (see [Se] , pp.158-159, also [BV 1] , p.58) that we can find a holomorphic h locally defined around $\zeta = 0$ with values in GL(U) such that $h(\zeta) h(\varepsilon\zeta)^{-1} = t(\varepsilon)$. With any such choice of h B_1 is determined from $B^{\sim}$ by

$$h^{-1}[B^{\sim}](\zeta) = b\zeta^{b-1}B_1(\zeta^b) \qquad (h^{-1}[B^{\sim}] = h^{-1}B^{\sim}h - h^{-1}dh/d\zeta).$$

A slight modification of the preceding analysis actually yields the following more precise result.

PROPOSITION 1.4.4 *Fix an integer* $b \geq 1$. *Then the assignment*

$$(\textit{formal data} + \textit{descent structure } t) \longrightarrow M_t^{\wedge}$$

is functorial and defines an equivalence of categories from the category of formal data with descent structures with the category of differential modules over $\mathfrak{F}$ *that are unramifired over* $\mathfrak{F}_b$.

REMARK Let M be a differential module over $\mathfrak{F}$ with canonical levels

$$r_1, \ldots, r_m.$$

Let M_ζ be the pull back module over $\mathfrak{F}_\zeta$ where $z = \zeta^b$. It is then immediate that the canonical levels of M_ζ are

$$br_1 + b-1, \ldots, br_m + b-1.$$

Indeed, it is enough to verify this for canonical forms for which it is obvious since the pull back of $\omega^{\#}(z^{1/b}).\, dz$ is $\omega^{\#}(\zeta)\, b\, \zeta^{b-1}.\, d\zeta$.

One can now get a complete description of the category of differential modules over $\mathfrak{F}$. We begin by constructing a functor $M \longrightarrow M_{cgt}$ from the extended category (cf. §1.3) of differential modules over $\mathfrak{F}^{cl}$ to the corresponding category over $\mathfrak{F}_{cgt}{}^{cl}$ that inverts formalization, namely, has the property $(M_{cgt})^{\wedge} \approx M$ for all M, $\approx$ denoting natural isomorphism of functors. First of all, as formalization is an equivalence for Fuchsian modules by Proposition 1.3.5, we can find such a functor on the subcategory of Fuchsian modules over $\mathfrak{F}^{cl}$. For any $\omega = \omega^{\#}.\, dz \in \mathfrak{L}^{cl}$ let $L(\omega) = \mathfrak{F}_{cgt}{}^{cl}$ be the one dimensional module over $\mathfrak{F}_{cgt}{}^{cl}$ equipped with the connection

$$u \longrightarrow du/dz - \omega^{\#} u \qquad (u \in \mathfrak{F}_{cgt}{}^{cl}),$$

and let $L(\omega)^{cl}$ be its extension to $\mathfrak{F}^{cl}$. Then it is a consequence of the Hukuhara-Turrittin- Levelt theory that any differential module M over $\mathfrak{F}^{cl}$ has a unique decomposition

$$M = \bigoplus_\omega M(\omega) \qquad (\ \omega \in \mathfrak{L}^{cl},\ M(\omega) = 0 \text{ for almost all } \omega\ )$$

where the $M(\omega)$ are differential submodules uniquely determined by the requirement that

$$F(M)(\omega) := L(-\omega)^{cl} \otimes M(\omega)$$

is Fuchsian for all ω. We now define

$$M_{cgt} = \bigoplus_\omega M_{cgt}(\omega), \quad M_{cgt}(\omega) = L(\omega) \otimes F(M)(\omega)_{cgt}.$$

It is easy to convince oneself that $M \longrightarrow M_{cgt}$ is a functor (relative to the extended categories) that inverts formalization. If now M is a differential module over $\mathfrak{F}$ and $M^{cl} = \mathfrak{F}^{cl} \otimes_{\mathfrak{F}} M$, then the above constructions are applicable to M^{cl}. But now we have in addition the Galois action of $\mu = \mathrm{Gal}(\mathfrak{F}^{cl}/\mathfrak{F})$ on M^{cl} and it is clear that $\sigma \in \mu$ is a morphism (extended) from $M^{cl}(\omega)$ (resp. $F(M)(\omega)$, $F(M)(\omega)_{cgt}$) to $M^{cl}(\sigma.\omega)$ (resp. $F(M)(\sigma.\omega)$, $F(M)(\sigma.\omega)_{cgt}$), and so the subspace of M^{cl} of elements fixed by μ defines a differential module over $\mathfrak{F}_{cgt}$. We denote this by M_{cgt}.

LEMMA 1.4.5 *The assignment*

$$M \longrightarrow M_{cgt}$$

is a functor from $\mathfrak{C}_0{}^\wedge$ *to* $\mathfrak{C}_0$ *that inverts formalization . Moreover the Fuchsian module*

$$F(M^{cl})_{cgt} := \bigoplus_\omega (F(M^{cl})(\omega))_{cgt}$$

over $\mathfrak{F}_{cgt}{}^{cl}$ *admits a Galois action whose fixed points define a Fuchsian module* $F(M)$.

PROOF This is clear from the preceding discussion. ◆

We now proceed in analogy with the Fuchsian case treated in §1.3. Let Φ_2 be the subalgebra of $\mathfrak{A}$ generated by Φ_1 and all the functions of the form

$$E(\omega)(z) = \exp\left(\int_1^z \omega^{\#} . dz\ \right), \qquad \omega \in \mathfrak{L}^{cl}.$$

We have an action of $\mathbf{Z}$ on Φ_2 by analytic continuation around S^1. For any differential module M over $\mathfrak{F}_{cgt}{}^{cl}$ we write $M(\Phi_2) = \Phi_2 \otimes_\Phi M$ and define the functor h from $\mathfrak{G}_0{}^\wedge$ to the category of complex finite dimensional vector spaces by

$$h(M) = H(M^{cl}{}_{cgt}(\Phi_2)) = \bigoplus_\omega H(M^{cl}{}_{cgt}(\omega)(\Phi_2))$$

For any $m \in \mathbf{Z} \hookrightarrow \mu$ the maps $M^{cl}{}_{cgt}(\omega) \longrightarrow M^{cl}{}_{cgt}(m.\omega)$ define maps

$$H(M^{cl}{}_{cgt})(\omega)(\Phi_2)) \longrightarrow H(M^{cl}{}_{cgt})(m.\omega)(\Phi_2)),$$

so that we have an action of $\mathbf{Z}$ on $H(M^{cl}{}_{cgt}(\Phi_2))$. In other words, we may view h as a functor with values in the category of finite dimensional vector spaces with $\mathfrak{X}^{cl}$ grading and a compatible $\mathbf{Z}$-action. On the other hand, as

$$M^{cl}{}_{cgt}(\omega) = L(\omega) \otimes F(M^{cl})(\omega)_{cgt},$$

we have the obvious relation

$$H(M^{cl}{}_{cgt}(\omega)(\Phi_2)) \approx E(\omega) \otimes H(F(M^{cl})(\omega)_{cgt}(\Phi_2)),$$

so that, we have the natural isomorphism

$$h(M) \approx h^F(M) := H(F(M^{cl})_{cgt}(\Phi_2)) = \bigoplus_\omega H(F(M^{cl})(\omega)_{cgt}(\Phi_2)) .$$

PROPOSITION 1.4.6 *The functor of formalization from* $\mathfrak{G}_0$ *to* $\mathfrak{G}_0{}^\wedge$ *is essentially surjective , namely ,every formal differential module over* $\mathfrak{F}$ *is isomorphic to the formalization of a meromorphic pair at* $z = 0$. *Furthermore ,the functor* h *is an equivalence of categories compatible with* $\otimes$, $*$, *and Hom from* $\mathfrak{G}_0{}^\wedge$ *to the category of* $\mathfrak{X}^{cl}$ *-graded vector spaces over* $\mathbf{C}$ *of finite dimension equipped with a compatible* $\mathbf{Z}$*-action .*

PROOF The first assertion has been proved already. For the second it is convenient to work with h^F rather than h. The compatibility with $\otimes$, $*$, and Hom is obvious. If we now observe that gradation preserving linear maps of two graded vector spaces U, U' are in natural bijection with the elements of the graded component corresponding to the zero element of $\mathrm{Hom}(U, U')$, we can, by the compatibility with Hom, reduce the proof that h^F is fully faithful to showing that $H(M) = H(F(M^{cl})_{cgt}(0)(\Phi_2))^{\mathbf{Z}}$ where 0 refers to the component of the grading corresponding to the zero element. We now observe that $F(M^{cl})_{cgt}(0) = (M^{cl})_{cgt}(0)$ is a differential submodule of $M^{cl}{}_{cgt}$ which is Fuchsian and invariant under the Galois group, so that it may be viewed as arising through extension of scalars from a Fuchsian submodule $M(0)$ of M. The proof of Proposition 1.3.4 now shows that $H(M(0)) = H(F(M^{cl})_{cgt}(0)(\Phi_2))^{\mathbf{Z}}$. Hence we see that $H((F(M^{cl})_{cgt}(0))^{\mathbf{Z}} \subset H(M)$. On the other hand, a direct calculation shows that $H(M^{cl}(\omega)) = 0$ if $\omega \neq 0$, so that $H(M) \subset H(M^{cl}(0))^{\mathbf{Z}}$; from this we get $H(M) \subset H((F(M^{cl})(0)(\Phi_2))^{\mathbf{Z}}$. For proving the essential surjectivity of the functor h^F, we start with a pair $(U = \bigoplus_\omega U_\omega, \gamma)$ where $\gamma \in GL(U)$ is the action of $1 \in \mathbf{Z}$ in U, and let $b \geq 1$ be an integer such that all the $\omega^\#$ for which $U_\omega \neq 0$ are in $\mathfrak{F}_b$. Clearly γ^b leaves each U_ω invariant and so we can find a b-reduced $C_\omega \in End(U_\omega)$ such that $\gamma^b = \exp(2\pi ibC)$ where $C = \bigoplus_\omega C_\omega$. Since bC is reduced, it follows that γ commutes with bC, hence with C, and that $\gamma^m \exp(-2\pi imC)$ depends only on the residue class of m mod b, so that there is a unique way to define a representation t of μ_b in U such that $t(\exp(2\pi im/b)) = \gamma^m \exp(-2\pi imC)$. It is then easy to see that t is a descent structure for the canonical form defined by the formal data U, Σ, $(U_\omega)_{\omega\in\Sigma}$, $(C_\omega)_{\omega\in\Sigma}$. We thus have a well defined module (M_t, ∇_t) over $\mathfrak{F}_{cgt}$. An elementary calculation shows that the monodromy action of $1 \in \mathbf{Z}$ (which corresponds to the Galois action $\sigma \otimes t(\sigma)$ for $\sigma = \exp(2i\pi/b)$ on the horizontal elements) is

$$\exp(\log z.C)u \longrightarrow \exp(\log z.C)\exp(2i\pi C)t(\sigma)u = \exp(\log z.C)(\gamma u) \quad (u \in U).$$

It folows from this that our functor takes (M_t, ∇_t) to $(U = \bigoplus_\omega U_\omega, \gamma)$ upto isomorphism. ◆

Given differential modules $M^\wedge$ over $\mathfrak{F}$ and M over $\mathfrak{F}_{cgt}$ it is natural to say that M is *formally isomorphic* to $M^\wedge$ if there is an isomorphism

$$\xi : \mathfrak{F} \bigotimes_{\mathfrak{F}_{cgt}} M \cong M^\wedge .$$

Any such isomorphism is called a *marking of* M *by* $M^\wedge$. For fixed $M^\wedge$ the pairs (M, ξ) form a category such that the morphisms $(M, \xi) \longrightarrow (M', \xi')$ are the morphisms u $(M \longrightarrow M')$ that are compatible with ξ and ξ'. If (V, ∇) is a meromorphic pair at $z = 0$ and M is the differential module over $\mathfrak{F}_{cgt}$ of the germs of meromorphic sections of V, then, a *marking of* (V, ∇) *by* $M^\wedge$ is, by definition, a marking of M by $M^\wedge$. The pairs marked by $M^\wedge$ form a category in the obvious fashion.

1.5 The theory of formal differential modules can be developed over fairly general rings (cf. [BV 2]). In this paragraph we shall confine ourselves to a few consequences of the work of [BV 2] that will be important in the context of the moduli problems treated in **III** as well as in the asymptotic theory of families of differential equations treated in the next section.

We begin by formalizing the notion of an analytic family of formal differential modules over $\mathfrak{F}$. Let $d \geq 1$ be an integer, and for any open set $\Omega \subset \mathbf{C}^d$ let $\mathcal{O}_d(\Omega)$ be the algebra of analytic functions on Ω. We use the symbol Δ with or without suffixes to denote polydiscs in $\mathbf{C}^d$ centered at the origin. We then define

$$\mathcal{O}_{d.b}(\Delta) = \mathcal{O}_d(\Delta)[[z^{1/b}]]\,[z^{-1}]\,, \quad \mathcal{O}_{d,b} = \cup_\Delta \mathcal{O}_{d.b}(\Delta)\,.$$

These are all differential algebras in the obvious sense with the elements of the coefficient rings $\mathcal{O}_d(\Delta)$ behaving like constants with respect to d/dz. A *family of differential modules of dimension* N *over* $\mathfrak{F}$ (resp. $\mathfrak{F}_b$) is by definition a differential module (M, ∇) over $\mathcal{O}_{d,1}$ (resp. $\mathcal{O}_{d,b}$) with the underlying module over $\mathcal{O}_{d,1}$ (resp. $\mathcal{O}_{d,b}$) being free of rank N. By choosing a basis for M we find that there is a Δ such that (M, ∇) is isomorphic to a module (M', ∇') defined over $\mathcal{O}_{d.1}(\Delta)$ (resp. $\mathcal{O}_{d.b}(\Delta)$). We can obviously specialise (M', ∇') at the points $\lambda \in \Delta$ to obtain an assignment $\lambda \longrightarrow (M'_\lambda\,, \nabla'_\lambda)$ $(\lambda \in \Delta)$ of differential modules over $\mathfrak{F}$ (resp. $\mathfrak{F}_b$) parametrized by Δ. If the isomorphism class of $(M'_\lambda, \nabla'_\lambda)$ over $\mathfrak{F}$ (resp. $\mathfrak{F}_b$) is *independent* of λ for all λ in some Δ', we shall say that (M, ∇) is an *isoformal family of differential modules over* $\mathfrak{F}$ (resp. $\mathfrak{F}_b$), and refer to the constant isomorphism class of $(M'_\lambda\,, \nabla'_\lambda)$ as *the class of the family*. This property is clearly independent of the choice of (M', ∇') or Δ'. The fundamental theorem on isoformal families is the following.

THEOREM 1.5.1 *Let* (M, ∇) *defined over* $\mathcal{O}_{d.1}$ *be an isoformal family of differential modules of dimension* N *over* $\mathfrak{F}$. *Suppose that* (M_0, ∇_0) *is a differential module over* $\mathfrak{F}$ *that represents the class of the family. Then* (M, ∇)

is isomorphic over $\mathcal{O}_{d,1}$ *to the module obtained from* (M_0, ∇_0) *by extension of scalars to* $\mathcal{O}_{d,1}$.

PROOF This is based on the theory of [BV 2]. We choose bases for (M, ∇) and (M_0, ∇_0) and work in the following situation. $(M_\lambda, \nabla_\lambda)$ $(\lambda \in \Delta)$ is a family of differential modules over $\mathcal{F}$ where

$$M_\lambda = \mathcal{F}^N, \quad \nabla_{\lambda,d/dz} = d/dz - A(\lambda : z)$$

A being an element of $\mathfrak{gl}(N, \mathcal{O}_{d,1}(\Delta))$; and for each $\lambda \in \Delta$ we have an isomorphism

$$(M_\lambda, \nabla_\lambda) \cong (M_0, \nabla_0).$$

We wish to find another such isomorphism but depending analtically on λ, namely, to find an element $g \in GL(N, \mathcal{O}_{d,1}(\Delta_1))$ $\Delta_1 \subset \Delta$, such that

$$g(\lambda : .) : M_\lambda \cong M_0 \qquad (\lambda \in \Delta_1).$$

If M_0 is unramified this is just theorem 10.3.4 of [BV 2]. In particular this takes care of the case when M_0 is regular. So we may suppose that M_0 is irregular and ramified. We shall now proceed to give the arguments that reduce this to the unramified case; they depend on the descent theory discussed in §1.4.

Let $b \geq 1$ be an integer such that all the M_λ are unramified over $\mathcal{F}_b$ and let

$$M_{\lambda,b} = \mathcal{F}_b{}^N, \quad \nabla_{\lambda,b} = d/dz - A(\lambda : z) .$$

Then by Theorem 10.3.4 of [BV 2] we can find a $h \in GL(N, \mathcal{O}_{d,b}(\Delta_1))$ for a suitable $\Delta_1 \subset \Delta$ such that for each $\lambda \in \Delta_1$,

$$h(\lambda :) : M_{\lambda,b} \cong M_{0,b}.$$

We shall assume, as we may, that $M_{0,b}$ is a b-reduced canonical form and that M_0 arises from it through formal data and descent structure t_0 on $U = \mathbf{C}^N$. By the descent theory discussed in §1.4 we can find a representation $t(\lambda : .)$ of μ_b in $\mathbf{C}^N$ such that for each $\lambda \in \Delta_1$,

$$h(\lambda : .)\,(\varepsilon \otimes 1)\, h(\lambda : .)^{-1} = \varepsilon \otimes t(\lambda : \varepsilon) \qquad \varepsilon = \exp(2i\pi/b).$$

This equation makes it obvious that $t(\lambda : \varepsilon)$ is holomorphic in λ. Furthermore, as M_λ and M_0 are isomorphic, the descent structures $t(\lambda : .)$ and $t(0 : .)$ are isomorphic for each λ. Let G be the subgroup of $GL(N, \mathbf{C})$ commuting with all P_ω and C. Then we can find $\tau(\lambda) \in G$ such that $\lambda \in \Delta_1$,

$$t(\lambda : \varepsilon) = \tau(\lambda)\ t(0 : \varepsilon)\ \tau(\lambda)^{-1}, \qquad \tau(0) = 1.$$

Let H be the subgroup of G centralizing $t(0 : \varepsilon)$. Although $\tau(\lambda)$ is not unique, the above relation makes it clear that it is uniquely defined in G/H and hence may be viewed as an analytic map of Δ_1 into G/H. By the theorem of existence of local sections in Lie groups we may choose an analytic map k on a suitable $\Delta_2 \subset \Delta_1$ such that $k(\lambda)$ and $\tau(\lambda)$ have the same image in G/H for $\lambda \in \Delta_2$. Define now

$$h_1 = k^{-1} h.$$

Then $h_1 \in GL(N, \mathcal{O}_{d,b}(\Delta_2))$ and for all $\lambda \in \Delta_2$,

$$h_1(\lambda : .) : M_{\lambda,b} \cong M_{0,b},$$

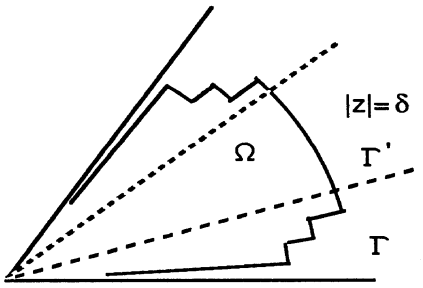
$$h_1(\lambda : .)\ (\varepsilon \otimes 1)\ h_1(\lambda : .)^{-1} = \varepsilon \otimes t(0 : \varepsilon).$$

In other words, $h_1(0 : .)^{-1} h_1(\lambda : .)$ commutes with $(\varepsilon \otimes 1)$, showing that it is defined over $\mathfrak{F}$ itself. If we write g for the element of $GL(N, \mathcal{O}_{d,1}(\Delta_2))$ thus defined, we see that $g(\lambda : .)$ is an isomorphism of M_λ with M_0, and this is what we wanted to prove. ◆

2 ASYMPTOTIC THEORY OF ISOFORMAL FAMILIES OF DIFFERENTIAL MODULES

2.1 The point of departure for the analytic theory of systems of linear meromorphic differential equations with irregular singularities is the result, going back in a generic form to Poincaré, that any formal solution of such a system is asymptotic to an analytic solution on a sector with vertex at the singularity provided the angle of the sector at the vertex is sufficiently small. The analytic solution is of course not unique, a fact that was the point of departure of the cohomological treatment of these questions due to Malgrange and Sibuya. In this section we shall discuss those aspects of the asymptotic theory that will be needed by us. To get additional perspective the reader should consult the classic treatise of Wasow [W] as well as [Ra-Si]. We shall however consider not only the case of a *single* system of differential equations but also the case of *families* which is deeper and involves other issues. It will be needed in **III** when we treat the problem of moduli.

We recall the definitions of sectors and sectorial domains given in §1.3. If A and B are subsets of a topological space, we write $A \subset\subset B$ to mean $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$. If Γ is a sector, an open subset Ω of Γ is said to be *asymptotic to* Γ ($\Omega \sim \Gamma$) if for any sector $\Gamma'' \subset\subset \Gamma'$, there exists a $\delta > 0$ such that $\Gamma'_\delta \subset \Omega$ (see figure below)):



For any set $A \subset \mathbf{C}^\times$ we write $A_\delta = A \cap \Delta_\delta$, Δ_δ being the disc $\{z : |z| < \delta\}$.

Let Γ be a sector in $\mathbf{C}^\times$. We consider analytic functions defined on open sets $\Omega \subset \Gamma$ that are asymptotic to Γ, two such functions being regarded equivalent if they coincide on an open subset of Γ that is asymptotic to Γ. The equivalence classes are the *germs* , but we shall allow ourselves as usual to abuse the notation and work with the functions rather than the germs. Let a be such an analytic function. If $\alpha = \Sigma_r c_r z^r \in \mathfrak{F}$, we say that a is *asymptotic to* α *in* Γ(or Ω) $a \sim \alpha\ (\Gamma)$, if for any sector $\Gamma' \subset\subset \Gamma$ and any integer $N \geq 0$,

$$a(z) = \Sigma_{r \leq N}\, c_r z^r + O(|z|^{N+1}) \qquad (z \in \Gamma', z \longrightarrow 0).$$

The element α is then uniquely determined by a and is denoted by $a^\wedge$. In this case, we have for any integer $r \geq 0$,

$$(d/dz)^r a \sim (d/dz)^r a^\wedge \ (\Gamma)$$

The set of all germs of such a is thus a differential $\mathbf{C}$-algebra, which we denote by $A_1(\Gamma)$. The map $a \longrightarrow a^\wedge$ is a homomorphism from $A_1(\Gamma)$ to $\mathfrak{F}$. If $\Gamma = \mathbf{C}^\times$, then $A_1(\Gamma) = \mathfrak{F}_{cgt}$, $a^\wedge = a$; if $\Gamma \neq \mathbf{C}^\times$, then the classical theorem of Borel-Ritt asserts that the map $a \longrightarrow a^\wedge$ is a *surjective* homomorphism of $A_1(\Gamma)$ into $\mathfrak{F}$. The kernel of this map, denoted by $A_0(\Gamma)$, consists of the so-called germs of *flat* functions in Γ, namely, germs of functions a such that

$$a \sim 0 \ (\Gamma).$$

Let $\Gamma \neq \mathbf{C}^\times$ and let $\log_\Gamma$ be a branch of log in Γ. For any $t \in \mathbf{C}$, we define $z^t = \exp(t \log_\Gamma z)$. For any integer $b \geq 1$ fix a choice of $z^{1/b}$ in $\mathfrak{F}_b$ and define $A_b(\Gamma)$ to be the set of all germs of functions a defined on some $\Omega \sim \Gamma$ with the following property: there is an element (necessarily unique) $a^\wedge = \Sigma_r c_{r/b} z^{r/b} \in \mathfrak{F}_b$ such that $a \sim a^\wedge\ (\Gamma)$, i. e.,

$$a(z) = \Sigma_{r \leq N}\, c_{r/b} z^{r/b} + O(|z|^{(N+1)/b}) \quad (z \in \Gamma', z \longrightarrow 0),$$

for any integer $N \geq 0$, and any sector $\Gamma' \subset\subset \Gamma$. It is clear that $A_b(\Gamma)$ is defined independently of the choice of $z^{1/b}$ and of the branch $\log_\Gamma$. The theorem of

Borel-Ritt implies that the map $a \longrightarrow a^\wedge$ is a surjective homomorphism of $A_b(\Gamma)$ into $\mathfrak{F}_b$ with the same kernel $A_0(\Gamma)$ as before. We shall say that a is *asymptotic to* $a^\wedge$ *on* Γ. We put

$$A(\Gamma) = \cup_{b \geq 1} A_b(\Gamma)$$

and view $a \longrightarrow a^\wedge$ as a homomorphism of $A(\Gamma)$ onto $\mathfrak{F}^{cl}$ with kernel $A_0(\Gamma)$. The elements a of $A(\Gamma)$ are the germs that *have an asymptotic expansion on* Γ, $a^\wedge$ being the *asymptotic expansion* of a.

We also need to consider asymptotic expansions when parameters are present. Fix an integer $d \geq 1$ and let notations and conventions be as in §1.5. Let Γ be a sector in $\mathbf{C}_z^\times$. An open subset Ω of $\mathbf{C}^d \times \Gamma$ is said to be *associated to* Γ if for any sector $\Gamma' \subset\subset \Gamma$ there is a polydisc $\Delta(\Gamma')$ and $\delta = \delta(\Gamma') > 0$ such that $\Delta(\Gamma') \times \Gamma'_\delta \subset \Omega$; we write $\Omega \sim \Gamma$. Note that for any such Ω, the set of points z in Γ for which $(0, z) \in \Omega$ is asymptotic to Γ. We consider germs of analytic functions defined on open sets Ω associated to Γ, germs being equivalence classes for the obvious equivalence : f' defined on Ω' is equivalent to f defined on Ω if there is an Ω'' associated to Γ on which $f = f'$. If f is defined on Ω, f is said to *have an asymptotic expansion* if there is an integer $b \geq 1$ and $f^\wedge = \Sigma_r a_{r/b} z^{r/b} \in \mathcal{O}_{d,b}$ (cf. §1.5) with the following property: for any sector $\Gamma' \subset\subset \Gamma$ there are $\Delta(\Gamma')$ and $\delta = \delta(\Gamma') > 0$ for which $\Delta(\Gamma') \times \Gamma'_\delta \subset \Omega$ and $f^\wedge \in \mathcal{O}_{d,b}(\Delta(\Gamma'))$, such that for any integer $N \geq 0$ we have

$$f(\lambda : z) = \Sigma_{r \leq N} a_{r/b}(\lambda) z^{r/b} + O(|z|^{(N+1)/b}) \quad (z \in \Gamma', z \longrightarrow 0)$$

the O being uniform in $\lambda \in \Delta(\Gamma')$, and $z^t = \exp(t \log_\Gamma z)$, $\log_\Gamma$ being a branch of log in Γ. We denote this by

$$f \sim f^\wedge(\Gamma),$$

the element $f^\wedge$ being uniquely determined by f. We write $A_{d,b}(\Gamma)$ for the differential $\mathcal{O}_d$ -algebra of (the germs of) those f for which $f^\wedge$ exists and belongs to $\mathcal{O}_{d,b}$, and put

$$A_d(\Gamma) = \cup_{b \geq 1} A_{d,b}(\Gamma).$$

When there are no parameters, i. e., when $d = 0$, $A_{d,b}(\Gamma)$ and $A_d(\Gamma)$ reduce to the algebras $A_b(\Gamma)$ and $A(\Gamma)$ considered earlier. The map $f \longrightarrow f^\wedge$ is a homomorphism of $\mathcal{O}_{d,b}$-algebras on $A_{d,b}(\Gamma)$; and for any differential operator $D = (\partial/\partial\lambda)^m (d/dz)^r$, with $\lambda = (\lambda_1, \dots, \lambda_d)$ and $m = (m_1, \dots, m_d)$, we have,

$$Df \sim Df^\wedge \quad (\Gamma)$$

The Borel-Ritt theorem continues to be true, and we formulate it in the following sharp form : if Γ is a sector $\neq \mathbf{C}^\times$, and $\varphi = \Sigma_{r\in\mathbf{Z}}\, a_{r/b}\, z^{r/b} \in \mathcal{O}_{d,b}(\Delta)$, then, for any $\Delta' \subset\subset \Delta$ and any $\alpha > 0$, we can find f defined and analytic on $\Delta' \times \Gamma_\alpha$ such that $f \sim \varphi \ (\Delta' \times \Gamma)$. Indeed, we may assume $b = 1$ and define t_m to be 0 when $a_m = 0$ and to be $(\sup_{\lambda\in\Delta'} | a_m(\lambda)|)^{-1}$ otherwise. If $\alpha > 0$, and $0 < \beta < 1$ is so small that $\cos(\beta \arg \log_\Gamma z) \geq 1/2$ for all $z \in \Gamma$, then the function

$$f(\lambda : z) = \Sigma_m\, a_m(\lambda)\, (1-\exp(-\, t_m \alpha^{-m} z^{-\beta}))\, z^m$$

is analytic on $\Delta' \times \Gamma_\alpha$ and $f \sim \varphi\ (\Gamma)$ (see [W], pp. 41-42; the O-estimates for the differences between f and the initial segments of φ are actually uniform in all of $\Delta' \times \Gamma_\alpha$). In particular the map $f \longrightarrow f^\wedge$ from $A_{d,b}(\Gamma)$ to $\mathcal{O}_{d,b}$ is surjective.

The extension of the notion of order (in z) to the rings $A_{d,b}(\Gamma)$ is immediate ; if $f \in A_{d,b}(\Gamma)$ and $f \sim f^\wedge(\Gamma)$, then the order of f is defined as the order of $f^\wedge$, namely, the smallest of the numbers r/b such that $a_{r/b} \neq 0$.

If M_b is a free differential module over $A_{d,b}(\Gamma)$ and $\Gamma' \subset \Gamma$, we have the module $M_b \otimes A_{d,b}(\Gamma')$, the tensor product corresponding to the restriction map $A_{d,b}(\Gamma) \longrightarrow A_{d,b}(\Gamma')$; we call this restriction to Γ' and denote it by $M_b(\Gamma')$. Similarly we associate to M_b the module $M_b^\wedge$ corresponding to the map $f \longrightarrow f^\wedge$ of $A_{d,b}(\Gamma)$ into $\mathcal{O}_{d,b}$; $M_b^\wedge$ is called the *formalization* of M_b, and if M_b is free, $M_b^\wedge$ is also a free differential module over $\mathcal{O}_{d,b}$ of the same rank. The maps $M_b \longrightarrow M_b(\Gamma')$ and $M_b \longrightarrow M_b^\wedge$ commute with ∇. In particular, we obtain the map

$$H(M_b) \longrightarrow H(M_b^\wedge),$$

$H(N)$ being the space of horizontal elements of the module N It is clear that the assignments $M_b \longrightarrow M_b{}^\wedge$ and $M_b \longrightarrow M_b(\Gamma')$ are functorial.

Let V be a (holomorphic) vector bundle defined and trivializable on a sectorial domain Γ_ε. If $b \geq 1$ is an integer, an *asymptotic structure of level* b for V on Γ is a maximal set of trivializations of V with the property that the transition matrices between any two of them is in $GL(N, A_b(\Gamma))$, N being the rank of V. The trivializations are then called the *asymptotic trivializations of level* b of V. If V is defined on a neighbourhood of 0, then on any sector Γ there is a canonical asymptotic structure of level b, namely the one that contains the meromorphic trivializations. We shall always suppose that in this case V is equipped with this asymptotic structure for all b. Given V on Γ_ε with an asymptotic structure of level b, a section of V on an open subset $\Omega \sim \Gamma$ is said to be *asymptotic of level* b if its components with respect to some (any) asymptotic trivialization are in $A_b(\Gamma)$. The germs of sectorial asymptotic sections then form a free module over $A_b(\Gamma)$ of rank N. *A pair* (V, ∇) *defined over* $A_b(\Gamma)$ is then V together with a connection ∇ defined on V such that $\nabla_{d/dz}$ maps the module of asymptotic sections of level b into itself; this is equivalent to requiring that the entries of the connection matrix in any asymptotic trivialization are in $A_b(\Gamma)$. If we replace $A_b(\Gamma)$ by $A(\Gamma)$ we obtain the corresponding asymptotic notions without any reference to the level.

2.2 We shall now formulate the fundamental results of the asymptotic theory in the parametric context. These are classical when there are no parameters (see [W], §§12-19, especially Theorems 12.3 and 19.1). But for analytic families of differential equations that are *isoformal*, namely *whose formal invariants do not change with the parameter*, they are new, at least in the generality considered here.

THEOREM 2.2.1 *Let* Γ *be a sector in* $\mathbf{C}_z^\times$, $d \geq 1$, *and let* M *be a free differential module over* $A_{d,1}(\Gamma)$ *with the property that its formalization* $M^\wedge$ *is an isoformal family of differential modules over* $\mathfrak{F}$ *(see* §1.5 *). Let* r_1 *be the principal level of the class of the family. If the vertex angle of* Γ *is* $\leq \pi/(|r_1| - 1)$, *then the map*

$$H(M) \longrightarrow H(M^\wedge)$$

is surjective.

REMARK If $M^\wedge$, or rather its class, is Fuchsian, then $r_1 = -1$, and the restriction on the angles is interpreted, here and elsewhere, as imposing no condition at all, except that the sector is proper. If the class is not Fuchsian, then it is quite possible that $\pi/(|r_1| - 1) \geq 2\pi$ because the levels are not necessarily integers. In this case the sector can be arbitrary except that it be proper. However, the point is that the asymptotic result should be formulated in the plane $\mathbf{C}_\zeta$ where $\zeta = z^{1/b}$, b being such that $b\, r_i \in \mathbf{Z}$. The principal level of the pull back class is $r_1' = b\, r_1 + b - 1$, and the condition on the sector Γ_ζ is that its angle be $\leq \pi/(|r_1'| - 1) = \pi/b(|r_1| - 1) \leq \pi$. In other words, the sectors Γ_ζ in the plane $\mathbf{C}_\zeta$ of angle $\pi/(|r_1'| - 1)$ are the natural domains for the sections that are asymptotic to the elements of $H(M^\wedge)$.

Going over from M to Hom (M, M') in the usual manner we get the following consequences of this theorem.

THEOREM 2.2.2 *Let assumptions and notation be as above and let* M' *be another free differential module over* $A_{d,1}(\Gamma)$ *whose formalization* $M'^\wedge$ *is isoformal with principal level* r_1'. *Let* $r = min\ (r_1, r_1')$ *and let* ξ *be any morphism (resp. isomorphism)* $M^\wedge \longrightarrow M'^\wedge$. *Suppose that the vertex angle of* Γ *is* $\leq \pi/(|r| - 1)$. *Then we can find a morphism (resp. isomorphism)*

$$x : M \longrightarrow M'$$

such that $x^\wedge = \xi$.

PROOF We note that if s is the principal level of Hom $(M^\wedge, M'^\wedge)$, then s $\geq$ r ; the assertion for the morphisms is now an immediate consequence of Theorem 2.2.1 since $|s| - 1 \leq |r| - 1$. If ξ is an isomorphism, we use the theorem to find $x \in$ Morph (M, M'), $u \in$ Morph (M', M) with $x^\wedge = \xi$, $u^\wedge = \xi^{-1}$. Using bases for M and M' we may assume that x and u are $N \times N$ matrices over $A_{d,1}(\Gamma)$ with $ux \sim 1$ and $xu \sim 1$ on (Γ). To prove that x and u are in $GL(N, A_{d,1}(\Gamma))$ we must show that xu and ux are in $GL(N, A_{d,1}(\Gamma))$. Going over to determinants we must show that if $a \in A_{d,1}(\Gamma)$ and $a^\wedge = 1$, then a is a unit. By definition we can find $\Gamma_n \subset\subset \Gamma$, $\cup_n \Gamma_n = \Gamma$, and polydisks Δ_n in $\mathbf{C}^d$

such that $a \sim 1$ on $\Delta_n \times \Gamma_n$ for all n. It is then clear that $|a| \geq 1/2$ on $\Delta_n \times \Gamma_{n,\delta(n)}$ for some $\delta(n) > 0$, and so, if $\Omega = \cup_n \Delta_n \times \Gamma_{n,\delta(n)}$, we have $\Omega \sim \Gamma$ and $a^{-1} \sim 1$. ◆

REMARK The last argument actually proves that if y is an $N \times N$ matrix over $A_{d,1}(\Gamma)$ which satisfies $y \sim \xi\ (\Gamma)$ where $\xi \in GL(N, \mathcal{O}_{d,1})$, then $y \in GL(N, A_{d,1}(\Gamma))$.

THEOREM 2.2.3 *Let assumptions and notation be as in Theorem 2.2.1. Let* M_0 *be a differential module over* $\mathfrak{F}_{cgt}$ *such that* $M_0^\wedge$ *represents the class of the isoformal family* $M^\wedge$. *Suppose that the vertex angle of* Γ *is* $\leq \pi/(|r_1| - 1)$. *If* ξ *is any isomorphism of* $M^\wedge$ *with* $\mathcal{O}_{d,1} \otimes_{\mathfrak{F}} M_0^\wedge$, *then we can find an isomorphism* x *of* M with $A_{d,1}(\Gamma) \otimes_{\mathfrak{F}_{cgt}} M_0$ *such that* $x^\wedge = \xi$. *In particular, if* A_0 *is the matrix of the connection on* M_0 *with respect to some basis of* M_0, *there is a basis of* M *such that the matrix of the connection on* M *relative to this basis is* A_0.

PROOF This is immediate from Theorem 2.2.2. Note that the existence of ξ with the properties stated in this theorem is a consequence of Theorem 1.5.1. ◆

When there are no parameters these existence theorems can also be formulated in terms of vector bundles.

THEOREM 2.2.4 *Let* Γ *be a sector in* $\mathbf{C}_z^\times$ *and let* (V, ∇), (V', ∇') *be two pairs on* Γ, *defined over* $A_b(\Gamma)$ *for some* $b \geq 1$, *with respective formalizations* $M_b^\wedge$, $M'^\wedge_b$, *and principal levels* r_1, r_1'. *Let* $r = \min(r_1, r_1')$. *If the vertex angle of* Γ *is* $\leq \pi/(|r_1| - 1)$, *the map*

$$H(M_b) \longrightarrow H(M_b^\wedge)$$

is surjective. Moreover, if the vertex angle of Γ *is* $\leq \pi/(|r| - 1)$, *then, for any* $\xi \in$ *Morph* $(M_b^\wedge, M'^\wedge_b)$, *we can choose* x *from Morph* $((V, \nabla), (V', \nabla'))$ *such that* x *preserves the asymptotic structures (of level* b*) and* $x^\wedge = \xi$. *If* ξ *is an isomorphism then* x *can be chosen to be an isomorphism also. Finally, suppose* (V, ∇) *is a meromorphic pair whose formalization is represented by a* *canonoical form* B *of ramification index* b. *Then there is an asymptotic trivialization for* (V, ∇) *of level* b *with respect to which* $\nabla_{d/dz} = d/dz - B$.

PROOF The results for $b = 1$ follow from the preceding theorems applied to the constant families defined by M and M'. To extend this to the case of arbitrary b it is a question of going over to the plane $\mathbf{C}_\zeta$ where $\zeta = z^{1/b}$. For any differential module over $\mathfrak{F}_b$ with principal level s, its pull back has principal level $s' = bs + b - 1$; if Γ^b is a sector in $\mathbf{C}_\zeta^\times$ above Γ, the angle of Γ^b is $\leq \pi/b(|s| - 1) = \pi/(|s'| - 1)$, showing that the situation in $\mathbf{C}_z$ persists in $\mathbf{C}_\zeta$. This shows that we are reduced to the case $b = 1$. The results then follow from the preceding ones. The only point to note is that although the preceding results permit us to define the bundle morphisms and trivializations only on an open subset of Γ that is asymptotic to Γ, analytic continuation (which is available because we are dealing with solutions of *linear* differential equations) extends them to the whole of Γ_ε for some $\varepsilon > 0$. ◆

REMARKS 1 The basic result to be established is therefore Theorem 2.2.1. We shall prove it in §§ 2.4 – 2.6. Suppose now there are no parameters and $f \in H(M(\Gamma'))$ with $f \sim f^\wedge(\Gamma')$, $\Gamma' \subset \Gamma$. Since f is a solution of a family of *linear* differential equations on Γ_ε, f can be continued analytically to an element of $H(M(\Gamma))$. The question is whether the asymptotic relation $f \sim f^\wedge$ persists on Γ also. It can be shown that this is true, provided *we start with a* Γ' *that is sufficiently close to* Γ. The proof of this is postponed to §3.2 since it requires the theory of the Stokes sheaf. Note that this is stronger than Theorem 2.2.4 which asserts only that there is *some* solution f defined over Γ which is asymptotic to $f^\wedge$.

2 Although the asymptotic context in which these theorems are formulated is the natural setting for them, one may wonder whether the proofs might become simpler if we assume that we work in the *meromorphic* situation. This is certainly not possible with the methods we use. In our method which is inductive, even if we start with a meromorphic connection, the inductive step leads to a connection that lives only on a sector. However this connection will still preserve the asymptotic structure, so that the method will go forward smoothly if we work from the beginning with bundles and connections on sectors admitting asymptotic structures.

3 The proof of Theorem 2.2.1 has two fundamental ingredients. The first is a far-reaching theorem asserting the existence of asymptotic solutions to certain *nonlinear* differential equations. We shall formulate this result in §2.3. The second ingredient is a result asserting that if M is a family of differential modules defined on a sector Γ whose formalizations $M^\wedge$ are isoformal, then certain splittings of $M^\wedge$ can be "lifted" to splittings of M provided the vertex angle of Γ is small enough. We do this in §2.4. Using Theorem 1.5.1 we complete the proof of Theorem 2.2.1 in §2.6 by induction on the rank of the bundle. To start the induction it is necessary to take care of the case when the spectrum of the class of $M^\wedge$ consists of a single element. This is done in §2.5.

The reader will notice that this approach is very close to that followed in [W] (§§ 12 – 19). However, in the parametric context the formal theory becomes much more complicated, and one has to use the deeper results of the theory of formal differential modules over the rings $\mathcal{O}_{d,b}$ that flow from the work of [BV 2].

2.3 We fix a sector Γ in the z-plane, an integer $m \geq 1$, and consider a system of n ordinary differential equations in $u = (u_1, \ldots, u_n)$ of the form

$$(1) \qquad z^{m+1}\, du_i/dz = \delta_i u_i + f_i(z : u_1, \ldots, u_n) \qquad (1 \leq i \leq n),$$

where the following conditions are satisfied:

(a) the δ_i are units of $\mathcal{O}_d$

(b) the f_i are polynomials in $u_1, \ldots, u_n$ with coefficients in $A_{d,1}(\Gamma)$

(c) the coefficients of the f_i have order ≥ 0; and those of the terms of degree (in the u_i) ≤ 1 are of order > 0.

We say that $v = (v_1, \ldots, v_n)$, $v_i \in \mathcal{O}_{d,1}$ is a formal solution to (1) if it is a solution to

$$(1_f) \qquad z^{m+1}\, dv_i/dz = \delta_i v_i + f_i^\wedge(z : v_1, \ldots, v_n) \quad (1 \leq i \leq n),$$

where $f_i^\wedge$ is the polynomial in $v_1, \ldots, v_n$ whose coefficients are the elements of $\mathcal{O}_{d,1}$ that are the asymptotic expansions of the corresponding coefficients of the f_i.

THEOREM 2.3.1 *Assume that the angle of* Γ *is* $\leq \pi/m$, *and that the system* (1) *has a formal solution* $v = (v_1, \ldots, v_n)$, *with* *ord* $(v_i) > 0$, $1 \leq i \leq n$. *Then , we can find* $u_i \in A_{d,1}(\Gamma)$ *such that*

(a) $u = (u_1, \ldots, u_n)$ *satisfies* (1)

(b) $u_i \sim v_i$ (Γ).

We remark that this is essentially the version with parameters of Theorem 12.1 of [W] or the corresponding theorem in [Ra-Si]. We do not prove this theorem here because the ideas used in its proof are not needed anywhere in the sequel. The interested reader can either work out the modifications needed in the proofs in [Ra-Si] to take care of the presence of parameters, or else refer to [BV 6] where a detailed proof of this theorem is given.

2.4 We shall now take up the lifting of differential module decompositions from $\mathcal{O}_{d,1}$ to $A_{d,1}(\Gamma)$.

Let us begin with an irregular (= non Fuchsian) differential module M_0 over $\mathfrak{F}$ and let $M_0 = M_{01} \oplus M_{02}$ be a decomposition into differential submodules of M_0. We shall say that the M_{0i} are *spectrally disjoint* if their spectra are disjoint in $\mathfrak{X}^{cl}$. More generally, let $M^\wedge$ be a free isoformal differential module over $\mathcal{O}_{d,1}$, and let $M^\wedge = M^\wedge{}_1 \oplus M^\wedge{}_2$ be a decomposition into differential submodules *each one of which is free , irregular , and isoformal* . We shall say that the $M^\wedge{}_i$ are *spectrally disjoint* if the spectra of the modules over $\mathfrak{F}$ that represent the classes of the $M^\wedge{}_i$ are disjoint. Let the class of $M^\wedge$ be M_0 and let ξ be an isomorphism of $M^\wedge$ with $\mathcal{O}_{d,1} \otimes_{\mathfrak{F}} M_0$; such a ξ exists by Theorem 1.5.1, and all decompositions of $M^\wedge$ into free, isoformal submodules are obtained by applying ξ^{-1} to the decompositions

$$\mathcal{O}_{d,1} \otimes_{\mathfrak{F}} M_0 = (\mathcal{O}_{d,1} \otimes_{\mathfrak{F}} M_{01}) \oplus (\mathcal{O}_{d,1} \otimes_{\mathfrak{F}} M_{02})$$

determined by decompositions

$$M_0 = M_{01} \oplus M_{02}$$

of M_0 into spectrally disjoint submodules M_{0i} $(i = 1,2)$.

PROPOSITION 2.4.1 *With the above notation let* $M^\wedge = M^\wedge{}_1 \oplus M^\wedge{}_2$ *be a decomposition that arises in the manner described above from a decomposition* $M_0 = M_{01} \oplus M_{02}$ *into spectrally disjoint submodules . Let* Γ *be a sector in* $\mathbf{C}_z{}^\times$ *and* M *a differential module over* $A_{d,1}(\Gamma)$ *with* $M^\wedge$ *as its formalization . If the vertex angle of* Γ *is* $\leq \pi/(|r_1| - 1)$ *where* r_1 *is the principal level of* M_0*, then there is a decomposition* $M = M_1 \oplus M_2$ *into free differential submodules such that* M_i *maps onto* $M^\wedge{}_i$ $(i = 1,2)$.

PROOF Let b be the ramification index of M_0. The argument given in the proof of Theorem 2.2.4 shows that we can work over the complex plane $\mathbf{C}_\zeta$ where $\zeta = z^{1/b}$. Without loss of generality we may therefore suppose that M_0 is unramified and is a canonical form : $M_0 = \mathfrak{F} \otimes_{\mathbf{C}} U$ where U is a vector space over $\mathbf{C}$, and $\nabla_{0,d/dz} = d/dz - B$,

$$B = \Sigma_{r \in L}\, z^r \otimes D_r + z^{-1} \otimes C \qquad L = \{ r_1, \ldots , r_m \},\ r_i \in \mathbf{Z},\ r_1 < \ldots < r_m$$

being a canonical form (cf. §1.4). The decompositions of M_0 into spectrally disjoint parts are then obtained in the obvious way from spectral decompositions of U that correspond to nontrivial partitions of the spectrum of $(D_r)_{r \in L}$. The complete decomposition of U is obtained iteratively, by first splitting with respect to the spectrum of D_{r_1} , then splittig each eigenspace of D_{r_1} with respect to D_{r_2} and so on. So it is enough to prove the proposition when the decomposition of U is defined by D_{r_j} where r_j is the first index for which the corresponding D is not a scalar. If $j > 1$, we can tensor by a differential module of dimension 1 with connection matrix $- \Sigma_{r<r_j} z^r D_r$ to come down to the case $j = 1$. Let $U = U_1 \oplus U_2$, $M_{0i} = \mathfrak{F} \otimes_{\mathbf{C}} U_i$. Let D_i denote the restriction of D_{r_1} to U_i. We may clearly suppose that $M^\wedge$ =

$\mathcal{O}_{d,1} \otimes_C U$, $M = A_{d,1}(\Gamma) \otimes_C U$, that $M \longrightarrow M^\wedge$ is the extension of the map $A_{d,1}(\Gamma) \longrightarrow \mathcal{O}_{d,1}$, and finally that $A^\wedge = B$ where A is the connection matrix of ∇ defined by $\nabla_{d/dz} = d/dz - A$ and B is as above, but now viewed as a connection matrix on $M^\wedge$. If P_i $(i = 1,2)$ are the projections $U \longrightarrow U_i$, we have the projections $1 \otimes P_i : M \longrightarrow M_i = A_{d,1}(\Gamma) \otimes_C U_i$, and the problem is to find an $A_{d,1}(\Gamma)$-module automorphism g of M such that the projections $g^{-1}(1 \otimes P_i)\, g$ are horizontal for $\mathrm{End}(M)$ $(i = 1,2)$. The condition for this is

$$d/dz\, (g^{-1}(1 \otimes P_i)\, g) + [\, g^{-1}(1 \otimes P_i)\, g, A\,] = 0 \quad (i = 1,2),$$

which is easily seen to be equivalent to

(1) $$[\, g[A]\,,\ 1 \otimes P_i\,] = 0 \quad (i = 1,2) \qquad (\, g[A] = (dg/dz)g^{-1} + gAg^{-1}).$$

If in addition we have $g \sim 1\ (\Gamma)$, we would be done because $Q_i \sim P_i$ for $i = 1,2$ in that case. If we write endomorphisms of M as partitioned matrices corresponding to the projections $1 \otimes P_i$, then (1) is equivalent to

(2) $$dg/dz + g\,A - Rg = 0,$$

for some R of the form

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}.$$

We shall seek g in the form

(3) $$g = \begin{pmatrix} 1 & t' \\ t'' & 1 \end{pmatrix}, \qquad t', t'' \sim 0\ (\Gamma).$$

Such a g would be $\sim 1\ (\Gamma)$ and so would be in $\mathrm{Aut}(M)$ by the remark following Theorem 2.2.2. If we write

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

we obtain the condition that (t', t'') must be a solution to

(4a) $$A_{11} + t' A_{21} = R_1\,, \qquad dt'/dz + A_{12} + t' A_{22} = R_1 t',$$

(4b) $$A_{22} + t'' A_{12} = R_2, \qquad dt''/dz + A_{21} + t'' A_{11} = R_2 t''.$$

Eliminating R_1 and R_2 we get

(5a) $$dt'/dz + A_{12} + t' A_{22} - A_{11}t' - t' A_{21} t' = 0$$

(5b) $$dt''/dz + A_{21} + t'' A_{11} - A_{22} t'' - t'' A_{12} t'' = 0 .$$

Conversely, the system (5), in the presence of

(6) $$t', t'' \sim 0,$$

implies (4). In fact, if

$$g[A] = \begin{pmatrix} R_1 & R' \\ R'' & R_2 \end{pmatrix}$$

then the equations (4) are satisfied but with the proviso that we must replace R_1 and $R_1 t'$ by $R_1 + R' t''$ and $R_1 t' + R'$, and R_2 and $R_2 t''$ by $R_2 + R'' t'$ and $R_2 t'' + R''$ respectively. So, if (5) are satisfied, then,

$$R'(1 - t'' t') = 0, \quad R'' (1 - t' t'') = 0.$$

These imply that $R' = 0$, and $R'' = 0$, because $(1 - t'' t')$ and $(1 - t' t'')$ are invertible.

We shall now give the argument for proving the existence of a t' that satisfies (5a) and (6); the case of t'' is entirely similar. The equations (5a) are nonlinear and we shall apply Theorem 2.3.1. Let us select a basis of U_i in which D_i is diagonal with eigenvalues λ_{iq} $(1 \le q \le n_i)$. Also, as B is a Laurent *polynomial* it makes sense as an endomorphism of M, so that, as $A^\wedge = B$, we may write

$$A_{11} = B_1 + \beta_1, \ A_{22} = B_2 + \beta_2 , \qquad A_{12}, A_{21}, \beta_1, \beta_2 \sim 0 \ (\Gamma).$$

Let $r_1 = -m - 1$ where $m \ge 1$; then,we have,

$$z^{m+1} B_i = D_i + F_i , \qquad F_i \in End(U)[X], \ F_i(0) = 0.$$

Thus the equation (5a) may be rewritten as follows:

(7) $$z^{m+1} \, dt'/dz = D_1 t' - t' D_2 + G$$

where G is given by

(8) $$G = -z^{m+1}A_{12} + (F_1 + z^{m+1}\beta_1)t' - t'(F_2 + z^{m+1}\beta_2) + t'(z^{m+1}A_{21})t'.$$

In the corresponding basis of $\mathrm{End}(U_i)$ we get the following equations for the entries t'_{rs} of the matrix of t' $(1 \le r \le n_1, 1 \le s \le n_2)$:

(7*) $$z^{m+1}dt'_{rs}/dz = (\lambda_{1r} - \lambda_{2s})t'_{rs} + G_{rs}$$

where G_{rs} is the rs-th entry of G. The $(\lambda_{1r} - \lambda_{2s})$ are *constants* $\neq 0$ while the G_{rs} are polynomials in the t'_{kl} with coefficients in $A_{d,1}(\Gamma)$ such that the constant term and the coefficients of the linear terms are of order > 0. On the other hand, replacing (7) by its asymptotic form we get the equation

(9) $$dt'^{\wedge}/dz = B_1 t'^{\wedge} - t'^{\wedge} B_2$$

for which $t'^{\wedge} = 0$ is a soluion. Theorem 2.3.1 is now applicable to (7*) and gives us the existence of a solution t' to (5a) which satisfies the flatness condition (6).

2.5 We now take up the Fuchsian case.

PROPOSITION 2.5.1 *Suppose* Γ *is a sector in* $\mathbf{C}_z^{\times}$ *and* M *a free differential module over* $A_{d,1}(\Gamma)$ *whose formalization* $M^{\wedge}$ *is isoformal and Fuchsian , Let* M_0 *be a differential module over* $\mathfrak{F}_{cgt}$ *such that its formalization* $M_0^{\wedge}$ *represents the class of the family. Then*

$$M \cong A_{d,1}(\Gamma) \otimes_{\mathfrak{F}_{cgt}} M_0.$$

PROOF We may assume that for some vector space U over $\mathbf{C}$, $M_0 = \mathfrak{F}_{cgt} \otimes_{\mathbf{C}} U$, $\nabla_{0,d/dz} = d/dz - z^{-1} \otimes C$ where $C \in \mathrm{End}(U)$. Then $M_0^{\wedge} = \mathfrak{F} \otimes_{\mathbf{C}} U$, $\nabla^{\wedge}_{0,d/dz} = d/dz - z^{-1} \otimes C$, and, in view of Theorem 1.5.1, we may suppose that $M^{\wedge}$ and M are given by $M^{\wedge} = \mathcal{O}_{d,1} \otimes_{\mathbf{C}} U$, $M = A_{d,1}(\Gamma) \otimes_{\mathbf{C}} U$, with $\nabla_{d/dz} = d/dz - A$, $A \in \mathrm{End}(M)$, and $A^{\wedge} = z^{-1} \otimes C$. So we can write

(1) $$A = z^{-1} \otimes C + F, \quad F \sim 0 \ (\Gamma)$$

We now consider the family of differential equations

(2) $du/dz = Fu$

We suppose for definiteness that F is defined on an open set $\Omega \sim \Gamma$ in $\mathbf{C}^d \times \mathbf{C}^{\times}$. We need a lemma.

LEMMA 2.5.2 *There is a unique germ of a fundamental solution* G *to* (2) *such that* $G \sim 1$ (Γ). *In particular, if* H *is any fundamental solution to* (2), *then there is an analytic map* $h : \Omega \longrightarrow GL(N, \mathbf{C})$ *such that* $H = Gh$ *(as germs).*

PROOF For the uniqueness we may assume that the real axis is contained in Γ. If G_i $(i = 1,2)$ are two fundamental solutions to (2), both ~ 1 (Γ), write $K = G_1 - G_2$. Take Δ and $\delta > 0$ so that $\Delta \times (0, \delta) \subset \Omega$, and for any fixed $\lambda \in \Delta$ extend $K(\lambda : t)$ and $F(\lambda : t)$ to C^∞ functions on $(-\infty, \delta)$ by defining them to be 0 for $t \leq 0$. Then $dK/dt = KF$ and K vanishes for $t = 0$, so that $K = 0$. By analytic continuation it follows that the germ of $G_1 - G_2$ is 0.

The existence of G is proved by the method used in Theorem 2.3.1; but the present situation is vastly simpler as it is linear. Fix a sector $\Gamma' \subset\subset \Gamma$ and choose Δ' and $0 < \alpha < 1$ so that $\Delta' \times \Gamma'_\alpha \subset \Omega$. Let C_k $(k = 0,1,\ldots)$ be constants > 0 such that

$$|F(\lambda : z)| \leq C_k |z|^k \qquad \text{on } \Delta' \times \Gamma'_\alpha$$

Let $|\,.\,|$ be a norm on $\mathrm{End}(U)$ so that $|XY| \leq |X|\,|Y|$ for all X, Y. We seek G in the form $G = 1 + V$, $V \sim 0$ (Γ) ; the equation for V is

$$dV/dz = FV + F.$$

For any $k \geq 0$ and $0 \leq \delta \leq \alpha$ let $B_k(\delta)$ be the Banach space of analytic functions on $\Delta' \times \Gamma'_\delta$ with values in $\mathrm{End}(U)$ such that

$$\| g \| := \sup (| g(\lambda : z) |\,|z|^{-k}) < \infty$$

where the sup is over $\Delta' \times \Gamma'_\delta$. We consider the linear operators $J_k(\delta)$ $(\delta \leq \alpha)$ on $B_k(\delta)$ defined by

$$(J_k(\delta)\, v)(\lambda : z) = \int_0^z F(\lambda : \zeta) v(\lambda : \zeta)\, d\zeta \, .$$

Then

$$|(J_k(\delta)\,v)(\lambda : z)| \;=\; |\,z\,|\,.\,|\int_0^1 F(\lambda\;:\;t\;z)v(\lambda\;:t\;z)\,dt\;\;|$$

$$\leq\; \delta C_k\,|z|^k\,\|\,v\,\|\,.$$

So $J_k(\delta)$ is bounded with norm $\leq\; \delta C_k$. Note also that if

$$F_1(\lambda : z) \;=\; \int_0^z F(\lambda\;:\;\zeta)\;d\zeta\,,$$

then $\|\,F_1\,\| \leq \delta C_k$ so that $F_1 \in B_{k(\delta)}$ also. If we choose $(\delta_k)_{k\,\geq\,0}$, $0 < \delta_k \leq \delta_0$

and $\delta C_k < 1$ for all k, then $J_k(\delta)$ is a contraction operator on $B_k(\delta)$, and hence we can find a unique $v_k \in B_k(\delta_k)$ such that

(3) $\qquad v_k \;=\; F_1 \;+\; J_k(\delta_k)\,v_k\,.$

If we write $V = v_0$, it follows from the uniqueness of v_0 that v_0 restricts to v_k on $\Delta'\times\Gamma'_\delta$, $\delta = \delta_k$. Differentiating (3) we see that $G = 1 + V$ is the required fundamental solution to (2) on $\Delta'\times\Gamma'_\delta$, $\delta = \delta_0$. Using the uniqueness of G and replacing Γ' by a sequence $\Gamma^{(n)} \subset\subset \Gamma$ with $\cup_n \Gamma^{(n)} \;=\; \Gamma$ we see that G is defined on an open set $\sim\;\Gamma$. ◆

We now come to the proof of Priposition 2.5.1. By (1) it is a question of finding $g \in \mathrm{Aut}(M)$ with $g[A] \;=\; z^{-1}\otimes C$. If $g_1(z) \;=\; z^{-C} := \exp(-\log_\Gamma z.\,C)$, we can view g_1 as an element of $\mathrm{Aut}(M)$ and $g_1[A] \;=\; F_1 \;=\; z^{-\,C}\,F\,z^{-\,C}$. Since $z^{\,\pm\,C}$ has moderate growth at $z = 0$ it is clear that $F_1 \sim 0\;(\Gamma)$. By the above Lemma we can find a fundamental solution G_1 to the system of differential equations $du/dz \;=\; F_1 u$ with the property that $G_1 \sim 1\;(\Gamma)$. Then $G_1^{-1}\,[F_1]$ $=\; 0$ and so $(z^{-\,C}\,G_1^{-1})\,[F_1] \;=\; z^{-1}\otimes C$. If $g \;=\; z^{\,C}\,G_1^{-1}\,g_1$, then $g[A] \;=\; z^{-1}\otimes C$. ◆

2.6 PROOF OF THEOREM 2.2.1 Let M_0 be as in Proposition 2.4.1. If b is the ramification index of M_0, it is easily seen that it is sufficient to prove the theorem for the pull back connection obtained from M in $\mathbf{C}_\zeta{}^\times$ where $\zeta = z^{1/b}$. We may therefore assume that M_0 is unramified.

If M_0 is Fuchsian, Proposition 2.5.1 allows us to assume that $M = A_{d,1}(\Gamma)\otimes_{\mathbf{C}} U$ and $\nabla_{d/dz} = d/dz - z^{-1}\otimes C$. If U_0 is the null space of C, we have $H(M) = A_{d,1}(\Gamma)\otimes_{\mathbf{C}} U_0$, $H(M^\wedge) = \mathcal{O}_{d,1}\otimes_{\mathbf{C}} U_0$, and the theorem is obvious.

If M_0 is irregular, we use induction on $\dim_{\mathfrak{F}} M_0$. If the spectrum of M_0 has at least two elements we can find a nontrivial decomposition of M_0 into two spectraly disjoint submodules. Proposition 2.4.1 and the induction hypothesis now give the result for M.

So we are left with the case when the spectrum of M_0 has a single element, say, $\omega = \omega^{\#}.dz$, where $\omega^{\#} = \Sigma_{r\in\mathbf{Z},\, r<-1}\, c_r z^r$. If L_ω is the one dimensional differential module with $\nabla_{d/dz} = d/dz + \omega^{\#}.1$, it is obvious that $L_\omega\otimes M_0$ is Fuchsian. By our earlier result, $H(L_\omega\otimes M) \longrightarrow H(L_\omega\otimes M^\wedge)$ is surjective. Tensoring by $L_{(-\omega)}$ now gives the result for M. ◆

3 THE STOKES SHEAF OF A MEROMORPHIC CONNECTION

3.1 Let (V, ∇) be a meromorphic pair at $z = 0$ and let M be the differential module over $\mathcal{F}_{cgt}$ of the germs of meromorphic sections of (V, ∇). If $M^\wedge$ is its formalization and Γ is a sector in $\mathbf{C}_z^\times$, we know from Theorem 2.2.4 that the map $H(M(\Gamma)) \longrightarrow H(M^\wedge)$ is surjective, provided the vertex angle of Γ is $< \theta_0$, where we can choose θ_0 to depend only on the (formal) isomorphism class of $M^\wedge$. In general the map $H(M(\Gamma)) \longrightarrow H(M^\wedge)$ has a nonzero kernel, namely the space of germs of sections on Γ that are flat on Γ, i. e., $\sim 0\ (\Gamma)$. It is clear that for a deeper understanding of the structure of (V, ∇) it is essential to make a closer investigation of these flat sections. By varying Γ these kernels will define a sheaf ; it is this sheaf and its variants that are the subject of study in this section. For the language and basic results of sheaves and their cohomology we follow [G].

Given (V, ∇) we define $\mathcal{H}_0(V, \nabla) = \mathcal{H}_0(V)$ as the sheaf on S^1 whose stalk at $u \in S^1$ consists of the vector space of germs of sections f of V defined on some sectorial domain Γ_δ, Γ being a sector containing u, such that

(a) f is horizontal for (V, ∇)

(b) $f \sim 0\ (\Gamma)$.

We often write the condiion (b) as

(b') $f \sim 0\ (\Gamma(u))$.

If W is a proper open arc of S^1, then the space of sections of $\mathcal{H}_0(V)$ on W is the space of germs of sections of V on $\Gamma(W)_\delta$ for some $\delta > 0$ which are horizontal for ∇ and flat on $\Gamma(W)$.

If f is a global section of $\mathcal{H}_0(V)$, it is immediate from the theorem of Riemann on removable singularities that f is meromorphic and hence that it is zero. Thus, for the zeroth cohomology, we have,

$$H^0(S^1, \mathcal{H}_0(V)) = 0.$$

The dimension of the first cohomology $H^1(S^1, \mathcal{H}_0(V))$ is called the *irregularity* of the pair (V, ∇), denoted by $Irr(V, \nabla)$ or, $Irr(V)$. Of course this depends only on the germ of the pair.

REMARK If (V, ∇) is Fuchsian, there is a meromorphic trivialization of it such that in this basis $\nabla_{d/dz} = d/dz - z^{-1}C$ where $C \in End(\mathbf{C}^N)$. The local horizontal sections of V are now identified with the local $\mathbf{C}^N$-valued functions of the form $\exp(\log z. C)\, u$, $u \in \mathbf{C}^N$. It is easy to see that this is flat if and only if u is zero. Indeed, the assertion is trivial when $N = 1$, for then the section is a constant multiple of z^λ for some $\lambda \in \mathbf{C}$; if $N > 1$, we take a flag of C-stable subspaces of $\mathbf{C}^N$ whose successive quotients are one dimensional, and use the result in dimension 1 in conjunction with the induction hypothesis to obtain the result in the general case. Thus

$$\mathcal{H}_0(V) = 0 \qquad ((V, \nabla) \text{ Fuchsian}).$$

In particular $Irr(V, \nabla) = 0$. The converse is also true, as we shall prove in §3.3, as a consequence of Deligne's formula for the irregularity ([De 2] [Be]).

3.2 If (V, ∇) and (V', ∇') are two pairs which are formally isomorphic, it follows from Theorem 2.2.4 that on any sector Γ with a sufficiently small vertex angle the two pairs are isomorphic via an isomorphism that is defined over $A_1(\Gamma)$. Hence the two sheaves $\mathcal{H}_0(V)$ and $\mathcal{H}_0(V')$ are locally isomorphic. In other words the local structure of $\mathcal{H}_0(V)$ depends only on the formal isomorphism class of (V, ∇), and so should be quite easy to describe. This is actually true, and we shall now proceed to give a description of it. We begin with some preparation.

For any $u \in S^1$ let $\mathcal{D}(u)$ be the space of germs of holomorphic differential forms ω defined on sectors containing u of the form

$$\omega = \Sigma_{a \in \mathbf{Q},\, a < -1}\, c_a z^a . dz$$

where the sum is finite and the analytic branches z^a are arbitrary. For convenience we shall write $\omega^\#$ for the coefficient of dz in ω so that

$$\omega = \omega^{\#} . dz.$$

On any sector containing u we then have a well defined primitive of ω, normalized by the requirement that it is zero at u. Let us write $E(\omega)$ for the exponential of this primitive, so that,

$$d/dz\, E(\omega) = \omega^{\#} E(\omega), \quad E(\omega)(u) = 1.$$

LEMMA 3.2.1 *Let* $u \in S^1$, $\omega \in \mathcal{D}(u)$. *We then have the following* :
(a) *If* $\omega^{\#} = c_a z^a$ + *terms of order* > a, $c_a \neq 0$, *then* $\mathrm{Re}(c_a u^{a+1}/(a+1)) := \rho_\omega(u)$ *depends only on* ω *and not on its representation* . *And*

$$\rho_\omega(u) < 0 \Leftrightarrow E(\omega) \sim 0\,(\Gamma(u))$$

(b) *Suppose* $F \subset \mathcal{D}(u)$ *is a finite set and* F^- *is the subset of all* $\omega \in F$ *such that* $\rho_\omega(u) < 0$. *Let*

$$\varphi = \Sigma_{\omega \in F}\, E(\omega)\, g_\omega$$

where the g_ω *are polynomials in* log z *and the* z^λ. *If* $\varphi \sim 0\,(\Gamma(u))$, *then*

$$g_\omega = 0 \text{ for all } \omega \in F \setminus F^- .$$

PROOF (a) That $\rho_\omega(u)$ is independent of the representation of ω is easy to see, and we omit the proof. To prove its asymptotic interpretation we begin by supposing that $\rho_\omega(u) \neq 0$; then there is a sufficiently small open arc W of S^1 containing u such that $|\rho_\omega(v)| \geq (1/2)|\rho_\omega(u)|$ for all $v \in W$. Then, there is a $\beta > 0$ such that for $z = r v$, $r > 0$, $v \in W$,

$$(*) \qquad \mathrm{Re}\Big(\int_u^z \omega^{\#}\, dz \Big) = \rho_\omega(v) \Big\{\frac{1}{r^{|a|-1}}\Big\} \Big\{1 + O(r^\beta)\Big\}$$

as $z \longrightarrow 0$. If $\rho_\omega(u) < 0$ it is then immediate from (*) that $E(\omega) \sim 0\,(\Gamma(u))$. Suppose conversely that the exponential is flat. Then it is clear from (*) that for r sufficiently small,

$$|E(\omega)| \geq (1/2) \exp(\alpha / r^{[|a|-1]})$$

for some constant $\alpha > 0$, so that the exponential cannot be flat. If $\rho_\omega(u) = 0$, and W is now an open arc containing u such that $E(\omega) \sim 0\ (\Gamma(W))$, we can move u slightly to a position u' within W so as to have $\rho_\omega(u') > 0$. The previous argument then applies and contradicts the flatness of the exponential. So we must have $\rho_\omega(u) < 0$.

(b) The result is proved by induction on $|F|$, the cardinality of F. If $|F| = 1$, then $\varphi = E(\omega)\, g_\omega$ and we may suppose that $\rho_\omega(u) \geq 0$. Then we can find a non empty sector Γ with $\varphi \sim 0\ (\Gamma)$ and $\rho_\omega(v) \geq \alpha > 0$ for $v \in \Gamma \cap S^1$. But then, as $E(-\omega) \sim 0\ (\Gamma)$, we must have $g_\omega \sim 0\ (\Gamma)$. But it is known that g_ω satisfies a Fuchsian differential equation meromorphic at $z = 0$, and so its flatness implies that it must vanish (since $\mathcal{H}_0(V) = 0$ for a Fuchsian V; cf. the remark at the end of §3.1). Let now $|F| > 1$. We may suppose that F^- is empty. If for some $\omega \in F$ we have $\rho_\omega(u) = 0$, $\exists$ u' arbitrarily close to u such that $\rho_\omega(u') < 0$ so that $E(\omega)\, g_\omega \sim 0\ (\Gamma(u'))$, and so the corresponding term may be dropped; the result then follows by induction. If all $\rho_\omega(u) > 0$, we choose ω such that $\rho_\omega(u) \geq \rho_{\omega'}(u)$ for all $\omega' \in F$. Then $E(-\omega)\, g_\omega \sim 0\ (\Gamma(u))$, and so,

$$(*) \qquad g_\omega + \Sigma_{\omega' \neq \omega} E(\omega' - \omega)\, g_{\omega'} \sim 0\ (\Gamma(u)).$$

If for all the $\omega' \neq \omega$ we have $\rho_{\omega'-\omega}(u) < 0$, then all the corresponding terms may be dropped to obtain $g_\omega \sim 0\ (\Gamma(u))$, giving $g_\omega = 0$; we are then finished by induction. If for some $\omega' \neq \omega$ we have $\rho_{\omega'-\omega}(u) = 0$, we can find u" arbitrarily close to u such that $\rho_{\omega'-\omega}(u'') < 0$ for that ω'. As before we can drop the corresponding term from (*), so that the argument is finished by induction once again. ♦

Recall from §1.4 the space $\mathfrak{L}(\mathcal{F}^{cl})$ of differential forms over $\mathcal{F}^{cl}$ of the form $\omega = \Sigma\, c_r z^r.\, dz$ where the sum is finite and $c_r = 0$ unless $r < -1$; and let $\mathfrak{L}(\mathcal{F}_b)$ be the subspace of those forms whose coefficients are in $\mathcal{F}_b$. Let us now fix a branch $z^{1/b}$ in $\mathcal{F}$ and consider the complex plane $\mathbf{C}_\zeta$ $(z = \zeta^b)$ with the covering map f $(\mathbf{C}_\zeta \longrightarrow \mathbf{C})$ that takes $t \in \mathbf{C}_\zeta$ to $t^b = z$ in $\mathbf{C}$. Let $S^{1,b}$ be the unit circle in $\mathbf{C}_\zeta$. For any $\omega \in \mathfrak{L}(\mathcal{F}_b)$ let $\omega_\zeta \in \mathfrak{L}(\mathcal{F}_\zeta)$ be obtained from ω by the substitution $z^{1/b} \longrightarrow \zeta$. If $\omega = \omega^\#.\, dz$ where $\omega^\# = \Sigma_r\, c_{r/b}\, z^{r/b}$, then $\omega_\zeta = \omega_\zeta^\#.\, d\zeta$, where

$$\omega_\zeta^{\#} = b\zeta^{b-1}\omega^{\#}(\zeta) \in \mathfrak{F}_\zeta .$$

We thus have an identification of $\mathfrak{L}(\mathfrak{F}_b) \cong \mathfrak{L}(\mathfrak{F}_\zeta)$, the latter being naturally imbedded in $\mathfrak{D}(v)$ for any $v \in S^{1,b}$. If we now choose v on $S^{1,b}$ to lie above u, the map f_* allows us to imbed $\mathfrak{L}(\mathfrak{F}_b)$ inside $\mathfrak{D}(u)$.

Returning to the pair (V, ∇) in $\mathbf{C}_z$ let $b \geq 1$ be its ramification index. The pull back of (V, ∇) to $\mathbf{C}_\zeta$ $(z = \zeta^b)$ is now unramified and we associate to its canonical form a pair $(V_{B'}, \nabla_{B'})$ defined as follows: $V_{B'}$ is the trivial bundle $\mathbf{C}_\zeta \times U$, U being a vector space of dimension N over $\mathbf{C}$ graded by $\omega \in \Sigma$, $U = \bigoplus U_\omega$; $\nabla_{B',d/d\zeta} = d/d\zeta - B'(\zeta)$, where $\Sigma \subset \mathfrak{L}(\mathfrak{F}_\zeta)$ and

$$B'(\zeta) = \Sigma_{\omega\in\Sigma}\ \omega_\zeta^{\#}.1 \otimes P_\omega + \zeta^{-1} \otimes C, \qquad C = \bigoplus_\omega C_\omega ,$$

P_ω being the projections $U \longrightarrow U_\omega$ and C_ω an endomorphism of U_ω. Fix a point $u \in S^1$ and choose a point $v \in S^{1,b}$ above u. The pair $(V_{B'}, \nabla_{B'})$ may now be viewed, locally on a sector containing v, as the pull back through the covering map f $(\mathbf{C}_\zeta \longrightarrow \mathbf{C})$ of a pair (V_B, ∇_B) defined on a sector $\Gamma \subset \mathbf{C}_z^\times$ containing u ; $V_B = \Gamma \times U$, $\nabla_{B,d/dz} = d/dz - B(z)$, and

$$B(z) = \Sigma_{\omega\in\Sigma}\ \omega^{\#}.1 \otimes P_\omega + z^{-1} \otimes C .$$

Here we use the identification of $\mathfrak{L}(\mathfrak{F}_b)$ with $\mathfrak{L}(\mathfrak{F}_\zeta)$ via f_*. As (V_B, ∇_B) and (V, ∇) are formally isomorphic over $\mathfrak{F}_b$ we see from Theorem 2.2.4 that if the vertex angle of Γ is small enough, there is an isomorphism y of (V,∇) with (V_B, ∇_B) that preserves the asymptotic structures and hence induces an isomorphism of $\mathcal{H}_0(V)$ with $\mathcal{H}_0(V_B)$ on $\Gamma \cap S^1$. The local structure of $\mathcal{H}_0(V_B)$ is easy to determine because the differential equations

$$ds/dz = B(z)s$$

have the solutions

$$(*) \qquad s = \Sigma_{\omega\in\Sigma}\ E(\omega)(z).\exp(\log_\Gamma z.\, C_\omega)\, t_\omega , \quad t_\omega \in U_\omega .$$

Write $W = \Gamma \cap S^1$, and for any $w \in W$, let $\Sigma(w : -)$ (resp. $\Sigma(w : +)$) be the set of $\omega \in \Sigma$ such that $\rho_\omega(w)$ is < 0 (resp. > 0).

PROPOSITION 3.2.2 *The isomorphism* y *induces an isomorphism on* W *of* $\mathcal{H}_0(V)$ *with the sheaf* $\mathcal{U}$ *of linear subspaces of* U *whose stalk at any* $w \in W$ *is the linear span of the* U_ω *with* $\omega \in \Sigma(w : -)$.

PROOF This is immediate ; for, by Lemma 3.2.1, the solution (*) is flat around w if and only if all the t_ω for which ω does not belong to $\Sigma(w : -)$ are zero. ◆

Fix a choice of $z^{1/b}$ and let $\omega \in \mathcal{E}(\mathcal{F}_b)$, $\omega^\# = c_{r/b}\, z^{r/b} +$ terms of higher order, $c_{r/b} \neq 0$. Then $\omega_\zeta = \omega_\zeta^\# \, d\zeta$, where $\omega_\zeta^\# = b\, c_{r/b}\, \zeta^{r+b-1} +$ terms of higher order, so that $r + b-1 = a \in \mathbf{Z}$ is the order of ω_ζ, and $c_a = bc_{r/b}$ is the coefficient in $\omega_\zeta^\#$ of ζ^a. Write $\mathbf{C}^b$ for $\mathbf{C}_\zeta$. The points v on $S^{1,b}$ in $\mathbf{C}^b$ such that $\mathrm{Re}(c_a\zeta^{a+1}/(a + 1)) = 0$ form a division of $S^{1,b}$ into $2(|r| - b)$ arcs of equal length ; the rays in $\mathbf{C}^b$ issuing from the origin and going through these points are called the *Stokes lines* (*rays*) *in* $\mathbf{C}^b$ *associated to* ω. They form a finite set that will be denoted by $S(\omega, b)$. If we use another choice of $z^{1/b}$, say $\gamma z^{1/b}$ where $\gamma \in \mu_b$, then $S(\omega, b)$ changes to $\gamma S(\omega, b)$. If b' is divisible by b, b' = bd, then $\omega \in \mathcal{E}(\mathcal{F}_{b'})$ also, and if $z^{1/b'}$ is chosen so that $(z^{1/b'})^d = z^{1/b}$, it is easy to see that

$$S(\omega, b) = f\,(S(\omega, b'))\ ,\quad S(\omega, b') = f^{-1}(S(\omega, b)),$$

f being the covering map $\mathbf{C}^{b'} \longrightarrow \mathbf{C}^b$ that takes $\zeta' \in \mathbf{C}^{b'}$ to $\zeta'^{b'/b} = \zeta$ in $\mathbf{C}^b$. The Stokes lines associated to ω thus form a configuration consisting of finite subsets of the various $S^{1,b'}$ $(b \mid b')$ that are compatible with the covering maps that link the corresponding complex planes. The Galois group μ_b acts on both $\mathcal{E}(\mathcal{F}_b)$ and $\mathbf{C}^b$, and it is easy to see that the actions are compatible in the sense that for $\sigma = \exp(2\pi i m/b)$ acting via

$$z^{1/b} \longrightarrow \sigma z^{1/b},\ \ \zeta \longrightarrow \sigma\zeta,\ \text{and}\ d\zeta \longrightarrow \sigma d\zeta,$$

we have $(\sigma.\omega)_\zeta = \sigma.\omega_\zeta$, $(\omega \in \mathcal{F}_b)$. This implies that

$$S(\sigma.\omega, b) = \sigma^{-1}S(\omega, b)\ .$$

Suppose now that (V, ∇) is a meromorphic pair at $z = 0$, and let $\Sigma = \Sigma(V) \subset \mathcal{E}(\mathcal{F}_b)$ be the spectrum of the associated formal differential module over $\mathcal{F}$. We define

$$S(V, \nabla)_b = \cup_{\omega \in \Sigma} S(\omega, b)$$

and refer to the elements of $S(V, \nabla)_b$ as the *Stokes lines* (*rays*) *of* (V, ∇) *in* $\mathbf{C}^b$. Since Σ is stable under μ_b the remarks made above show that $S(V, \nabla)_b$ is well defined independently of the choice of $z^{1/b}$ and is invariant under μ_b. Hence it is the preimage of a well defined subset $S(V, \nabla)$ of S^1 :

$$S(V, \nabla)_b = f_b^{-1}(S(V, \nabla)),$$

f_b ($\mathbf{C}^b \longrightarrow \mathbf{C}$) being the covering map that takes $\zeta \in \mathbf{C}^b$ to $z = \zeta^b \in \mathbf{C}$. The elements of $S(V, \nabla)$ (or $S(V)$ if there is no ambiguity about ∇) are called the *Stokes lines* (*rays*) *of* (V, ∇). Note that they are determined entirely by the formalization of (V, ∇).

PROPOSITION 3.2.3 *Let* (V, ∇) *be a meromorphic pair at* $z = 0$. *Then, for any open arc* $Y \subset S^1$, *the restriction of* $\mathcal{H}_0(V)$ *to* Y *is a local system* (*necessarily trivial on* Y) *if and only if* Y *does not meet any Stokes line of* (V, ∇), *or, equivalently, if and only if the function*

$$N_V : u \longrightarrow \dim_{\mathbf{C}} \mathcal{H}_0(V)(u) \qquad (u \in S^1)$$

is constant on Y. *Moreover, the function* N_V *is lower semicontinuous* (*value goes down at special points*), *and is discontinuous at* u *if and only if* u *lies on a Stokes line of* (V, ∇).

PROOF If Y does not meet any Stokes line, then the set $\Sigma(w : -)$ does not change when w varies over Y. It is obvious from this that $\mathcal{U}$ of Proposition 3.2.2 is the constant sheaf on Y, and so the same is true of $\mathcal{H}_0(V)$ on Y. If the sheaf $\mathcal{H}_0(V)$ on Y is a local system, it is trivial on Y as Y is simply connected, and the function N_V is obviously constant on Y. Finally suppose $u \in Y$. If S_ℓ (resp. S_r) is the set of $\omega \in \Sigma$ for which $\rho_\omega(w)$ is < 0 for points w *to the left* (resp. *right*) of u and arbitrarily close to it, and U_ℓ (resp. U_r) is the linear span of the U_ω for $\omega \in S_\ell$ (resp. $\omega \in S_r$), then

$$\Sigma(u : -) = S_\ell \cap S_r,$$

so that $\mathcal{U}(u) \subset U_\ell \cap U_r$. On the other hand, if w_ℓ (resp. w_r) is to the left (resp. right) of u and arbitrarily close to it, then

$$N_V(w_\ell) = \dim (U_\ell), \qquad N_V(w_r) = \dim (U_r).$$

This shows that

$$N_V(u) \le N_V(w_\ell), \; N_V(u) \le N_V(w_r),$$

proving that N_V is lower semicontinuous. If now N_V is constant around u, it follows that $\mathcal{U}(u) = U_\ell = U_r$, from which we get $\Sigma(u:-) = S_\ell = S_r$. This shows that $\rho_\omega(u)$ cannot be zero for any $\omega \in \Sigma$. The assertions of the Proposition are now proved. ◆

Let M be the differential module of germs of meromorphic sections of V and let W be a proper open arc $\subset S^1$. Let $M(W)$ be the associated module of asymptotic sections of level 1 (over $\Gamma(W)$) :

COROLLARY 3.2.4 *If* W' *is an open arc* $\subset W$ *and* $W \setminus W'$ *does not meet any Stokes line , then the restriction map* $H(M(W)) \longrightarrow H(M(W'))$ *is an isomorphism .*

PROOF Proposition 3.2.3 implies that the result is true for the flat sections. Let $M^\wedge$ be the formalization of M, $m \in M^\wedge$, and let s be a horizontal element of $M(W)$ with the property that $s \sim m\ (\Gamma(W'))$; we must show that this asymptotic relation persists in $\Gamma(W)$ also. Let $\mathcal{W}$ be the set of all open arcs Y with $W' \subset Y \subset W$ such that $s \sim m\ (\Gamma(Y))$ and let W'' be the union of the arcs in $\mathcal{W}$. Then $W'' \in \mathcal{W}$ also, and we need to show that $W'' = W$. Suppose this is not true, let $W'' = (a'', b'') \subset W = (a, b)$, and assume for definiteness that $a < a''$. By Theorem 2.2.4 we can find a sufficiently small open arc $(a_1, b_1) \subset W$ containing a'' and a horizontal sction $t \in M(W)$ such that $t \sim m\ (\Gamma((a_1, b_1)))$. Then $s - t \sim 0\ (\Gamma((a'', b_1)))$ while there are no Stokes lines through points of $(a_1, a'']$, so that $s - t \sim 0\ (\Gamma((a_1, b_1)))$ by the remark at the beginning of the proof. But then $s = (s-t) + t$ is $\sim m\ (\Gamma((a_1, b_1)))$, showing that $(a_1, b'') \in \mathcal{W}$. Since this is an arc strictly bigger than W'', we have a contradiction to the maximality of W''. ◆

REMARK It is clear from this proposition that the Stokes lines mark the boundaries of sectors where the asymptotic expansion of a horizontal section ceases to be valid. This type of behaviour, where the asymptotic structure of a solution of a system of linear differential equations changes with the sector on which the solution is defined, is known as the *Stokes phenomenon* , named after Stokes who seems to have been the first to have observed it. This is the also the reason for naming the lines as *Stokes lines* . The reader should also recall Remark 1 following the proof of Theorem 2.2.4.

3.3 We shall now obtain a formula, due to Deligne ([De 2]; see also [Be]) for the irrgularity of a meromorphic pair at $z = 0$.

If f is any integer valued function defined on S^1 such that $\lim f(u\pm)$ exist, its *jump* $j(f : u)$ at a point $u \in S^1$ is dfined as

$$j(f : u) = |f(u+) - f(u)| + |f(u) - f(u-)|.$$

If f is lower semicontinuous, we have,

$$j(f : u) = (f(u +) - f(u)) + (f(u -) - f(u)).$$

The *variation* $\mathrm{var}(f)$ of f is then defined by

$$\mathrm{var}(f) = \textstyle\sum_u j(f : u)$$

where the summation is over all the points of S^1. This is finite if f has only finitely many points of discontinuity. We shall suppose that this is true and also that f is lower semicontinuous from now on. Let $u_1, \dots, u_m$ be the points of discontinuity and let us define the u_i for all $i \in \mathbf{Z}$ by $u_{i+m} = u_i$; then

$$\mathrm{var}(f) = \textstyle\sum_{i \in \mathbf{Z}/(m)} j(f : u_i)$$

Observe that (in all summations below i varies over $\mathbf{Z}/(m)$)

$$0 = \sum (f(u_i +) - f(u_i -)) = \sum f(u_i +) - f(u_i)) + \sum f(u_i) - f(u_i -)$$

while

$$\mathrm{var}(f) = \sum (f(u_i +) - f(u_i)) - \sum (f(u_i) - f(u_i -))$$

So $\mathrm{var}(f)$ must be an *even* integer. Let us write $n_{2i} = f(u_i)$ and n_{2i+1} for the constant value of f in the open arc $]\, u_i, u_{i+1} [$. Then it is clear that

$$\mathrm{var}(f) = 2\sum n_{2i+1} - 2\sum n_{2i}$$

This expression for var(f) shows that it is *additive* in f.

Let us consider now the sheaf $\mathcal{H}_0(V)$, and for any $u \in S^1$ let

$$N_V(u) = \dim \mathcal{H}_0(V)$$

as in §3.2. We have seen that N_V is lower semicontinuous and that its points of discontinuity are precisely the points of S^1 that lie on Stokes lines.

PROPOSITION 3.3.1(Deligne) *We have*

$$\mathrm{Irr}(V, \nabla) = (1/2)\,\mathrm{var}(N_V)$$

In particular, Irr(V,∇) *depends only on the formal isomorphism class of* (V, ∇).

The second assertion follows from the first; indeed, we note that if (V',∇') and (V, ∇) are in the same formal isomorphism class, Theorem 2.2.4 allows us to conclude that the sheaves $\mathcal{H}_0(V)$ and $\mathcal{H}_0(V')$ are locally isomorphic, and hence that $N_V = N_{V'}$. For proving the formula we shall follow [De 2] and deduce this as a consequence of a more general formula for the Euler characteristic of certain sheaves of vector spaces in terms of the dimensions of their stalks. Let $\mathcal{S}$ be any sheaf of complex vector spaces on S^1 satisfying the following three conditions:

(a) $N(u) = \dim \mathcal{S}(u) < \infty$ for all $u \in S^1$

(b) for any open arc U and $u \in S^1$, the restriction $\mathcal{S}(U) \longrightarrow \mathcal{S}(u)$ is injective

(c) N has only finitely many points of discontinuity.

Given $u \in S^1$ the conditions (a) and (b) imply that for a sufficiently small arc U containing u, the map $\mathcal{S}(U) \longrightarrow \mathcal{S}(u)$ is an isomorphism. Hence for $u' \in U$, we have a map $\mathcal{S}(u) \longrightarrow \mathcal{S}(u')$ which is injective, showing that $N(u') \geq N(u)$, and therefore that N is lower semicontinuous. If N is constant on U, we have a canonical isomorphism $\mathcal{S}(u) \longrightarrow \mathcal{S}(u')$. Hence on any open arc on which N is constant, $\mathcal{S}$ is the constant sheaf.

PROPOSITION 3.3.2 *Under the above assumptions*, $H^i(S^1, \mathcal{A})$ *is finite dimensional*, $i = 0,1$. *If* $\chi = \dim H^0(S^1, \mathcal{A}) - \dim H^1(S^1, \mathcal{A})$, *then,*

$$\chi = -(1/2)\mathrm{var}(N).$$

It is clear that the sheaf $\mathcal{H}_0(V)$ satisfies the conditions on $\mathcal{A}$. Further we have seen in §3.2 that $H^0(S^1, \mathcal{H}_0(V)) = 0$. So, $\chi = -\dim H^1(S^1, \mathcal{H}_0(V))$, and Proposition 3.3.1 follows from Proposition 3.3.2.

We shall now prove Proposition 3.3.2 assuming the following lemma.

LEMMA 3.3.3 *If* $]A, B[$ *is an open arc of* S^1 *on which* N *takes the constant value* n, *and* i $(]A, B[\longrightarrow S^1)$ *is the natural inclusion, then for the sheaf* $\mathcal{A}_{]A,B[} = i_*$(*restriction of* $\mathcal{A}$ *to* $]A, B[$), *we have,*

$$H^0(S^1, \mathcal{A}_{]A,B[}) = 0, \quad H^1(S^1, \mathcal{A}_{]A,B[}) = H^0(]A, B[, \mathcal{A}).$$

In particular,

$$\chi = -n.$$

PROOF OF PROPOSITION 3.3.2 Assuming this Lemma we shall prove Proposition 3.3.2. Let $u_1, \dots, u_m$ be the points of discontinuity of N and let J_i be the arc $]u_i, u_{i+1}[$. We then have the exact sequence

$$0 \longrightarrow \bigoplus_i \mathcal{A}_{J_i} \longrightarrow \mathcal{A} \longrightarrow \bigoplus_i \mathcal{A}_{\{u_i\}} \longrightarrow 0$$

where $\mathcal{A}_{\{u\}}$ for any $u \in S^1$ is the sheaf whose stalk at v is 0 for $v \neq u$ and $\mathcal{A}(u)$ for $v = u$. Since $H^0(S^1, \mathcal{A}_{\{u\}}) = \mathcal{A}(u)$ and $H^1((S^1, \mathcal{A}_{\{u\}}) = 0$ we have $\chi(\bigoplus_i \mathcal{A}_{\{u_i\}}) = \Sigma_i n_{2i}$, while the lemma shows that $\chi(\bigoplus_i \mathcal{A}_{J_i}) = -\Sigma_i n_{2i+1}$.

Since the sheaves on either side of $\mathcal{A}$ have finite dimensional H^0 and H^1, the same is true for $\mathcal{A}$. Hence

$$\chi(\mathcal{A}) = \Sigma_i n_{2i} - \Sigma_i n_{2i+1} = -(1/2)\mathrm{var}(N).$$

This proves the proposition. ♦

PROOF OF LEMMA 3.3.3 A section s of $\mathcal{A}_{]A, B[}$ may be viewed as a section of $\mathcal{A}$ on S^1 which vanishes outside a closed arc contained in $]A, B[$, and is hence zero. This gives $H^0(S^1, \mathcal{A}_{]A, B[}) = 0$. To determine H^1, we note that as $\mathcal{A}_{]A, B[}$ is 0 at all points not in $]A, B[$, any $\mathcal{A}_{]A, B[}$– torsor $\mathfrak{T}$ is uniquely trivializable on $S^1 \setminus [A', B']$ where $[A', B'] \subset]A, B[$, thus giving a section t' on $S^1 \setminus [A', B']$ (cf. **II**,§1) :

—|A—|A'—————————|B'—|B———

On the other hand, $\mathcal{A}$ is the constant sheaf on $]A, B[$, hence it is trivializable on it, hence so is $\mathfrak{T}$. Let t be a section of $\mathfrak{T}$ on $]A, B[$. We then have sections s', s'' of $\mathcal{A}$ on $]A, A'[$ and $]B', B[$ respectively such that $s'[t] = t'$ on $]A, A'[$ and $s''[t] = t'$ on $]B', B[$. By our assumptions s' and s'' may be viewed as global sections on $]A, B[$ and so we see that $s = s' - s'' \in H^0(]A, B[, \mathcal{A})$. Had we chosen another section t_1 of $\mathfrak{T}$, $t_1 = s[t]$ for a unique element s_1 from $H^0(]A, B[, \mathcal{A})$; then s' and s'' change to $s' - s_1$ and $s'' - s_1$, so that s remains the same. We thus get a map $H^1(S^1, \mathcal{A}_{]A, B[}) \longrightarrow H^0(]A, B[, \mathcal{A})$. To show that it is a linear isomorphism we use the covering $\{S^1 \setminus [A', B'],]A, B[\}$ of S^1. The corresponding cocycle is s' on $]A, A'[$ and s'' on $]B', B[$; subtracting from this the coboundary which is 0 on $S^1 \setminus [A', B']$ and s'' on $]A, B[$, we see that the map in question is $(s, 0) \longrightarrow s$. Since $(s, 0)$ depends linearly on s and is a cocycle for all s, we are done. ◆

COROLLARY 3.3.4 *(V,∇) is Fuchsian if and only if* $\mathrm{Irr}(V,\nabla) = 0$.

PROOF If the irregularity is 0, Proposition 3.3.1 shows that N_V is constant and so, by Proposition 3.2.3, there are no Stokes lines. This means that (V,∇) is Fuchsian. ◆

COROLLARY 3.3.5 *If* f_b *is the covering map* $\mathbf{C}^b \longrightarrow \mathbf{C}$, *then* ,

$$\mathrm{Irr}(f_b^*(V,\nabla)) = b.\,\mathrm{Irr}(V,\nabla).$$

PROOF Let $(V',\nabla') = f_b^*(V,\nabla)$. Then $N_{V'} = f_b^* N_V$, and the Corollary follows at once. ◆

We shall now use Deligne's formula to obtain an explcit expression for the irregularity in terms of the formal data provided by the connection, namely, the spectrum $\Sigma \subset \mathfrak{L}(\mathfrak{F}^{cl})$ and the complex vector space U, equipped with a gradation by $\Sigma : U = \bigoplus_{\omega \in \Sigma} U_\omega$, and a compatible action by **Z**; the compatibility is with respect to the action of the Galois group $\mathrm{Gal}(\mathfrak{F}^{cl}/\mathfrak{F})$ (in which **Z** is imbedded naturally; see §1.4) on Σ. Actually, only the action of the Galois group on Σ is needed for the computation of $\mathrm{Irr}(V, \nabla)$; the **Z**-action on U is not needed except to ensure that $\dim(U_\omega)$ is constant on the Galois orbits. For any $\omega \in \Sigma \setminus (0)$, let us put

$$i(\omega) = -\operatorname{ord}(\omega) - 1, \quad b(\omega) = \inf\{b \geq 1 : \omega \in \mathfrak{L}(\mathfrak{F}_b)\}, \quad d(\omega) = \dim(U_\omega)$$

Here ord refers to the order and is < -1. The definition of $b(\omega)$ implies that

$$(*) \qquad i(\omega)\, b(\omega) \in \mathbf{Z}.$$

We write $[\omega]$ for the Galois orbit of ω; its cardinality is of course $b(\omega)$. It is obvious that the functions i, b, and d are constant on the orbits. We then have the following result in which the integrality of $\mathrm{Irr}(V, \nabla)$ is manifest in view of $(*)$.

PROPOSITION 3.3.6 *With the above notation, we have,*

$$\mathrm{Irr}(V, \nabla) = \sum_{[\omega] \neq [0]} i(\omega)\, b(\omega)\, d(\omega).$$

PROOF Choose $b \geq 1$ such that all the ω are in $\mathfrak{L}(\mathfrak{F}_b)$. We shall work in $\mathbf{C}^b$ with the pull back pair (V', ∇'). Then by Corollary 3.3.5, $\mathrm{Irr}(V, \nabla) = (1/b)\,\mathrm{Irr}(V', \nabla')$. Now, if v is a point on a Stokes line for V', the contribution to $\mathrm{var}(N_{V'})$ from v is precisely $\Sigma_{\omega : v \in S(\omega, b)}\, d(\omega_\zeta)$. Hence,

$$\begin{aligned} \mathrm{Irr}(V, \nabla) &= (1/2b)\, \Sigma_v\, \Sigma_{\omega : v \in S(\omega, b)}\, d(\omega_\zeta) \\ &= (1/2b)\, \Sigma_{\omega \neq 0}\, \Sigma_{v \in S(\omega, b)}\, d(\omega_\zeta) \\ &= (1/b) \Sigma_{\omega \neq 0}\, i(\omega_\zeta)\, d(\omega_\zeta) \\ &= (1/b)\, \Sigma_{[\omega] \neq [0]}\, i(\omega_\zeta)\, d(\omega_\zeta)\, b(\omega). \end{aligned}$$

But, if $\omega = (c\, z^{r/b} + \ldots).\, dz$, then, $\omega_\zeta = (bc\zeta^{r+b-1} + \ldots)\, d\zeta$, so that $i(\omega_\zeta) = b.i(\omega)$, while, trivially, $d(\omega_\zeta) = d(\omega)$. This gives the required formula. ♦

3..4 The *Stokes sheaf* St(V, ∇) or St(V) of a meromorphic pair (V, ∇) at $z = 0$ is the sheaf of groups of units of the sheaf $\mathcal{H}_0$(End V). More explicitly, for any $u \in S^1$, the stalk St(V,∇)(u) is the group of germs of automorphisms g of (V,∇) defined on sectorial domains Γ_δ for some $\delta > 0$ and some sector Γ containing u, satisfying the asymptotic condition of *multiplicative flatness* :

$$g \sim 1\ (\Gamma(u)).$$

Clearly,

$$g \in \mathrm{St}(V, \nabla)(u) \Leftrightarrow g - 1 \in \mathcal{H}_0(\mathrm{End}\ V).$$

The stalks are in general non commutative. Guided by the analogy with Lie groups and Lie algebras we shall think of $\mathcal{H}_0$(End V) as the *infinitesimal Stokes sheaf* and denote it alternately as $\mathfrak{st}$(V, ∇). If we choose an asymptotic trivialization of level 1 for (V, ∇) in a sector Γ_0 containing u so that $\nabla_{d/dz} = d/dz - A$ where $A \in \mathfrak{gl}(N, A_1(\Gamma_0))$, thun for any $v \in \Gamma \cap S^1$, St(V, ∇)(v) may be identified with the group of germs of GL(N, $A_1(\Gamma)$)-valued functions on sectorial domains Γ_δ for some $\delta > 0$ and sectors Γ containing v, such that

(a) $dg/dz + [g, A] = 0$

(b) $g - 1 \sim 0\ (\Gamma(v))$.

On replacing V by End(V) in the results of the preceding sections one is led to the basic results on the Stokes sheaf. The Stokes lines of End(V) are usually referred to as *the* Stokes lines when one is dealing with the Stokes sheaf.

The first basic result is that the local structure of St(V, ∇) is entirely determined by the formal isomorphism class of End(V). We shall now give a description of it using the formal data of (V, ∇), namely, the spectrum Σ, the complex vector space $U = \bigoplus_{\omega \in \Sigma} U_\omega$, and the pair (V_B, ∇_B) defined on $\mathbf{C}^{b\times}$ as in §3.2. The endomorphisms of U may be viewed as matrices $(g_{\sigma\tau})_{\sigma,\tau \in \Sigma}$ where $g_{\sigma\tau} \in \mathrm{Hom}(U_\tau, U_\sigma)$. Let $u \in S^1$ and let $v \in S^{1,b}$ be above u. Let W be an open arc $\subset S^1$ of length $\leq \pi/(|r_1| - 1)$ so that there is an isomorphism, say y, of (V, ∇) on Γ(W) with the pair (V_B, ∇_B) whose pull back to $\mathbf{C}^b$ is isomorphic to $(V_{B'}, \nabla_{B'})$ on the connected component of the preimage of Γ(W) that contains v, y preserving the asymptotic structures. The local horizontal sections of $\text{End}(V_B)$ are of the form

$$s = \Sigma_{\sigma,\tau}\ E(\sigma-\tau)(z)\,.\,\exp(\log_\Gamma z.\, C_{\sigma\tau})\, T_{\sigma\tau} \exp(-\log_\Gamma z.\, C_{\sigma\tau}),$$

where the $T_{\sigma\tau}$ are in $\text{Hom}(U_\tau, U_\sigma)$ for all σ, τ. Such a solution is flat if and only if all the $T_{\sigma\tau}$ are zero except those for which $\rho_{\sigma-\tau}(u) < 0$. This suggests the introduction of a partial order on $\mathcal{D}(u)$ as follows :

$$(*) \qquad \sigma <_u \tau \Leftrightarrow \rho_{\sigma-\tau}(u) < 0.$$

We shall explore this ordering (note that it varies with the point u) and its implications in the next chapter. At this time we limit ourselves to the following proposition that is an immediate consequence of the preceding remarks(cf. Proposition 3.2.2)

PROPOSITION 3.4.1 *The isomorphism* y *induces an isomorphism of* $\text{St}(V)$ (*resp* . $\mathfrak{st}(V)$) *on* W *with the sheaf* $\mathcal{S} = \mathcal{S}(B)$ (*resp* . $\mathfrak{s}(B)$) *of subgroups of* $GL(U)$ (*resp . Lie subalgebras of* $\mathfrak{gl}(U)$) *whose stalk at any* $w \in W$ *is the group* (*Lie algebra*) *of all* $g = (g_{\sigma\tau}) \in \text{End}(U)$ *such that*

$$g_{\sigma\sigma} = 1 \quad (\textit{resp}.\ g_{\sigma\sigma} = 0),$$

$$g_{\sigma\tau} = 0 \quad \textit{unless}\ \sigma <_w \tau \quad (\sigma \neq \tau)\,.$$

PROOF The only point that is not obvious at once is that any such g is invertible. But if we extend the ordering $<_w$ to a *linear ordering* $\prec_w$ in any manner (this is always possible since Σ is a finite set), then $g_{\sigma\tau} = 0$ whenever $\tau \prec_w \sigma$, so that g is " upper triangular with 1's on the diagonal ". This shows that $g \in GL(U)$. ♦

PROPOSITION 3.4.2 *Let* W, W' *be proper open arcs of* S^1, $W' \subset W$, *and suppose that* $W \setminus W'$ *does not meet any Stokes line* (*of* $\text{End}(V)$). *Then the restriction map* $\text{St}(V)(W) \longrightarrow \text{St}(V)(W')$ *is an isomorphism . Moreover, if* (V', ∇') *is another pair and* ξ *is an isomorphism of the formalization of* (V, ∇) *with that of* (V', ∇'), *and* x *is an isomorphism of* (V, ∇) *with* (V', ∇') *on* $\Gamma(W)_\delta$ *such that* $x \sim \xi\ (\Gamma(W'))$, *then* $x \sim \xi\ (\Gamma(W))$ *also .*

PROOF This is immediate from Proposition 3.2.4. ♦

For Irr(End(V)) we have a formula analogous to the one in Proposition 3.3.6. For $\sigma, \tau \in \Sigma$, let

$$b(\sigma, \tau) = \inf\{ b \geq 1 : \sigma, \tau \in \mathfrak{X}(\mathcal{F}_b) \}.$$

It is obvious that

$$i(\sigma - \tau)\, b(\sigma - \tau) \in \mathbf{Z}, \qquad b(\sigma - \tau) \mid b(\sigma, \tau).$$

PROPOSITION 3.4.3 *We have ,*

$$\mathrm{Irr}(\mathrm{End}(V)) = \dim H^1(S^1, \mathcal{S}t(V)) = \sum_{[\sigma,\tau] : \sigma \neq \tau} i(\sigma - \tau)\, b(\sigma, \tau)\, d(\sigma)\, d(\tau)$$

where the summation is over the Galois orbits of pairs (σ, τ) *for the diagonal action of the Galois group .*

PROOF It is obvious that $b(\sigma, \tau)$ is the cardinality of the Galois orbit of (σ, τ). From the proof of Proposition 3.3.6 we have,

$$\begin{aligned} \mathrm{Irr}(\mathrm{End}(V)) &= (1/b) \Sigma_{\sigma \neq \tau}\, i(\sigma_\zeta - \tau_\zeta)\, d(\sigma) d(\tau) \\ &= (1/b) \Sigma_{[\sigma,\tau] : \sigma \neq \tau}\, i(\sigma_\zeta - \tau_\zeta)\, b(\sigma, \tau)\, d(\sigma) d(\tau) \\ &= \Sigma_{[\sigma,\tau] : \sigma \neq \tau}\, i(\sigma - \tau)\, b(\sigma, \tau)\, d(\sigma) d(\tau) \end{aligned}$$

which is the desired expression. ◆

4. THEOREMS OF MALGRANGE-SIBUYA AND DELIGNE

4.1 In this chapter we shall formulate and prove the theorems of Malgrange-Sibuya and Deligne which allow us to get a deep underdtanding of the category of meromorphic pairs at $z = 0$. The Malgrange-Sibuya theorem gives a cohomological description of the set of isomorphism classes of marked pairs (cf. §1.4), namely, pairs equipped with an isomorphism of their formalizations with a given differential module over $\mathfrak{F}$. It is an easy consequence of another result of Malgrange and Sibuya that describes the first cohomology of certain sheaves of flat holomorphic matrices. The theorem of Deligne, which we shall obtain as a consequence of the theorem of Malgrange-Sibuya, gives a complete description of the category of meromorphic pairs at $z = 0$. These two theorems are the fundamental results of the subject .

4.2 We begin with the Malgrange-Sibuya theorem. Let $\mathbf{C}[[z]]$ (resp. $\mathbf{C}\{z\}$) be the ring of formal (resp. convergent) power series in z. If $u \in GL(n, \mathbf{C}[[z]])$ then by the classical Borel-Ritt theorem we can find, for any open arc $W \subset S^1$, an $\varepsilon > 0$ and a holomorphic map $g\ (\Gamma(W)_\varepsilon \longrightarrow GL(n, \mathbf{C}))$ such that $g \sim u$ $(\Gamma(W))$. In general the map g will neither be unique nor wil extend to a meromorphic map around $z = 0$. The obstruction to the meromorphic extendability is measured by the H^1 of a certain sheaf $\mathcal{G}$ of groups on S^1 defined as follows. For $t \in S^1$ the stalk $\mathcal{G}(t)$ at t of $\mathcal{G}$ is the group of germs of holomorphic maps $g\ (\Gamma(W)_\varepsilon \longrightarrow GL(n, \mathbf{C}))$ which are *multiplicatively flat* in the sense that $g \sim 1\ (\Gamma(W))$, where W is some arc around u and ε is > 0. For open arcs $W' \subset W \subset S^1$, the restriction map from W to W' is an injection; and if $W' \subset\subset W$, then for any section s of $\mathcal{G}$ on W there is an $\varepsilon > 0$ such that the restriction of s to W' is defined by a holomorphic map $g \sim 1$ of $\Gamma(W')_\varepsilon$ into $GL(n, \mathbf{C})$. It follows from this remark that any 1-cohomology class for $\mathcal{G}$ is represented by a cocycle (g_{ij}) associated to a finite covering (U_i) of S^1 by open arcs (U_i) such that for some $\varepsilon > 0$ all the g_{ij} are defined by holomorphic multiplicatively flat maps of $\Gamma(U_i \cap U_j)_\varepsilon$ into $GL(n, \mathbf{C})$.

We shall now set up a natural map

$$\Theta \ : \ GL(n, \mathbf{C}[[z]]) \ / \ GL(n, \mathbf{C}\{z\}) \longrightarrow H^1(S^1, \mathcal{G})$$

Let $u \in GL(n, \mathbf{C}[[z]])$ and let (U_i) be a finite covering of S^1 by open arcs. Then we can find $\varepsilon > 0$ and holomorphic maps $g_i \ (\Gamma(U_i)_\varepsilon \longrightarrow GL(n, \mathbf{C}))$ such that $g_i \sim u \ (\Gamma(U_i))$ for all i. Clearly $(g_i g_j^{-1})$ is an 1-cocycle for $\mathcal{G}$. If we write $\Theta(u)$ for the corresponding cohomology class, it is easy to verify that this class depends only on the image $[u]$ of u in the space $GL(n, \mathbf{C}[[z]]) \ / \ GL(n, \mathbf{C}\{z\})$ and not on the choice of the covering or the g_i. We note that Θ is one-one. Indeed, let $u, u' \in GL(n,\mathbf{C}[[z]])$ and $\Theta([u]) = \Theta([u'])$. Then we can find a finite covering (U_i) of S^1 by open arcs, $\varepsilon > 0$, holomorphic g_i, g_i' mapping $\Gamma(U_i)$ into $GL(n, \mathbf{C})$ with $g_i \sim u, g_i' \sim u' \ (\Gamma(U_i))$ such that $(g_i g_j^{-1})$ and $(g_i' g_j'^{-1})$ define the classes $\Theta([u])$ and $\Theta([u'])$ respectively and $g_i g_j^{-1} = c_i g'_i \, g_j'^{-1} c_j^{-1}$ on $\Gamma(U_i \cap U_j)_\varepsilon$ for all i, j, where c_i are holomorphic maps of $\Gamma(U_i)_\varepsilon$ into $GL(n, \mathbf{C})$ with $c_i \sim 1$ $(\Gamma(U_i)$ for all i. Then $g_i'^{-1} c_i^{-1} g_i = g_j'^{-1} c_j^{-1} g_j$ on $\Gamma(U_i \cap U_j)_\varepsilon$ for all i, j. This gives a holomorphic map g from a punctured disc into $GL(n, \mathbf{C})$ such that $g = g_i^{-1} c_i^{-1} g_i$ on $\Gamma(U_i)_\varepsilon$ for all i. As $g_i'^{-1} c_i^{-1} g_i \sim 1 \ (\Gamma(U_i))$ for all i, we have $g \sim 1$ and hence, by Riemann's theorem on removable singularities g is meromorphic at the origin. In particular this gives $g = u'^{-1} u \in GL(n, \mathbf{C}\{z\})$, so that $[u] = [u']$.

THEOREM 4.2.1 (Malgrange-Sibuya) *The map Θ defined above is a bijection* :

$$\Theta \ : \ GL(n, \mathbf{C}[[z]]) \ / \ GL(n, \mathbf{C}\{z\}) \cong H^1(S^1, \mathcal{G})$$

We shall prove this theorem in the next two sections. Our proof follows essentially the sketch outlined by Malgrange in [Mal 3].

4.3 We prove two technical lemmas in this section which will be used in the next section in the proof of the Malgrange-Sibuya theorem. Let us denote, for $z = x + i y$,

$$\partial = (\partial/\partial x - i\partial/\partial y)/2 \ , \quad \partial^* = (\partial/\partial x + i\partial/\partial y)/2 \qquad (z = x + iy)$$

LEMMA 4.3.1 *Let* Δ *be a disc around* $z = 0$ *and let* a *be an* $n \times n$ *matrix of* C^∞ *functions on* Δ. *Then we can find a concentric disc* $\Delta_1 \subset \Delta$ *and a* C^∞ *map* v *of* Δ_1 *into* $GL(n, \mathbf{C})$ *such that*

$$\partial^* v = v\,a, \qquad v(0) = 1.$$

PROOF It is enough to find v satisfying the differential equation with $v(0)$ invertible; for then we can replace v by $v(0)^{-1} v$. Replacing a by αa where $\alpha \in C_c^\infty(\Delta)$ (the suffix c means compact support) and $\alpha = 1$ around $z = 0$ we may assume that a itself has compact support and is defined on all of $\mathbf{C}$. The case $n = 1$ is classical and the case $n > 1$ is the noncommutative generalization of it. We shall treat the case of arbitrary n by suitably adapting the classical,arguments.

When $n = 1$ we define, for any $h \in C_c(\mathbf{C})$, $z \in \mathbf{C}$,

$$y_h(z) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{h(w+z)}{w}\, dw \wedge dw^* \cdot$$

Then y_h is a continuous function and $\partial^* y_h = h$ in the sense of distributions ; y_h is smooth if h is, and $y_h(\infty) = 0$. Moreover y_h is the unique continuous solution of the equation $\partial^* y_h = h$ which is bounded at ∞ ; the uniqueness follows from Liouville's theorem as the difference k of two solutions of this type satisfies $\partial^* k = 0$ and so is a holomorphic function bounded at ∞. If now $a \in C_c^\infty(\mathbf{C})$, $v = \exp(y_a)$ is then a solution to $\partial^* v = v\,a$, $v(\infty) = 1$, and as before is the unique solution with this property. Note that it does not vanish anywhere.

Let us now consider the general case $n > 1$. If v is a matrix satisfying

$$(*) \qquad \partial^* v = v\,a, \qquad v(\infty) = 1,$$

then a classical calculation shows that $w = \det(v)$ satisfies the one-dimensional equation $\partial^* w = w\,\mathrm{tr}(a)$ with $w(\infty) = 1$, hence is nowhere zero, showing that v is everywhere invertible. As before such a solution is unique. Moreover, when a is C^∞ but v is only continuous and satisfies $(*)$ only as a distribution, v has to be smooth because the system of equations $(*)$ for the entries of v is elliptic with smooth coefficients. Thus for proving the lemma it is enough to construct a continuous matrix function v satisfying

$$(**) \qquad \partial^* v = v.(fa), \quad v(\infty) = 1$$

in the weak sense, i. e., as a distribution, for a suitable $f \in C_c^\infty(\mathbf{C})$ with $f = 1$ around $z = 0$. Fix a as above, a smooth matrix with compact support defined on **C**.

We shall now show that there is a number $\delta = \delta(a) > 0$ with the following property: if $f \in C_c^\infty(\mathbf{C})$, $|f| \le 1$, and supp(f) is contained in a disc of radius δ about $z = 0$, then the equation (**) has a continuous distribution solution Let $|.|$ be a matrix norm with $|XY| \le |X||Y|$ for all $n \times n$ matrices X, Y, and let **B** be the Banach space of all continuous maps g from **C** to the space of $n \times n$ matrices with

$$\| g \| = \sup_z | g(z) | < \infty, \qquad g(\infty) = 0.$$

If $b \in \mathbf{B}$ is compactly supported we consider the map J_b of **B** into itself defined by

$$(J_b g)(z) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{g(w+z)b(w+z)}{w} dw \wedge dw^* \qquad (z \in \mathbf{C})$$

If Δ_ν is the disc around the origin of radius ν and the support of b is contained in Δ_ν, a simple argument shows that $\| J_b \| \le 7\nu \| b \|$. In fact, the integration is only over the region $|w+z| < \nu$, and may be split as the sum of integrals over the subregions where $|z| \le 2\nu$ and $|z| > 2\nu$; in the former one has $|w| < 3\nu$ and the integral is majorized by

$$\pi^{-1} \|gb\| \int\int_{r<\nu} dr\, d\theta \le 6\nu \|g\| \|b\| ;$$

in the latter where $|w| > \nu$, the integral is majorized by

$$\pi^{-1} \|gb\| \nu^{-1} \int\int_{|w+z|<\nu} dx\, dy \le \nu \|g\| \|b\|,$$

verifying our claim. If $b = fa$ where $f \in C_c^\infty(\Delta_\nu)$, $|f| \le 1$, and $\nu = 1/8 \|a\|$, we have $\| J_{fa} \| < 1$ and so the operator $I - J_{fa}$ is invertible on **B**. So $\exists$ $u \in \mathbf{B}$ such that $(I - J_{fa})\, u = J_{fa}\, 1$; this relation is just

$$u(z) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{u(w+z)\ (fa)(w+z) + (fa)(w+z)}{w}\, dw \wedge dw^* \qquad (z \in \mathbf{C}).$$

This shows that $\partial^* u = f a + u fa$ as distributions and so $v = 1 + u$ is a solution to $(**)$. ◆

For functions which are merely C^∞ and are defined on sectorial domains, we shall use the following definition of flatness : for $\alpha \in C^\infty(\Gamma_\varepsilon)$ where Γ_ε is an open sector, $\alpha \sim 0\ (\Gamma_\varepsilon)$ means that for arbitrary integers $m, n, p \ge 0$ and any sector $\Gamma' \subset\subset \Gamma$,

$$(\partial^m \partial^{*\,n} \alpha)(z) = O(|z|^p) \qquad (z \longrightarrow 0 \text{ in } \Gamma').$$

This definition makes sense even when $\Gamma = \mathbf{C}^\times$; then $\alpha \in C^\infty(\Delta_\varepsilon^\times)$, and α extends to an element of $C^\infty(\Delta_\varepsilon)$; we then simply write $\alpha \sim 0$. We now define the sheaf $\mathcal{G}_s$ as the C^∞ - analogue of the sheaf $\mathcal{G}$: for any $u \in S^1$, its stalk $\mathcal{G}_s(u)$ is the group of germs of C^∞ maps $g\ (\Gamma(W)_\varepsilon \longrightarrow GL(n, \mathbf{C}))$ which are multiplicatively flat i. e., $g \sim 1\ (\Gamma(W))$, where W is some arc around u and ε is > 0.

LEMMA 4.3.2 $H^1(S^1, \mathcal{G}_s) = 0$.

PROOF We start with the easily verified fact : if m, n are integers ≥ 0, there are differential operators D_{mn} on S^1 of order $\le m+n$ in $d/d\theta$ with coefficients that are polynomials in $e^{2\pi i\theta}$ such that for $\gamma \in C^\infty(S^1)$, and γ' defined by $\gamma'(r\, e^{2\pi i\theta}) = \gamma(e^{2\pi i\theta})$, one has

$$(\partial^m \partial^{*\,n} \gamma')(r\, e^{2\pi i\theta}) = r^{-(m+n)}\, (D_{mn}\gamma)(e^{2\pi i\theta}).$$

It follows from this that if $U \subset S^1$ is an open arc, $\alpha \in C^\infty(\Gamma(U)_\varepsilon)$, and $\alpha \sim 0\ (\Gamma(U))$, then for any $\gamma \in C^\infty(S^1)$, one has $\gamma'\alpha \sim 0\ (\Gamma(U))$; in particular, if $\mathrm{supp}(\alpha) \subset U$, $\gamma'\alpha \in C^\infty(\Delta_\varepsilon)$. A simple argument now gives the following extension principle for $n \times n$ matrices of functions: let U_1, U_2 be open arcs in S^1 with disjoint closures and g_i is a C^∞ map of $\Gamma(U_i)_\varepsilon$ into $GL(n, \mathbf{C})$ with

$g_i \sim 1\ (\Gamma(U_i))$, $\exists\ \eta$ with $0 < \eta \leq \varepsilon$ and a C^∞ map $g\ (\Delta_\eta \longrightarrow GL(n, \mathbf{C}))$ such that $g \sim 1$ and $g = g_i$ on $\Gamma(U_i)_\eta$.

Consider now a cocycle (g_{ij}) for the sheaf $\mathcal{G}_s$ associated to the finite covering (U_i) of S^1 by open arcs, with $g_{ij}\ (\Gamma(U_i \cap U_j)_\varepsilon \longrightarrow GL(n, \mathbf{C}))$ and $g_{ij} \sim 1(\Gamma(U_i \cap U_j))$. In view of the above remarks we may assume, by passing to a refinement of the covering that the g_{ij} are actually smooth maps of a disc Δ_ε into $GL(n, \mathbf{C})$ and $g_{ij} \sim 1$. Passing to a further refinement we may assume that we are working with a covering $(V_m)_{0 \leq m < N}$ where N is an integer ≥ 4, $V_m = (e_m, e_{m+2})$, $e_m = \exp(2\pi i m/N)$. Then $V_m \cap V_{m+1} = (e_m, e_{m+1})$ and we may take the cocycle as given by $(g_m)_{0 \leq m < N}$ where $g_m\ (= g_{N+m})$ is a map $\Gamma((e_m, e_{m+1}))_\varepsilon \longrightarrow GL(n, \mathbf{C})$ and is a restriction of a smooth map g'_m of the disc Δ_ε into $GL(n, \mathbf{C})$ flat at $z = 0$, i. e., $g'_m \sim 1$. We modify g'_m to h_m without changing g_m as follows: $h_m = g'_m$ for $m = 0, 1, N-1$; for $3 \leq m \leq N-2$, $h_m = 1$ on $\Gamma((e_0, e_1)_\varepsilon)$, $= g'_m$ on $\Gamma((e_m, e_{m+1})_\varepsilon)$; and $h_2 = g'_2$ on $\Gamma((e_2, e_3))_\varepsilon)$, $= g_1^{-1} g_0^{-1} g_{N-1}^{-1}$ on $\Gamma((e_0, e_1)_\varepsilon)$; we may have to make ε smaller to ensure this. The cocycle is not changed but now $h_0 h_1 \ldots h_{N-1} = 1$ on $\Gamma((e_0, e_1)_\varepsilon)$. We now define the $f_m\ (= f_{N+m})$by

$$f_0 = 1,\ f_1 = h_1^{-1},\ f_2 = (h_1 h_2)^{-1}, \ldots\ f_{N-1} = (h_1 h_2 \ldots h_{N-1})^{-1}$$

It is then easy to verify that $f_{m-1} f_m^{-1} = g_m$ for all m, so that the cocycle (g_m) is a coboundary. ◆

4.4 PROOF OF THEOREM 4.2.1 It is a question of the surjectivity of the map Θ. Let us consider a cohomology class σ for the sheaf $\mathcal{G}$ and assume that it is represented by a cocycle (f_{ij}) associated to a finite covering (U_i) of S^1 by open arcs, f_{ij} being a holomorphic map of $\Gamma((U_i \cap U_j)_\varepsilon)$ into $GL(n, \mathbf{C})$ with $f_{ij} \sim 1(\Gamma(U_i \cap U_j))$. Lemma 4.3.2 allows us to write this , by passing to a refinement if necessary, in the form $f_{ij} = f_i f_j^{-1}$, f_i being a C^∞ map of $\Gamma(U_i)_\varepsilon$ into $GL(n, \mathbf{C})$ and $f_i \sim 1\ (\Gamma(U_i))$ for all i. As $\partial^* f_{ij} = 0$, we find that $f_i^{-1}(\partial^* f_i) = f_j^{-1}(\partial^* f_j)$ for all i, j, giving a C^∞ map $f\ (\Delta_\varepsilon^\times \longrightarrow \mathrm{End}(\mathbf{C}^n))$ coinciding with $f_i^{-1}(\partial^* f_i)$ on $\Gamma(U_i)_\varepsilon$ for all i. As $f_i \sim 1\ (\Gamma(U_i))$ it follows that $f \sim 0$, so that we may assume that f is C^∞ on the full disc Δ_ε. By Lemma 4.3.1(and diminishing ε if necessary) we can find a smooth $g\ (\Delta_\varepsilon \longrightarrow GL(n, \mathbf{C}))$ with $g(0) = 1$, $g^{-1}(\partial^* g) = f$.

Let $g_i = f_i g^{-1}$ on $\Gamma(U_i)_\varepsilon$ for all i. Then $\partial^* g_i = 0$ so that g_i is *holomorphic* and $g_i g_j^{-1} = f_{ij}$ on $\Gamma(U_i \cap U_j)_\varepsilon$; at the same time as $f_i \sim 1$ $(\Gamma(U_i))$, we also have $(\partial^{*n} \partial^m g^{-1})(0) = 0$ for all $n \geq 1$. Thus we see that $g_i \sim u$ $(\Gamma(U_i))$ for all i , $u \in GL(n, \mathbf{C}[[z]])$ being defined by

$$u = 1 + \sum_{m \geq 1} (\partial^m g^{-1})(0) z^m / m! .$$

It is now clear that $\Theta([u]) = \sigma$. ◆

REMARK The *additive* version of the Malgrange-Sibuya theorem is also true but is much easier to prove since it is in the *commutative* context. We work with the sheaf $\mathcal{G}_{mn}$ on S^1 whose stalk at $t \in S^1$ is the vector space of germs of flat holomorphic maps g $(\Gamma(U)_\varepsilon \longrightarrow M_{mn}(\mathbf{C}))$; here, for any ring R, $M_{mn}(R))$ is the space of $m \times n$ matrices over the ring R and flatness means that $g \sim 0$ $(\Gamma(U))$. As before we have a natural map

$$\Theta_+ : M_{mn}(\mathbf{C}[[z]]) \,/\, M_{mn}(\mathbf{C}\{z\}) \longrightarrow H^1(S^1, \mathcal{G}_{mn})$$

The basic theorem is the following whose proof is left to the reader.

THEOREM 4.4.1 *The map* Θ_+ *is a linear isomorphism.*

4.5 We shall now use the Malgrange-Sibuya theorem to obtain the fundamental cohomological description of the set of all isomorphism classes of marked meromorphic pairs $((V, \nabla), \xi)$ at $z = 0$ (cf. §1.4). We fix (V_0, ∇_0) and introduce the category of germs of pairs $((V, \nabla), \xi)$, ξ being an isomorphism of the formalization of (V, ∇) with that of (V_0, ∇_0), $\mathrm{Morph}((V, \nabla), \xi)\,((V', \nabla'), \xi'))$ being the set of all morphisms u $((V, \nabla) \longrightarrow (V', \nabla'))$ such that $\xi' \circ u^\wedge = \xi$. Let $\mathfrak{M}(V_0, \nabla_0)$ be the set of all isomorphism classes of objects in this category.

THEOREM 4.5.1(Malgrange-Sibuya) *We have a canonical bijection*

$$\Phi : \mathfrak{M}(V_0, \nabla_0) \cong H^1(S^1, St_0)$$

where St_0 *is the Stokes sheaf of* (V_0, ∇_0).

PROOF Given $((V, \nabla), \xi) \in \mathfrak{M}(V_0, \nabla_0)$ we can find by Theorem 2.2.4 a finite covering (U_i) of S^1 by open arcs, an $\varepsilon > 0$, and holomorphic isomorphisms x_i of (V, ∇) with (V_0, ∇_0) on the sector $\Gamma(U_i)_\varepsilon$ that preserve the asymptotic structures and lie above ξ, i. e., $x_i{}^\wedge = \xi$. Then $(x_i x_j^{-1})$ is a cocycle for the sheaf St_0, and it is easy to check that the corresponding cohomology class depends only on the isomorphism class of $((V, \Delta), \xi)$ and not on the x_i or the (U_i). We thus get a map

$$\Phi : \mathfrak{M}(V_0, \nabla_0) \longrightarrow H^1(S^1, St_0),$$

and it is a question of proving that this is a bijection.

Φ is injective Let x'_i correspond to $((V, \nabla), \xi') \in \mathfrak{M}(V_0, \nabla_0)$ as above, for the same covering (U_i). We suppose that there are $c_i \in St_0(U_i)$ such that $x'_i x'_j{}^{-1} = c_i (x_i x_j^{-1}) c_j^{-1}$ on $\Gamma(U_i \cap U_j)_\varepsilon$ for all i, j. Writing $y_i = c_i^{-1} x'_i$, we find that $y_i^{-1} x_i = y_j^{-1} x_j$ on $\Gamma(U_i \cap U_j)_\varepsilon$ for all i, j. So there is a holomorphic isomorphism v of (V, ∇) with (V', ∇') on $\Delta_\varepsilon{}^\times$ coinciding with $y_i^{-1} x_i$ on $\Gamma(U_i)_\varepsilon$ for all i. Since v preserves the asymptotic structures at 0, it follows that v is meromorphic at $z = 0$. As $\xi'^{-1} \xi = v^\wedge$, $((V, \nabla), \xi)$ and $((V', \nabla'), \xi')$ are isomorphic.

Φ is surjective We may assume that V_0 is the trivial bundle $\Delta_\varepsilon \times C^n$ so that ∇_0 is the connection with the matrix $A_0 \in \mathfrak{gl}(n, \mathfrak{F}_{cgt})$. Let the cohomology class $\sigma \in H^1(S^1, St_0)$ be represented by an 1-cocycle (g_{ij}) associated with the finite covering (U_i) of S^1 by open arcs:

$$g_{ij}\,(\Gamma(U_i \cap U_j)_\varepsilon \longrightarrow GL(n, \mathbf{C}))\text{ holomorphic, } g_{ij} \sim 1\ (\Gamma(U_i \cap U_j)),\ \ g_{ij}[A_0] = A_0.$$

Clearly (g_{ij}) determines an element c of $H^1(S^1, \mathcal{G})$ and so we can find, by Theorem 4.2.1, an element $\xi \in GL(n, \mathbf{C}[[z]])$ that determines c. This means that (passing to a refinement of (U_i) if necessary) we can find holomorphic maps $x_i\,(\Gamma(U_i)_\varepsilon \longrightarrow GL(n, \mathbf{C}))$, with $x_i \sim \xi\ (\Gamma(U_i))$, $x_i x_j^{-1} = g_{ij}$ on $\Gamma(U_i \cap U_j)_\varepsilon$, for all i, j.

Let us now define for each i the connection ∇_i on the restriction of V_0 to $\Gamma(U_i)_\varepsilon$ by requiring that its matrix is $A_i = x_i^{-1}[A_0]$. Since $x_i x_j^{-1}[A_0] = A_0$ for all i, j, it is immediate that $A_i = A_j$ on $\Gamma(U_i \cap U_j)_\varepsilon$ for all i, j, so that there is a connection ∇ on the restriction of V_0 to $\Delta_\varepsilon^\times$ that coincides with ∇_i on $\Gamma(U_i)_\varepsilon$ for all i. Since $A_i \sim \xi^{-1}[A_0]$ for all i, we see that ∇ is meromorphic at $z = 0$. From our construction it is clear that x_i is an isomorphism of (V_0, ∇) with (V_0, ∇_0) on $\Gamma(U_i)_\varepsilon$ that lies above ξ for all i. In other words, the isomorphism class of $((V_0, \nabla), \xi)$ lies in $\mathfrak{M}(V_0, \nabla_0)$ and its image under Φ is σ. This finishes the proof. ♦

For any fixed (V_0, ∇_0), the pairs (V, ∇) whose formalizations are isomorphic to that of (V_0, ∇_0) form a subcategory of $\mathfrak{C}_0$, and it is natural to want to have a description of the set of isomorphism classes of the objects in it. Let us write $\mathfrak{I}(V_0, \nabla_0)$ for this set. We have an obvious surjective map

$$P : \mathfrak{M}(V_0, \nabla_0) \longrightarrow \mathfrak{I}(V_0, \nabla_0),$$

and it is a question of describing the fibers of this map. Let

$$G^\wedge(V_0, \nabla_0) = G^\wedge(V_0) = \mathrm{Aut}\,(M_0^\wedge)$$

where $M_0^\wedge$ is the formalization of (V_0, ∇_0). We have an action of $G^\wedge(V_0)$ on $\mathfrak{M}(V_0, \nabla_0)$ given by

$$u\,[((V, \nabla), \xi)] = (\,(V, \nabla), u\,\xi),$$

where [...] refers to isomorphism classes.

THEOREM 4.5.2 *The fibers of* P *are precisely the orbits of* $G^\wedge(V_0)$. *In other words ,*

$$\mathfrak{I}(V_0, \nabla_0) \cong G^\wedge(V_0)\backslash\mathfrak{M}(V_0, \nabla_0).$$

PROOF Routine. ♦

4.6 We shall now take up the theorem of Deligne which gives a complete description of the category of germs of meromorphic pairs at $z = 0$. In a letter to Malgrange [De 2] Deligne gave the formulation of this theorem and sketched the outlines of the proof (see [Be] and [Mal 4] for brief discussions of Deligne's proof). In view of the fundamental nature of Deligne's result it may be worthwhile to discuss it in more detail than given in [De 2] [Mal 4] or [Be].

Let $\mathcal{D}$ be the local system defined on S^1 as follows: for any open arc $U \subset S^1$, $\mathcal{D}(U)$ is the vector space of all holomorphic differential forms on $\mathbf{C}^\times$ of the form

$$\omega = \omega^{\#}.dz, \qquad \omega^{\#} = \sum\nolimits_{a \in \mathbf{Q},\, a < -1} c_a z^a$$

where the sum is finite and the branches z^a are chosen arbitrarily. The stalk $\mathcal{D}(u)$ at $u \in S^1$ comes equipped with the monodromy action $m, \omega \longrightarrow m.\omega$ of $\mathbf{Z}$, the action of $1 \in \mathbf{Z}$ being the result of analytic continuation of the elements along the unit circle described in the counter-clockwise direction starting at u. For $u \in S^1$, and $\omega, \omega' \in \mathcal{D}(u)$, write $\omega <_u \omega'$ if there is an arc W containing u such that

$$\exp\Big(\int_u^z (\omega^{\#} - \omega'^{\#})\, dz\Big) \sim 0\, (\Gamma(W)) .$$

If $\omega, \omega' \in \mathcal{D}(U)$, we write $\omega <_U \omega'$ if $\omega <_u \omega'$ for all $u \in U$. This is the same as requiring the above flatness condition on $\Gamma(U)$. If $\omega^{\#} - \omega'^{\#} = c_a z^a$ + terms of higher order where $c_a \neq 0$, then

$$\omega <_u \omega' \Leftrightarrow \rho_{\omega-\omega'}(u) = \mathrm{Re}(c_a u^{a+1}/a+1) < 0 .$$

We have already observed in Lemma 3.2.1 that $\rho_{\omega-\omega'}(u)$ is independent of the choice of the branche z^a. If $\omega <_u \omega'$ it is clear that $\omega <_U \omega'$ for an open arc U containing u ; ω and ω' are not comparable at u if and only if $\rho_{\omega-\omega'}(u) = 0$; then there is an arc $U' = (u', u)$ (resp. $U'' = (u, u'')$) such that $\omega <_{U'} \omega'$ and $\omega' <_{U''} \omega$ or vice versa.

We now introduce the notions of local systems of finite rank on open subsets U of S^1 that are graded and filtered by $\mathcal{D}$. To say that a local system $\mathcal{V}$ on $U \subset S^1$ is *$\mathcal{D}$-graded* is to require that for each $u \in U$ there is a grading of $\mathcal{V}(u)$ by $\mathcal{D}(u)$, $\mathcal{V}(u) = \bigoplus_{\omega \in \mathcal{D}(u)} \mathcal{V}_\omega(u)$, such that for some open arc W containing u, the grading at any point $u' \in W$ is the one induced by the grading at u through analytic continuation from u to u'; in this case this is true for any open arc W with $u \in W \subset U$. Clearly, for any $\mathcal{D}$-graded local system defined on all of S^1, the grading at any point $u \in S^1$ is compatible with the monodromy actions on $\mathcal{V}(u)$ and on $\mathcal{D}(u)$; this means that the monodromy action of $m \in \mathbf{Z}$ on $\mathcal{V}(u)$ induces an isomorphism of $\mathcal{V}_\omega(u)$ with $\mathcal{V}_{m.\omega}(u)$ for all ω. It is obvious that for any fixed u, the assignment $\mathcal{V} \longrightarrow \mathcal{V}(u)$ is an equivalence of categories, from the category of $\mathcal{D}$-graded local systems on S^1 into the category of $\mathcal{D}(u)$-graded vector spaces equipped with a $\mathbf{Z}$-action compatible with the monodromy action of $\mathbf{Z}$ on $\mathcal{D}(u)$. For any open $U \subset S^1$ the category of $\mathcal{D}$-graded local systems on U comes equipped with $\otimes$, $*$, and Hom; we note that

$$(\mathcal{V}^1 \otimes \mathcal{V}^2)_\omega = \Sigma_{\sigma + \xi = \omega} (\mathcal{V}^1)_\sigma \otimes (\mathcal{V}^2)_\xi$$

$$\mathcal{V}^*_\omega(u) = (\bigoplus\nolimits_{\xi \neq -\omega} \mathcal{V}_\xi(u))^\perp, \ \perp \text{ being the annihilator.}$$

Of course $\mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2) \cong \mathcal{V}^{1*} \otimes \mathcal{V}^2$, and if $h \in \mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2)$ is represented by the matrix $(h_{\sigma\tau})$, then $h \in \mathrm{Hom}\,(\mathcal{V}^1, \mathcal{V}^2)_\omega$ if and only if $h_{\sigma\tau} = 0$ when $\sigma - \tau \neq \omega$. If $U = S^1$, the equivalence $\mathcal{V} \longrightarrow \mathcal{V}(u)$ is compatible with $\otimes$, $*$, and Hom.

Given a $\mathcal{D}$-graded local system $\mathcal{V}$ on the open set $U \subset S^1$ one can introduce the subspaces $\mathcal{V}(w)^\omega$ of $\mathcal{V}(w)$ $(w \in U)$ defined as follows:

$$\mathcal{V}(w)^\omega = \mathcal{V}(w)_\omega \oplus \bigoplus\nolimits_{\omega' <_w \omega} \mathcal{V}(w)_{\omega'}$$

It is then clear that one has the following properties:

(i) $\{\mathcal{V}(w)^\omega\}_{\omega \in \mathcal{D}(w)}$ is a filtration: $\omega <_w \omega' \Rightarrow \mathcal{V}(w)^\omega \subset \mathcal{V}(w)^{\omega'}$
(ii) if s is a local section of $\mathcal{V}$ and $s(v) \in \mathcal{V}(w)^\omega$, then $s(w') \in \mathcal{V}(w')^\omega$ for all w' sufficiently close to w.

We call this the *$\mathcal{D}$-filtration induced by the $\mathcal{D}$-* gradation. An arbitrary local system $\mathcal{V}$ of finite rank on U is said to be *$\mathcal{D}$-filtered* if its stalks $\mathcal{V}(w)$ are filtered by $\mathcal{D}(w)$ and locally on U this filtration is induced by a $\mathcal{D}$-gradation. This means that properties (i) and (ii) above are satisfied, and that for each $u \in U$, there is an open arc W and the structure of a $\mathcal{D}$-graded local system on the restriction of $\mathcal{V}$ to W such that for all w in W,

$$\mathcal{V}(w)^\omega = \mathcal{V}(w)_\omega \oplus \bigoplus\nolimits_{\omega' <_w \omega} \mathcal{V}(w)_{\omega'}$$

In other words, $\mathcal{S}$-filtered local systems are obtained by gluing together $\mathcal{S}$-graded local systems taking care to preserve the $\mathcal{S}$-filtrations of the latter. We note that if U is an open set on which the $\mathcal{S}$-filtration on $\mathcal{V}$ is induced by a $\mathcal{S}$-gradation, and for any open arc $W' \subset U$ and $\omega \in \mathcal{S}(W')$ we write $\mathcal{V}(W')^{\omega}$ (resp. $\mathcal{V}(W')_{\omega}$) for the space of all sections of $\mathcal{V}$ on W' that are in $\mathcal{V}(w)^{\omega}$ (resp. $\mathcal{V}(w)_{\omega}$) for all $w \in W'$, then

$$\mathcal{V}(W')^{\omega} = \mathcal{V}(W')_{\omega} \bigoplus \bigoplus_{\omega' <_W \omega} \mathcal{V}(W)_{\omega'}$$

The category of $\mathcal{S}$-graded local systems is equipped with the operations of $\bigoplus$, $\bigotimes$, $*$, and Hom in a natural manner. To define these operations in an unambiguous way it is enough to check that the $\mathcal{S}$-filtrations arising out of the local $\mathcal{S}$-graded structures on $\mathcal{V}^*$, $\mathcal{V}^1 \bigotimes \mathcal{V}^2$, and $\mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2)$ may be described entirely in terms of the $\mathcal{S}$-filtered structures on $\mathcal{V}$, $\mathcal{V}^1$, and $\mathcal{V}^2$. This is seen easily from the following: write $\lambda(\omega)$ for the set of $\xi \neq -\omega$ such that either $\xi < -\omega$ or ξ is not comparable with $-\omega$; then,

$$\mathcal{V}^*(u)^{\omega} = \left(\Sigma_{\xi \in \lambda(\omega)} \mathcal{V}(u)^{\xi} \right)^{\perp}$$

$$(\mathcal{V}^1 \bigotimes \mathcal{V}^2)(u)^{\omega} = \Sigma_{\sigma + \tau \leq \omega} \left(\mathcal{V}^1(u)^{\sigma} \bigotimes \mathcal{V}^2(u)^{\tau}\right)$$

Finally, the subsheaf $\mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2)^0$ is nothing but the subsheaf of $\sigma \in \mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2)$ that preserves the filtered structures; this is immediate from the local identifications

$$\begin{aligned} \mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2)^0 &= \bigoplus_{\xi \leq 0} \mathrm{Hom}(\mathcal{V}^1, \mathcal{V}^2)_{\xi} \\ &= \bigoplus_{\xi \leq 0} \bigoplus_{\sigma - \omega = \xi} \mathrm{Hom}(\mathcal{V}^1_{\omega}, \mathcal{V}^2_{\sigma}) \end{aligned}$$

For any open $U \subset S^1$ and any $\mathcal{S}$-filtered local system $\mathcal{V}$ defined on U we shall now associate a $\mathcal{S}$-graded local system $\mathrm{Gr}\,\mathcal{V}$ on U defined as follows. For $u \in U$, and $\omega \in \mathcal{S}(u)$,

$$(\mathrm{Gr}\,\mathcal{V})_{\omega}(u) = \mathcal{V}(u)^{\omega} / \Sigma_{\omega' <_u \omega} \mathcal{V}(u)^{\omega}$$

$$\mathrm{Gr}\,\mathcal{V}(u) = \bigoplus_{\omega} (\mathrm{Gr}\,\mathcal{V})_{\omega}(u)$$

If $W \subset U$ is an open arc on which the $\mathcal{I}$-filtered structure of $\mathcal{V}$ arises from a $\mathcal{I}$-graded structure, then for $u \in W$, the natural map

$$\mathcal{V}(u) \longrightarrow (\mathrm{Gr}\,\mathcal{V})_\omega(u)$$

is an isomorphism when restricted to $\mathcal{V}_\omega(u)$. This permits us to transfer the $\mathcal{I}$-graded structure of $\mathcal{V}$ to $\mathrm{Gr}\,\mathcal{V}$ on W, and regard $\mathrm{Gr}\,\mathcal{V}$ as a $\mathcal{I}$-graded local system on W. If we consider another $\mathcal{I}$-graded structure for $\mathcal{V}$ on W compatible with its $\mathcal{I}$-filtered structure, say with components ${}_\omega\mathcal{V}(u)$ ($u \in W$), there is a unique isomorphism $v \longrightarrow (v)$ of $\mathcal{V}_\omega(u)$ with ${}_\omega\mathcal{V}(u)$ such that

$$(*) \qquad v \equiv (v) \mod \Sigma_{\omega' <_u \omega} \mathcal{V}(u)^{\omega'} \qquad (u \in W).$$

It is clear from (*) that $v \longrightarrow (v)$ defines an isomorphism of the two $\mathcal{I}$-graded structures on the restriction of $\mathcal{V}$ to W that induces the identity on $\mathrm{Gr}\,\mathcal{V}$, showing that the $\mathcal{I}$-graded structure of $\mathrm{Gr}\,\mathcal{V}$ on W defined above is independent of the $\mathcal{I}$-graded structure of $\mathcal{V}$ on W used in its construction. The assignment

$$\mathcal{V} \longrightarrow \mathrm{Gr}\,\mathcal{V}$$

is a covariant functor from the category of $\mathcal{I}$-filtered local systems on U to the category of $\mathcal{I}$-graded local systems on U.

Observe that the $\mathcal{I}$-filtered local system associated to $\mathrm{Gr}\,\mathcal{V}$ will in general be only *locally* isomorphic to $\mathcal{V}$ on U. Indeed, it is clear that for any $u \in U$ there is an open arc $W \subset U$ containing u and an isomorphism α of $\mathcal{V}$ with $\mathrm{Gr}\,\mathcal{V}$ as $\mathcal{I}$-filtered local systems on W such that for $w \in W$, $\omega \in \mathcal{I}(w)$, $v \in \mathcal{V}(w)^\omega$,

$$\alpha(v) = [v] + \dots$$

where [v] is the image of v in $(\mathrm{Gr}\,\mathcal{V})_\omega(w)$ and +... are terms in $(\mathrm{Gr}\,\mathcal{V})_{\omega'}(w)$ with $\omega' <_W \omega$. We call such isomorphisms *admissible* .

Suppose $\mathcal{V}_0$ is a $\mathcal{I}$-filtered local system on $U \subset S^1$. If $\mathcal{V}$ is a $\mathcal{I}$-filtered local system on U, a *marking* of $\mathcal{V}$ (by $\mathcal{V}_0$) is an isomorphism

$$\xi : \mathrm{Gr}\,\mathcal{V} \cong \mathrm{Gr}\,\mathcal{V}_0$$

We shall say that $\mathcal{V}$ is *marked* and write $\mathcal{L}(\mathcal{V}_0)$ for the set of all isomorphism classes of marked pairs $(\mathcal{V}, \xi)$, where $(\mathcal{V}, \xi)$ and $(\mathcal{V}', \xi')$ are isomorphic if there is an isomorphism $j\ (\mathcal{V} \longrightarrow \mathcal{V}')$ such that

$$\xi' \circ \mathrm{Gr}\,(j) = \xi$$

The set $\mathcal{L}(\mathcal{V}_0)$ has a cohomological description which we shall now elucidate. For this purpose we introduce the sheaf $\mathcal{G}_0 = \mathcal{G}(\mathcal{V}_0)$ of groups of germs of automorphisms of the $\mathcal{D}$-filtered local system $\mathcal{V}_0$ that induce the identity on $\mathrm{Gr}\,\mathcal{V}_0$.

LEMMA 4.6.1 *If the class of* $(\mathcal{V}, \xi)$ *lies in* $\mathcal{L}(\mathcal{V}_0)$ *and* $u \in U$, *we can find an open arc* $W \subset U$ *containing* u *and an isomorphism* x *of* $\mathcal{V}$ *with* $\mathcal{V}_0$ *on* W *such that* x *lies above* ξ, *i. e.,* $\mathrm{Gr}(x) = \xi$.

PROOF If we choose W containing u and admissible isomorphisms α of $\mathcal{V}$ with $\mathrm{Gr}\,\mathcal{V}$ and α_0 of $\mathcal{V}_0$ with $\mathrm{Gr}\,\mathcal{V}_0$ on W, and take $x = \alpha_0^{-1}\xi\alpha$, then it is clear that $\mathrm{Gr}(x) = \xi$. ♦

PROPOSITION 4.6.2 *There is a canonical bijection*

$$\Psi : \mathcal{L}(\mathcal{V}_0) \cong H^1(U, \mathcal{V}_0)$$

that takes the isomorphism class of $(\mathcal{V}_0, \mathrm{id})$ *to the zero element.* .

PROOF Let $(\mathcal{V}, \xi)$ be a marked pair whose class is in $\mathcal{L}(\mathcal{V}_0)$. By the above lemma we can find a covering (W_i) of U by open arcs, and for each i, an isomorphism $x_i\ (\mathcal{V} \cong \mathcal{V}_0)$ on W_i such that $\mathrm{Gr}(x_i) = \xi$. If $g_{ij} = x_i x_j^{-1}$, (g_{ij}) is an 1-cocycle for the sheaf $\mathcal{G}_0$. We leave it to the reader to make the routine verification that the cohomology class of this cocycle depends only on the isomorphism class of $(\mathcal{V}, \xi)$ and that the map $\Psi : \mathcal{L}(\mathcal{V}_0) \cong H^1(U, \mathcal{V}_0)$ thus defined takes the class of $(\mathcal{V}_0, \mathrm{id})$ to the zero element. If $(\mathcal{V}', \xi')$ is another pair, (x_i') the associated isomorphisms, and if $x_i' x_j'^{-1} = c_i x_i x_j^{-1} c_j^{-1}$ where c_i are sections of $\mathcal{G}_0$ on W_i, we have $x_i^{-1}c_i^{-1}x_i' = x_j^{-1}c_j^{-1}x_j'$ on $W_i \cap W_j$, so that there is an isomorphism $t\ (\mathcal{V}' \cong \mathcal{V})$ coinciding with $x_i^{-1}c_i^{-1}x_i'$ on W_i for all i. This proves that Ψ is injective. For proving the surjectivity let (g_{ij}) be any 1-cocycle for $\mathcal{G}_0$. We write $\mathcal{V}_{0i}$ for the restriction of $\mathcal{V}_0$ to W_i and glue the sheaves $\mathcal{V}_{0i}$ along the intersections $W_i \cap W_j$ via the identifications $v \longrightarrow g_{ij}(v)$ of $\mathcal{V}_{0j}(u)$ with $\mathcal{V}_{0i}(u)$, $u \in W_i \cap W_j$ ($v \in \mathcal{V}_{0j}(u)$); this is just the *twisting* of the sheaf $\mathcal{V}_0$ by the cocycle (g_{ij}) (cf. **II**, §1). The cocycle identities show that the gluing process is self-consistent and leads to a local system $\mathcal{V}$ and isomorphisms θ_i ($\mathcal{V}_i \cong \mathcal{V}_{0i}$), $\mathcal{V}_i$ being the restriction of $\mathcal{V}$ to W_i. $\mathcal{V}$ is naturally $\mathcal{D}$-filtered since the g_{ij} preserve the filtration. Since $Gr(g_{ij}) = 1$, there is an isomorphism ξ ($Gr\,\mathcal{V} \cong Gr\,\mathcal{V}_0$) such that $Gr(\theta_i) = \xi$ on W_i for all i. Since $g_{ij} = \theta_i\,\theta_j^{-1}$, the cocycle (g_{ij}) corresponds to the pair $(\mathcal{V}, \xi)$. ◆

4.7 Following Deligne we shall now introduce the basic functor

$$\mathcal{G}_0 \longrightarrow \text{category of } \mathcal{D}\text{-filtered local systems on } S^1$$

For any open set $U \subset S^1$ let us consider the assignment that takes germs of holomorphic pairs (V, ∇) defined on some sector $\Gamma(U)_\varepsilon$ to the local system $\mathcal{H}(V, \nabla)$ defined on U of germs of sectorial horizontal sections of (V, ∇). This assignment is a fully faithful functor compatible with $\otimes$, $*$, and Hom. If we take $U = S^1$ and consider the *meromorphic* pairs at $z = 0$, it is necessary to give the local systems $\mathcal{H}(V, \nabla)$ an additional structure to maintain the fully faithful nature of this functor. We shall view $\mathcal{H} = \mathcal{H}(V, \nabla)$ as a $\mathcal{D}$-filtered local system as follows: for any $u \in S^1$ and $\omega \in \mathcal{D}(u)$, $\mathcal{H}^\omega$ is defined by

$$v \in \mathcal{H}^\omega \Leftrightarrow \exp(-\int_u^z \omega^\#\, dz)\,.\,v(z) = O(|z|^{-N}) \qquad (\Gamma)$$

for some integer $N \geq 0$ and some sector Γ containing u; the O refers to the components of the section v with respect to some (every) meromorphic trivialization of (V, ∇) at $z = 0$.

PROPOSITION 4.7.1 *The* $\{\mathcal{H}(u)^\omega\}$ *define the structure of a* $\mathcal{D}$*-filtered local system on* $\mathcal{H}(V, \nabla)$; *and the assignment*

$$\textit{germ of } (V, \nabla) \longrightarrow \mathcal{H}$$

is a fully faithful covariant functor from $\mathcal{G}_0$ *into the category of* $\mathcal{D}$*-filtered local systems on* S^1 *compatible with* $\otimes$, $*$, *and Hom . Furthermore, under this cor-*

respondence, the sheaf St(V, ∇) *goes over to the sheaf* $\mathcal{G}(\mathcal{H})$ *of groups of germs of automorphisms of* $\mathcal{H}$ *that induce the identity on* Gr $\mathcal{H}$.

PROOF It is obvious that the $\{\mathcal{H}(u)^{\omega}\}$ defines a filtration by $\mathcal{D}(u)$ and that if $v \in \mathcal{H}(u)^{\omega}$, then $v \in \mathcal{H}(u')^{\omega}$ for all u' sufficiently close to u. We shall now verify, using the asymptotic theory of §2, that around any $u \in S^1$ this filtration is induced by a $\mathcal{D}$-graded structure. Fix $u \in S^1$. Then (cf. §3.2) we can find an arc $W_0 \subset S^1$ containing u and an asymptotic trivialization on $\Gamma(U)_\varepsilon$ of (V, ∇) such that $\nabla_{d/dz} = d/dz - B$ where the matrix B is a canonical form

$$B = \sum_{\omega \in \Sigma} \omega^{\#}.\, 1 \otimes P_\omega + z^{-1} \otimes C .$$

If we fix a branch of log on $\Gamma(W_0)$ we can identify $\mathcal{H}(W)$ ($W \subset W_0$ any open arc) with the space of U-valued analytic functions on $\Gamma(W)$ such that

$$du/dz = B(z)u$$

on $\Gamma(W)$. Clearly

$$\mathcal{H}(W) = \bigoplus_\omega \mathcal{H}_\omega(W)$$

where, for $\omega \in \Sigma$,

$$\mathcal{H}_\omega(W) = \left\{ \exp\left(\left(\int_u^z \omega^{\#}\, dz\right) + \log\ z.\, C_\omega\right)v : v \in U_\omega \right\}$$

and $\mathcal{H}_\omega(W) = 0$ for $\omega \notin \Sigma$. We shall now show that for any $\omega \in \mathcal{D}(u)$,

$$\mathcal{H}(W)^{\omega} = \mathcal{H}_\omega(W) \oplus \bigoplus_{\omega' <_W \omega} \mathcal{H}_{\omega'}(W) .$$

Observe that if u is such a horizontal section so are the $P_\sigma u$. Hence the $\mathcal{H}(W)^{\omega}$ are stable under the P_σ so that it is enough to prove the following: $\mathcal{H}_\sigma(W) \subset \mathcal{H}(W)^{\omega}$ if $\sigma = \omega$ or if $\sigma <_W \omega$, and $\mathcal{H}_\sigma(W) \cap \mathcal{H}(W)^{\omega} = 0$ otherwise. The first relation is obvious. For the second, if v belongs to the intersection, we have, for some $t \in V_\sigma$,

$$\exp\left(-\int_u^z \omega^\# \, dz\right) v(z) = \exp\left(\int_u^z (\sigma^\# - \omega^\#)\, dz + \log z \,.C_\sigma\right) t$$

is of moderate growth on $\Gamma(W)$. Hence $\exp\left(\int_u^z (\sigma^\# - \omega^\#)\, dz\right) t$ is of moderate growth on $\Gamma(W)$. By the assumption on σ we can find a nonempty open arc $W' \subset W$ such that $\omega <_{W'} \sigma$, so that

$$|z|^N \operatorname{Re}\left(\int_u^z (\sigma^\# - \omega^\#)\, dz\right) \longrightarrow \infty \qquad (z \in \Gamma(W'),\ z \longrightarrow 0)$$

for any $N \geq 0$. So $\exp\left(\int_u^z (\sigma^\# - \omega^\#)\, dz\right).\, t$ can be of moderate growth only when $t = 0$, i.e., $v = 0$.

The assignment

$$\text{germ of } (V, \nabla) \longrightarrow \mathcal{H}(V, \nabla)$$

is clearly functorial and compatible with $\otimes$, $*$, and Hom. We shall now verify that it is fully faithful, i.e., the maps

$$\text{Morph}\,(\,(V, \nabla), (V', \nabla')\,) \longrightarrow \text{Morph}\,(\,\mathcal{H}(V, \nabla), \mathcal{H}(V', \nabla')\,)$$

are bijections. But in view of the compatibility with Hom this comes down to proving that if $\mathcal{M}_V$ is the space of germs of meromorphic sections of (V, ∇), the natural map

$$\mathcal{M}_V \longrightarrow H^0(S^1, \mathcal{H}(V, \nabla)\,),$$

which is obviously injective, is a bijection of $\mathcal{M}_V$ with the subspace of sections that lie in $\mathcal{H}(V, \nabla)^0$ everywhere. If $s \in \mathcal{M}_V$, then $s(z) = O(|z|^{-N})$ for some $N \geq 0$ as $z \longrightarrow 0$, so that the corresponding section lies in $\mathcal{H}(V, \nabla)^0$ everywhere. Conversely, suppose $\sigma \in H^0(S^1, \mathcal{H}(V, \nabla)^0)$. Then σ arises from a horizontal section s of V on a punctured disc at $z = 0$; by assumption the section is of moderate growth on some sector around each point of S^1 and so is a meromorphic section.

It remains to determine what happens to the Stokes sheaf. Let us fix a meromorphic pair (V, ∇) at $z = 0$ and consider a horizontal section v defined on a sector around $u \in S^1$. Clearly $v(z) = O(|z|^{-N})$ for some $N \geq 0$ if and only if $v \in \mathcal{H}(u)^0$. We now have the following.

LEMMA 4.7.2 *A horizontal section* v *is flat if and only if* $v \in \mathcal{H}(u)^0$ *and maps to* 0 *in* $\mathrm{Gr}\,\mathcal{H}(u)$.

PROOF Indeed, if v satisfies the latter condition, then, with respect to a compatible $\mathcal{D}$-graded structure for $\mathcal{H}$ near u, v is a sum of sections $v' \in \mathcal{H}(u)_{\omega'}$, $\omega' <_u 0$; this implies that v is flat. Conversely suppose that v is flat. Then proceeding as above with $\nabla_{d/dz} = d/dz - B$, it suffices to show that $P_\sigma v = 0$ if σ is not $<_u 0$. But

$$P_\sigma v = \exp\left(\int^z \sigma^\# \, dz + \log_u z . C_\sigma\right) t$$

for some $t \in V_\sigma$; if this is flat, we find as before that $t = 0$ or $P_\sigma v = 0$. ◆

To complete the proof of the prposition we apply this lemma to the endomorphism bundle E to get the following : $L \in \mathcal{H}_0(\mathrm{End}(V))(u) \Leftrightarrow L$ preserves the filtration of $\mathcal{H}(V)(u)$ and induces 0 on $\mathrm{Gr}\,\mathcal{H}(V)$. Since

$$St(V)(u) = 1 + \mathcal{H}_0(\mathrm{End}(V))(u),$$

the last assertion concerning $St(V)(u)$ follows at once. ◆

To formulate Deligne's theorem we introduce the following diagram :

$$\begin{array}{lccc} & \mathcal{C}_0 & \longrightarrow & \textit{category of } \mathcal{D}\textit{-filtered local systems on } S^1 \\ (\mathbf{D}) & \downarrow \textit{formalization} & & \downarrow \\ & \mathcal{C}_0^\wedge & \longrightarrow & \textit{category of } \mathcal{D}\textit{-graded local systems on } S^1 \end{array}$$

We shall say that this diagram is commutative for a choice of a covariant functor F representing the bottom arrow if the two compositions are naturally equivalent; the functor F is then called *admissible* . Note that the vertical functors are essentially surjective, and all functors are compatible with $\otimes$, $*$, and Hom.

THEOREM 4.7.3 (Deligne) *There are functors that make* **(D)** *commutative ,and all of them are naturally equivalent.. In particular , the functor* h *of Propostion* 1.4.6 *is admissible (after a suitable identification of* $\mathfrak{X}^{cl}$ *with* $\mathcal{D}(1)$).

We shall prove this theorem in the next paragraph.

4.8 PROOF OF DELIGNE'S THEOREM

We begin with

LEMMA 4.8.1 *Every* $\mathcal{D}$*-graded local system on* S^1 *is isomorphic to one that arises from a meromorphic pair* (V, ∇).

PROOF Let $\mathcal{U}$ be a $\mathcal{D}$-graded local system on S^1 and let $\mathcal{U}(1) = U = \bigoplus_{\omega \in \mathcal{D}(1)} U_\omega$. Let P_ω be the projections $U \longrightarrow U_\omega$. Let $\gamma \in GL(U)$ be such that $m \in \mathbf{Z}$ acts via γ^m . Let Σ be the set of all ω such that $U_\omega \neq 0$ and let $b \geq 1$ be an integer such that $\omega^\# \in \mathcal{F}_{b,cgt}$ for $\omega \in \Sigma$; here we choose for the $z^{m/n}$ the branches that are equal to 1 at $z = 1$. By hypothesis $\mathbf{Z}$ acts on Σ through μ_b and γ is an isomorphim of U_ω witth $U_{1.\omega}$. We now proceed as in the proof of Proposition 1.4.5 and the remarks following it. Since γ^b preserves the grading we can select a *b- reduced* endomorphism C of U preserving the grading such that $\gamma^b = \exp(2\pi i bC)$. We now go over to the ζ-plane and consider the pair (V_ζ, ∇_ζ) where V_ζ is the trivial bundle $\mathbf{C}_\zeta \times U$ and $\nabla_{\zeta, d/d\zeta} = d/d\zeta - B^\sim(\zeta)$ with

$$B^\sim(\zeta) = \sum_{\omega \in \Sigma} \omega_\zeta^\#. 1 \otimes P_\omega \quad + \zeta^{-1} \otimes bC .$$

The discussion loc. cit shows that there is a meromorphic pair (V_1, ∇_1) at $z = 0$ and a local isomorphism h of its pull back to $\mathbf{C}_\zeta$ with the pair (V_ζ, ∇_ζ) around $\zeta = 0$. We wish to show that $\mathcal{U}$ is isomorphic to the $\mathcal{D}$–graded local system defined by (V_1, ∇_1).

The horizontal sections u of (V_1, ∇_1) near $z = 1$ are solutions to $du/dz = B_1(z)u$; taking $z = \zeta^b$ and $v(\zeta) = g(\zeta)u(z)$, this equation becomes $dv/d\zeta = B(\zeta)v(\zeta)$. Hence (cf. §4.7) the space $\mathcal{H}(V_1, \nabla_1)(1) = \mathcal{H}(1)$ can be written as the direct sum $\bigoplus_\omega \mathcal{H}_\omega(1)$ where $\mathcal{H}_\omega(1)$ is the space of U-valued functions of the form

$$u_{\omega,t}(z) = h(\zeta)^{-1}\exp\left(\int_1^{\zeta} \omega_{\zeta}{}^{\#}.d\zeta + \log \zeta \, . \, bC_{\omega}\right)t \qquad (t \in U_{\omega})$$

If we observe that $h(\zeta) = O(|\zeta|^{-N})$ as $\zeta \longrightarrow 0$, we may conclude that $t \longrightarrow u_{\omega,t}$ is an isomorphism that gives rise to an isomorphism of $\mathcal{V}(1)$ with $\mathcal{H}(1)$ as $\mathcal{D}(1)$-graded vector spaces. Moreover, analytic continuation around the circle S^1 in the z-plane changes the solution $u_{\omega,t}$ to the solution

$$h(\varepsilon\zeta)^{-1} \exp\left(\int_1^{\zeta}(1.\omega)_{\zeta}{}^{\#}.d\zeta + \log \zeta \, . \, bC_{\omega} + 2\pi i C_{\omega} \right)t$$

which simplifies, in view of the relation $h(\zeta)\, h(\varepsilon\zeta)^{-1} = t(\varepsilon) = \gamma \exp(-2i\pi C)$, to

$$h(\zeta)^{-1} \exp\left(\int_1^{\zeta} (1.\omega)_{\zeta}{}^{\#}\, d\zeta + \log \zeta. \, b\, \gamma C_{\omega} \right)t$$

$$= h(\zeta)^{-1} \exp\left(\int_1^{\zeta}(1.\omega)_{\zeta}{}^{\#}\, d\zeta + \log \zeta. \, bC_{1.\omega} \right)\gamma t$$

$$= u_{1.\omega,\, \gamma t} \, .$$

This proves that the above isomorphism commutes with the $\mathbf{Z}$-actions. Hence the $\mathcal{D}$-graded local system on S^1 of (V, ∇) is isomorphic to $\mathcal{V}$. ◆

We shall now prove that the top horizontal functor is an equivalence of categories. Since it is fully faithful by Proposition 4.7.1, we need only prove that it is essentially surjective. Let $\mathcal{V}$ be a $\mathcal{D}$-filtered local system on S^1. By the above lemma there is a meromorphic pair (V_0, ∇_0) such that Gr $\mathcal{V}$ is isomorphic to Gr $\mathcal{H}$; here we write $\mathcal{H}$ for $\mathcal{H}(V_0, \nabla_0)$. Choose an isomorphism t of Gr $\mathcal{V}$ with Gr $\mathcal{H}$). Let $\mathcal{G}$ be the sheaf of groups of germs of sectorial automorphisms of $\mathcal{H}$ that are multiplicatively flat at 0, preserve the filtration, and induce the identity on Gr $\mathcal{H}$. The pair $(\mathcal{V}, t)$ then corresponds to an element α of $H^1(S^1, \mathcal{G})$ by Proposition 4.6.2. As $\mathcal{G}$ is canonically isomorphic by Proposition 4.7.1 to St (V_0, ∇_0) = St, α defines an α^* of $H^1(S^1, \text{St})$ which in turn corresponds by Theorem 4.5.1 to a pair $((V, \nabla), \xi)$. We shall show that $\mathcal{H}(V, \nabla)$ is isomorphic to $\mathcal{V}$. We choose a finite covering (W_i) of S^1 by open arcs and isomorphisms x_i of (V, ∇) with (V_0, ∇_0) on W_i such that $x_i{}^\wedge = \xi$ for all i and the cocycle $(x_i x_j^{-1})$ belongs to α^*. Then x_i corresponds to an isomorphism y_i of $\mathcal{H}(V, \nabla)$ with $\mathcal{H}(V_0, \nabla_0)$ on W_i, and it is obvious that the cocycle $(y_i y_j^{-1})$ belongs to α. But, as $y_i y_j^{-1}$ induces the identity on the graded local system, $Gr(y_i) = Gr(y_j)$ on $W_i \cap W_j$ for all i,j. So there is an isomorphism t' ($Gr\, \mathcal{H}(V, \nabla) \cong Gr\, \mathcal{H}(V_0, \nabla_0)$) coinciding with $Gr(y_i)$ on W_i for all i. As ($\mathcal{H}(V, \nabla)$, t') and $(\mathcal{V}, t)$ both define the same cohomology class α, they must be isomorphic.

The construction of admissible functors from the category of differential modules over $\mathcal{F}$ to the category of $\mathcal{D}$-graded local systems on S^1 is now accomplished with the help of the following lemma.

LEMMA 4.8.2 *Let* (V, ∇) *be a meromorphic pair at* $z = 0$ *with formalization* $M^\wedge$, *and let* $H^\wedge$ *be the space of horizontal elements of* $M^\wedge$. *To any* $f \in H^\wedge$ *we can then associate a unique* $f_0 \in H^0(S^1, Gr\, \mathcal{H}(V, \nabla)_0)$ *with the following propoerty : if* $U \subset S^1$ *is an open arc and* s *is a horizontal section of* (V, ∇) *on* U *such that* $s^\wedge = f$, *then* s *induces* f_0 *on* $\Gamma(U)$. *Then the map* $f \longrightarrow f_0$ *is an isomorphism of* $H^\wedge$ *with* $H^0(S^1, Gr\, \mathcal{H}(V, \nabla)_0)$.

PROOF We note that by Theorem 2.2.4, if U is sufficiently small, there exixt s with the properties described above. If s_i (i = 1,2) are two such, then $y = s_1 - s_2 \sim 0\ (\Gamma(U))$ and so, by Lemma 4.7.2 y maps to 0 in $Gr\, \mathcal{H}(V, \nabla)$, so that s_1 and s_2 induce the same element of $H^0(U, Gr\, \mathcal{H}(V, \nabla)_0)$. This means that $f \longrightarrow f_0$ is a well defined map. Its linearity is obvious and its injectivity is also immediate since 0 is the only flat section of $Gr\, \mathcal{H}(V, \nabla)_0$. For the surjectivity we suppose that V is $\Delta_\varepsilon \times \mathbf{C}^N$ and that $\nabla_{d/dz} = d/dz - A(z)$. Let us now consider $\varphi \in H^0(S^1, (Gr\, \mathcal{H})_0)$ where we write $\mathcal{H}$ for $\mathcal{H}(V, \nabla)$. We can then find a finite covering (U_i) of S^1 by open arcs and sections s_i of V on $\Gamma(U_i)_\varepsilon$ such that $s_i \in \mathcal{H}^0$ and s_i projects to φ. If $s_{ij} = s_i - s_j$, it follows from Lemma 4.7.2 again that (s_{ij}) is an 1-cocycle for the sheaf $\mathcal{H}_0(V)$ of flat sectorial horizontal sectorial sections of (V, ∇). By theorem 4.4.1 the cohomology class of this cocycle is defined by an element of $f \in \mathcal{F}^N$: $s_i - s_j = t_i - t_j$ where t_i is a holomorphic map of $\Gamma(U_i)_\varepsilon$ into $\mathbf{C}^N$ and $t_i \sim f\ (\Gamma(U_i))$. As $s_i - t_i = s_j - t_j$ there is a holomorphic map u of the punctured disc $\Delta_\varepsilon{}^\times$ into $\mathbf{C}^N$ such that $u = s_i - t_i$ on $\Gamma(U_i)_\varepsilon$ for all i; as u is of moderate growth at 0, u is actually in $\mathcal{F}_{cgt}{}^N$. If $h = f + u$, then $s_i = t_i + u \sim h\ (\Gamma(U_i))$ for all i so that

$$0 = \nabla s_i \sim \nabla h \ \ (\Gamma(U_i)),$$

showing that $h \in H^\wedge$. It is clear that φ is associated to h. ◆

If we apply this to Hom ((V, ∇), (V', ∇')) where (V, ∇) and (V', ∇') are two pairs at $z = 0$, we get the following corollary.

COROLLARY 4.8.3 *Let* (V, ∇) *and* (V', ∇') *be meromorphic pairs at* $z = 0$, $M^\wedge$ *and* $M'^\wedge$ *their formalizations, and* $\mathcal{H} = \mathcal{H}(V, \nabla)$, $\mathcal{H}' = \mathcal{H}(V', \nabla')$. *Then there is a bijection* $\sim (\beta \longrightarrow \beta^\sim)$,

$$\sim \ : \text{Morph}\,(M^\wedge, M'^\wedge) \longrightarrow \text{Morph}\,(\text{Gr}\,\mathcal{H}, \text{Gr}\,\mathcal{H}')$$

characterized by the following property: if U *is a sufficiently small arc and* x *is a morphism from* (V, ∇) *to* (V', ∇') *on* $\Gamma(U)$ *that is asymptotic to* β, *then* x *induces* $\beta^\sim$.

To construct admissible functors we proceed as follows. For each differential module over E over $\mathcal{F}$ choose some pair $((V, \nabla), \xi)$ where (V, ∇) is a meromorphic pair at $z = 0$ with formalization $M^\wedge$ and ξ is an isomorphism : $M^\wedge \cong E$. This is possible in view of Proposition 1.4.6. If E' is another differential module over $\mathcal{F}$ and $((V', \nabla'), \xi')$ is the corresponding selected pair, $\beta \longrightarrow \xi'^{-1}\beta\xi$ is a bijection of Morph(E, E') with Morph$(M^\wedge, M'^\wedge)$, and so Corollary 4.8.3 allows us to associate to any $\beta \in$ Morph(E, E') an element $\beta' \in$ Morph (Gr $\mathcal{H}$, Gr $\mathcal{H}'$) ($\mathcal{H}' = \mathcal{H}(V', \nabla')$) defined by $\beta' = (\xi'^{-1}\,\beta\,\xi)^\sim$. It is then clear that $\beta \longrightarrow \beta'$ is a bijection. It is now quite straightforward to verify that the assignments

$$E \longrightarrow \text{Gr}\,\mathcal{H}(V, \nabla), \quad \beta \longrightarrow \beta^\sim$$

define an admissible functor from $\mathcal{C}_0{}^\wedge$ to the category of $\mathcal{D}$–graded local systems on S^1.

To complete the proof of Deligne's Theorem it remains to show that the functor of Proposition 1.4.5, say h, is the unique admissible functor upto natural equivalence. Write λ for the functor from $\mathcal{C}_0$ to the category of $\mathcal{D}$–graded local systems on S^1 obtained by composing the top horizontal arrow in (**D**) with Gr. it is clear from Proposition 1.4.6 that there are functors κ from $\mathcal{C}_0{}^\wedge$ to $\mathcal{C}_0$

with $M \approx \kappa(M)^\wedge$ for all M. It is obvious that the following Lemma is sufficient to complete the proof of Deligne's Theorem.

LEMMA 4.8.4 *Let* $\kappa\ (\mathfrak{C}_0{}^\wedge \longrightarrow \mathfrak{C}_0)$ *be any functor with* $M \approx \kappa(M)^\wedge$ *for all* M. *Then* F *is admissible if and only if* $F \approx \lambda \circ \kappa$. *This is true in particular for the functor* $M \longrightarrow M_{cgt}$ *defined in* § 1.4.

PROOF If F is admissible, we have $F(N^\wedge) \approx \lambda(N)$ for all $N \in \mathfrak{C}_0$. Hence $F(\kappa(M)^\wedge) \approx \lambda(\kappa(M))$ for all $M \in \mathfrak{C}_0{}^\wedge$. But $\kappa(M)^\wedge) \approx M$, and so one has $F(\kappa(M)^\wedge) \approx F(M)$, showing that $F(M) \approx \lambda(\kappa(M))$ for all $M \in \mathfrak{C}_0$. If conversely we assume that $F \approx \lambda \circ \kappa$, we start with $F(N^\wedge) \approx \lambda(\kappa(N^\wedge))$ for all $N \in \mathfrak{C}_0$. But $(\kappa(N^\wedge))^\wedge \approx N^\wedge$, and so, by Corollary 4.8.3, $\lambda(\kappa(N^\wedge)) \approx \lambda(N)$, showing that $F(N^\wedge) \approx \lambda(N)$, i. e., F is admissible.

We now prove that the functor $M \longrightarrow M_{cgt}$ defined in §1.4 is a possible choice for the functor κ. Write, for any differential module N over $\mathfrak{F}_{cgt}$,

$$N^{cl} = \mathfrak{F}_{cgt}{}^{cl} \otimes_{\mathfrak{F}_{cgt}} N, \quad N^{cl}(\mathfrak{a}) = \mathfrak{a} \otimes_{\Phi} N^{cl},$$

where $\mathfrak{a}$ and Φ have the same meaning as in §1.3. Here we are also identifying $\mathfrak{F}_{cgt}{}^{cl}$ with Φ so that $\mathfrak{C}^{cl}$ gets identified with $\mathfrak{D}(1)$. But, from the definition of M_{cgt} it is clear that

$$H(M^{cl}{}_{cgt})(\mathfrak{a}) = H(M^{cl}{}_{cgt})(\Phi_2),$$

so that

$$\lambda(M_{cgt}) \approx H(M^{cl}{}_{cgt}(\Phi_2)) = h(M).$$

This finishes the proof of the lemma and that of Deligne's theorem. ◆

5 EXAMPLES

5.1 In this section we shall discuss some examples that illustrate many of the themes treated in this paper. We treat the Bessel and Whittaker differential equations and analyse the differential modules that they give rise to. The two cases are very similar, but the Bessel theory is a little simpler since the Bessel equation is a limiting case of the Whittaker equation. We therefore begin with the Bessel equation.

5.2 The Bessel connections The Bessel differential equations are

$$y'' + \frac{1}{t}y' + \left(1 - \frac{\nu^2}{t^2}\right) = 0$$

where $\nu \in \mathbf{C}$ is a complex parameter. The equation is considered on $\mathbb{P}^1$ and it is well known that 0 and ∞ are its only singular points, 0 being regular and ∞ irregular. We go over to the associated first order system and then change over to $z = t^{-1}$ so that $z = 0$ becomes the irregular singular point and ∞ regular. The resulting family of first order differential equations is

$$\frac{du}{dz} = A_\nu(z)u, \quad u(z) = \begin{pmatrix} y\,(t^{-1}) \\ y'(t^{-1}) \end{pmatrix}$$

where

$$A_\nu(z) = z^{-2}\begin{pmatrix} 0 & -1 \\ (1-\nu^2 z^2) & z \end{pmatrix}$$

We also define

$$B = z^{-2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + z^{-1}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

The matrices A_ν and B define connections ∇_ν and ∇_B on the trivial bundle $V = \mathbf{C}^2 \times \mathbf{C}$ in the usual manner :

$$\nabla_{\nu,d/dz} = d/dz - A_\nu, \quad \nabla_{B,d/dz} = d/dz - B.$$

Actually ∇_ν and ∇_B are defined on $\mathbf{C}^2 \times \mathbb{P}^1$ and we are interested in the germs determined by them at $z = 0$. The family of connections ∇_ν on $\mathbf{C}^2 \times \mathbb{P}^1$ is called the *Bessel family*. B is a reduced unramified canonical form.

The fundamental fact is that the Bessel family is *isoformal* at $z = 0$.

LEMMA 5.2.1 *For any ν the formalizations of* (V, ∇_ν) *and* (V, ∇_B) *are isomorphic. More precisely, let* $L = \begin{pmatrix} -1 & -i \\ -i & -1 \end{pmatrix}$. *Then* $L \in GL(2, \mathbf{C})$ *and there is a unique* $u_\nu \in GL(2, \mathbf{C}[[z]])$ *such that*

$$u_\nu(0) = L, \quad u_\nu[A_\nu] = B.$$

Moreover, the family (u_ν) *belongs to* $GL(2, \mathbf{C}[\nu][[z]])$.

PROOF It is a trivial calculation that

$$L \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} L^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

If we write $B_\nu = L[A_\nu]$, then

$$B_\nu = D z^{-2} + z^{-1} R + K_0$$

where

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad R = (1/2)\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad K_0 = (\nu^2/2)\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix}.$$

It is now a question of proving the existence of a unique $g_\nu \in GL(2, \mathbf{C}[[z]])$ such that $g_\nu(0) = 1$ and $g_\nu[B_\nu] = B$; u_ν will then be $g_\nu L$. The existence of g_ν is an example of formal splitting of a differential module along the spectral subspaces of the leading coefficient of its connection matrix that goes back to the beginnings of our subject. The uniqueness will follow from the fact that the automorphism group of (V, ∇_B) is just thegroup of diagonal matrices in $GL(2, \mathbf{C})$. The present situation is however sufficiently simple that one can do this in an elementary manner. We seek g_ν in the form

$$g_\nu = 1 + zT_1 + z^2T_2 + \ldots, \quad T_k \in \mathfrak{gl}(2, \mathbf{C})$$

and wish to solve uniquely for a diagonal matrix C and T_k $(k \geq 1)$ so that

$$g_\nu[B_\nu] = D z^{-2} + z^{-1}C.$$

The relations

$$(1 + zT_1 + \dots)(D z^{-2} + z^{-1}R + K_0) + T_1 + 2zT_2 + \dots = (D z^{-2} + z^{-1}C)(1 + zT_1 + \dots)$$

yield the recursion formulae :

$$[D, T_1] = R - C$$

$$[D, T_{k+2}] = T_{k+1} R - CT_{k+1} + (k+1)T_{k+1} + T_k K_0 \quad (k \geq -1,\ T_0 = 1).$$

Since D is diagonal with distinct eigenvalues, the space of matrices with 0 on the diagonal is the range of $\operatorname{ad} D$, and $\operatorname{ad} D$ is an isomorphism on this range. Hence C must be $(1/2)1$, the diagonal part of R, and the off diagonal part of T_1 is then determined by the first relation. Suppose that $T_1, \dots T_k$ and the off diagonal part of T_{k+1} are known $(k \geq 0)$. Equating the diagonal part of the right side of the second relation above to zero leads to an equation of the form

$$(k+1)\ (\text{diagonal part of } T_{k+1}) = \text{known quantity},$$

and so, as $k \geq 0$, the diagonal part of T_{k+1} is determined. Thus T_{k+1} is known, and the off diagonal part of T_{k+2} is then determined since the right side of the second relation is known completely. The induction thus goes forward. It is moreover easily seen by a similar induction that the entries of the T_k are polynomials in ν. ♦

We consider the Stokes sheaf St of (V, ∇_B). If

$$\psi(z) = \begin{pmatrix} e^{-i/z} & 0 \\ 0 & e^{i/z} \end{pmatrix},$$

then for an open set $U \subset S^1$, a holomorphic map $g(\Gamma(U) \longrightarrow GL(2, \mathbf{C}))$ belongs to $St(U)$ if and only if

(a) $g \sim 1\ (\Gamma(U))$

(b) $d\,(\psi^{-1} g \psi)/dz = 0.$

The map

$$g \longrightarrow \psi^{-1} g \psi$$

thus defines an isomorphism of the sheaf St with the sheaf of subgroups of GL(2, **C**) for which the group of sections over an open arc U is the subgroup of all $h = (h_{ij}) \in GL(2, \mathbf{C})$ such that

$$h_{11} = h_{22} = 1, \quad h_{12}\, e^{-2i/z} \sim 0\ (\Gamma(U)), \quad h_{21}\, e^{2i/z} \sim 0\ (\Gamma(U)).$$

The Stokes lines are thus the rays through the points $z = \pm 1$, and the stalk at $u \in S^1$ is canonically isomorphic to the upper (resp. lower) triangular group of matrices of the form

$$\begin{pmatrix} 1 & t_+ \\ 0 & 1 \end{pmatrix} \qquad \left(\text{resp.} \begin{pmatrix} 1 & 0 \\ t_- & 1 \end{pmatrix}\right)$$

for $u \in S^{1,+}$ (resp. $S^{1,-}$), the upper (resp. lower) semicircular half of S^1. We thus obtain a sheaf of vector spaces of dimension ≤ 1 on which $t_\pm$ are linear coordinates. The cohomology $H^1(S^1, St)$ is thus a vector space over **C**. It follows from Proposition 3.3.2 that $H^1(S^1, St)$ is two dimensional. To describe it explicitly we use the covering $\mathfrak{A}$:

$$S^1 = U_+ \cup U_-\,, \quad U_\pm = S^1 \setminus \{\mp 1\}, \quad \mathfrak{A} = \{U_+, U_-\}.$$

Since $H^0(U_\pm, St) = 0$, we have

$$H^1(\mathfrak{A}, St) = H^0(U_+ \cap U_-, St) = H^0(S^{1,+}, St) \times H^0(S^{1,-}, St),$$

and as the last written space is two dimensional, we have

$$H^1(S^1, St) \cong H^0(S^{1,+}, St) \times H^0(S^{1,-}, St).$$

In particular $t_\pm$ are linear coordinates on $H^1(S^1, St)$.

The main point of interest is of course the calculation of the Malgrange-Sibuya isomorphism Φ that takes the set $\mathfrak{M}(V, \nabla_B)$ to $H^1(S^1, St)$. We shall now prove that

(MS) $$\Phi(A_\nu, u_\nu) = \pm\,(2\cos \pi\nu, -2\cos \pi\nu),$$

where the sign $\pm$ is independent of ν. By Theorem 2.2.4 we can find isomorphisms $x_{\nu,\pm}$ of (V, ∇_ν) with (V, ∇_B) on sectors $\Gamma_\pm$ around $z = \pm 1$ such that $x_{\nu,\pm} \sim u_\nu \ (\Gamma_\pm)$. On the other hand the only Stokes lines here are the rays through $z = \pm 1$, and so, by Proposition 3.4.2, $x_{\nu,\pm} \sim u_\nu \ (\Gamma(U_\pm))$. Moreover, as the stalks at $z = \pm 1$ are trivial, the $x_{\nu,\pm}$ are uniquely determined by

$$x_{\nu,\pm} \sim u_\nu \ (\Gamma(U_\pm)), \quad x_{\nu,\pm}[A_\nu] = B.$$

A more precise knowledge of the $x_{\nu,\pm}$ will clearly lead to the determination of the class $\Phi(A_\nu, u_\nu)$. Now $z^{1/2} x_{\nu,\pm}(z)^{-1}\psi(z)$ is a fundamental solution of the Bessel equation with specific asymptotic properties and so may be computed explicitly (see [W], §15.2). However we can do this (almost) in a less painful manner using the *symmetry* properties of the Bessel connections.

Let us write, for any $A \in \mathfrak{gl}(2, \mathfrak{F})$, $x \in GL(2, \mathfrak{F})$, $y \ (\Gamma \longrightarrow GL(2, \mathbf{C}))$,

$$A^\vee(z) = -A(-z), \ x^\vee(z) = x(-z), \ y^\vee(z) = y(-z).$$

A simple calculation shows that if

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then

$$A_\nu^\vee = W[A_\nu], \quad B^\vee = S[B], \quad S^{-1}\psi S = \psi^{-1} = \psi^\vee,$$

and hence, as $S^{-1}LW = -iL$, the uniqueness of the $x_{\nu,\pm}$ shows that

$$u_\nu = -iSu_\nu^\vee W, \quad x_{\nu,\mp} = -iSx_{\nu,\pm}{}^\vee W \ (\text{on } \Gamma(U_\mp)).$$

The cocycle associated to the covering $\mathfrak{A} = \{U_+, U_-\}$ is the map

$$g : \Gamma(U_+ \cap U_-) \longrightarrow GL(2, \mathbf{C}), \quad g(z) = x_{\nu,-}(z)\, x_{\nu,+}(z)^{-1}.$$

It follows from the relation above linking $x_{\nu,+}$ and $x_{\nu,-}$ that

$$g(z) = S\, g(-z)^{-1} S^{-1} \quad (z \in \Gamma(U_+ \cap U_-)).$$

It is thus enough to determine $g(z)$ for z in one of the sectors $\Gamma(S^{1,\pm})$. Now,

$$g(z) = -iS\, x_{\nu,+}(-z)\, W\, x_{\nu,+}(z)^{-1},$$

and so it is a question of relating $x_{\nu,+}(-z)$ to $x_{\nu,+}(z)$. Now, if $z_+^{1/2}$ denotes the branch of $z^{1/2}$ on $\Gamma(U_+)$ that is 1 at $z = 1$,

$$z_+^{1/2}\, x_{\nu,+}(z)^{-1}\, \psi(z)$$

is a fundamental matrix of the connection ∇_ν. So, going to the w–plane that is a universal covering of $\mathbf{C}_z^\times$ via the map $w \longrightarrow z = e^w$, we see that

$$(*) \qquad F(w) = e^{w/2}\, x_{\nu,+}(e^w)^{-1}\, \psi(e^w)$$

is a fundamental matrix for the equation

$$du/dw = e^w A_\nu(e^w) u.$$

This is of course initially defined on the domain $\{w : -\pi < \mathrm{Im}(w) < \pi\}$ by $(*)$, and then extended to the whole w–plane. The monodromy of the solution is the matrix $M \in GL(2, \mathbf{C})$ defined by the relation

$$F(w + 2i\pi) = F(w)M.$$

But, $A_\nu^\vee = W[A_\nu]$, and so, as $w \longrightarrow w + i\pi$ corresponds to $z \longrightarrow -z$, we have

$$dF(w + i\pi)/dw = e^w\, W\, A_\nu(e^w)\, W^{-1} F,$$

so that, for some $M_{1/2} \in GL(2, \mathbf{C})$,

$$F(w + i\pi) = WF(w)M_{1/2}$$

for all w. The notation is justified because $M_{1/2}{}^2 = M$. From this we obtain,

$$x_{\nu,+}(z)^{-1}\, \psi(z) = i\, W\, x_{\nu,+}(-z)^{-1}\, \psi(z)^{-1} M_{1/2}{}^{-1} \quad (-\pi < \arg(z) < 0).$$

So finally get

$$g(z) = \psi(z)\, (SM_{1/2}{}^{-1})\, \psi(z)^{-1} \quad (-\pi < \arg(z) < 0),$$

and

$$g(z) = \psi(z)\, (-SM_{1/2})\, \psi(z)^{-1} \quad (0 < \arg(z) < \pi).$$

Since $\psi^{-1}(z)\, g(z)\psi(z)$ is upper triangular for $0 < \arg(z) < \pi$ it follows from the second of these relations that $M_{1/2}$ must be of the form

$$M_{1/2} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}$$

T determne λ we compute $M_{1/2}{}^2 = M$, the monodromy of the fundamental solution considered. A simple calculation gives

$$\mathrm{tr}\,(M_{1/2}{}^2) = \lambda^2 - 2, \quad \det\,(M_{1/2}{}^2) = 1.$$

On the other hand, tr(M) is independent of the choice of the fundamental solution ; moreover, as we have remarked at the beginning, the Bessel connection ∇_ν is really a global one defined on $\mathbf{C}^2 \times \mathbb{P}^1$, and it is well known that its monodromy at $z = \infty$ is the conjugacy class of

$$\begin{pmatrix} e^{2\pi i \nu} & 0 \\ 0 & e^{-2\pi i \nu} \end{pmatrix}$$

As the monodromy at $z = 0$ is the inverse of the one at ∞ and has determinant 1, we find

$$\mathrm{tr}(M) = 2\cos 2\pi\nu = \lambda^2 - 2$$

giving

$$\lambda = \pm 2 \cos \pi\nu.$$

We now observe that the uniqueness of $x_{\nu,+}$, in conjunction with Theorem 2.2.1, implies of the analyticity of g in ν. This is a special case of the general result that we shall prove in **III** that the Stokes class of an analytic isoformal family of marked pairs is analytic in the parameter. So the sign $\pm$ above is independent of ν. We thus finally get, in the coordinates $t_\pm$,

$$\Phi(A_\nu, u_\nu) = \pm\, (2\cos \pi\nu, -2\cos \pi\nu),$$

the signs being independent of ν. The sign can be determined to be + by looking more closely at $x_{\nu,\pm}$ (for one ν), but we shall not do this here.

The above formula is remarkable because it shows that even for an *algebraic* family such as the (V, ∇_ν) the map $\nu \longrightarrow \Phi(A_\nu, u_\nu)$ can be transcendental. That this map cannot be algebraic can be seen trivially from the fact that its fibers are infinite :

$$(V, \nabla_\nu) \text{ is isomorphic to } (V, \nabla_{\nu'}) \iff \nu + \nu' \text{ or } \nu - \nu' \text{ is an integer.}$$

For determining the isomorphism classes of the unmarked pairs we appeal toTheorem 4.5.2 which gives the bijection

$$\mathcal{I}(V, \nabla_B) \cong G_B \backslash H^1(S^1, St),$$

where G_B is the group of automorphisms of (V, ∇_B). Since B is a reduced canonical form this is a subgroup of $GL(2, \mathbf{C})$, and is in fact the subgroup of diagonal matrices. We use the bijection

$$H^1(S^1, St) \cong H^0(S^{1,+}, St) \times H^0(S^{1,-}, St)$$

to identify $H^1(S^1, St)$ with $\mathbf{C}^2$ via the linear coordinates $t_\pm$. The diagonal matrix $\mathrm{diag}(\alpha, \beta)$ acts on $\mathbf{C}^2$ via

$$(\alpha, \beta), (t_+, t_-) \longrightarrow (\lambda\, t_+, \lambda^{-1} t_-) \quad (\lambda = \alpha\beta^{-1}).$$

The orbits are

$$H_c = \{t_+ t_- = c\}\ (c \neq 0), \quad H_{0,\pm} = \{t_\pm = 0\}\backslash\{(0,0)\}, \quad \{(0,0)\}.$$

The hyperbolae H_c $(c \neq 0)$ are *stable* in the sense that they have maximal dimension and are closed; the punctured axes $H_{0,\pm}$ are smooth but not stable. For the set of stable orbits we have

$$(G_B \backslash \mathbf{C}^2)^{\text{stable}} \cong \mathbf{C}^\times .$$

THe space of smooth orbits of dimension 1 is not separated ; it is the complex line with the origin doubled. More precisely, it is obtained by gluing two copies of $\mathbf{C}$ along $\mathbf{C}^\times$ with the identification $t \approx t$. If we omit one of the punctured axes we obtain the space $\mathbf{C}$. The interest in the stable orbits is due to the fact that in their neighbourhoods one can construct a geometric quotient for the action of the group and hence a moduli space for the set of equivalence classes of the meromorphic pairs themselves (without any markings); we shall see this in detail in **III**.

Interestingly enough the Bessel family does not fill out the orbit space ; the orbits $H_{0,\pm}$ are not in the image of the Bessel family. If we define the connection ∇_-^α on $\mathbf{C}^2 \times \mathbb{P}^1$ by $\nabla^\alpha_{-,d/dz} = d/dz - A^\alpha_-(z)$ where

$$A^\alpha_-(z) = z^{-2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + z^{-1}\begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix} \; (\alpha \neq 0),$$

then it is not difficult to show that the formalization of (V, ∇^α_-) at $z = 0$ is isomorphic to that of (V, ∇_B) and that its analytic isomorphism class goes over to the orbit $H_{0,-}$. To see this we begin by asking whether we can choose an *upper triangular* $x = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ such that $x[A^\alpha_-] = B$, formally or anlytically. A simple calculation shows that this is possible if and only if u is a solution, formal or analytical, of

$$du/dz = 2i\, z^{-2} u - \alpha\, z^{-1}.$$

In the formal case we choose the solution

$$u^\wedge = \Sigma_{k \geq 1}\ \alpha(2i)^{-k}(k-1)!\, z^k .$$

As before we find unique $y_\pm$ analytic on $\Gamma(U_\pm)$ such that

$$y_\pm \sim \begin{pmatrix} 1 & u^\wedge \\ 0 & 1 \end{pmatrix} (\Gamma(U_\pm), \quad y_\pm[A^\alpha_-] = B.$$

The corresponding cocycle is upper triangular for *all* $u \in S^1 \setminus \{\pm 1\}$, not just for u in the upper arc $S^{1,+}$. Hence $t_- = 0$, showing that the orbit corresponding to this connection is $H_{0,-}$. Let us now define A^α_+ and ∇_+ by

$$A^\alpha_+ = S(A^\alpha_-)^\vee S^{-1} \quad , \quad \nabla^\alpha_{+,d/dz} = d/dz - A^\alpha_+,$$

then it can be shown that the formalization of (V, ∇^α_+) is isomorphic to that of (V, ∇) and that the associated orbit is $H_{0,+}$.

Note that conjugation by a suitable constant diagonal matrix takes A^α_- to A^β_- for any $\beta \neq 0$, and so, the isomorphism class of A^α_- is unchanged as $\alpha \longrightarrow 0$. But when $\alpha = 0$, $A^\alpha_- = B$ which corresponds to the orbit $\{(0,0)\}$. In other words, the image in the orbit space *jumps* from $H_{0,-}$ to $\{(0,0)\}$. This is the familiar *jump phenomenon* in the theory of deformations and shows that *no* *reasonable deformation theory exists for the pair* (V, ∇_B). At the points corresponding to the orbits H_c the Bessel family is locally universal, essentially because the map $\nu \longrightarrow \Phi(A_\nu, u_\nu)$ is a complex analytic isomorphism locally.

The monodromy at $z = 0$ of ∇_ν is obtainedby computing $M_{1/2}{}^2$ and so is the conjugacy class in $GL(2, \mathbf{C})$ of the $SL(2, \mathbf{C})$-matrix

$$M(\nu) = \begin{pmatrix} -1 & 2\cos\pi\nu \\ -2\cos\pi\nu & -1 + 4\cos^2\pi\nu \end{pmatrix}.$$

This shows that ∇_ν and ∇_μ are isomorphic at $z = 0$ if and only if they have the same monodromy there. Moreover all the conjugacy classes of determinant 1 in $GL(2, \mathbf{C})$ occur except the classes $\{-u : u \neq 1$ and unipotent$\}$ and $\{(1)\}$. We can show that the missing nontrivial class comes from the orbits $H_{0,\pm}$. Indeed, going over to $t = 1/z$, $A^{\alpha}{}_-$ becomes $A^\sim$ where

$$A^\sim = -t^{-1}\begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is of the first kind, hence regular, at $t = 0$. It follows from [BV 1] (cf. example (1) following Theorem 3.3.1, and Proposition 3.11) that $A^\sim$ is equivalent under $GL(2, \mathbf{C}\{z\})$ to the connection marix $-t^{-1}\begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix}$,

and so, its monodromy is the class of

$$\exp\left(-2\pi i\begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix}\right) = -\exp\left(-2\pi i\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}\right).$$

Since this class is its own inverse, it is also the class of the monodromy of $\nabla^{\alpha}{}_-$ at $z = 0$. From the definition of $\nabla^{\alpha}{}_+$ it is clear that its monodromy at $z = 0$ is also the same class. So, *unlike the Bessel connections , the two exceptional connections have the same monodromy but are not isomorphic at* $z = 0$.

5.3 THE WHITTAKER CONNECTIONS These depend on two parameters k and m and are defined as before on $\mathbf{C}^2 \times \mathbb{P}^1$. We shall be interested in the pairs defined at $z = 0$ by their restrictions to $V = \mathbf{C}^2 \times \mathbf{C}$. The differential equations satisfied by the Whittaker functions are ([WW], p 206)

$$\text{(W)} \qquad \frac{d^2W}{dt^2} + \frac{(1-4m^2) + 4kt - t^2}{4t^2} W = 0.$$

If $k = 0$, $m = \nu$, and $t = 2i\tau$, then the equation for $W = t^{1/2}J$ goes over to the Bessel equation for J_ν (see [WW], p360). By the Theorem of Fuchs, $t = 0$ is regular, and $t = \infty$ is irregular. We go over to the first order system and the variable $z = t^{-1}$ to get the equations

$$\frac{du}{dz} = A(k, m)u,$$

where

$$A(k, m)(z) = z^{-2}\left(-\tfrac{1}{4}Y - X\right) + z^{-1}kY + \left(\tfrac{1}{4} - m^2\right)Y.$$

Here we are using the notations

$$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad X = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad Y = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

If $L = \begin{pmatrix}1/2 & -1\\ 1/2 & 1\end{pmatrix}$, then

$$L\left(-\tfrac{1}{4}Y - X\right)L^{-1} = (1/2)H.$$

We define the connection $\nabla(k, m)$ on $\mathbf{C}^2 \times \mathbb{P}^1$ by

$$\nabla(k, m)_{d/dz} = d/dz - A(k, m).$$

Let

$$B(k) = (1/2)H\, z^{-2} - z^{-1}\, kH.$$

Then $B(k)$ is a canonical form and defines a connection $\nabla_{B(k)}$ on $\mathbf{C}^2 \times \mathbb{P}^1$ by the usual formula. We have

LEMMA 5.3.1 *The formalizations of* $(V, \nabla(k, m))$ *and* $(V, \nabla_{B(k)})$ *at* $z = 0$ *are isomorphic for any* k. *More precisely, there is a unique* $u = u(k, m) \in GL(2, \mathbf{C}[[z]])$ *such that*

$$u(0) = L, \quad u[A(k, m)] = B(k).$$

Moreover, $u \in GL(2, \mathbf{C}[k, m][[z]])$.

PROOF This is proved essentially as Lemma 5.2.1 and so we omit the proof. ◆

The meaning of the parameters is thus clear ; in particular, for fixed k, the family $(V, \nabla(k, m))_{m \in \mathbf{C}}$ is isoformal and $((V, \nabla(k, m)), u_\nu)$ lies in $\mathcal{M}(V, \nabla_{B(k)})$ for all m.

For $R \in \mathfrak{gl}(2, \mathfrak{F})$ (resp. $r \in GL(2, \mathfrak{F})$ write

$$R^\iota = -R^t, \quad r^\iota = (r^{-1})^t \quad (t = \text{transpose}).$$

Then ι is an involution on the respective spaces, and $r[R] = S \Leftrightarrow r^\iota[R^\iota] = S^\iota$. An easy calculation shows that if

$$T = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then,

$$A(k, m) = T\,[A(-k, m)^{\vee\iota}], \quad S[B(k)] = -B(k), \quad B(k) = -B(-k)^{\vee}.$$

We introduce the function

$$\varphi(z) = \exp(-(1/2)\, z^{-1} H\,).$$

If $St(k)$ is the sheaf $St(V, \nabla_B)$, then, for any open arc $U \subset S^1$, $g \in St(k)(U)$ if and only if $h = \varphi^{-1} g \varphi = (h_{ij})$ is holomorphic from $\Gamma(U)$ into $GL(2, \mathbf{C})$ that satisfies

(a) $\varphi\, h\, \varphi^{-1} \sim 1\ (\Gamma(U)$

(b) $dh/dz + [h, -k\, z^{-1} H] = o$.

The relation (a) is equivalent to

$$h_{11} = h_{22} = 1, \quad e^{-1/z} h_{12} \sim 0\ (\Gamma(U)), \quad e^{1/z} h_{21} \sim 0\ (\Gamma(U)).$$

Hence, by (b) if we fix the branches $z^{\pm 2k}$ on $\Gamma(U)$, we have

$$h_{12} = c_{12}\, z^{-2k}, \quad h_{21} = c_{21}\, z^{2k} \quad (c_{ij} \in \mathbf{C}).$$

The stalks of $St(k)$ are trivial at $z = \pm i$; if $u \in S^{1,r}$, the right half of $S^1 \setminus \{i, -i\}$, $\varphi^{-1}St(k)(u)\varphi$ is the group of upper triangular matrices of the form

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \beta = \text{const. } z^{-2k}.$$

For $u \in S^{1,\ell}$, the left half of $S^1 \setminus \{i, -i\}$, $\varphi^{-1}St(k)(u)\varphi$ is the group of lower triangular matrices of the form

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad \alpha = \text{const. } z^{2k}.$$

The Stokes lines are the rays through $z = \pm i$. $St(k)$ is thus a sheaf of vector spaces and Proposition 3.3.2 shows that $H^1(S^1, St(k))$ is two dimensional over $\mathbf{C}$. If $U_\pm = S^1 \setminus \{\mp i\}$, then the covering $\mathfrak{U} = \{U_+, U_-\}$ of S^1 may be used to compute H^1. We have, as in the Bessel case,

$$H^1(S^1, St(k)) \cong H^0(S^{1,\ell}, St(k)) \times H^0(S^{1,r}, St(k)) \cong \mathbf{C}^{k,+} \times \mathbf{C}^{k,-},$$

where $\mathbf{C}^{k,+}$ (resp. $\mathbf{C}^{k,-}$) is the one dimensional space spanned by the branches of z^{+2k} (resp. z^{-2k}) on $\Gamma(S^{1,\ell})$ (resp. $\Gamma(S^{1,r})$. Although $B(k)$ is not in general reduced, it is true that the group $G(k)$ of automorphisms of $(V, \nabla_{B(k)})$ is the diagonal subgroup of $GL(2, \mathbf{C})$. Indeed, if $u \in GL(2, \mathfrak{F})$ and $u[B(k)] = B(k)$, then u is diagonal, and fixes the connection defined by $-z^{-1}kH$; thus $(du/dz)u^{-1} = 0$, so that u is constant. The action of $G(k)$ on $H^1(S^1, St(k)) \cong \mathbf{C}^{k,+} \times \mathbf{C}^{k,-}$ is analogous to what it is in the Bessel case ; if $u = \mathrm{diag}(\alpha, \beta)$ and $\lambda = \beta\alpha^{-1}$, then u acts by

$$(a, b) \longrightarrow (\lambda a, \lambda^{-1} b) \qquad (a \in \mathbf{C}^{k,+}, b \in \mathbf{C}^{k,-}).$$

LEMMA 5.3.2 *There are unique isomorphisms*

$$x_\pm(k, m) : (V, \nabla(k, m)) \cong (V, \nabla_{B(k)})$$

preserving the asymptotic structures of level 1 *such that*

$$x_\pm(k, m) \sim u(k, m)\,(\Gamma(U_\pm)).$$

Moreover, if $\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ *and* T *as defined above, then*

$$x_-(k, m) = \delta^{-1} x_+(-k, m)^{\vee\iota} T^{-1} \quad (\text{on } \Gamma(U_-))$$

$$u(k, m) = \delta^{-1} u(-k, m)^{\vee\iota} T^{-1}.$$

Finally, $x_\pm(k, m)$ *and* $u(k, m)$ *depend analytically on* k *and* m.

PROOF The existence of $x_\pm(k, m)$ follows from Theorem 2.2.4 while the uniqueness follows from Proposition 3.4.2 since $\Gamma(U_\pm)$ do not contain more than one Stokes line. The relations linking $U_\pm$ follow from the uniqueness and the easily verified formulae :

$$\delta^{-1} x_+(-k, m)^{\vee\iota} T^{-1}[A(k, m)] = B(k), \quad \delta^{-1}(L^{-1})^t T^{-1} = L.$$

Finally, the analyticity in k and m follows from Theorem 2.2.1. ♦

We define $\log_+$ to be the branch of the logarithm on $\Gamma(U_+)$ which is $i\pi/2$ at $z = i$. For any $M \in \mathrm{End}(\mathbf{C}^2)$ we put $z_+^M = \exp(\log_+ z.\, M)$. Define

$$F_+(k, m) = x_+(k, m)^{-1} z_+^{-kH} \varphi.$$

Then $F_+(k, m)$ is a fundamental matrix for the connection $\nabla(k, m)$ on $\Gamma(U_+)$. We now associate to $A(k, m)$ the cocycle defined by the $x_\pm(k, m)$ which may be identified with the map

$$g(k, m) = x_-(k, m)\, x_+(k, m)^{-1}$$

from $\Gamma(U_+ \cap U_-)$ into G. By Lemma 5.3.2 we get

$$g(k, m) = \delta^{-1} x_+(-k, m)^{\vee\iota} T^{-1} x_+(k, m)^{-1} \quad (\text{on } \Gamma(U_+ \cap U_-).$$

From this we obtain, on $\Gamma(U_+ \cap U_-)$,

$$g(k, m)^{\vee\iota} = \delta^{-1} x_+(-k, m)\, T\, (x_+(k, m)^{\vee\iota})^{-1}$$

$$g(k, m)^{-1} = x_+(k, m)\, T\, (x_+(-k, m)^{\vee\iota})^{-1} \delta\,.$$

In particular, we get, on $\Gamma(U_+ \cap U_-)$,

$$g(k, m)^{\vee\iota} = \delta\, g(-k, m)^{-1} \delta^{-1}\,.$$

The idea is now to determine $g(k, m)$ in terms of $F_+(k, m)$. Observe that

$$x_+(k, m) = z_+^{-kH} \varphi F_+(k, m)^{-1}.$$

On the other hand, it is obvious that

$$(\log_+ z)^\vee = \log_+ z \pm i\pi \quad (z \in S^{1,r} \text{ or } z \in S^{1,\ell}),$$

and so,

$$g(k, m)(z) = \delta^{-1} z_+^{-kH} \exp(\mp i\pi kH) \varphi(z) (F_+(-k, m)^{\vee\iota})^{-1} T^{-1} F_+(k, m) \varphi(z)^{-1} z_+^{kH}$$

according as $z \in S^{1,r}$ or $z \in S^{1,\ell}$.

Let us now consider the direct sum of the bundle $\mathbf{C}^2 \times \mathbb{P}^1$ with itself equpped with the connection

$$\nabla(k, m) = \nabla(k, m) \oplus \nabla(k, m)$$

whose connection matrix is

$$\mathbf{A}(k, m) = A(k, m) \oplus A(k, m)^\vee.$$

We go over to the w-plane covering $\mathbf{C}_z^\times$ via $w \longrightarrow z = e^w$ and denote by $h^\sim$ the lift to the w-plane of the function h on the z-plane. Put

$$\mathbf{F}^\sim(k, m) = F_+^\sim(k, m) \oplus T F_+^\sim(-k, m)^\iota.$$

In view of the relation $A(k, m) = T[A(-k, m)^{\vee\iota}]$ it is clear that $\mathbf{F}^\sim(k, m)$ is a fundamental matrix for $\mathbf{A}(k, m)$. On the other hand the matrix $J = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$, where 1_2 is the identity endomorphism of $\mathbf{C}^2$, may be viewed as an automorphism of the bundle and it takes $\nabla(k, m)$ to $\nabla(k, m)^\vee$ because $\mathbf{A}^\vee = J[\mathbf{A}]$. Hence we conclude that there is an element $\Gamma \in GL(4, \mathbf{C})$ such that

$$\mathbf{F}^\sim(k, m)^\vee = J\, \mathbf{F}^\sim(k, m)\, \Gamma.$$

The diagonal nature of both $\mathbf{F}^\sim(k, m)$ and $\mathbf{F}^\sim(k, m)^\vee$ implies that Γ is zero on the diagonal, i. e.,

$$\Gamma = \begin{pmatrix} 0 & \Gamma_{12} \\ \Gamma_{21} & 0 \end{pmatrix}$$

The previous relation then reduces to the relations

$$F_+^{\sim}(k, m)(w - i\pi) = T\,F_+^{\sim}(-k, m)^{\iota}\,(w)\,\Gamma_{21}$$

$$T\,F_+^{\sim}(-k, m)^{\iota}\,(w - i\pi) = F_+^{\sim}(k, m)(w)\,\Gamma_{12}.$$

From the first of these we get

$$T\,F_+^{\sim}(k, m)^{\iota}\,(w - i\pi) = F_+^{\sim}(-k, m)(w)\,\Gamma_{21}{}^{\iota}$$

leading to the identity

$$\Gamma_{21}(k, m)^{\iota} = \Gamma_{12}(-k, m).$$

Moreover we also get

$$F_+^{\sim}(k, m)(w - 2i\pi) = F_+^{\sim}(k, m)(w)\,\Gamma_{12}\Gamma_{21},$$

so that, if $M \in GL(4, \mathbf{C})$ is the monodromy of $F_+^{\sim}(k, m)$ defined by

$$F_+^{\sim}(k, m)(w + 2i\pi) = F_+^{\sim}(k, m)\,M,$$

we have,

$$M = (\Gamma_{12}\Gamma_{21})^{-1},$$

or,

$$M(k, m) = \Gamma_{21}(k, m)^{-1}\,\Gamma_{21}(-k, m)^{t} \quad (t = \text{transpose}).$$

These relations become, on the z-plane,

$$F_+(k, m)(-z) = T\,F_+(-k, m)^{\iota}\,(z)\,\Gamma_{21},\ \ T\,F_+(-k, m)^{\iota}\,(-z) = F_+(k, m)(z)\,\Gamma_{12},$$

for $z \in S^{1,\ell}$. Hence,

$$F_+(k, m)(z) = T\,F_+(-k, m)^{\iota}\,(-z)\,\Gamma_{21} \quad (z \in S^{1,r}).$$

Substituting this in the formula for $g(k, m)$ we obtain

$$\varphi(z)^{-1}g(k, m)(z)\varphi(z) = \delta^{-1}\,z_+^{-kH}\,e^{-i\pi kH}\,\Gamma_{21}\,z_+^{kH} \quad (z \in S^{1,r}).$$

But the left side is the matrix

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \beta = \text{const. } z^{-2k},$$

so that we get the identity

$$\Gamma_{21}(k, m) = \begin{pmatrix} ie^{i\pi k} & ie^{i\pi k}\, c(k, m) \\ 0 & -ie^{-i\pi k} \end{pmatrix}$$

where $c(k, m)$ is the constant defined by the equation

$$\beta = c(k, m)\, z_+^{-2k}.$$

On the other hand the formula linking the values of $g(k, m)$ on U_+ and U_- shows that

$$\varphi(z)^{-1} g(k, m)(z)\varphi(z) = \delta^{-1}\, \varphi(z)^{-1} g(-k, m)(-z)^t \varphi(z)\, \delta \qquad (z \in S^{1,\ell}),$$

which simplifies to

$$\varphi(z)^{-1} g(k, m)(z)\varphi(z) = \begin{pmatrix} 1 & 0 \\ -e^{-2i\pi k} c(k,m)\, z_+^{2k} & 1 \end{pmatrix} \qquad (z \in S^{1,\ell}).$$

From the formula for Γ_{21} we obtain the following formula for M^{-1} :

$$M^{-1} = \begin{pmatrix} e^{2i\pi k} & e^{2i\pi k}\, c(k,m) \\ c(-k,m) & e^{-2i\pi k} + c(k,m)c(-k,m) \end{pmatrix}$$

In particular we see that

$$\mathrm{tr}\,(M^{-1}) = 2\cos 2\pi k + c(k, m)c(-k, m).$$

On the other hand M^{-1} is the monodromy at ∞ and so its trace is the trace of the monodromy M' of the original system at $t = 0$. Now the connection matrix at $t = 0$ is

$$A' = -t^{-2}(\tfrac{1}{4} - m^2)Y - kt^{-1}Y + (X + \tfrac{1}{4}Y)$$

and a simple calculation shows that if $y = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$, then

$$y[A'] = Ct^{-1} + \ldots.., \qquad C = \begin{pmatrix} 0 & 1 \\ \tfrac{1}{4} - m^2 & 1 \end{pmatrix}$$

From standard results (cf. [BV 1]) we then find that

$$\mathrm{tr}(M') = \mathrm{tr}(\exp(2i\pi C) = -2\cos 2\pi m.$$

Hence

$$c(k, m)c(-k, m) = -2\cos 2\pi m - 2\cos 2\pi k.$$

Let us identify $\mathbf{C}^{k,\pm}$ with $\mathbf{C}$ using $z_+^{\pm 2k}$ as bases. Then the Malgrange-Sibuya map becomes

$$(\nabla(k, m), u(k, m)) \longrightarrow (-e^{-2i\pi k} c(-k, m), c(k, m)).$$

The action of $G(k) \cong \mathbf{C}^\times$ is $\lambda, (a, b) \longrightarrow (\lambda a, \lambda^{-1} b)$ and so the map taking isomorphism classes of pairs to $G(k)$-orbits becomes

$$[\nabla(k, m)]_0 \longrightarrow H_{c(m)}, \quad d(k, m) = e^{-2i\pi k}(2\cos 2\pi m + 2\cos 2\pi k),$$

($[..]_0$ refers to the isomorphism class of thepair defined at $z = 0$) at least when

(ST) $$d(k, m) \neq 0 \iff k \pm m - \frac{1}{2} \notin \mathbf{Z}.$$

When $\pm m \in \mathbf{Z} - k + \frac{1}{2}$ we can only say that the image of $[\nabla(k, m)]_0$ is either the trivial orbit $\{(0, 0)\}$ or one of $H_{0,\pm}$. The condition for stability is (ST).

To proceed further it is necessary to make a deeper use of the properties of the Whittaker functions, and this comes down essentially to the use of their integral representations. We shall now give a brief sketch of the arguments that are needed to obtain explicitly the Malgrange-Sibuya map itself. Our main reference is [WW] (pp. 337-346). It is enough to work with generic values of m since the Malgrange-Sibuya map is analytic in m.

The differential equations (**W**) have a basis of solutions $M_{k,\pm m}$, where

($\mathbb{M}$) $$M_{k,\pm m} = t^{\pm m + 1/2} e^{-t/2}\{1 + \ldots\ldots\},$$

where the expression within $\{\ldots\}$ is an everywhere convergent power series in t whose coefficients are polynomials in k and m. The branch $t^{\pm m + 1/2}$ is the principal one, so that in reality, the $M_{k,\pm m}$ are functions on the w-plane which covers $C_z^{\times}$ through the map $w \longrightarrow z = e^w$. It is then immediate that the monodromy transformation corresponding to a circuit around $t = 0$ is given by

$$M_{k, \pm m} \longrightarrow -e^{\pm 2i\pi m} M_{k, \pm m}.$$

Let us write, for any function h of w, $h^{\vee}$ for the function defined by

$$h^{\vee}(w) = h(w + i\pi).$$

Then the $M_{k, \pm m}$ are related by Kummer's formulae([WW], p 338) :

$$M_{-k, m}^{\vee} = e^{i\pi(m + 1/2)} M_{k, m}.$$

For studying the behaviour at $t = \infty$ or $z = 0$ one uses the functions $W_{k, m}$ which are defined through certain contour integrals ([WW], p 339). The integral representation leads to an asymptotic expansion, while a second integral representation, going back to Barnes, allows one to determine the relations between $W_{k, m}$ and $M_{k, \pm m}$ ([WW], pp. 343-346). Thus we have,

$$W_{k, m} = \frac{\Gamma(-2m)}{\Gamma(-m- k+1/2)} M_{k, m} + \frac{\Gamma(2m)}{\Gamma(m-k+1/2)} M_{k, -m},$$

and

$$W_{k, m} \sim t^{k} e^{-t/2} \{1 + ...\}, \quad (t = e^w, \quad |\mathrm{Im}\, w| < \pi, \quad \mathrm{Re}\, w \longrightarrow \infty).$$

On the other hand $W_{-k, m}^{\vee}$ is also a solution of (**W**) and so we have

$$e^{ik\pi} W_{-k, m}^{\vee} \sim t^{-k} e^{t/2} \{1 + ...\}, \quad (t = e^w, \quad -2\pi < \mathrm{Im}\, w < 0, \quad \mathrm{Re}\, w \longrightarrow \infty).$$

These relations may of course be differentiated formally with respect to t, and so we get the following asymptotic expansion of a fundamenatl matrix for (**W**):

$$\mathbf{G} = \begin{pmatrix} W_{km} & e^{ik\pi} W_{-km}^{\vee} \\ W_{km}' & e^{ik\pi} W_{-km}^{\vee\prime} \end{pmatrix} \sim H(t) \begin{pmatrix} t^k e^{-t/2} & 0 \\ 0 & t^k e^{t/2} \end{pmatrix} \quad (' = d/dt)$$

$$H(t) \sim \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} + ... \in GL(2, \mathbf{C}[[1/t]]),$$

for the regime

$$-\pi < \mathrm{Im}\, w < 0, \quad \mathrm{Re}\, w \longrightarrow \infty, \quad t = e^w.$$

Let us now go over to $z = 1/t$ which corresponds to making the transformation $w \longrightarrow -w$ so that $z = e^{-w}$. Write

$$\mathbf{R}(w) = \mathbf{G}(-w).$$

Then

$$\mathbf{R}(w) \sim K(z)\begin{pmatrix} z^{-k}\, e^{-1/2z} & 0 \\ 0 & z^{k}\, e^{1/2z} \end{pmatrix} \quad (z \longrightarrow 0, \quad 0 < \arg(z) < \pi),$$

where K is asymptotic to an element of $GL(2, \mathbf{C}[[z]])$ in the same sector. As $\begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} = L^{-1}$, we have,

$$\begin{pmatrix} z^{-k}\, e^{-1/2z} & 0 \\ 0 & z^{k}\, e^{1/2z} \end{pmatrix}\mathbf{R}(w)^{-1} \sim K_1 \quad (z \longrightarrow 0,\ 0 < \arg(z) < \pi),$$

where $K_1 \in GL(2, \mathbf{C}[[z]])$ and $K_1(0) = L$. Since $\mathbf{R}$ is a fundamental matrix for $\nabla(k, m)$ it follows that the gauge transformation above must be the same as the $x_+(k, m)$ we have been working with earlier. But then, as

$$F_+(k, m) = x_+(k, m)^{-1}\, z_+^{-kH}\, \varphi\ ,$$

we have,

$$F_+(k, m)(z) = \mathbf{R}(w) \quad (z = e^{-w} \in \Gamma(U_+)).$$

Let M be as before the monodromy of $F_+(k, m)$ so that $\mathbf{R}(w + 2i\pi) = \mathbf{R}(w)\, M$ which leads to the relation $\mathbf{G}(w + 2i\pi) = \mathbf{G}(w)\, M^{-1}$, showing that M^{-1} is the monodromy of $\mathbf{G}$. On the other hand the relation between $W_{k,\, m}$ and the $M_{k,\, \pm m}$ implies the following relation:

$$e^{ik\pi} W_{-k,\, m}{}^{\vee} = \frac{\Gamma(-2m)}{\Gamma(-m+k+1/2)}\, e^{i\pi(m + k + 1/2)}\, M_{k,\, m} + \frac{\Gamma(2m)}{\Gamma(m+k+1/2)}\, e^{i\pi(-m + k + 1/2)}\, M_{k,\, -m}\, .$$

Hence, if

$$\mathbf{M} = \begin{pmatrix} M_{km} & M_{-k\,m} \\ M'_{km} & M'_{k-m} \end{pmatrix},$$

we have,

$$\mathbf{G} = \mathbf{M}\gamma, \quad \gamma = (\gamma_{ij})_{i,j=1,2},$$

where,

$$\gamma_{11} = \frac{\Gamma(-2m)}{\Gamma(-m-k+1/2)}, \quad \gamma_{12} = \frac{\Gamma(-2m)}{\Gamma(-m+k+1/2)} e^{i\pi(m+k+1/2)}$$

$$\gamma_{21} = \frac{\Gamma(2m)}{\Gamma(m-k+1/2)}, \quad \gamma_{22} = \frac{\Gamma(2m)}{\Gamma(m+k+1/2)} e^{i\pi(-m+k+1/2)}.$$

Hence

$$M^{-1} = \gamma^{-1}\begin{pmatrix} -e^{2i\pi m} & 0 \\ 0 & -e^{-2i\pi m} \end{pmatrix}\gamma,$$

which gives, after a simple but tedious calculation, the following formula for $M^{-1} = (\mu_{ij})_{i,j=1,2}$:

$$\mu_{11} = e^{2ik\pi}, \qquad \mu_{12} = \frac{2i\pi\, e^{2ik\pi}}{\Gamma(m+k+1/2)\Gamma(-m+k+1/2)},$$

$$\mu_{21} = \frac{2i\pi}{\Gamma(m-k+1/2)\Gamma(-m-k+1/2)}, \quad \mu_{22} = -e^{2ik\pi} - 2\cos 2m\pi.$$

If we compare this with the formula derived earlier for M^{-1}, we get,

$$c(k, m) = \frac{2i\pi}{\Gamma(m+k+1/2)\Gamma(-m+k+1/2)}.$$

It follows from the above formulae that the cohomology class which is the image of $((A(k, m), u(k, m))$ under the Malgrange-Sibuya map is represented by the cocycle $g(k, m)$ attached to the covering $\mathfrak{U} = \{U_+, U_-\}$:

$$g(k, m) = \begin{pmatrix} 1 & \frac{2i\pi}{\Gamma(m+k+1/2)\Gamma(-m+k+1/2)} z_+^{-2k} \\ 0 & 1 \end{pmatrix}, \quad (z \in S^{1,r}),$$

$$g(k, m) = \begin{pmatrix} 1 & 0 \\ \dfrac{-2i\pi e^{-2ik\pi}}{\Gamma(m-k+1/2)\Gamma(-m-k+1/2)} z_+^{2k} & 1 \end{pmatrix}, \quad (z \in S^{1,\ell}).$$

These formulae allow us to determine the orbits that arise from the connections $\nabla(k, m)$. Since $1/\Gamma(z)$ is entire and vanishes only at the points $z = 0, -1, -2, ...$, we get the following conclusions:

$2k \notin \mathbf{Z}$:	All orbits except $\{(0,0)\}$ arise.
$2k \in \mathbf{Z}$, $k < 0$:	All orbits except $H_{0,+}$ arise.
$2k \in \mathbf{Z}$, $k > 0$:	All orbits except $H_{0,-}$ arise.
$k = 0$:	All orbits except $H_{0,\pm}$ arise.

5.4 GENERALIZATIONS These calculations have been generalized in a far-reaching manner in a recent work of Duval and Mitschi [DM]. Their object of study is the family of differential equations

$$D_{q,p} = (-1)^{q-p} z \prod_{1 \le j \le p} (\partial + \mu_j) - \prod_{1 \le j \le q} (\partial + \nu_j - 1)$$

wher ∂ is the Euler operator zd/dz, and $\mu_1, \dots, \mu_p$ and $\nu_1, \dots, \nu_q$ are complex parameters. When $q \ge p + 1 \ge 2$, the associated connections have an irregular singular point at ∞ and their formalizations have only one level, namely, $1/q-p$. The reader is referred to their paper for the details involved in the calculation of the Stokes multipliers and the Malgrange-Sibuya maps.

PART II : THE COHOMOLOGY OF STOKES SHEAF

1 COHOMOLOGY OF GROUPS

1.1 Let X be a topological space. We assume that the reader is familiar with the language of sheaf theory, as in [G] for instance. We work with sheaves of sets and groups which are not necessarily abelian. If $\mathcal{A}$ is a sheaf of groups (resp. sets) on X, $\mathcal{A}(U)$ or $H^0(U,\mathcal{A})$ will denote the group (resp. set) of sections of $\mathcal{A}$ on U. If U and V are open sets and s and t are two sections on U and V respectively, we write $s = t$ on $U \cap V$ to mean that s and t have the same restriction on $U \cap V$.

Let $\mathcal{G}$ be a sheaf of groups and $\mathcal{T}$ a sheaf of sets, both defined on X. We say that $\mathcal{T}$ is a *(left)* $\mathcal{G}$*-sheaf* if $\mathcal{G}$ acts on $\mathcal{T}$. More precisely, this means that we have left actions of $\mathcal{G}(U)$ on $\mathcal{T}(U)$ for each open set $U \subset X$ that are compatible with the restriction maps. We then have an action of the stalk $\mathcal{G}(x)$ on the stalk $\mathcal{T}(x)$ for each x in X. We regard $\mathcal{G}$ itself as a $\mathcal{G}$-sheaf by letting $\mathcal{G}(U)$ act on itself by left translations. A $\mathcal{G}$*-torsor* is a $\mathcal{G}$-sheaf $\mathcal{T}$ that is locally isomorphic to $\mathcal{G}$, i.e., there is a covering (U_i) of X by open sets U_i such that the restrictions of $\mathcal{T}$ and $\mathcal{G}$ to U_i are isomorphic as $\mathcal{G}$-sheaves for all i. Torsors generalize principal bundles.

1.2. If $\mathcal{G}$ is a sheaf of groups on X, we write as usual

$$H^0(X, \mathcal{G}) = \mathcal{G}(X) = \text{the group of sections of } \mathcal{G} \text{ on } X.$$

Let $\mathfrak{U} = (U_i)$ be an open covering of X. The *1-cochains* associated to $\mathfrak{U}$ with values in $\mathcal{G}$ are systems $g = (g_{ij})$ where g_{ij} is a section of $\mathcal{G}$ on $U_i \cap U_j$; g is called *alternating* if $g_{ij}\, g_{ji} = 1$; g is called an *1-cocycle* if $g_{ij}\, g_{jk}\, g_{ki} = 1$ on $U_i \cap U_j \cap U_k$. Note that if the covering is such that *all distinct triple intersections are empty,* i.e., if $U_i \cap U_j \cap U_k = \varnothing$ whenever i, j, and k are distinct, then all alternating cochains are automatically cocycles. The set of 1-cocycles associated to $\mathcal{U}$ is denoted by $Z(\mathcal{U}, \mathcal{G})$. The *coboundary group* $C(\mathcal{U}, \mathcal{G})$ *associated to* $\mathcal{U}$ is the *full direct product* of the $\mathcal{G}(U_i)$:

$$C(\mathcal{U}, \mathcal{G}) = \prod_i \mathcal{G}(U_i).$$

We have an action $c, g \longrightarrow c[g]$ of the coboundary group on the space of cocycles given in the usual manner for $c = (c_i)$, $g = (g_{ij})$, by

$$c[g] = h \quad \text{where } h = (h_{ij}),\ h_{ij} = c_i\, g_{ij}\, c_j^{-1} \text{ on } U_i \cap U_j .$$

The space of orbits for this action is the *cohomology associated to* $\mathcal{U}$:

$$H^1(\mathcal{U}, \mathcal{G}) = C(\mathcal{U}, \mathcal{G}) \setminus Z(\mathcal{U}, \mathcal{G}) .$$

The system $g_{ij} = 1$ defines the trivial cohomology class 0 ; thus $H^1(\mathcal{U}, \mathcal{G})$ is a *pointed set.* If $\mathcal{B} = (V_\alpha)$ is an open refinement of $\mathcal{U}$ with refinement inclusion $V_\alpha \subset U_{i(\alpha)}$, we have induced maps $g \longrightarrow g^r$ of $Z(\mathcal{U}, \mathcal{G})$ into $Z(\mathcal{B}, \mathcal{G})$ and $c \longrightarrow c^r$ of $C(\mathcal{U}, \mathcal{G})$ into $C(\mathcal{B}, \mathcal{G})$ given by

$$g^r{}_{\alpha\beta} = g_{i(\alpha)\, i(\beta)} \quad \text{on } U_\alpha \cap U_\beta\ , \quad c^r{}_\alpha = c_{i(\alpha)} \quad \text{on } U_\alpha$$

The first is a morphism of pointed sets and the second is a homomorphism of groups. It is immediate that $c[g]^r = c^r[g^r]$ and so we have an induced map on the cohomology spaces $H^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(\mathcal{B}, \mathcal{G})$. This map is independent of the refinement inclusion chosen. Indeed if $V_\alpha \subset U_{i'(\alpha)}$ is another refinement inclusion, we have (with obvious notation) $g^{r'} = c[g^r]$ where c is the element of $C(\mathcal{B}, \mathcal{G})$ given by $c_\alpha = g_{i'(\alpha)\, i(\alpha)}$ on V_α .

A special feature of first cohomology is that the *refinement maps are always injective.* More generally, we have

PROPOSITION 1.2.1 *If* $g, h \in Z(\mathcal{U}, \mathcal{G})$ *and* $h^r = c[g^r]$ *for some* $c \in C(\mathcal{B}, \mathcal{G})$, *then we can find* $d \in C(\mathcal{U}, \mathcal{G})$ *such that* $h = d[g]$ *and* $d^r = c$. *In particular, the refinement map of the cohomology spaces is injective.*

PROOF Define, for indices i, α,

$$d_{i\alpha} = h_{i\, i(\alpha)}\, c_\alpha\, g_{i(\alpha)\, i} \quad \text{on } U_i \cap V_\alpha\ .$$

As $h_{i\,i(\alpha)} = h_{i\,i(\beta)}\, h_{i(\beta)\,i(\alpha)}$, $g_{i(\alpha)\,i} = g_{i(\alpha)\,i(\beta)}\, g_{i(\beta)\,i}$, $h_{i(\beta)\,i(\alpha)}\, c_\alpha\, g_{i(\alpha)\,i(\beta)} = c_\beta$ on $U_i \cap V_\alpha \cap V_\beta$, we see that $d_{i\alpha} = d_{i\beta}$ on $U_i \cap V_\alpha \cap V_\beta$. So there are $d_i \in \mathcal{G}(U_i)$ such that $d_i = d_{i\alpha}$ on $U_i \cap V_\alpha$ for all α, i ; taking $i = i(\alpha)$ in the definition of $d_{i\,\alpha}$ we see that $d_{i(\alpha)} = c_\alpha$ on V_α. So h_{ij} and $d_i g_{ij} d_j^{-1}$ restrict to the same element on $U_i \cap U_j \cap V_\alpha \cap V_\beta$, and hence $h = c[g$. ◆

As usual we define $H^1(X, \mathcal{G})$ as an inductive limit :

$$H^1(X, \mathcal{G}) = \lim_{\mathfrak{a}} H^1(\mathfrak{a}, \mathcal{G}) .$$

In view of the above proposition each $H^1(\mathfrak{a}, \mathcal{G})$ imbeds naturally in $H^1(X, \mathcal{G})$, so that we can write

$$H^1(X, \mathcal{G}) = \bigcup_{\mathfrak{a}} H^1(\mathfrak{a}, \mathcal{G}) .$$

The covering $\mathfrak{a}$ is called *good* if $H^1(X, \mathcal{G}) = H^1(\mathfrak{a}, \mathcal{G})$. From our definitions it is immediate that $C(\mathfrak{a}, \mathcal{G})$, $Z(\mathfrak{a}, \mathcal{G})$, $H^0(X, \mathcal{G})$, and $H^1(X, \mathcal{G})$ are all covariant functors of $\mathcal{G}$.

PROPOSITION 1.2.2 *Let $\mathcal{G}$ be a sheaf of groups on X. Then the elements of $H^1(X, \mathcal{G})$ classify the $\mathcal{G}$–torsors on X. More precisely, there is a natural bijection from the pointed set of isomorphism classes of $\mathcal{G}$–torsors on X to the pointed set $H^1(X, \mathcal{G})$.*

PROOF For any $\mathcal{G}$–torsor $\mathcal{T}$ on X, select an open covering $\mathfrak{a} = (U_i)$ of X and $\mathcal{G}_i$-isomorphisms $\varphi_i : \mathcal{G}_i \simeq \mathcal{T}_i$, the suffix i denoting (here and elsewhere) restriction to U_i. The identity section of $\mathcal{G}_i$ maps to a section t_i of $\mathcal{T}_i$ and there are *unique* $g_{ij} \in \mathcal{G}(U_i \cap U_j)$ such that $g_{ij}[t_j] = t_i$ on $U_i \cap U_j$. Clearly $g = (g_{ij})$ is an element of $Z(\mathfrak{a}, \mathcal{G})$. It is a standard verification that the image of g in $H^1(X, \mathcal{G})$ depends only on the isomorphism class of $\mathcal{T}$ and not on the choice of $\mathcal{T}$, $\mathfrak{a}$, or the φ_i. If $\mathcal{T}$, $\mathcal{T}'$ are such that $g'_{ij} = c_i g_{ij} c_j^{-1}$ for suitable $c_i \in \mathcal{G}(U_i)$, one can construct a *global* isomorphism of $\mathcal{T}$ with $\mathcal{T}'$. Indeed, replacing φ_i by $\varphi_i \circ r(c_i)$ ($r(c_i)$ is right translation by c_i) we may assume that $g'_{ij} = g_{ij}$ for all i,j. If θ_i is the isomorphism of $\mathcal{T}_i$ with $\mathcal{T}'_i$ that takes t_i to t'_i, it is immediate that $\theta_i = \theta_j$ on $U_i \cap U_j$. Thus the θ_i are the restrictions of a global isomorphism of $\mathcal{T}$ with $\mathcal{T}'$. In other words, the isomorphism classes form a set, and we have a natural injection of this set into $H^1(X, \mathcal{G})$. To see that this is surjective let $g = (g_{ij})$ be an element of $Z(\mathfrak{a}, \mathcal{G})$. We can define a $\mathcal{G}$–torsor $\mathcal{T}$

by gluing the $\mathcal{G}_i$ along the $U_i \cap U_j$ via the bijections $t \longrightarrow t g_{ij}(x)$ of the stalks $\mathcal{G}_i(x)$ with $\mathcal{G}_j(x)$, $x \in U_i \cap U_j$. It is easy to check that the resulting $\mathcal{G}$–torsor $\mathcal{T}$ is the one that gives rise to g. ♦

COROLLARY 1.2.3 *The image of a* $\mathcal{G}$*–torsor* $\mathcal{T}$ *in* $H^1(X, \mathcal{G})$ *is trivial if and only if* $\mathcal{T}$ *is trivializable , i.e., if and only if* $\mathcal{T}$ *admits a global section. If moreover* $H^0(X, \mathcal{G}) = 0$, *any such* $\mathcal{T}$ *has a unique section, i.e.,* $\mathcal{T}$ *is uniquely trivializable.*

PROOF Obvious. ♦

If $U \subset X$ is open, then we have a natural restriction map taking $\mathcal{G}$–torsors on X to $\mathcal{G}$–torsors on U . Thus we have a natural map from $H^1(X, \mathcal{G})$ to $H^1(U, \mathcal{G})$. At the level of cocycles this is the map that associates to the cocycle (g_{ij}) coming from the covering (U_i) the cocycle g_U = (restriction of g_{ij} to $U \cap U_i \cap U_j$) .

COROLLARY 1.2.4 *Suppose* $\mathfrak{U} = (U_i)$ *is a covering of* X *such that for any* i *the restriction map* $H^1(X, \mathcal{G}) \longrightarrow H^1(U_i, \mathcal{G})$ *is the zero map. Then* $\mathfrak{U}$ *is a good covering.*

PROOF By assumption any $\mathcal{G}$–torsor $\mathcal{T}$ on X trivializes on U_i and hence it follows from our discussion that the cohomology class associated to $\mathcal{T}$ is already represented by a cocycle from $\mathfrak{U}$. ♦

The geometric interpretation of $H^1(X, \mathcal{G})$ furnished by Proposition 1.2.2 behaves well from the functorial point of view also . More precisely, let $\mathcal{G}'$ be another sheaf of groups on X and $\mathcal{G} \longrightarrow \mathcal{G}'$ a sheaf morphism . Then the corresponding map $H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{G}')$ can be viewed geometrically as follows . Let $\mathcal{T}$ be a $\mathcal{G}$–torsor on X, and let us identify $\mathcal{G}'$, $\mathcal{G}$, and $\mathcal{T}$ with their etale spaces above X, with $\mathcal{G}$ acting from the right on $\mathcal{G}'$ via $\mathcal{G} \longrightarrow \mathcal{G}'$. We form the fibre product $\mathcal{G}' \times_X \mathcal{T}$ on which $\mathcal{G}$ acts freely from the right via $(g', t), g \longrightarrow (g'g, g^{-1}[t])$ and define $\mathcal{T}'$ as the quotient space for this action with the quotient topology . The action $h, (g',t) \longrightarrow (h g', t)$ of $\mathcal{G}'$ from the left induces an action of $\mathcal{G}'$ on $\mathcal{T}'$. It is easy to verify that $\mathcal{T}'$ is a $\mathcal{G}'$-torsor and that the map of $H^1(X, \mathcal{G}) \longrightarrow H^1(X, \mathcal{G}')$ defined by $\mathcal{T} \longrightarrow \mathcal{T}'$ is in fact the natural map corresponding to the map $\mathcal{G} \longrightarrow \mathcal{G}'$.

If X is compact one needs to work only with finite coverings. If X is not compact this may not be enough. In general, if Γ is any class of open coverings directed with respect to the refinement ordering, $H^1{}_\Gamma(X, \mathcal{G})$ will denote the union of all the $H^1(\mathfrak{U}, \mathcal{G})$ where $\mathfrak{U}$ runs through the coverings from Γ. Clearly under the identification of Proposition 1.2.2, $H^1{}_\Gamma(X, \mathcal{G})$ corresponds to the set of isomorphism classes of torsors $\mathfrak{T}$ on X for which there exists a covering (U_i) from Γ with the property thet $\mathfrak{T}$ trivializes on the U_i for all i.

1.3 Let $\mathcal{G}$ be a sheaf of groups on X, and let g be a cocycle for $\mathcal{G}$. Our purpose now is to define a sheaf of groups $\mathcal{G}^{(g)}$, the so-called *twist* of $\mathcal{G}$ by g. We shall be a little more general and suppose only that g is a cocycle for $\mathfrak{F}$, $\mathfrak{F}$ being a sheaf of groups on X that contains $\mathcal{G}$ as a *normal subsheaf*.

Let us write $\mathfrak{U} = (U_i)$ for the open covering such that the cocycle $g = (g_{ij})$ belongs to $Z(\mathfrak{U}, \mathfrak{F})$. Let $\mathcal{G}_i$ be the restriction of $\mathcal{G}$ to U_i. The sheaf $\mathcal{G}^{(g)}$ is then obtained by gluing the $\mathcal{G}_i$ along $U_i \cap U_j$ by identifying the stalk $\mathcal{G}_j(x)$ at $x \in U_i \cap U_j$ with the the stalk $\mathcal{G}_i(x)$ via the isomorphism $t \longrightarrow g_{ij}(x)[t]$ where $u[t]$ denotes $u\,t\,u^{-1}$. It is obvious that $\mathcal{G}^{(g)}$ is locally isomorphic to $\mathcal{G}$ and that for any open $U \subset X$ the sections of $\mathcal{G}^{(g)}$ on U may be identified with families (s_i) where $s_i \in \mathcal{G}(U \cap U_i)$ and $g_{ij}[s_j] = s_i$ on $U \cap U_i \cap U_j$. In particular, if g trivializes on U, there is no twisting on U, i.e., $\mathcal{G}$ and $\mathcal{G}^{(g)}$ are isomorphic on U. Indeed, if $g_{ij} = c_i c_j^{-1}$ on $U \cap U_i \cap U_j$ where c_i are sections of $\mathcal{G}$ on $U \cap U_i$, the isomorphism takes the section s of $\mathcal{G}$ on an open set $V \subset U$ to the section of $\mathcal{G}^{(g)}$ on V given by the family $(c_i[s])$. If $c = (c_i)$ is in $C(\mathfrak{U}, \mathfrak{F})$ and $h = c[g]$ we have an isomorphism θ_c of $\mathcal{G}^{(g)}$ with $\mathcal{G}^{(h)}$ that is defined by the requirement that it takes the section (s_i) of $\mathcal{G}^{(g)}$ on U to the section $(c_i[s_i])$ of $\mathcal{G}^{(h)}$ on U. We check easily that $\theta_{cc'} = \theta_c \circ \theta_{c'}$. If $\mathfrak{B} = (V_\alpha)$ is a refinement of $\mathfrak{U}$ and $V_\alpha \subset U_{i(\alpha)}$ is the refinement inclusion, and we replace g by $h = g^r$, we have an isomorphism φ_r of $\mathcal{G}^{(g)}$ with $\mathcal{G}^{(h)}$ that takes the section (s_i) on U to the section (σ_α) on U where $\sigma_\alpha = s_{i(\alpha)}$ on $U \cap V_\alpha$. If $U_\alpha \subset V_{i'(\alpha)}$ is another refinement inclusion, $k = g^{r'}$, and $\varphi_{r'}$ is the corresponding isomorphism of $\mathcal{G}^{(g)}$ with $\mathcal{G}^{(k)}$, we have $\varphi_{r'} = \theta_d \circ \varphi_r$; here d is in $C(\mathfrak{B}, \mathfrak{F})$ and is given by $d_\alpha = g_{i'(\alpha)\, i(\alpha)}$, so that $k = d[h]$. It is clear from these remarks that the isomorphism class of $\mathcal{G}^{(g)}$ depends only on the cohomology class of g, say γ. We therefore often write $\mathcal{G}^{(\gamma)}$ instead of $\mathcal{G}^{(g)}$.

PROPOSITION 1.3.1 *Let* $\mathfrak{F} = \mathcal{G}$, $\gamma \in H^1(X, \mathcal{G})$, *and let* $g \in Z(\mathfrak{A}, \mathcal{G})$ *represent* γ. *Then*

$$H^0(X, \mathcal{G}^{(g)}) \cong \text{ the stabilizer of } g \text{ in } C(\mathfrak{A}, \mathcal{G}) .$$

In particular $H^0(X, \mathcal{G}^{(\gamma)}) = 0$ *if and only if the stabilizer of* g *in* $C(\mathfrak{A}, \mathcal{G})$ *is trivial.*

PROOF The group $H^0(X, \mathcal{G}^{(g)})$ is isomorphic to the group of all systems $c = (c_i)$ such that $c_i \in \mathcal{G}(U_i)$ and $g_{ij}c_jg_{ij}^{-1} = c_i$, i.e., to the subgroup of all c in $C(\mathfrak{A}, \mathcal{G})$ with $c[g] = g$. ♦

REMARK Suppose that $H^0(X, \mathcal{G}^{(\gamma)}) = 0$. Then the sheaf $\mathcal{G}^{(\gamma)}$ itself is canonically defined (not just its isomorphism class) . For in this case the isomorphisms $\mathcal{G}^{(g)} \approx \mathcal{G}^{(h)}$ constructed in the above discussion are *uniquely determined.* Indeed, by the above result, when we go from g to $h = c[g]$ the coboundary c itself is uniquely determined by g and h , so that we have a canonically defined sheaf corresponding to the choice of $\mathfrak{A}$; when we change over to a refinement $\mathfrak{B}$, the formula $\varphi_{r'} = \theta_d \circ \varphi_r$ shows that the sheaves associated to $\mathfrak{A}$ and $\mathfrak{B}$ are canonically isomorphic. ♦

Suppose now that $\mathfrak{F} = \mathcal{G}$. We shall now show that given any cocycle g for $\mathcal{G}$ one can naturally define a twist $\mathfrak{X}^{(g)}$ of any $\mathcal{G}$–sheaf $\mathfrak{X}$, which will be a $\mathcal{G}^{(g)}$–sheaf defined upto isomorphism . Let $\mathfrak{A} = (U_i)$ and $g = (g_{ij})$ be as before and define the sheaf $\mathfrak{X}^{(g)}$ by gluing the $\mathfrak{X}_i$ and $\mathfrak{X}_j$ along $U_i \cap U_j$ via the identification $t \longrightarrow g_{ij}(x)\,[t]$ of the stalk $\mathfrak{X}_j(x)$ with the stalk $\mathfrak{X}_i(x)$, for all $x \in U_i \cap U_j$. The sections of $\mathfrak{X}^{(g)}$ over an open set U are families (s_k) , $s_k \in \mathfrak{X}(U \cap U_k)$, such that $g_{kl}\,[s_l] = s_k$ on $U \cap U_k \cap U_l$ for all k,l ; if (g_k) is a section of $\mathcal{G}^{(g)}$, it is then immediate that $(g_k\,[s_k])$ is also a section of $\mathfrak{X}^{(g)}$.Thus $\mathfrak{X}^{(g)}$ is a $\mathcal{G}^{(g)}$-sheaf, and $\mathfrak{X} \longrightarrow \mathfrak{X}^{(g)}$ is a covariant functor . As before we have isomorphisms $\theta_c\,(\mathfrak{X}^{(g)} \longrightarrow \mathfrak{X}^{(h)})$ when $h = c[g]$, and isomorphisms $\varphi_r\,(\mathfrak{X}^{(g)} \longrightarrow \mathfrak{X}^{(h)})$ when $h = g^r$, with the same relations . We thus obtain the $\mathcal{G}^{(\gamma)}$–sheaf $\mathfrak{X}^{(\gamma)}$ defined upto isomorphism , the sheaf itself being canonically determined if $H^0(X, \mathcal{G}^{(\gamma)}) = 0$, γ being the cohomology class of g.

PROPOSITION 1.3.2 *Twisting by* g *induces a bijection of* $H^1(X, \mathcal{G})$ *with* $H^1(X, \mathcal{G}^{(g)})$ *and takes* γ *to* 0, γ *being the cohomology class of* g.

PROOF It is clear from our construction that $(\mathfrak{X}, \mathcal{G})$ is locally isomorphic to $(\mathfrak{X}^{(g)}, \mathcal{G}^{(g)})$, and so, if $\mathfrak{X}$ is a $\mathcal{G}$–torsor, $\mathfrak{X}^{(g)}$ is a $\mathcal{G}^{(g)}$–torsor. Twisting by g thus defines a map from $H^1(X, \mathcal{G})$ to $H^1(X, \mathcal{G}^{(g)})$. Since we can replace g by refinements coming from a cofinal family of coverings of X, it is sufficient to show that for any $g \in Z(\mathfrak{a}, \mathcal{G})$, twisting by g gives a bijection $H^1(\mathfrak{a}, \mathcal{G}) \cong H^1(\mathfrak{a}, \mathcal{G}^{(g)})$. Let $\mathfrak{X}$ be a $\mathcal{G}$–torsor with sections s_i on U_i which are related on $U_i \cap U_j$ by $h_{ij}[s_j] = s_i$, $h = (h_{ij})$ being the cocycle that corresponds to $\mathfrak{X}$. Clearly $\mathfrak{X}^{(g)}$ also trivializes on the U_i. We shall now compute the cocycle corresponding to the sheaf $\mathfrak{X}^{(g)}$.

To this end we shall construct sections for $\mathfrak{X}^{(g)}$ over the U_i. Define the family $s_i^\wedge = (s_{ik})$ by setting $s_{ik} = g_{ki}[s_i]$ on $U_i \cap U_k$. As $s_{il} = g_{lk}[s_{ik}]$ on $U_i \cap U_k \cap U_l$, $s_i^\wedge$ is a section of $\mathfrak{X}^{(g)}$ on U_i; the cocycle corresponding to $\mathfrak{X}^{(g)}$ is then given by the system $h^\wedge = (h_{ij}^\wedge)$ where $h_{ij}^\wedge$ is the element of $\mathcal{G}^{(g)}(U_i \cap U_j)$ that satisfies $h_{ij}^\wedge[s_j^\wedge] = s_i^\wedge$ on $U_i \cap U_j$. An easy calculation shows that $h_{ij}^\wedge = (h_{ijk})$ is given by

$$\text{(T)} \qquad h_{ijk} = g_{ki}\, h_{ij}\, g_{jk}.$$

This is the basic formula for our purposes. If we take $h_{ij} = g_{ij}$ here, we see that $h_{ijk} = 1$, showing that $\mathfrak{X}^{(g)}$ is trivial as a $\mathcal{G}^{(g)}$–torsor. Let $\mathfrak{M}$ be another $\mathcal{G}$–torsor represented by the cocycle $m = (m_{ij})$, such that $\mathfrak{X}^{(g)}$ is isomorphic with $\mathfrak{M}^{(g)}$. Writing the cocycle for $\mathfrak{M}^{(g)}$ as $m_{ij}^\wedge = (m_{ijk})$ as above, we have sections $c_i^\wedge = (c_{ik})$ of $\mathcal{G}^{(g)}$ on U_i such that $m_{ij}^\wedge = c_i^\wedge h_{ij}^\wedge c_j^{\wedge -1}$ on $U_i \cap U_j$. So $m_{ijk} = c_{ik}\, h_{ijk}\, c_{jk}^{-1}$ on $U_i \cap U_j \cap U_k$, or $m_{ij} = d_{ik}\, h_{ij}\, d_{jk}^{-1}$ on $U_i \cap U_j \cap U_k$, where $d_{ik} = g_{ik}\, c_{ik}\, g_{ki}$. The relations $c_{il} = g_{kl}^{-1}\, c_{ik}\, g_{kl}$ imply that $d_{il} = d_{ik}$ on $U_i \cap U_k \cap U_l$, so that there are sections d_i of $\mathcal{G}$ on U_i that restrict to d_{ik} on $U_i \cap U_k$; and we have $m_{ij} = d_i\, h_{ij}\, d_j^{-1}$ on $U_i \cap U_j$. This proves that $\mathfrak{X}$ and $\mathfrak{M}$ are isomorphic.

It remains to establish the surjectivity. Let $m^\wedge = (m_{ij}^\wedge)$ be a cocycle for $\mathcal{G}^{(g)}$ and write $m_{ij}^\wedge = (m_{ijk})$. Define $h_{ijk} = g_{ik}\, m_{ijk}\, g_{kj}$ on $U_i \cap U_j \cap U_k$. Then an easy calculation, based on the relations $m_{ijk} = g_{lk}^{-1}\, m_{ijl}\, g_{lk}$ shows that $h_{ijl} = h_{ijk}$ on $U_i \cap U_j \cap U_k \cap U_l$. So there are h_{ij} in $\mathcal{G}(U_i \cap U_j)$ that restrict to the h_{ijk}, and it is an easy matter to verify that the (h_{ij}) define a cocycle h for $\mathcal{G}$. It is obvious that the twist of h is $m^\wedge$. ♦

DIRECT AND INVERSE IMAGES. Let X, Y be topological spaces, and f ($X \longrightarrow Y$) a continuous map. If $\mathcal{G}$ is a sheaf of groups or sets on Y (resp. X) its inverse (resp. direct) image is $f^*\mathcal{G}$ (resp. $f_*\mathcal{G}$) on X (resp. on Y). If $\mathcal{G}$ is on Y, $(f^*\mathcal{G})(x) = \mathcal{G}_{f(x)}(x \in X)$; for open $U \subset X$, $f^*\mathcal{G}(U)$ is the set of continuous maps s of U into the etale space of $\mathcal{G}$ such that $s(x) \in \mathcal{G}_{f(x)}$ for all $x \in U$. If $\mathcal{G}$ is on X, then for any open $V \subset Y$, $f_*\mathcal{G}(V) = \mathcal{G}(f^{-1}(V))$. Let us now suppose that X and Y are compact metric spaces, Y is the space of orbits of a finite group G with a *free* action on X, and f ($X \longrightarrow Y$) is locally trivial so that each point of Y has an open neighbourhood U such that $f^{-1}(U) \cong G \times U$, G acting on the first component by left translation; in particular, $f^{-1}(U) = \coprod_{g \in G} U_g$ where U_g are disjoint, $gU_h = U_{gh}$, and f ($U_g \longrightarrow U$) is a homeomorphism. For a sheaf $\mathcal{G}$ of groups on X, $(f_*\mathcal{G})(U) = \bigoplus_{g \in G} \mathcal{G}(U_g)$ so that we have $(f_*\mathcal{G})(y) = \bigoplus_{f(x)=y} \mathcal{G}(x)$. Assume now that G operates on $\mathcal{G}$ compatibly with its action on X. The stalks of $f_*\mathcal{G}$ are stable under G, so that the subsheaf of invariants $(f_*\mathcal{G})^G$ is well defined. The inclusion $(f_*\mathcal{G})^G \hookrightarrow f_*\mathcal{G}$ gives a natural map

$$i : H^1(Y, (f_*\mathcal{G})^G) \longrightarrow H^1(Y, f_*\mathcal{G}).$$

On the other hand, we have a natural imbedding $j : H^1(Y, f_*\mathcal{G}) \hookrightarrow H^1(X, \mathcal{G})$; indeed, if $\mathcal{U}_Y = (V_i)$ is an open covering of Y, and $\mathcal{U}_X = f^*\mathcal{U}_Y = (U_i)$ where $U_i = f^{-1}(V_i)$, $H^1(\mathcal{U}_Y, f_*\mathcal{G}) = H^1(\mathcal{U}_X, \mathcal{G})$. So we have a natural map

$$F = j \circ i : H^1(Y, (f_*\mathcal{G})^G) \longrightarrow H^1(X, \mathcal{G}).$$

Finally, as G acts on $\mathcal{G}$, it acts on $H^1(X, \mathcal{G})$ also, and we write $H^1(X, \mathcal{G})^G$ for the pointed subset of its G-invariant elements. If $\mathcal{H}$ is a sheaf of groups on Y and $\mathcal{G} = f^*\mathcal{H}$, the stalks of $\mathcal{G}$ at points above $y \in Y$ are canonically identified with $\mathcal{H}(y)$ so that we have a natural action of G on $\mathcal{G}$ that commutes with this identification; and $\mathcal{H} \cong (f^*\mathcal{G})^G$ canonically.

PROPOSITION 1.3.3 *Suppose* $H^0(X, \mathcal{G}^{(\alpha)}) = 0$ *for all* $\alpha \in H^1(X, \mathcal{G})$. *Then* j *is a bijection of* $H^1(Y, f_*\mathcal{G})$ *with* $H^1(X, \mathcal{G})$ *and* F *is a bijection of* $H^1(Y, (f_*\mathcal{G})^G)$ *with* $H^1(X, \mathcal{G})^G$. *If further* $\mathcal{G} = f^*\mathcal{H}$ *as above, then* $H^1(S^1, \mathcal{H}) \cong H^1(X, \mathcal{G})^G$ *canonically.*

PROOF To prove that i is injective let us consider (in the notation introduced above) $g,h \in Z(\mathfrak{A}_Y, (f_*\mathcal{G})^G)$ such that $h = c[g]$ for some $c = (c_i) \in C(\mathfrak{A}_Y, f_*\mathcal{G})$; it is a question of proving that $c_i \in \mathcal{G}(U_i)^G$ for all i. Clearly, as g_{ij} and h_{ij} are G-invariant sections of $\mathcal{G}$ on $U_i \cap U_j$, we have $c^t[g] = h$ also for any $t \in G$, and so $((c^t)^{-1}c)[g] = g$. As $H^0(X, G\mathcal{G}^{(\alpha)}) = 0$ for $\alpha = [g]$, we conclude from Proposition 1.3.1 that $(c^t)^{-1}c = 1$, i.e., $c^t = c$. It is obvious that F maps into $H^1(X, \mathcal{G})^G$. Suppose now that $\alpha \in H^1(X, \mathcal{G})$. We shall find a covering $\mathfrak{A}_Y$ of Y and a $g \in Z(f^*\mathfrak{A}_Y, \mathcal{G})$ that represents α; this will prove that j is surjective. To this end, we choose a G-invariant metric $dist_X$ for X; then $dist_Y(y, y') := dist_X(f^{-1}(y), f^{-1}(y'))$ is a metric for Y. For any $\varepsilon > 0$ let $\mathfrak{C}(\varepsilon)$ be the covering of X by all the open balls of radius ε. As X is compact these coverings are cofinal. Since G acts freely on X, there is an $\varepsilon_0 > 0$ such that $dist_X(x, t(x)) \geq \varepsilon_0$ for all $x \in X$, $1 \neq t \in G$, and so we can find ε, $0 < \varepsilon < \frac{1}{2}\varepsilon_0$, and a cocycle $h \in Z(\mathfrak{C}(\varepsilon), \mathcal{G})$ representing α. If B is in $\mathfrak{C}(\varepsilon)$ its transforms $t[B]$ ($t \in G$) are disjoint; if $B_Y = f(B)$ then $f^{-1}(B_Y) = \coprod_{t \in G} t[B]$, and $\mathfrak{A}_Y = \{B_Y \mid B \in \mathfrak{C}(\varepsilon)\}$ is a covering of Y. We may thus view $\mathfrak{C}(\varepsilon)$ as a refinement of $f^*\mathfrak{A}_Y$ via the inclusion $B \subset f^{-1}(B_Y) = G[B]$. If B, B' are in $\mathfrak{C}(\varepsilon)$, $G[B] \cap G[B']$ is the *disjoint* union of the $t[B] \cap t'[B']$ and so there is a unique section of $\mathcal{G}$ over $G[B] \cap G[B']$ that restricts on $t[B] \cap t'[B']$ to the section defined by h. So we obtain a cocycle k from $Z(f^*\mathfrak{A}_Y, \mathcal{G})$ that maps into h under the refinement map. Thus k represents α also. Suppose finally that α is invariant under G. The cocycle k constructed above may not be invariant; we shall now show that it can be modified so as to become invariant. If $t \in G$, the transform k^t will also represent α, i.e., $k^t = c_t[k]$ for a $c_t \in C(f^*\mathfrak{A}_Y, \mathcal{G})$; the element c_t is *unique* because $H^0(X, \mathcal{G}^{(\alpha)}) = 0$, exactly as in the earlier proof. Let us now write M for $C(f^*\mathfrak{A}_Y, \mathcal{G})$ viewed as a G-module; $c\ (t \longrightarrow c_t)$ is then a map of G into M satisfying $c_1 = 1$, $c_{st} = (c_t)^s c_s$, i.e., $c^{-1} \in H^1(M, G)$. We shall presently prove that $H^1(M, G) = 0$; assuming this for the moment we see that there is a $d \in M$ such that $c_t = d^t d^{-1}$ for all $t \in G$. If $g = d^{-1}[k]$, g represents α and $g^t = (d^t)^{-1}[k^t] = d^{-1}(c_t)^{-1}[k^t] = d^{-1}[k] = g$.

It remains to prove that $H^1(M, G) = 0$. Since M is the *complete direct sum* of the G-modules $M_B = \mathcal{G}(G[B])$, it is enough to show that $H^1(M_B, G) = 0$ for any B. We identify M_B with the G-module of maps from G to $\mathcal{G}(B)$ by identifying the map $t \longrightarrow b_t$ with the section of $\mathcal{G}$ on $G[B]$ that restricts to $(b_t)^t$

on t [B] ; the action of $s^{-1} \in G$ on the map $t \longrightarrow b_t$ is to send it to the map $t \longrightarrow b_{st}$. Suppose maps m_s $(G \longrightarrow \mathcal{G}(B))$ are given such that $m_1(u) = 1$, $m_{st}(u) = m_t(s^{-1}u)\, m_s(u)$, $s,t,u \in G$. Let $d(G \longrightarrow \mathcal{G}(B))$ be the map defined by $d(u^{-1}) = m_u(1)$. Then

$$d^t (u^{-1})\, d(u^{-1})^{-1} = d(t^{-1}u^{-1})\, d(u^{-1})^{-1} = m_{ut}(1)\, m_u(1)^{-1} = m_t(u^{-1})$$

proving that $m_t = d^t d^{-1}$, $t \in G$. This completes the proof. ◆

1.4. EXACT SEQUENCES OF SHEAVES ON S^1 AND THEIR COHOMOLOGIES From now now on we shall suppose that $X = S^1$. We begin by recalling that when we have a diagram $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ of *pointed sets* , exactness at V means that the fibre in V above the distinguished point of W is the image of U , while exactness at U (resp. W) means that the map from U to V (resp. V to W) is injective (resp. surjective) .Consider now an exact sequence of sheaves of groups

$$(E_1) \qquad 0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

on S^1 . The induced map from $H^1(S^1, \mathcal{G})$ to $H^1(S^1, \mathcal{G}'')$ is surjective; in fact, the standard construction defining the usual boundary map in the abelian case (see for example [MK] p 59) shows that if γ is in $H^1(S^1, \mathcal{G}'')$ we can find a finite open covering $\mathfrak{M} = (M_\lambda)$ and a representative cocycle of γ from this covering, say g, such that g lifts to an alternating cochain of $\mathcal{G}$ from $\mathfrak{M}$. As $\dim (S^1) = 1$, we can choose (see below) $\mathfrak{M} = (M_\lambda)$ so that triple intersections of the M_λ corresponding to distinct indices are all empty, and hence it follows that γ lifts to a cocycle for $\mathcal{G}$ .To (E_1) we can therefore associate the exact sequence

$$(E_2) \qquad \begin{array}{l} 0 \longrightarrow H^0(S^1, \mathcal{G}') \longrightarrow H^0(S^1, \mathcal{G}) \longrightarrow H^0(S^1,\mathcal{G}'') \longrightarrow \\ \longrightarrow H^1(S^1, \mathcal{G}') \longrightarrow H^1(S^1,\mathcal{G}) \longrightarrow H^1(S^1, \mathcal{G}'') \longrightarrow 0 . \end{array}$$

Suppose $\mathfrak{A} = (U_i)$ is a covering of S^1 and $b \in Z(\mathfrak{A}, \mathcal{G})$ and let a be the image cocycle in $Z(\mathfrak{A}, \mathcal{G}'')$. From the discussion of twisting we have given it is clear that we can twist $\mathcal{G}'$ and $\mathcal{G}$ by b and $\mathcal{G}''$ by a to obtain the exact sequence of the twisted sheaves

$$(E_3) \qquad 0 \longrightarrow \mathcal{G}'^{(b)} \longrightarrow \mathcal{G}^{(b)} \longrightarrow \mathcal{G}''^{(a)} \longrightarrow 0$$

which leads to the exact sequence

$$(E_4) \qquad \begin{array}{l} 0 \longrightarrow H^0(S^1,\mathcal{G}'^{(b)}) \longrightarrow H^0(S^1, \mathcal{G}^{(b)}) \longrightarrow H^0(S^1, \mathcal{G}''^{(a)}) \longrightarrow \\ \longrightarrow H^1(S^1, \mathcal{G}'^{(b)}) \longrightarrow H^1(S^1,\mathcal{G}^{(b)}) \longrightarrow H^1(S^1, \mathcal{G}''^{(a)}) \longrightarrow 0. \end{array}$$

Moreover it follows from our definition of twisting of cohomology classes that the square

$$\begin{array}{ccc} H^1(S^1, \mathcal{G}) & \longrightarrow & H^1(S^1, \mathcal{G}'') \\ \downarrow t_b & & \downarrow t_a \\ H^1(S^1, \mathcal{G}^{(b)}) & \longrightarrow & H^1(S^1, \mathcal{G}''^{(a)}) \end{array}$$

is commutative, the map t_b (resp. t_a) being the twist by b (resp. a) ; this is clear from the formulae (T) of the previous paragraph for the twists of cocycles .

We have used above the fact that there is a cofinal family of coverings whose triple intersections are empty, and for these all alternating cochains are cocycles . This is easy to see. Indeed, let $q \geq 1$ be any integer and $z_0 , z_1, \dots ,$ $z_{4q} = z_0$ be the division points of the circle S^1 into 4q arcs of length $\pi/2q$, the points being ordered in the counterclockwise direction; the open arcs $(z_0 , z_2) , (z_1 , z_3) , \dots ,(z_{4q-2} , z_0), (z_{4q-1} , z_1)$ then form a covering of S^1 whose distinct triple intersections are empty . For later use we note that by perturbing z_0 slightly it is possible to ensure that the z_i do not lie in any given finite set . It is also useful to remember that if $\mathcal{a}$ is any open covering of S^1 , there is an integer $p \geq 1$ such that any open covering of S^1 by open arcs of length $\leq \pi/p$ is a refinement of $\mathcal{a}$. The following proposition is now obvious.

PROPOSITION 1.4.1 *Let* $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$ *be sheaves of groups on* S^1 *satisfying* (E_1) *and let* $H^0(S^1, \mathcal{G}''^{(\alpha)}) = 0$ *for all* $\alpha \in H^1(S^1, \mathcal{G}'')$. *Fix a co-*

cycle a *representing* α *and a cocycle* b *for* $\mathcal{G}$ *that lies above* α. *Then the fibre above* α *of the map* $H^1(S^1, \mathcal{G}) \longrightarrow H^1(S^1, \mathcal{G}'')$ *is canonically isomorphic to* $H^1(S^1, \mathcal{G}'^{(b)})$. *More precisely, the diagram*

$$\begin{array}{ccccccccc} & & & & H^1(S^1, \mathcal{G}) & \longrightarrow & H^1(S^1, \mathcal{G}'') & & \\ & & & & \downarrow t_b & & \downarrow t_a & & \\ 0 & \longrightarrow & H^1(S^1, \mathcal{G}'^{(b)}) & \longrightarrow & H^1(S^1, \mathcal{G}^{(b)}) & \longrightarrow & H^1(S^1, \mathcal{G}''^{(a)}) & \longrightarrow & 0 \end{array}$$

is commutative , the bottom line is exact, and the fibre in $H^1(S^1,\mathcal{G})$ *above* α *gets mapped to the image of* $H^1(S^1, \mathcal{G}'^{(b)})$ *in* $H^1(S^1, \mathcal{G}^{(b)})$.

1.5 We shall conclude this section with the formulation and proof of a result of Deligne [De 3] that will be decisive in the proof of the representability theorem discussed in the next section. We fix a real number $a > 2\pi$ and consider the map f of the open interval $I = (0, a)$ onto S^1 that takes x to its residue class mod 2π. Let $\mathcal{E}$ be the category of sheaves of groups $\mathcal{G}$ on S^1 such that

$$H^0(I, f^*\mathcal{G}) = 0, \quad H^1(I, f^*\mathcal{G}) = 0.$$

PROPOSITION 1.5.1 (Deligne) *For any* $\mathcal{G}$ *in* $\mathcal{E}$, $H^0(S^1, \mathcal{G}) = 0$. *Moreover, if* $J = (0, a-2\pi)$, *we have a natural isomorphism (of functors with values in the category of pointed sets)*

$$H^1(S^1, \mathcal{G}) \cong H^0(J, f^*\mathcal{G}).$$

PROOF We shall give Deligne's proof [De 3] of this result that relies on the interpretation of H^1 as the set of isomorphism classes of torsors . To begin with, as we have the imbedding $H^0(S^1, \mathcal{G}) \hookrightarrow H^0(I, f^*\mathcal{G})$, we must have $H^0(S^1, \mathcal{G}) = 0$. So we are left with H^1. Let $\mathcal{G} \in \mathcal{E}$ and $\mathcal{T}$ be a $\mathcal{G}$–torsor on S^1. Since f is a local homeomorphism it is immediate that $f^*\mathcal{T}$ is a $f^*\mathcal{G}$–torsor . By assumption $f^*\mathcal{T}$ is trivial and uniquely trivializable on I. Let t be the unique section of $f^*\mathcal{T}$ on I. Since the stalks of $f^*\mathcal{T}$ at x and $x + 2\pi$ are the same there is a unique element $g(x) \in f^*\mathcal{G}(x)$ such that

$$(*) \qquad g(x)\,[t(x)] = t(x+2\pi) \qquad (0 < x < a - 2\pi)$$

In other words, we have an isomorphism above S^1 between the restrictions of $f^*\mathfrak{T}$ to $(0, a-2\pi)$ and $(2\pi, a)$ induced by g. Now $g \in H^0(J, f^*\mathcal{G})$, and we note first that g depends only on the isomorphism class of $\mathfrak{T}$. Indeed, if $\mathfrak{T}'$ is isomorphic to $\mathfrak{T}$, the lifted isomorphism $f^*\mathfrak{T}' \cong f^*\mathfrak{T}$ must take the unique section t' of $f^*\mathfrak{T}'$ to the section t of $f^*\mathfrak{T}$. This shows immediately that g does not change if we replace $\mathfrak{T}$ with $\mathfrak{T}'$. We thus have a map

$$H^1(S^1, \mathcal{G}) \longrightarrow H^0(J, f^*\mathcal{G}) .$$

If $\mathfrak{T}$ is already trivial on S^1, $t(x + 2\pi) = t(x)$ so that $g(x) = 1$. Hence the map above is defined in the category of pointed sets. We wish to prove that it is a bijection.

To prove the injectivity, let us consider two $\mathcal{G}$–torsors $\mathfrak{T}$, $\mathfrak{T}'$ for which the associated sections g are the same :

$$g(x)[t(x)] = t(x + 2\pi), \quad g(x)[t'(x)] = t'(x + 2\pi) .$$

It is a question of showing that the isomorphism Φ of $f^*\mathfrak{T}$ with $f^*\mathfrak{T}'$ that takes t to t' descends to S^1, i.e., $\Phi(x)$ depends only on $f(x)$. If x and $x + 2\pi$ are both in I, then $0 < x < a-2\pi$, and we have

$$\Phi(x)(t(x + 2\pi)) = \Phi(x)(g(x)[t(x)]) = g(x)[t'(x)] = t'(x + 2\pi) = \Phi(x+2\pi)(t(x + 2\pi)) .$$

This proves that $\Phi(x + 2\pi) = \Phi(x)$.

To prove the surjectivity, let $g \in H^0(J, f^*\mathcal{G})$. Write U for the trivial $f^*\mathcal{G}$–torsor on I with t as its unique section. To show that U descends to S^1 we first prove that there is a natural way to identify $U(x)$ and $U(x + 2\pi)$ for $0 < x < a - 2\pi$. We define $\theta(x)$ as the unique bijection of $U(x)$ with $U(x + 2\pi)$ such that

$$\theta(x)(g(x)[t(x)]) = t(x + 2\pi).$$

If $x, x + 2\pi, \dots, x + 2r\pi$ are in I, we define $\theta_r(x)$ $(U(x) \cong U(x + 2r\pi))$ by

$$\theta_r(x) = \theta(x + 2(r-1)\pi) \circ \dots \circ \theta(x + 2\pi).$$

It is obvious that we have a consistent scheme of identifying the stalks of U above S^1 so that $U = f^*\mathfrak{T}$ for a $\mathcal{G}$–torsor $\mathfrak{T}$ on S^1. The construction shows that $\mathfrak{T}$ gives rise to g .

It remains to show that the map $\mathfrak{T} \longrightarrow g$ is functorial . If $\mathcal{G}'$ is another sheaf from $\mathcal{E}$ and $\mathcal{G} \longrightarrow \mathcal{G}'$ is a sheaf map, the $\mathcal{G}'$–torsor $\mathfrak{T}'$ which is the image of $\mathfrak{T}$ has the stalks $\mathfrak{T}'(x) = \mathcal{G}'(x) \times \mathfrak{T}(x)/\mathcal{G}(x)$. Hence $f^*\mathfrak{T}'(x) = f^*\mathcal{G}'(x) \times f^*\mathfrak{T}(x)/f^*\mathcal{G}(x)$ so that we may represent t'(x) by (1, t (x)) . It follows easily from this that g'(x) is the image of g(x) in $f^*\mathcal{G}'(x)$. In other words, we have a commutative diagram

$$\begin{array}{ccc} H^1(S^1, \mathcal{G}) & \longrightarrow & H^0(J, f^*\mathcal{G}) \\ \downarrow & & \downarrow \\ H^1(S^1, \mathcal{G}') & \longrightarrow & H^0(J, f^*\mathcal{G}') \end{array}$$

This proves the functoriality. ♦

2 SHEAVES OF UNIPOTENT GROUP SCHEMES AND THE REPRESENTABILITY OF THEIR COHOMOLOGY

2.1. We begin by recalling some basic facts about affine group schemes and algebraic matrix groups ; for more details see [Wa] and [Bo] . We work over **C** and all our **C**-algebras are commutative and have units . For any **C**–algebra R we consider covariant functors $F : S \longrightarrow F(S)$ from the category of R-algebras to the category of sets. Two such functors F, G are naturally isomorphic if there is a bijection $F(S) \longrightarrow G(S))$ for each S such that for any homomorphism of R-algebras $S \longrightarrow S'$ the diagram

$$\begin{array}{ccc} F(S) & \longrightarrow & G(S) \\ \downarrow & & \downarrow \\ F(S') & \longrightarrow & G(S') \end{array}$$

commutes . F is said to be *representable over* R or an *affine scheme over* R if there exists an R- algebra A such that F and $\mathrm{Hom}_R(A, .)$ are naturally isomorphic functors. We then say that A *represents* F *over* R, and write A = R[F]. Given F, A is determined upto isomorphism .

An *affine group scheme over* R is a representable functor from the category of R-algebras to the category of groups. The algebra that represents this functor is then a *Hopf algebra;* and conversely, if A is a Hopf algebra over R, the sets $\mathrm{Hom}_R(A, S)$ become groups in a natural way for any R-algebra S, and $S \longrightarrow \mathrm{Hom}_R(A, S)$ is an affine group scheme over R . All these definitions are of course relative to R; when R = **C**, we shall generally omit any reference to **C** .

An affine scheme F is said to be of *finite type* or *algebraic* if A = **C**[F] is finitely generted ; *reduced* if A has no nonzero nilpotent elements. A general affine scheme need not be reduced, but an affine *group scheme* is always reduced. If F is a reduced affine scheme of finite type represented by A = **C**[F], one may view A as the **C**–algebra of polynomial functions on an algebraic subset V of some $\mathbf{C}^N$; we then write V(R) for F(R) for any **C**–algebra R . If F is an affine group scheme of finite type, we may take A to be of the form **C**[G] where G is an *algebraic matrix group over* **C** , i.e., an algebraic subgroup of some GL(N, **C**) (even SL(N, **C**)) ; we write G(R) for F(R) and say that G *generates* F . An arbitrary affine group scheme is the inverse limit of algebraic affine group schemes ; and conversely, the inverse limit of affine group schemes is an affine group scheme .

If G is an algebraic matrix group and K is a normal algebraic subgroup, there is a unique structure of an algebraic matrix group for G/K such that $G \longrightarrow G/K$ is a morphism . In particular, a bijective morphism $G \longrightarrow G'$ of algebraic matrix groups is an isomorphism . If $G \longrightarrow G'$ is a morphism of algebraic matrix groups, the image of G is Zariski closed in G' and so is an algebraic matrix group. For an algebraic matrix group $G \subset GL(N, \mathbf{C})$ and an element $x \in G$, x is semisimple (resp. unipotent) if it is mapped into a semisimple (resp. unipotent) element in all (rational) linear representations ; it is enough if this is so in a faithful representation . If all elements of G are unipotent G is called a *unipotent group* ; the corresponding group scheme is also called *unipotent* . An arbitrary affine group scheme is called unipotent if it is the inverse limit of algebraic affine group schemes that are unipotent . If G is a unipotent group scheme, so is any group scheme that is represented by a Hopf subalgebra of **C**[G] .

2.2 Let us consider a unipotent affine group scheme represented by a unipotent subgroup of SL(N, **C**) which we denote by G . If $\mathfrak{g}$ = Lie (G) , the map exp ($\mathfrak{g} \longrightarrow G$) is then an isomorphism of algebraic varieties ; in particular, in the *usual* topology G is always connected and simply connected, and the same is true of all the algebraic subgroups of G . If G' is another unipotent matrix group with $\mathfrak{g}'$ = Lie(G') , we have a *bijection* $\varphi \longrightarrow d\varphi$ of Morph(G, G') with Morph($\mathfrak{g}, \mathfrak{g}'$) such that $\varphi(\exp X) = \exp d\varphi(X)$ for all $X \in \mathfrak{g}$.

PROPOSITION 2.2.1 *Fix* G, G', *and* $\varphi(G \longrightarrow G')$ *and let* $H' \subset G'$ *be an algebraic subgroup with* $H = \varphi^{-1}(H')$. *Then, for any* **C**–*algebra* R ,

$$H(R) = \varphi(R)^{-1}(H'(R)) .$$

PROOF Going over to the corresponding Lie algebras and selecting suitable bases in them we come down to the following situation : $(\varphi_1, \ldots, \varphi_m)$ is a polynomial map of $\mathbf{C}^n$ into $\mathbf{C}^m$ and for suitable integers $p, q \geq 1$,

$$\varphi_i(x_1, \ldots, x_n) = 0,\ 1 \leq i \leq q \iff x_1 = \cdots = x_p = 0$$

We want to prove that this relation is true when $\mathbf{C}$ is replaced by R . But by the Nullstellensatz the above relation is equivalent to

$$\varphi_i^a = \Sigma_j\, p_{ij}\, x_j\,, \quad x_j^b = \Sigma_i\, q_{ji}\, \varphi_i$$

for suitable integers a , $b \geq 1$, and complex polynomials p_{ij} , q_{ij} . We can obviously substitute R-values for the x's in the last relation, and hence in the previous relation also . ◆

COROLLARY 2.2.2 *If* $K = \mathit{ker}(\varphi)$, *then* $K(R) = \mathit{ker}(\varphi(R))$.

PROOF Take $H' = (1)$. ◆

PROPOSITION 2.2.3 *If* φ *is surjective, there is a morphism of the underlying algebraic varieties* $s\ (G' \longrightarrow G)$ *such that* $\varphi \circ s = \mathrm{id}$, *i.e.,* G , *viewed as a fibre space over* G', *has a global section . In particular, if* K *is* $\mathit{ker}(\varphi)$,*the map* $f\ (G' \times K \longrightarrow G\ )$ *given by* $f(x', h) = s(x')h$, *is an isomorphism of algebraic varieties . Moreover* $\varphi(R)$ *is a surjective homomorphism from* $G(R)$ *to* $G'(R)$ *for any* **C**–*algebra* R .

PROOF The last assertion is immediate from the existence of the section s , since $\varphi(R) \circ s(R) = \mathrm{id}(R)$. The result is well known and the proof is a minor variant of the one given in [V] (Theorem 3.18.2, p238). ◆

COROLLARY 2.2.4 *Suppose* G_i $(i = 1,2,3)$ *are unipotent algebraic groups and*

$$G_1 \longrightarrow G_2 \longrightarrow G_3$$

is exact at G_2 . *Then*

$$G_1(R) \longrightarrow G_2(R) \longrightarrow G_3(R)$$

is exact at $G_2(R)$ *for any* **C**–*algebra* R .

PROOF Let K_1 = image(G_1), K_2 = ker ($G_2 \longrightarrow G_3$) , so that $K_1 = K_2 \subset G_2$. By the Proposition $G_1(R) \longrightarrow K_1(R)$ is surjective for all R so that $K_1(R)$ = image $G_1(R)$ for all R . Similarly, by Corollary 2, $K_2(R)$ coincides with ker($G_2(R) \longrightarrow G_3(R)$) for all R. As $K_1 = K_2$, we have $K_1(R) = K_2(R)$ for all R. ♦

COROLLARY 2.2.5 *Let* φ, ψ $(G \longrightarrow H)$ *be two homomorphisms . If* K *is the subgroup* $\{ x \mid \varphi(x) = \psi(x) \}$, *then*

$$K(R) = \{x \mid \varphi_R(x) = \psi_R(x)\}$$

for all R .

PROOF Consider $\theta = (\varphi, \psi)$ mapping G into $H \times H$. If H_1 is the diagonal subgroup of $H \times H$, then $K = \theta^{-1}(H_1)$. The corollary follows from Proposition 1. ♦

2.3 Let X be an arbitrary topological space and R any **C**–algebra . A *sheaf of affine group schemes over* R *on* X is a covariant functor $\mathcal{G}(S \longrightarrow G(S))$ from the category of R-algebras S to the category of sheaves of groups on X such that for any open set $U \subset X$,

$$\mathcal{G}(U) \ \{S \longrightarrow \mathcal{G}(U)(S) := \mathcal{G}(S)(U) = \text{group of sections of } \mathcal{G}(S) \text{ over } U\}$$

is an affine group scheme over R . A sheaf of *unipotent affine group schemes* is a sheaf $\mathcal{G}$ of group schemes such that the group schemes $\mathcal{G}(U)$ are unipotent for all open sets $U \subset X$. If $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$ are sheaves of affine group schemes over R with maps $\mathcal{G}' \longrightarrow \mathcal{G}$, $\mathcal{G} \longrightarrow \mathcal{G}''$, we say that

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

is an *exact sequence* if for each R-algebra S

$$0 \longrightarrow \mathcal{G}'(S) \longrightarrow \mathcal{G}(S) \longrightarrow \mathcal{G}''(S) \longrightarrow 0$$

is an exact sequence of sheaves of groups . We may then view $\mathcal{G}'$ as a *normal subsheaf* of $\mathcal{G}$ and $\mathcal{G}''$ as the *quotient* sheaf $\mathcal{G} / \mathcal{G}'$.

We shall now indicate a simple method of constructing sheaves of unipotent affine group shemes and associated exact sequences. Fix a basis $\mathbb{B}$ for the topology of X that is closed under finite intersections ; for instance, and this is especially important for us, when $X = S^1$ we may take $\mathbb{B}$ to be the set of all open arcs of length $\leq c$, c being a sufficiently small number . We consider the category $\mathbb{A}(\mathbb{B})$ whose objects are sheaves G of groups on X such that for each $U \in \mathbb{B}$, $G(U)$ is a complex *algebraic* unipotent matrix group, the restriction maps $G(U) \longrightarrow G(V)$ (when $V \subset U$) being morphisms of algebraic groups, and whose morphisms are maps $G \longrightarrow G'$ such that $G(U) \longrightarrow G'(U)$ is a morphism of algebraic groups for all $U \in \mathbb{B}$ (we shall say that G is *algebraic on* $\mathbb{B}$) . For any $U \in \mathbb{B}$, we write $G(U)$ for the group scheme over $\mathbf{C}$ defined by $G(U)$. For any $\mathbf{C}$–algebra R, $\{\mathcal{G}(U)(R)\}_{U \in \mathbb{B}}$ is a presheaf of groups; we denote by $\mathcal{G}(R)$ the associated sheaf. Then $R \longrightarrow \mathcal{G}(R)$ is a covariant functor from the category of $\mathbf{C}$–algebras to the category of sheaves of groups. We now have the following proposition.

PROPOSITION 2.3.1 $\mathcal{G}(R \longrightarrow G(R))$ *is a sheaf of unipotent group schemes on* X *; and for any* $\mathbf{C}$*–algebra* R *and* $U \in \mathbb{B}$, $\mathcal{G}(U)(R) \cong \mathcal{G}(R)(U)$. *Moreover* $\mathcal{G}$ *is unchanged if we replace* $\mathbb{B}$ *by a basis* $\mathbb{B}_1 \subset \mathbb{B}$ *that is also closed under finite intersections . Finally* $\mathcal{G}(U)$ *is algebraic for all* U *which are finite unions of sets in* $\mathbb{B}$.

PROOF We shall verify first that for any $\mathbf{C}$–algebra R, $\{\mathcal{G}(U)(R)\}_{U \in \mathbb{B}}$ is already the restriction to $\mathbb{B}$ of a sheaf of groups on X . So we must prove that if U, U_i $(i \in I)$ are in $\mathbb{B}$, $U = \bigcup_i U_i$, and $s_i \in \mathcal{G}(U_i)(R)$ are such that $s_i = s_j$ on $U_i \cap U_j$ for all i and j , then there is a unique $s \in \mathcal{G}(U)(R)$ that restricts to s_i on U_i for all i . Suppose first that I is finite. We then have the usual diagram of complex algebraic groups

$$\text{(E)} \qquad 0 \longrightarrow G(U) \longrightarrow \prod_{i \in I} G(U_i) \rightrightarrows \prod_{i,j \in I} G(U_i \cap U_j)$$

and the corresponding diagram over any $\mathbf{C}$–algebra R

$$(E_R) \quad 0 \longrightarrow G(U)\,(R) \longrightarrow \prod_{i\in I} G(U_i)\,(R) \rightrightarrows \prod_{i,j\in I} G(U_i \cap U_j)\,(R)$$

The required property is equivalent to the exactness of (E_R) at the second step together with its exactness at the third step in the sense that the image of $G(U)(R)$ is the subgroup where the two maps into the third group coincide. By assumption we know that this is true for the first diagram, and so we are through by Corollaries 2.2.2 and 2.2.5 .

Before taking up the case when I is infinite we note that if $\mathbb{B}'$ is the class of sets which are finite unions of sets in $\mathbb{B}$, the above argument shows that there is a unique way to regard the $G(U)$ for $U \in \mathbb{B}'$ as algebraic matrix groups and $R \longrightarrow \mathcal{G}(U)(R)$ as the corresponding group scheme .

Suppose now that I is infinite. For any finite $F \subset I$ let $U_F = \bigcup_{i\in F} U_i$. Then $U_F \in \mathbb{B}'$ for all F, and the algebraic groups $G(U_F)$ form an inverse system with $G(U)$ as their limit as *abstract groups* . We must prove that $\mathcal{G}(U)(R)$ is the inverse limit of the $\mathcal{G}(U_F)(R)$. For this it is enough to prove that if A_F and A are the **C**–algebras representing $\mathcal{G}(U_F)$ and $\mathcal{G}(U)$, then $A = \lim_F A_F$. But if $K_F = \ker(G(U) \longrightarrow G(U_F))$, then (K_F) is a directed family and $\bigcap_F K_F = (1)$, so that $K_{F'} = (1)$ for some F' i.e., $G(U) \cong G(U_{F'})$, which gives $A \cong A_{F'}$.

It remains to show that for any open set $V \subset X$, $R \longrightarrow \mathcal{G}(V)(R)$ is a unipotent affine group scheme. Since $\mathbb{B}'$ is closed under finite unions we can write $V = \bigcup_\alpha U_\alpha$ where the set of indices α is directed and $\alpha < \beta$ implies that $U_\alpha \subset U_\beta$, the U_α being in $\mathbb{B}'$. The sheaf property shows that $\mathcal{G}(V)(R) = \lim_\alpha \mathcal{G}(U_\alpha)(R)$. As the $\mathcal{G}(U_\alpha)$ are unipotent group schemes so is $\mathcal{G}(V)$. The remaining statements are obvious. ♦

PROPOSITION 2.3.2 *Suppose* G_i $(i = 1,2,3)$ *are sheaves in* $\mathbb{A}(\mathbb{B})$ *such that*

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

is exact . Let $\mathcal{G}_i$ *be the sheaf of unipotent group scheme associated to* G_i *as above. Then*

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3 \longrightarrow 0$$

is exact.

PROOF It is a question of showing that for any **C**–algebra R the sequence

$$0 \longrightarrow \mathcal{G}_1(R) \longrightarrow \mathcal{G}_2(R) \longrightarrow \mathcal{G}_3(R) \longrightarrow 0$$

is exact. The exactness at the first and second stages is immediate from the exactness of the sequence $0 \longrightarrow G_1(U) \longrightarrow G_2(U) \longrightarrow G_3(U)$ for all open $U \in \mathbb{B}$ (Corollary 2.2.4) . To prove exactness at the third stage let us fix $x \in X$ and define for any $U \in \mathbb{B}$, $x \in U$ the (algebraic) group $G_{3,2}(U)$ as the image of $G_2(U)$ in $G_3(U)$, and for any $V \subset U$, $V \in \mathbb{B}$, the (algebraic)group $G_3(U,V)$ as the image of $G_3(U)$ in $G_3(V)$. By our assumption, for any $U \in \mathbb{B}$ and any s in $G_3(U)$ we can find a $V \subset U$, $x \in V \in \mathbb{B}$, such that the image of s in $G_3(V)$ lies in $G_{3,2}(V)$. But, as a unipotent algebraic group is the result of successive extensions of (the additive group of) **C** by **C** , it is clear that such a group is the closure of the subgroup generated by a finite set of its elements. So, for given $U \in \mathbb{B}$ we can find $V \subset U$, $x \in V \in \mathbb{B}$ such that $G_3(U,V) \subset G_{3,2}(V)$. But the results of §2.2 now imply the same inclusion for the corresponding groups of R-points, thus giving the exactness we wanted at the third stage. ◆

In view of this Proposition it is natural to say that a sheaf $\mathcal{G}$ of unipotent group schemes is *algebraic on* $\mathbb{B}$ if for each $U \in \mathbb{B}$, the group scheme $\mathcal{G}(U)$ is algebraic . It is then clear that the assignment $G \longrightarrow \mathcal{G}$ is an equivalence of categories from $\mathbb{A}(\mathbb{B})$ to the category of sheaves of unipotent group schemes that are algebraic on $\mathbb{B}$ that takes exact sequences to exact sequences . We shall also say that G *generates* $\mathcal{G}$.

PROPOSITION 2.3.3 *Let* $\mathcal{F}$, $\mathcal{G}$ *be sheaves of unipotent group schemes on* X *that are algebraic on* $\mathbb{B}$ *with* $\mathcal{G}$ *a normal subsheaf of* $\mathcal{F}$. *Then for any* $U \in \mathbb{B}$ *the functor* $R \longrightarrow H^0(U, \mathcal{G}(R))$ *is representable by affine space . More generally, let* R *be a* **C**–*algebra,* $\gamma \in H^1(X, \mathcal{F}(R))$ *, and let* α *be a cocycle representing* γ. *Then* $\mathcal{G}^{(\alpha)}$ $(S \longrightarrow \mathcal{G}(S)^{(\alpha)})$ *is a sheaf of affine group schemes over* R *; and for any* $U \in \mathbb{B}$ *such that* γ *trivializes on* U, *the functor* $S \longrightarrow H^0(U, \mathcal{G}(S)^{(\alpha)})$ *is representable by affine space over* R.

PROOF For $U \in \mathbb{B}$ the group scheme $\mathcal{G}(U)$ is represented by a unipotent algebraic group $G(U)$. As the underlying variety of $G(U)$ is isomorphic to Lie(G(U)) , it is an affine space . This proves the first statement . For the second statement, fix R , γ, α, and U and let α be associated to the covering (U_i). For any R-algebra S, let $\mathcal{G}^{(\alpha)}(S) = \mathcal{G}(S)^{(\alpha')}$ where α' is the image of α induced by the map R $\longrightarrow$ S. Then it is clear from our discussion in §1.3 that $\mathcal{G}^{(\alpha)}$ is a covariant functor from the category of R–algebras to the category of sheaves of groups on X, and further that for any $U \in \mathbb{B}$ on which γ trivializes, the restriction to U of this functor is naturally isomorphic to the restriction of $\mathcal{G}$ to U. Hence the restriction of $\mathcal{G}^{(\alpha)}$ to U is an affine group scheme over R and the functor S $\longrightarrow$ $H^0(U, \mathcal{G}^{(\alpha)}(S))$ is representable by affine space over R. To complete the proof we must show that $\mathcal{G}^{(\alpha)}(V)$ is an affine group scheme for any open V. Now V is the union of the $V \cap U_i$ and γ is trivial on U_i, so trivial on $V \cap U_i$. The result is thus true for $V \cap U_i$, and the proof for V would follow if we show that the class of open sets V, for which $\mathcal{G}^{(\alpha)}(V)$ is an affine group scheme, is closed under unions. Since inverse limits of affine group schemes are affine group schemes, we are reduced to the case of finite unions and hence to a union of two open sets. If V and W are open sets such that $\mathcal{G}^{(\alpha)}(V)$ and $\mathcal{G}^{(\alpha)}(W)$ are affine group schemes and $U = V \cap W$, $\mathcal{G}^{(\alpha)}(U)$ may be viewed as the fiber product $\mathcal{G}^{(\alpha)}(V) \times_{\mathcal{H}} \mathcal{G}^{(\alpha)}(W)$ where $\mathcal{H} = \mathcal{G}^{(\alpha)}(V \cap W)$, so that $\mathcal{G}^{(\alpha)}(U)$ is an affine group scheme. ◆

If $\mathfrak{g}$ is a nilpotent (finite dimensional) Lie algebra and G is a simply connected complex Lie group with Lie algebra $\mathfrak{g}$, the exponential map is an isomorphism of complex manifolds from $\mathfrak{g}$ to G; and the multiplication law on G, when taken back to $\mathfrak{g}$ by the inverse of this isomorphism, becomes a polynomial map $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$. This shows that we can view G as a complex unipotent group in a natural and unique manner so that the exponential map is an isomorphism of algebraic varieties. Now $\mathfrak{g}$ is known to possess a faithful linear representation by nilpotent endomorphisms of a finite dimensional vector space ([V] p, 237, Cor. 3.17.6) from which it follows that G is an affine algebraic unipotent group over **C.** If we carry over this correspondence between nilpotent Lie algebras and unipotent matrix groups to the sheaves on X we obtain a correspondence between sheaves of unipotent groups and nilpotent Lie algebras. Suppose G is a sheaf of groups from $\mathbb{A}(\mathbb{B})$. For $U \in \mathbb{B}$ let $\mathfrak{g}(U)$ = Lie(G(U)). Then $\{\mathfrak{g}(U)\}_{U \in \mathbb{B}}$ is a presheaf of finite dimensional nilpotent Lie algebras over **C**, and the bijectivity of the exponential map shows at once that it is the restriction to $\mathbb{B}$ of a unique sheaf of Lie algebras over **C**. We write

Lie(G) for this sheaf. If $\mathfrak{g}$ is a sheaf of complex Lie algebras on X such that $\mathfrak{g}(U)$ is a finite dimensional nilpotent Lie algebra for all $U \in \mathbb{B}$, then there is a unique (upto isomorphism) sheaf G from $\mathbb{A}(\mathbb{B})$ such that $\mathfrak{g} = \mathrm{Lie}(G)$. The assignment $G \longrightarrow \mathrm{Lie}(G)$ is a functor which establishes an equivalence of categories from the category $\mathbb{A}(\mathbb{B})$ to the category of sheaves of complex Lie algebras on X whose sections on any $U \in \mathbb{B}$ form a finite dimensional nilpotent Lie algebra. If

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

is an exact sequence from $\mathbb{A}(\mathbb{B})$, it is then clear that

$$0 \longrightarrow \mathrm{Lie}(G') \longrightarrow \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G'') \longrightarrow 0$$

is exact and vice versa.

Suppose that G and G' are sheaves from $\mathbb{A}(\mathbb{B})$ with an imbedding $G' \hookrightarrow G$ that allows us to identify G' as a normal subsheaf of G. Let G" be the quotient sheaf G/G'. It is often useful to know when G" belongs to $\mathbb{A}(\mathbb{B})$. If $\mathcal{G}'$ and $\mathcal{G}$ are the sheaves of unipotent group schemes corresponding to G' and G respectively, we can define a covariant functor $\mathcal{G}''$ from the category of **C**–algebras to the category of sheaves of groups on X such that

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

is exact. It is then clear from Proposition 2.3.2 that there is at most one way to regard G" as a sheaf from $\mathbb{A}(\mathbb{B})$, and that this is possible if and only if for each $U \in \mathbb{B}$, $\mathcal{G}''(U)$ is an affine algebraic group scheme. The following result is often useful to decide this. Write $\mathfrak{g}'' = \mathrm{Lie}(G)/\mathrm{Lie}(G')$, so that

$$0 \longrightarrow \mathrm{Lie}(G') \longrightarrow \mathrm{Lie}(G) \longrightarrow \mathfrak{g}'' \longrightarrow 0$$

is an exact sequence.

PROPOSITION 2.3.4 G" *belongs to* $\mathbb{A}(\mathbb{B})$ *if and only if*

$$\dim H^0(U, \mathfrak{g}'') < \infty \qquad U \in \mathbb{B},$$

and this is automatically satisfied if

$$\dim H^1(U, \mathrm{Lie}(G')) < \infty.$$

PROOF The first assertion is a rephrasing of the discussion given just now. The exact sequence

$$H^0(U, \mathrm{Lie}(G)) \longrightarrow H^0(U, \mathfrak{g}'') \longrightarrow H^1(U, \mathrm{Lie}(G'))$$

leads to the second assertion immediately. ♦

COROLLARY 2.3.5 *If* G^0 *is a normal subsheaf of* G *from* $\mathbb{A}(\mathbb{B})$ *such that* $G' \subset G^0 \subset G$ *and* $G/G' \in \mathbb{A}(\mathbb{B})$, *then* G^0/G' *is also in* $\mathbb{A}(\mathbb{B})$.

PROOF We have $\mathrm{Lie}(G') \subset \mathrm{Lie}(G^0) \subset \mathrm{Lie}(G)$, and $\mathrm{Lie}(G^0)/\mathrm{Lie}(G')$ is a subsheaf of $\mathrm{Lie}(G)/\mathrm{Lie}(G') = \mathfrak{g}''$. So $\dim H^0(U, \mathrm{Lie}(G^0)/\mathrm{Lie}(G')) < \infty$. ♦

2.4 Given a sheaf $\mathcal{G}$ of affine group schemes over a **C**–algebra R defined on X , the assignments

$$H^i(X, \mathcal{G}) : S \longrightarrow H^i(X, \mathcal{G}(S)), \qquad i = 0,1$$

are covariant functors with values in the category of groups for i = 0, and in the category of pointed sets for i = 1. For i = 0 the functor is the group scheme $S \longrightarrow \mathcal{G}(X)(S)$. Clearly it makes sense to ask whether the functor $H^1(X, \mathcal{G})$ is represented by an affine scheme over R . In particular, when $\mathcal{G}$ is a sheaf of unipotent group schemes , the above Proposition suggests that it is natural to ask whether $H^1(X, \mathcal{G})$ is represented by an affine space, i.e., by the **C**–algebra $\mathbf{C}[T_1,...T_d]$ for some indeterminates $T_1,...,T_d$. We shall now formulate a result that asserts that this is the case for certain types of sheaves that arise in the theory of meromorphic differential equations . We shall give the proof of the theorem in the next paragraph.

So we suppose that $X = S^1$ and write $\mathbb{A}$ for the category of sheaves of unipotent group schemes on S^1 that are algebraic over the collection of all open arcs. Actually it is necessary to work with the unramified finite coverings $S^{1,d}$ of S^1 with the covering maps $f_d : e^{2i\pi\theta} \longrightarrow e^{2i\pi d\theta}$. Let $\mathbb{A}^d$ be the counterpart of $\mathbb{A}$ on $S^{1,d}$; it is clear that $f_d{}^*\mathbb{A} \subset \mathbb{A}^d$. On S^1 we use the usual arc length $|\,.\,|$, but on the $S^{1,d}$ we use the arc length normalized so that

f_d is an isometry (thus $|S^{1,d}| = 2\pi d$). Let us call a sheaf $\mathcal{G}$ from the category $\mathbb{A}$ *elementary* if it has the following property :

(𝔈) *there is an integer* $d \geq 1$, *a number* $a = a(\mathcal{G}), 0 < a(\mathcal{G}) < 2d\pi$ *and a finite subset* $\Phi = \Phi(\mathcal{G}) \subset S^{1,d}$ *such that, for any open arc* $I \subset S^{1,d}$ *whose length is* a *and whose endpoints are not in* Φ, *and any* **C**– *algebra* R, *we have*

$$H^0(I, f_d{}^*\mathcal{G}(R)) = 0 \text{ , } H^1(I, f_d{}^*\mathcal{G}(R)) = 0 \text{ .}$$

If $\mathcal{G}$ is elementary with d, a, Φ as above and if d' is an integer divisible by d, it is clear that the pullback of $\mathcal{G}$ to $S^{1,d'}$ also has the same property for the *same* a and the finite set Φ' which is the preimage of Φ in $S^{1,d'}$. Further if d" divides d and a is actually $\leq 2d''\pi$, then the decisive property (𝔈) is already satisfied on $S^{1,d''}$. Indeed, in this case, the covering map $S^{1,d} \longrightarrow S^{1,d''}$ is a *homeomorphism* on arcs of length $\leq 2d''\pi$ and we have (𝔈) with Φ'' as the image in $S^{1,d''}$ of Φ. In particular, the case when $a \leq 2\pi$ is especially interesting; for then we can take $d = 1$, i.e., $\mathcal{G}$ *already satisfies the defining condition* (𝔈) *on* S^1. We refer to such a $\mathcal{G}$ as *unramified .*

THEOREM 2.4.1 *Suppose* $\mathcal{U}$ *is a sheaf of unipotent group schemes on* S^1 *from the category* $\mathbb{A}$ *and that there is a finite filtration*

$$\mathcal{U} = \mathcal{U}^{(0)} \supset \mathcal{U}^{(1)} \supset \ldots \supset ..$$

of normal subsheaves of affine unipotent group schemes from $\mathbb{A}$ *such that the quotients* $\mathcal{U}/\mathcal{U}^{(i)}$ *are from* $\mathcal{U}$ *and*

(i) *the quotients* $\mathcal{U}^{(i)}/\mathcal{U}^{(i+1)} = \mathcal{V}^{(i)}$ *are from* $\mathbb{A}$ *and are all elementary*

(ii) *the arc lengths* $a_i = a(\mathcal{V}^{(i)})$ *are decreasing* , i.e.,

$$a_0 \geq \cdots a_i \geq a_{i+1} \geq \cdots \cdot$$

Then

(a) *for any* **C**–*algebra* R *and any* $\alpha \in H^1(S^1, \mathcal{U}(R))$,

$$H^0(S^1, \mathcal{U}(R)^{(\alpha)}) = 0$$

(b) *for any* $\lambda \in H^1(S^1, \mathcal{U}(\mathbf{C}))$, *the twisted sheaves of group schemes* $\mathcal{U}^{(i)(\lambda)}$ *form a filtration for* $\mathcal{U}^{(\lambda)}$ *that has the same properties as* $(\mathcal{U}^{(i)})$; *in particular the* $\mathcal{V}^{(i)(\lambda)}$ *are elementary for the same arc lengths* a_i

(c) *the functor*

$$H^1(S^1, \mathcal{G}) : R \longrightarrow H^1(S^1, \mathcal{G}(R))$$

is representable by affine space .

. The proof of the theorem divides itself naturally into two parts : the proof when $\mathcal{U}$ itself is elementary, and the inductive step of going from the sheaf $\mathcal{U}/\mathcal{U}^{(i)}$ to the sheaf $\mathcal{U}/\mathcal{U}^{(i+1)}$ using the exact sequence

$$(E_i) \qquad 0 \longrightarrow \mathcal{V}^{(i)} \longrightarrow \mathcal{U}/\mathcal{U}^{(i+1)} \longrightarrow \mathcal{U}/\mathcal{U}^{(i)} \longrightarrow 0$$

Before taking up the proof we shall obtain a criterion that will be very useful to us in verifying that certain sheaves that occur are elementary.

PROPOSITION 2.4.2 *Suppose that* $\mathcal{G}$ *is a sheaf of unipotent group schemes from* $\mathbb{A}$ *(generated by* G*)and that* $f_d{}^*G = G^*$ *satisfies the following condition : there is a finite set* $\Phi \subset S^{1,d}$ *such that if* $I \subset I'$ *are open arcs in* $S^{1,d}$ *with* $(I' \setminus I) \cap \Phi = \varnothing$, *the restriction map* $G^*(I') \longrightarrow G^*(I)$ *is an isomorphism. Then in order that* $\mathcal{G}$ *be elementary it is sufficient that the condition* **(**$\mathcal{E}$**)** *be satisfied when* R = **C** .

PROOF Replacing $S^{1,d}$ by S^1 we may assume that $d = 1$. Fix an open arc I of length a and endpoints not in Φ . By assumption $G(I) = 0$ so that the groups $\mathcal{G}(I)(R) = 0$ for all **C**–algebras R . We are thus left with the case of H^1.

Consider first a *finite* covering $\mathfrak{A} = (U_i)_{1 \le i \le N}$ of I by open arcs U_i , with distinct triple intersections empty. For any **C**–algebra R, the set of cocycles for

$\mathcal{G}(R)$ from $\mathfrak{a}$ may be identified with $\prod_{i<j} \mathcal{G}(U_i \cap U_j)(R)$ which is the set of R–points of the **C**–variety $\prod_{i<j} \mathcal{G}(U_i \cap U_j)$ whose underlying set is the set of cocycles for the sheaf G. Since G is essentially $\mathcal{G}(\mathbf{C})$ our assumption implies that $H^0(I, G) = 0$ and $H^1(I, G) = 0$, and so the map

$$t : \prod_i G(U_i) \longrightarrow \prod_{i<j} G(U_i \cap U_j)$$

given by

$$t((c_i)) = (c_i c_j^{-1})$$

is *bijective* . Indeed, it is surjective because the 1-cohomology for the arc is 0; if $c_i c_j^{-1} = c'_i c'_j{}^{-1}$, then $c'_i{}^{-1} c_i = c'_j{}^{-1} c_j$ on $U_i \cap U_j$ so that the $c'_i{}^{-1} c_i$ extend to a section of G on S^1, which must be trivial by the assumption of vanishing of the 0^{th} cohomology, giving $c'_i = c_i$ for all i. Now the varieties G(J) are affine spaces for any arc J . Hence t is a bijective morphism of affine spaces , and so must be an *isomorphism* of varieties by *Zariski's main theorem* (see[Di]) . So the corresponding maps t (R) of R–points must be bijective for all **C**–algebras R . Since t (R) is obviously given by the same formula, we can conclude that $H^1(\mathfrak{a}, \mathcal{G}(R)) = 0$ for all R . As the coverings such as $\mathfrak{a}$ are easily seen to be cofinal in the collection of all *finite* coverings of I by open arcs, we have proved that $H^1{}_{\Gamma}(X, \mathcal{G}(R)) = 0$ where Γ is the collection of all finite open coverings of I by open arcs . It thus remains only to extend this conclusion to the full cohomology of I .

Before taking up arbitrary coverings we shall extend the preceding result and show that for an open arc $J \subset I$ with $(I \setminus J) \cap \Phi = \varnothing$,

$$H^0(J, \mathcal{G}(R)) = 0 , \quad H^1{}_{\Gamma}(J, \mathcal{G}(R)) = 0$$

for any **C**–algebra R. In fact, by our assumption, if $K \subset K'$ are open arcs with $(K' \setminus K) \cap \Phi = \varnothing$, the isomorphism $G(K') \cong G(K)$ implies the isomorphisms $\mathcal{G}(K')(R) \cong \mathcal{G}(K)(R)$ for all R ; thus any section of $\mathcal{G}(R)$ on J extends to a section on I and so is zero. To prove that $H^1{}_{\Gamma}$ is 0, let $I = (u, v)$, $J = (u', v')$, and let $(U_i)_{1 \le i \le N}$ be a finite covering of J by open arcs U_i, the enumeration being such that $U_1 = (u', u'')$ and $U_N = (v'', v')$. Let $\mathfrak{T}$ be a $\mathcal{G}(R)$-torsor on J which is trivial on the U_i. We glue the trivial torsor on (u, u'') (resp. (v'', v)) to $\mathfrak{T}$ along U_1 (resp. U_N) by identifying the identity section on U_1 (resp.U_N) with some section of $\mathfrak{T}$ on U_1 (resp. U_N) . The resulting torsor on I is trivial because of our earlier result that $H^1{}_\Gamma(I, \mathcal{G}(R)) = 0$. Thus $\mathfrak{T}$ is trivial on J.

Fix the **C**–algebra R and let $\mathfrak{T}$ be a torsor for the sheaf $\mathcal{G}(R)$ on I. Let (V_j) be an open covering of I such that $\mathfrak{T}$ is trivial on V_j for all j. We wish to prove that $\mathfrak{T}$ is trivial. We may assume that the covering is countable and that the V_j are open arcs. We write I as the union of an increasing sequence of open arcs J_n such that $I \setminus J_1$ does not meet Φ, and go over to a subcovering (U_i) such that for some increasing sequence of integers (k_n), the U_i $(1 \le i \le k_n)$ form a covering of the closure of J_n . If I_n is the union of the U_i , $1 \le i \le k_n$, the I_n are increasing and we know from the previous result that the restriction of $\mathfrak{T}$ to I_n is *uniquely trivializable* ; let t_n be its unique section on I. The uniqueness implies that for $m > n$ t_m restricts to t_n . Hence the t_n build up to a section of $\mathfrak{T}$ on I, proving that $\mathfrak{T}$ is trivial. ◆

2.5 We shall now prove Theorem 2.4.1. We need a few lemmas.

LEMMA 2.5.1 *If* $\mathcal{G}$ *is a sheaf of unipotent group schemes defined on* S^1 *from the category* $\mathbb{A}$, *and if* $\mathcal{G}$ *is elementary, then* $H^1(S^1, \mathcal{G})$ *is representable by affine space* .

PROOF The definition of an elementary sheaf involves the covering space $S^{1,d}$, a number $a > 0$, and a finite subset $\Phi \subset S^{1,d}$. There are two cases to consider according as $\mathcal{G}$ is ramified or not, i.e., according as $a > 2\pi$ or $a \le 2\pi$. *Case* 1 : $a \le 2\pi$. We may assume that $d = 1$ so that $\mathcal{G}$ itself has the property:

$$(*) \qquad H^0(I, \mathcal{G}(R)) = 0, \quad H^1(I, \mathcal{G}(R)) = 0 \text{ for all } \mathbf{C}\text{–algebras } R,$$

for all open arcs I of length a whose endpoints are not in Φ. Fix a finite covering $\mathfrak{U} = (U_i)_{1 \le i \le N}$ of S^1 by open arcs of length a with endpoints not in Φ and with all triple intersections empty . It is then immediate from $(*)$ that $C(\mathfrak{U}, \mathcal{G}(R)) = 0$ and hence that $Z(\mathfrak{U}, \mathcal{G}(R)) \longrightarrow H^1(S^1, \mathcal{G}(R))$ is a bijective map. Since

$$Z(\mathfrak{U}, \mathcal{G}(R)) \cong \prod_{i<j} \mathcal{G}(U_i \cap U_j)(R)$$

and since $R \longrightarrow \mathcal{G}(U)(R)$ is represented by affine space for any open arc U by Proposition 2.3.3, we are through . *Case* 2 : $a > 2\pi$. If I is an open arc of length a on $S^{1,d}$ with endpoints not in Φ, the map $f_d : I \longrightarrow S^1$ is surjective. By Proposition 1.5.1 there is an arc J of $S^{1,d}$ such that we have a natural isomorphism of functors

$$H^1(S^1, \mathcal{G}) \cong H^0(J, f_d{}^*\mathcal{G}) .$$

As $R \longrightarrow H^0(J, f_d{}^*\mathcal{G}(R))$ is representable by affine space by Proposition 2.3.3, we are through. ♦

LEMMA 2.5.2 *Let assumptions be as in the previous lemma, but assume now in addition that* $\mathcal{G}$ *is a normal subsheaf of a sheaf* $\mathcal{F}$ *of unipotent group schemes from* $\mathbb{A}$. *Let* R *be a* $\mathbf{C}$*–algebra,* $\gamma \in H^1(S^1, \mathcal{F}(R))$, *and* α *a cocycle representing* γ . *Suppose that* $f_d{}^*\gamma$ *trivializes when restricted to any open arc of* $S^{1,d}$ *of length* a *whose endpoints are not in* Φ . *Then the functor* $S \longrightarrow H^1(S^1, \mathcal{G}(S)^{(\alpha)})$ *on* R*–algebras is representable by affine space over* R .

PROOF If β is the pull back of α then the pull back of $\mathcal{G}(S)^{(\alpha)}$ is the twisted sheaf $(f_d{}^*\mathcal{G}(S))^{(\beta)}$. Since β trivializes on arcs of length a whose endpoints are not in Φ, this twisted sheaf is isomorphic to the untwisted sheaf when restricted to such arcs . Hence we have

$$(*) \qquad H^0(I, f_d{}^*(\mathcal{G}(S)^{(\beta)})) = 0 , \quad H^1(I, f_d{}^*(\mathcal{G}(S)^{(\beta)})) = 0$$

for all such arcs I . As before we distinguish between the two cases $a > 2\pi$ and $a \le 2\pi$. If $a > 2\pi$, Proposition 1.5.1 applies and so we can select an arc $J \subset I$, with J of length $a - 2\pi$ and I as above of length a such that $S \longrightarrow H^1(S^1, \mathcal{G}(S)^{(\alpha)})$ and $S \longrightarrow H^0(J, (f_d{}^*\mathcal{G}(S))^{(\beta)})$ are naturally isomorphic functors . But β trivializes on I, hence on J, and hence the functor $S \longrightarrow H^0(J, (f_d{}^*\mathcal{G}(S))^{(\beta)})$ is representable by affine space over R by Proposition 2.3.3 . In the other case, when $a \le 2\pi$, the condition $(*)$ descends to S^1 itself; in particular, α itself trivializes on arcs of length a whose endpoints are not in a suitable finite set $\Phi' \subset S^1$. So, if we take a covering $\mathfrak{U} = (U_i)_{1 \le i \le N}$ of S^1 by open arcs of length a with endpoints not in Φ', we have

$$H^1(S^1, \mathcal{G}(S)^{(\alpha)}) \cong \prod_{i<j} \mathcal{G}(S)^{(\alpha)}(U_i \cap U_j) \cong \prod_{i<j} \mathcal{G}(S)(U_i \cap U_j),$$

the second isomorphism arising because of the fact that α trivializes on all the $U_i \cap U_j$. Proposition 2.3.3 now implies that these functors are representable by affine space over R. ◆

LEMMA 2.5.3 *Let* F, G *be two functors from the category of* **C**–*algebras to the category of sets. Suppose* $u : F \longrightarrow G$ *is a natural transformation and that the following conditions are staisfied*:

(i) G *is representable by affine space*

(ii) *For any* **C**–*algebra* R *and* $g \in G(R)$, *the functor*

$$u_g\ (S \longrightarrow u_S^{-1}(g) = \textit{fibre above } g),$$

on the category of R–*algebras ,is representable by affine space over* R *.Then* F *itself is representable by affine space .*

PROOF Let $A = \mathbf{C}[T_1, \ldots, T_f]$ represent G so that $G(R) = Hom_{\mathbf{C}}(A, R)$. We now take in condition (ii) above $R = A$ and $g \in Hom_{\mathbf{C}}(A, A)$ as the identity map, and obtain $A' = A[S_1,\ldots,S_h] = \mathbf{C}[T_1,\ldots,T_f, S_1,\ldots,S_h]$ representing u_g. Write $F'(R) = Hom_{\mathbf{C}}(A', R)$ for any **C**–algebra R . We shall prove now that $F' \cong F$. For this it is a question of proving that for any $g \in G(R)$, $F'_g(R) \cong F_g(R)$, the suffixes referring to the fibres . But from the definition of A' it is clear that for any $g \in G(R)$, as R is an A–algebra via $g(A \longrightarrow R)$, we have,

$$F_g(R) \cong Hom_A(A[S_1,\ldots,S_h], R) \cong F'_g(R),$$

as the middle term consists of all homomorphisms of $\mathbf{C}[T_1,\ldots T_f][S_1,\ldots,S_h]$ into R that restrict to g on $\mathbf{C}[T_1,\ldots T_f]$. ◆

We now begin the proof of the Theorem proper. There is no loss of generality in assuming that the condition $(\mathfrak{E})$ of being elementary refers to the same covering space $S^{1,d}$ for all the $\mathcal{V}^{(i)}$. Let us also write f instead of f_d for the map $S^{1,d} \longrightarrow S^1$. Thus we assume that there is a finite set $\Phi \subset S^{1,d}$ such that for any i , any **C**–algebra R, and any open arc I of length a_i whose endpoints are not in Φ, we have,

$$H^0(I, f^*\mathcal{V}^{(i)}(R)) = 0, \quad H^1(I, f^*\mathcal{V}^{(i)}(R)) = 0 .$$

LEMMA 2.5.4 *We have, for all* i, *and all* **C**–*algebras* R,

$$H^0(S^1, \mathcal{V}^{(i)}(R)) = 0, \quad H^0(S^1, (\mathcal{U}/\mathcal{U}^{(i)})(R)) = 0 .$$

PROOF Since $\mathcal{V}^{(i)}$ is elementary, the first statement is immediate from Proposition 1.5.1. For the second we use induction on i. If $i = 1$, we are through since $\mathcal{U}/\mathcal{U}^{(1)} = \mathcal{V}^{(0)}$. If the result is true for $\mathcal{U}/\mathcal{U}^{(i)}$, we get from **($E_i$)** the exact sequence

$$H^0(S^1, \mathcal{V}^{(i)}(R)) \longrightarrow H^0(S^1, (\mathcal{U}/\mathcal{U}^{(i+1)})(R)) \longrightarrow H^0(S^1, (\mathcal{U}/\mathcal{U}^{(i)})(R)) .$$

Since the extreme terms are zero, the middle term is also zero. ◆

LEMMA 2.5.5 *For all* $i \geq 1$, *all* **C**–*algebras* R, *and for any open arc* I *of* $S^{1,d}$ *of length* a_{i-1} *whose endpoints are not in* Φ, *the restriction map*

$$H^1(S^{1,d}, f^*(\mathcal{U}/\mathcal{U}^{(i)})(R)) \longrightarrow H^1(I, f^*\mathcal{U}/\mathcal{U}^{(i)}(R))$$

is the zero map.

PROOF For $i = 1$ $\mathcal{U}/\mathcal{U}^{(1)} = \mathcal{V}^{(0)}$ is elementary and the lemma is immediate from the definition. For $i > 1$ we shall use induction on i. Assume the lemma for $(\mathcal{U}/\mathcal{U}^{(i)})(R)$ and consider the exact sequence

$$0 \longrightarrow f^*\mathcal{V}^{(i)}(R) \longrightarrow f^*\mathcal{U}/\mathcal{U}^{(i+1)}(R) \longrightarrow f^*\mathcal{U}/\mathcal{U}^{(i)}(R) \longrightarrow 0 .$$

If I is an arc of length a_i with ends not in Φ, and $\beta \in H^1(S^{1,d}, f^*\mathcal{U}/\mathcal{U}^{(i+1)}(R))$, the image of β in $H^1(S^{1,d}, f^*(\mathcal{U}/\mathcal{U}^{(i)}(R))$ restricts to zero on open arcs of length a_{i-1} whose endpoints are not on Φ, and hence also on such arcs of length $a_i \leq a_{i-1}$. So, if I is such an open arc of length a_i, the exact sequence

$$0 = H^1(I, f^*\mathcal{V}^{(i)}(R)) \longrightarrow H^1(I, f^*\mathcal{U}/\mathcal{U}^{(i+1)}(R)) \longrightarrow H^1(I, f^*\mathcal{U}/\mathcal{U}^{(i)}(R))$$

shows that β must restrict to zero on I. ◆

LEMMA 2.5.6 *For any* R *and any* $\beta \in H^1(S^{1,d}, f^*\mathcal{U}/\mathcal{U}^{(i)}(R))$, *we have*

$$H^0(S^{1,d}, (f^*\mathcal{U}/\mathcal{U}^{(i)}(R))^{(\beta)}) = 0 .$$

In particular, for any $\alpha \in H^1(S^1, \mathcal{U}/\mathcal{U}^{(i)}(R)^{(\alpha)})$,

$$H^0(S^1, \mathcal{U}/\mathcal{U}^{(i)}(R)^{(\alpha)}) = 0 .$$

PROOF We use induction on i. To start the induction we must prove that $H^0(S^{1,d}, (f^*\mathcal{V}^{(0)}(R))^{(\beta)}) = 0$. It is obviously enough to do this with $S^{1,d}$ replaced by an open arc I of length a_0 whose endpoints are not in Φ. But for such an arc I, β trivializes on I, so that the twisted sheaf is isomorphic to the untwisted one on I; and as the definition of being elementary implies that $H^0(I, f^*\mathcal{V}^{(0)}(R)) = 0$, we are done. Assume now the result for i and consider the exact sequence

$$0 \longrightarrow f^*\mathcal{V}^{(i)}(R) \longrightarrow f^*\mathcal{U}/\mathcal{U}^{(i+1)}(R) \longrightarrow f^*\mathcal{U}/\mathcal{U}^{(i)}(R) \longrightarrow 0 .$$

Let β be an element of $H^1(S^{1,d}, f^*\mathcal{U}/\mathcal{U}^{(i+1)}(R))$ with image α. Then we have the exact sequence of twisted sheaves

$$(*) \qquad 0 \longrightarrow f^*\mathcal{V}^{(i)}(R)^{(\beta)} \longrightarrow (f^*\mathcal{U}/\mathcal{U}^{(i+1)}(R))^{(\beta)} \longrightarrow (f^*\mathcal{U}/\mathcal{U}^{(i)}(R))^{(\alpha)} \longrightarrow 0$$

To prove the result for $i+1$ we must show that $H^0(S^{1,d}, (f^*\mathcal{V}^{(i)}(R))^{(\beta)}) = 0$. But by Lemma 2.5.5, β trivializes on arcs of length a_i whose ends are not in Φ. So the argument given above for $\mathcal{V}^0$ goes through without any change whatsoever. ◆

LEMMA 2.5.7 *Fix a* **C***–algebra* R, *and let* β *be a cocycle for* $\mathcal{U}/\mathcal{U}^{(i+1)}(R)$. *Then the functor*

$$S \longrightarrow H^1(S^1, \mathcal{V}^{(i)}(S)^{(\beta)})$$

on R*–algebras is representable by affine space over* R.

PROOF In view of Lemma 2.5.2 it is enough to verify that $f^*\beta$ trivializes on open arcs of length a_i whose endpoints are not in Φ. But this is precisely what is proved in Lemma 2.5.5. ◆

Essentially the same arguments yield

LEMMA 2.5.8 *For* β *as in Lemma 2.5.6* $f^*\mathcal{V}^{(i)(\beta)}$ *is isomorphic to* $f^*\mathcal{V}^{(i)}$ *as sheaves of affine group schemes over* R *when restricted to arcs of length* a_i *whose endpoints are not in* Φ. *In particular if* $\lambda \in H^1(S^1, \mathcal{U}(\mathbf{C}))$, $\mathcal{V}^{(i)(\lambda)} \cong \mathcal{U}^{(i)(\lambda)}/\mathcal{U}^{(i+1)(\lambda)}$, *and* $\mathcal{V}^{(i)(\lambda)}$ *is elementary for the same arc length and covering as* $\mathcal{V}^{(i)}$.

PROOF We consider the exact sequence $(*)$ in the proof of Lemma 2.5.6, but restricted to an arc J of length a_i whose endpoints are not in Φ. By Lemma 2.5.5 β and α trivialize on J so that the second and third members of the exact sequence may be replaced by the untwisted sheaves. But then it is clear that $f^*\mathcal{V}^{(i)(\beta)}$ and $f^*\mathcal{V}^{(i)}$ must be isomorphic on J. ♦

PROOF OF THEOREM We have already proved (a) in Lemma 2.5.6 and (b) in Lemma 2.5.8. We prove the representability of the functor $R \longrightarrow H^1(S^1, \mathcal{U}/\mathcal{U}^{(i)}(R))$ by induction on i; as $\mathcal{U} = \mathcal{U}/\mathcal{U}^{(i)}$ for $i >> 0$, this will be enough. For $i = 0$ this is just Lemma 2.5.1. Suppose $i \geq 0$ and that the result is true for i. We consider the exact sequence

$$0 \longrightarrow \mathcal{V}^{(i)}(R) \longrightarrow \mathcal{U}/\mathcal{U}^{(i+1)}(R) \longrightarrow \mathcal{U}/\mathcal{U}^{(i)}(R) \longrightarrow 0$$

as well as its twists

$$0 \longrightarrow \mathcal{V}^{(i)}(R)^{(\beta)} \longrightarrow (\mathcal{U}/\mathcal{U}^{(i+1)}(R))^{(\beta)} \longrightarrow (\mathcal{U}/\mathcal{U}^{(i)}(R))^{(\alpha)} \longrightarrow 0$$

where α is a cocycle for $\mathcal{U}/\mathcal{U}^{(i)}(R)$ and β is the cocycle for $\mathcal{U}/\mathcal{U}^{(i+1)}(R)$ that maps into α. By Lemma 2.5.6 the sheaf $\mathcal{U}/\mathcal{U}^{(i)}(R)^{(\alpha)}$ has only the zero section on S^1, and hence Proposition 1.4.1 is applicable. So if we consider the exact sequence (for R–algebras S)

$$0 \longrightarrow H^1(S^1, \mathcal{V}^{(i)}(S)) \longrightarrow H^1(S^1, \mathcal{U}/\mathcal{U}^{(i+1)}(S))$$
$$\longrightarrow H^1(S^1, \mathcal{U}/\mathcal{U}^{(i)}(S)) \longrightarrow 0,$$

the fibre above α is canonically isomorphic to $H^1(S^1, (\mathcal{V}^{(i)}(S))^{(\beta)})$. So by Lemma 2.5.7 this is representable by affine space over R. But by the induction hypothesis $R \longrightarrow H^1(S^1, \mathcal{U}/\mathcal{U}^{(i)}(R))$ is representable by affine space and so Lemma 2.5.3 leads us to the same conclusion with i replaced by i+1. ♦

2.6 It remains to compute the dimension of the affine space that represents the cohomology $H^1(S^1,\mathcal{U})$ in Theorem 2.4.1. We shall devote this paragraph to this question.

We begin reviewing briefly how the tangent spaces of a variety are determined in terms of the associated schemes. Let $\mathbf{A}$ be an affine scheme represented by an affine algebra A over $\mathbf{C}$ that is reduced. One introduces the algebra R of dual numbers, $R = \mathbf{C}[\varepsilon]$, $\varepsilon^2 = 0$. We have the maps $\mathbf{A}(R) \longrightarrow \mathbf{A}(\mathbf{C})$ corresponding to the homomorphism $R \longrightarrow \mathbf{C}$ $(a + b\varepsilon \rightarrow a)$, and $\mathbf{A}(\mathbf{C}) \longrightarrow \mathbf{A}(\mathbf{R})$ corresponding to the injection $\mathbf{C} \longrightarrow R$; the composition $\mathbf{A}(\mathbf{C}) \longrightarrow \mathbf{A}(R) \longrightarrow \mathbf{A}(\mathbf{C})$ is the identity. If $p \in \mathbf{A}(\mathbf{C})$, the fibre of $\mathbf{A}(R) \longrightarrow \mathbf{A}(\mathbf{C})$ above p is easily seen to be the complex vector space of all "p-derivations" of A, i.e., $\mathbf{C}$-linear maps $v(A \longrightarrow \mathbf{C})$ such that $v(ab) = p(a)v(b) + p(b)v(a)$ for all $a,b \in A$, and hence may be identified with the Zariski tangent space to the complex variety $\mathbf{A}(\mathbf{C})$ at p . The map $\mathbf{A}(\mathbf{C}) \longrightarrow \mathbf{A}(R)$ is the "zero section" $p \longrightarrow (p,0)$. If $\mathbf{B}$ is another affine scheme represented by the reduced affine $\mathbf{C}$-algebra B, and $A \longrightarrow B$ a homomorphism, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{B}(R) & \longrightarrow & \mathbf{A}(R) \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathbf{C}) & \longrightarrow & \mathbf{A}(\mathbf{C}) \end{array}$$

and the top map is linear on the fibers. If $\mathbf{A}(\mathbf{C})$ is a smooth connected variety of dimension d , in particular if $A = \mathbf{C}[T_1,\dots, T_d]$ where T_j are indeterminates, then the tangent spaces have dimension d everywhere.

Suppose A is the affine algebra of an affine algebraic group scheme G over $\mathbf{C}$ so that A is a Hopf algebra. The commutator map sending (a,b) to $aba^{-1}b^{-1}$ gives rise to a bilinear map $\mathfrak{g}(\mathbf{C}) \times \mathfrak{g}(\mathbf{C}) \longrightarrow \mathfrak{g}(\mathbf{C})$ where $\mathfrak{g}(\mathbf{C})$ is the tangent space to $G(\mathbf{C})$ at 1, and it is a standard fact that this defines the structure of a Lie algebra on $\mathfrak{g}(\mathbf{C})$. On the other hand, $G(R)$ and $G(\mathbf{C})$ are groups, and the maps $G(\mathbf{C}) \longrightarrow G(R) \longrightarrow G(\mathbf{C})$ are homomorphisms, so that the fiber of $G(R)$ above 1 is a group. The group structure is determined by the following lemma.

LEMMA 2.6.1 *Under the identification of the fiber of* $G(R)$ *above* 1 *with* $\mathfrak{g}(\mathbf{C})$ *described above, the group multiplication corresponds to addition in* $\mathfrak{g}(\mathbf{C})$.

PROOF We write $a \longrightarrow 1(a)$ for the homomorphism $A \longrightarrow \mathbf{C}$ defined by the identity element $1 \in G(\mathbf{C})$. The image of 1 in $G(R)$ is $(1,0)$ and so $(1,0)$ is the (multiplicative) identity of $G(R)$. Let Δ $(A \longrightarrow A \otimes A)$ be the comultiplication and write $\Delta(a) = \Sigma\, a_i \otimes b_i$. For any $h \in \mathfrak{g}(\mathbf{C})$, the relation $(1,h)(1,0) = (1,h)$ gives, as $(1,h)(a) = 1(a) + \varepsilon h(a)$, $a \in A$,

$$
\begin{aligned}
1(a) + \varepsilon h(a) &= ((1,h).(1,0))(a) \\
&= \Sigma\, (1,h)(a_i)(1,0)(b_i) \\
&= \Sigma\, (1(a_i) + h(a_i))1(b_i) \\
&= \Sigma\, 1(a_i)1(b_i) + \varepsilon\, \Sigma\, h(a_i)1(b_i)
\end{aligned}
$$

Hence, and after a similar calculation based on $(1,0)(1,h) = (1,h)$

$$
1(a) = \Sigma\, 1(a_i)1(b_i), \quad h(a) = \Sigma h(a_i)1(b_i) = \Sigma\, 1(a_i)h(b_i)
$$

But then

$$
\begin{aligned}
((1,h).(1,h')(a) &= \Sigma(1,h)(a_i)(1,h')(b_i) \\
&= \Sigma\, (1(a_i) + \varepsilon h(a_i))(1(b_i) + \varepsilon h'(b_i)) \\
&= \Sigma\, 1(a_i)1(b_i) + \varepsilon\, \Sigma h(a_i)1(b_i) + \varepsilon \Sigma\, h'(b_i)1(a_i) \\
&= 1(a) + \varepsilon(h + h')(a)
\end{aligned}
$$

This proves the lemma. ♦

We shall now go over to the context of §2.2.. Let X be a topological space, $\mathbb{B}$, a basis for the topology of X, closed under finite intersections; $\mathcal{G}$, a sheaf of unipotent group schemes, algebraic on $\mathbb{B}$. We write $\mathrm{Lie}(\mathcal{G})$ for the associated sheaf of Lie algebras, so that, for $U \in \mathbb{B}$, $\mathrm{Lie}(\mathcal{G})(U) = \mathrm{Lie}(\mathfrak{g}(\mathbf{C})(U))$.

LEMMA 2.6.2 *Assume that* (a) $H^0(X, \mathcal{G}) = 0$ (b) $H^1(X, \mathcal{G})$ *is representable by a reduced affine scheme. Then there is a canonical linear isomor-*

phism of the tangent space to $H^1(X, \mathcal{G}(\mathbf{C}))$ *at the trivial class* 0 *with* $H^1(X, \mathrm{Lie}(\mathcal{G}))$. *In particular, if* $H^1(X, \mathcal{G})$ *is smooth and connected, its dimension is equal to the dimension of the complex vector space* $H^1(X, \mathrm{Lie}(\mathcal{G}))$.

PROOF For any covering $\mathcal{U} = (U_i)$, $U_i \in \mathbb{B}$, we shall set up a linear isomorphism of the fiber above 0 of $H^1(\mathcal{U}: \mathcal{G}(R)) \longrightarrow H^1(\mathcal{U}: \mathcal{G}(\mathbf{C}))$ with $H^1(\mathcal{U}: \mathrm{Lie}(\mathcal{G}))$, in such a way that the isomorphisms are compatible under refinement. We have the commutative diagram:

$$\begin{array}{ccc} Z(\mathcal{U}: \mathcal{G}(R)) & \longrightarrow & z(\mathcal{U}: \mathcal{G}(\mathbf{C})) \\ \downarrow & & \downarrow \\ H^1(\mathcal{U}; \mathcal{G}(R)) & \longrightarrow & H^1(\mathcal{U}: \mathcal{G}(\mathbf{C})) \end{array}$$

The cohomology classes in $H^1(\mathcal{U}: \mathcal{G}(R))$ above 0 are represented by (g_{ij}, h'_{ij}) where $(g_{ij}) \in Z(\mathcal{U}: \mathcal{G}(\mathbf{C}))$, $g_{ij} = c_i c_j^{-1}$ for suitable $c_i \in H^0(U_i, \mathcal{G}(\mathbf{C}))$. Replacing (g_{ij}, h'_{ij}) by $(c_i, 0)^{-1} (g_{ij}, h'_{ij}) (c_j, 0)$, we may assume that the representative cocycles are of the form $(1, h_{ij})$. It now follows from Lemma 1 that (h_{ij}) is a cocycle for the sheaf $\mathrm{Lie}(\mathcal{G})$ associated to the covering $\mathcal{U}$, and that $(h_{ij}) \longrightarrow (1, h_{ij})$ is a linear isomorphism of $Z(\mathcal{U}: \mathrm{Lie}(\mathcal{G}))$ with the fiber of $Z(\mathcal{U}: \mathcal{G}(R))$ above $1 \in Z(\mathcal{U}: \mathcal{G}(\mathbf{C}))$. We now claim that this descends to an isomorphism of $H^1(\mathcal{U}: \mathrm{Lie}(\mathcal{G}))$ with the fiber of $H^1(\mathcal{U}: \mathcal{G}(R))$ above 0. For this it is only necessary to show that $(1, h_{ij})$ and $(1, h'_{ij})$ define the same element of $H^1(\mathcal{U}: \mathcal{G}(R))$ if and only if (h_{ij}) and (h'_{ij}) define the same element of $H^1(\mathcal{U}: \mathrm{Lie}(\mathcal{G}))$. If $h'_{ij} = h_{ij} + k_i - k_j$, $(1, h'_{ij}) = (1, k_i)(1, h_{ij})(1, k_j)^{-1}$ by Lemma 1; on the other hand, if $(1, h'_{ij}) = (c_i, k_i)(1, h_{ij})(c_j, k_j)^{-1}$, we must have $c_i c_j^{-1} = 1$, showing that the c_i define an element of $H^0(X, \mathcal{G}(\mathbf{C}))$; since this group is 0, we must have $c_i = 1$ for all i so that $(1, h'_{ij}) = (1, k_i)(1, h_{ij}).\ (1, k_j)^{-1}$, i.e., $h'_{ij} = h_{ij} + k_i - k_j$, by Lemma 1 again. It is obvious that the maps $h_{ij} \longrightarrow (1, h_{ij})$ are compatible with refinements. ♦

PROPOSITION 2.6.3 *Let* $\mathcal{U}$ *be a sheaf of unipotent group schemes on* S^1 *as in Theorem 2.4.1. Then*

$$\dim H^1(S^1, \mathcal{U}) = \dim H^1(S^1, \mathrm{Lie}(\mathcal{U})) = \Sigma_{i \geq 0} \dim H^1(S^1, \mathcal{V}^{(i)})$$

PROOF $\mathcal{U}$ and $\mathcal{V}^{(i)}$ satisfy the conditions needed for the validity of Lemma 2.6.2 Hence

$$\dim H^1(S^1,\mathcal{U}) = \dim H^1(S^1,\mathrm{Lie}(\mathcal{U})), \quad \dim H^1(S^1,\mathcal{V}^{(i)}) = \dim H^1(S^1, \mathrm{Lie}(\mathcal{V}^{(i)})$$

On the other hand, the exactness of the sequence

$$0 \longrightarrow \mathrm{Lie}(\mathcal{V}^{(i)}) \longrightarrow \mathrm{Lie}(\mathcal{U}/\mathcal{U}^{(i+1)}) \longrightarrow \mathrm{Lie}\ (\mathcal{U}/\mathcal{U}^{(i)}) \longrightarrow 0$$

coupled with the vanishing of $H^0(S^1,\mathrm{Lie}(\mathcal{U}/\mathcal{U}^{(i)}))$ (which follows from the vanishing of $H^0(S^1,\mathcal{U}/\mathcal{U}^{(i)}))$ implies the exactness of

$$0 \longrightarrow H^1(S^1,\mathrm{Lie}(\mathcal{V}^{(i)}) \longrightarrow H^1(S^1,\mathrm{Lie}(\mathcal{U}/\mathcal{U}^{(i+1)}))$$

$$\longrightarrow H^1(S^1,\mathrm{Lie}(\mathcal{U}/\mathcal{U}^{(i)})) \longrightarrow 0$$

Hence

$$\dim H^1(S^1,\mathrm{Lie}(\mathcal{U}/\mathcal{U}^{(i+1)})) = \dim H^1(S^1,\mathrm{Lie}(\mathcal{U}/\mathcal{U}^{(i)})) + \dim H^1(S^1,\mathrm{Lie}(\mathcal{V}^{(i)}))$$

which leads to

$$\dim H^1(S^1, \mathrm{Lie}(\mathcal{U})) = \Sigma_{i\geq 0} \dim H^1(S^1, \mathrm{Lie}\ (\mathcal{V}^{(i)})$$

The proposition is now immediate. ◆

2.7 Let $X = S^1$ and let $\mathcal{U}$ be a sheaf of unipotent schemes on X satisfying the conditions of Theorem 2.4.1.

DEFINITION A covering $\mathcal{a} = (U_i)$ of S^1 such that $H^1(\mathcal{a} : \mathcal{U}(R)) = H^1(\mathcal{U}(R))$ for all **C**–algebras R is called a *good covering.*

Let us put

$$a(\mathcal{U}) = \min_{i \geq 0} a_i$$

$$\Phi(\mathcal{U}) = \bigcup_{i \geq 0} \Phi(\mathcal{V}^{(i)})$$

PROPOSITION 2.7.1 *Let* $\mathcal{A} = (U_\alpha)$ *be any finite covering such that* (a) *the* (U_α) *are open arcs of length* $\leq a(\mathcal{U})$; (b) *if* U_α *has length equal to* $a(\mathcal{U})$, *its end points are not in* $f[\Phi]$. *Then* $\mathcal{A}$ *is a good covering .*

PROOF We prove this for $\mathcal{U}/\mathcal{U}^{(i)}$ in place of $\mathcal{U}$, $i \geq p$. This is enough since $\mathcal{U}^{(i)} = 0$ for $i >> 0$. Clearly

$$a(\mathcal{U}/\mathcal{U}^{(i)}) = a_{i-1}, \quad \Phi(\mathcal{U}/\mathcal{U}^{(i)}) = \bigcup_{0\leq j\leq i-1} \Phi(\mathcal{V}^{(j)})$$

In view of corollary 1.2.4 it is sufficient to show that the map

$$H^1(S^1, \mathcal{U}/\mathcal{U}^{(i)})(R)) \longrightarrow H^1(U_\alpha, \mathcal{U}/\mathcal{U}^{(i)}(R))$$

is the zero map for any α . It is enough to prove this for the pull back sheaves on $S^{1,d}$ and for an arc V_α which maps homeorphically under f onto U_α. Since the ends of V_α are not in $\Phi(\mathcal{U}/\mathcal{U}^{(i)})$ we can enlarge V_α to an open arc I of length a_{i-1} whose ends are not in $\Phi(\mathcal{U}/\mathcal{U}^{(i)})$. The assertion to be proved now follows from Lemma 2.5.5. ◆

Let us now fix a good finite covering $\mathcal{A} = (U_i)$, each U_i being a finite union of arcs. Let

$$C(\mathcal{A} : \mathcal{U}) = C(\mathcal{U}) : R \longrightarrow C(\mathcal{A} : \mathcal{U}(R)) = C(\mathcal{U}(R))$$

$$Z(\mathcal{A} : \mathcal{U}) = Z(\mathcal{U}) : R \longrightarrow Z(\mathcal{A} : \mathcal{U}(R)) = Z(\mathcal{U}(R))$$

have their usual meanings. $C(\mathcal{U})$ is a unipotent algebraic group scheme and $Z(\mathcal{U})$ is an affine scheme. The latter is defined by the equaitons

$$(*) \qquad g_{ii} = 1, \ g_{ij}\, g_{ji} = 1, \ g_{ij}\, g_{jk}\, g_{ki} = 1.$$

PROPOSITION 2.7.2 $C(\mathcal{U}(R))$ *acts freely on* $Z(\mathcal{U}(R))$ *for all* R .

PROOF By Theorem 2.4.1, $H^0(\mathcal{U}(R)^{(\alpha)}) = 0$ for all $\alpha \in Z(\mathcal{U}(R))$. The result now follows from Proposition 1.3.1. ◆

LEMMA 2.7.3 *Let* E *and* F *be affine schemes over* R *and* φ (E ---→ F) *a natural map. If* $\varphi(S)(E(S) \longrightarrow F(S))$ *is surjective for all* R*–algebras* S , *there is a natural map* ψ (F ⟶ E) *such that* $\varphi \circ \psi = \mathrm{id}_F$.

PROOF Let A (resp. B) be the R–algebra that represents E (resp. F). By a standard result ([Wa] p. 6) we know that there is an R–algebra map f $(B \longrightarrow A)$ that gives rise φ . It is a question of constructing an R–algebra map g $(A \longrightarrow B)$ such that $g \circ f = id_B$. Since $\varphi(B)$ $(E(B) \longrightarrow F(B))$ is surjective, we can find $g \in E(B)$ such that $\varphi(B)(g) = id_B$. Then $g(A \longrightarrow B)$ is an R–algebra map; as f gives rise to φ , $\varphi(B)(g) = g \circ f$,and so $g \circ f = id_B$. ◆

PROPOSITION 2.7.4 $Z(\mathcal{U})$ *is representable by an affine space over* **C**.

PROOF $Z(\mathcal{U})$ and $H^1(S^1: \mathcal{U}) = H^1(\mathcal{U})$ are affine schemes and we have a natural map

$$\pi : Z(\mathcal{U}) \longrightarrow H^1(\mathcal{U})$$

that is surjective for all R . Hence by Lemma 2.7.3 there is a natural map

$$\sigma : H^1(\mathcal{U}) \longrightarrow Z(\mathcal{U})$$

such that

$$\pi \circ \sigma = \text{id}.$$

So for any R the map

$$\Theta(R) : C(\mathcal{U}(R)) \times H^1(\mathcal{U}(R)) \longrightarrow Z(\mathcal{U}(R))$$

defined by

$$\Theta(R)(c, \gamma) = c\,[\,\sigma(\gamma)]$$

is bijective. This shows that we have an isomorphism of functors

$$\Theta : C(\mathcal{U}) \times H^1(\mathcal{U}) \longrightarrow Z(\mathcal{U})$$

as $C(\mathcal{U}) \times H^1(\mathcal{U})$ is representable by an affine space, we are done. ◆

COROLLARY 2.7.5 *The ideal generated by* $(*)$ *is prime and is the ideal for all regular functions vanishing on* $Z(\mathcal{U}(\mathbf{C}))$; *the latter is an algebraic variety isomorphic to complex affine space.*

THEOREM 2.7.6 *The action of* $C(\mathcal{U}(\mathbf{C}))$ *on* $Z(\mathcal{U}(\mathbf{C}))$ *is free in the algebraic geometric sense, and the map*

$$Z(\mathcal{U}(\mathbf{C})) \longrightarrow H^1(\mathcal{U}(\mathbf{C})) = C(\mathcal{U}(\mathbf{C})) \setminus Z(\mathcal{U}(\mathbf{C}))$$

is the quotient map.

PROOF The first statement is immediate since

$$\theta(\mathbf{C}) : C(\mathcal{U}(\mathbf{C})) \times H^1(\mathcal{U}(\mathbf{C})) \longrightarrow Z(\mathcal{U}(\mathbf{C}))$$

is an isomorphism of varieties which is equivariant with respect to $C(\mathcal{U}(\mathbf{C}))$, the action on the left being left trranslation on the first component. The second statement follows trivially from the first. ♦

In practice the sheaves of group schemes one encounters are often unramified and possess additional features. We shall now make a few remarks that may be helpful in getting a further understanding of the representability theorem 2.4.1 in these special cases. We shall use the good coverings in the unramified case to get a more explicit description of the scheme structure on $H^1(S^1,\mathcal{U})$. We assume that $\mathcal{U}$ satisifes the following conditions :

(a) $\mathcal{U}$ is unramified.

(b) for each $i \geq 0$, the exact sequence

$$0 \longrightarrow \mathcal{V}^{(i)} \longrightarrow \mathcal{U}/\mathcal{U}^{(i+1)} \longrightarrow \mathcal{U}/\mathcal{U}^{(i)} \longrightarrow 0$$

splits, i.e., there is a map

$$\gamma_i : \mathcal{U}/\mathcal{U}^{(i)} \longrightarrow \mathcal{U}/\mathcal{U}^{(i+1)}$$

such that $\beta_i \circ \gamma_i = \text{id}$, β_i being the map $\mathcal{U}/\mathcal{U}^{(i+1)} \longrightarrow \mathcal{U}/\mathcal{U}^{(i)}$. We wish to prove

THEOREM 2.7.7 *If* $\mathcal{U}$ *satisfies these conditions, we have*

$$H^1(S^1,\mathcal{U}) \cong \prod_{i\geq 0} H^1(S^1, \mathcal{V}^{(i)})$$

We shall prove this for $\mathcal{U}/\mathcal{U}^{(i)}$ in place of $\mathcal{U}$; as $\mathcal{U}^{(i)} = 0$ for $i >> 0$, the result for $\mathcal{U}$ will follow at once. Fix $i \geq 0$ and assume the theorem for $\mathcal{U}/\mathcal{U}^{(i+1)}$. Let us fix a finite covering (U_α) of S^1 by open arcs of length a_i whole ends are not in $\Phi(\mathcal{U}/\mathcal{U}^{(i+1)})$ and whose triple intersections are empty. This is certainly a good covering for $\mathcal{U}/\mathcal{U}^{(i)}$ in view of Proposition 2.7.1. Since there will be no other covering involved in the discussion below we shall omit references to it. We may enumerate the indices α as 1, ..., M and identify, for any sheaf $\mathcal{S}$ for which the covering is good, the cocycles for $\mathcal{S}$ by systems $(s_{\alpha\beta})_{\alpha<\beta}$, $s_{\alpha\beta} \in \mathcal{S}(U_\alpha \cap U_\beta)$.

LEMMA 2.7.8 *The natural map*

$$\beta_i : C(\mathcal{U}/\mathcal{U}^{(i+1)}) \longrightarrow C(\mathcal{U}/\mathcal{U}^{(i)})$$

induced by β_i *is an isomorphism, and its inverse is the map*

$$C(\mathcal{U}/\mathcal{U}^{(i)}) \longrightarrow C(\mathcal{U}/\mathcal{U}^{(i+1)})$$

induced by γ_i.

PROOF It is enough to prove that β_i is bijective. Fix a **C**–algebra R and consider $c = (c_\alpha) \in C(\mathcal{U}/\mathcal{U}^{(i+1)}(R))$ such that $\beta_i(c_\alpha) = 1$ for all α. Then c_α is a section of $\mathcal{V}^{(i)}(R)$ on U_α, hence $c_\alpha = 1$. ◆

Suppose $g \in Z(\mathcal{V}^{(i)}(R))$ and $h \in Z(\mathcal{U}/\mathcal{U}^{(i)}(R))$. We define $g*h \in Z(\mathcal{U}/\mathcal{U}^{(i+1)}(R))$ by

$$(g*h)_{\alpha\beta} = g_{\alpha\beta}\,\gamma_i(h_{\alpha\beta}) \qquad (\alpha < \beta)$$

For any cocycle s, we write [s] for its cohomology class.

LEMMA 2.7.9 Fix R.

(a) If $[g*h] = [g'*h']$, *then* $[h] = [h']$

(b) If $[g*h] = [g'*h]$, then $g = g'$.

(c) If $E \subset Z(\mathcal{U}/\mathcal{U}^{(i)}(R))$ *maps onto* $H^1(\mathcal{U}/\mathcal{U}^{(i)}(R))$, *then*

$Z(\mathcal{V}^{(i)}(R)) * E$ maps onto $H^1(\mathcal{U}/\mathcal{U}^{(i+1)}(R))$.

PROOF a) Applying β_i, we get $[h] = [h']$.

b) Suppose $g'*h = c[g*h]$ for some $c \in C(\mathcal{U}/\mathcal{U}^{(i+1)}(R))$. Applying β_i we get $h = \beta_i(c)[h]$. As the action of $C(\mathcal{U}/\mathcal{U}^{(i)}(R))$ is free, $\beta_i(c) = 1$. By Lemma 2.7.8 this gives $c = 1$, i.e., $g'*h = g*h$. But then $g' = g$.

Note that by the results of § 2.5, the covering is good for $\mathcal{U}/\mathcal{U}^{(i+1)}(R))$ also, since each cocycle for it trivializes on arcs of length a_i.

c) Let $u \in Z(\mathcal{U}/\mathcal{U}^{(i+1)}(R))$. We can write $u = k*c\,[e]$ where $e \in E$ and $c \in C(\mathcal{U}/\mathcal{U}^{(i)}(R))$. It is a question of finding $k' \in Z(\mathcal{V}^{(i)}(R))$ with $k*c[e] = \gamma_i(c)[k'*e]$. But

$$u_{\alpha\beta} = k_{\alpha\beta}\,\gamma_i\,(c_\alpha\, e_{\alpha\beta}\, c_\beta^{-1})$$

$$= k_{\alpha\beta}\,\gamma_i(c_\alpha)\,\gamma_i(e_{\alpha\beta})\gamma_i\,(c_\beta)^{-1}$$

$$= \gamma_i\,(c_\alpha)k_{\alpha\beta}\,\gamma_i\,(e_{\alpha\beta})\,\gamma_i(c_\beta)^{-1}$$

where

$$k'_{\alpha\beta} = \gamma_i\,(c_\alpha)^{-1}\,k_{\alpha\beta}\,\gamma_i(c_\alpha)$$

Since $\mathcal{V}^{(i)}(R)(U_\alpha \cap U_\beta)$ is a normal subgroup of $\mathcal{U}/\mathcal{U}^{(i+1)}(R)(U_\alpha \cap U_\beta)$, $k'_{\alpha\beta}$ is again a section of $\mathcal{V}^{(i)}(R)$ on $U_\alpha \cap U_\beta$, so that k' is an element of $Z(\mathcal{V}^{(i)}(R))$. ♦

PROOF OF THEOREM 2.7.7 Let σ_i be the natural map

$$\sigma_i : H^1(S^1, \mathcal{U}/\mathcal{U}^{(i)}) \longrightarrow Z(\mathcal{U}/\mathcal{U}^{(i)})$$

such that $\pi_i \circ \sigma_i = id$ where π_i is the real natural projection

$$\pi_i : Z(\mathcal{U}/\mathcal{U}^{(i)}) \longrightarrow H^1(S^1,\mathcal{U}/\mathcal{U}^{(i)})$$

We now define the natural map

$$\xi : Z(\mathcal{V}^{(i)}) \times H^1(S^1, \mathcal{U}/\mathcal{U}^{(i)}) \longrightarrow H^1(S^1,\mathcal{U}/\mathcal{U}^{(i+1)})$$

by

$$\xi(R)\,(g,\gamma) = [g * \sigma_i(\gamma)]$$

The naturality of ξ is obvious. By Lemma 2.7.9 ξ is an isomorphism. Since the natural map

$$Z(\mathcal{V}^{(i)}) \longrightarrow H^1(S^1, \mathcal{V}^{(i)})$$

is an isomorphism, the proof is complete. ◆

REMARK If $\mathcal{V}$ is elementary (and unramified), the representability of $H^1(S^1, \mathcal{V})$ by affine space is immediate from Lemma 2.5.1. Hence it is clear that the above argument leads to an elementary proof of Theorem 2.4.1 in the special situation treated here. The Stokes sheaf of an unramified element of $\mathfrak{G}_0$ satisfies these conditions as we shall see below and so the above argument gives a proof of the affine structure on the cohomology of the Stokes sheaf and its explicit decomposition as a product of the cohomologies of Stokes sheaves of unramified connections with a single level. Such a proof is actually very close to the one discussed in [BV4] and [BV5]. It would be interesting if a similar proof could be found in the ramified case.

Let us consider the general situation of Theorem 2.4.1. Let us suppose, as in §2.5, that all the arclengths a_i are $< 2d\pi$, so that the condition ($\mathfrak{C}$) refers to the same covering space $S^{1,d}$ for all $\mathcal{V}^{(i)}$; let $f(S^{1,d} \longrightarrow S^1)$ be the covering map. We are then in the situation discussed in §1.3. Proposition 1.3.3 then gives rise, for all **C**-algebras R, to natural isomorphisms

$$H^1(S^1, \mathcal{U}(R)) \cong H^1(S^1, f^*\mathcal{U}(R))^{\mathrm{Inv}},$$

the superfix denoting the subspace of μ_d-invariant elements. This leads to

THEOREM 2.7.10 *There is a natural isomorphism of varieties*

$$H^1(S^1, \mathcal{U}(\mathbf{C})) \cong H^1(S^1, f^*\mathcal{U}(\mathbf{C}))^{\mathrm{Inv}}$$

REMARK Since $f^*\mathcal{U}$ is unramified, the affineness of its H^1 has an elementary proof, and so, it is natural to ask whether we can use the above Theorem to get an elementary proof of Theorem 2.4.1. This seems difficult to do, although the above thorem shows that $H^1(S^1, \mathcal{U}(\mathbf{C}))$ is *smooth*, in view of the classical theorem that the variety of fixed points of a complex analytic action of a finite group on a complex manifold is smooth [BM].

3 AFFINE STRUCTURE FOR THE COHOMOLOGY OF THE STOKES SHEAF OF A MEROMORPHIC PAIR

3.1 We shall now apply the theory of §§1-2 to the local study of meromorphic pairs. In order to do this it is necessary to show that the Stokes sheaf of a meromorphic pair (V, ∇) at $z = 0$ satisfies the conditions assumed in Theorem 2.4.1.

PROPOSITION 3.1.1 *Let* (V, ∇) *be a meromorphic pair at* $0 \in \mathbf{C}$ *and let* $\mathrm{St}(V,\nabla)$ *(resp.* $\mathfrak{st}(V,\nabla)$*) be its Stokes sheaf (resp. infinitesimal Stokes sheaf). Then* $\mathfrak{st}(V,\nabla)$ *is a sheaf of complex nilpotent Lie algebras and* $\mathrm{St}(V,\nabla)$ *is the corresponding sheaf of complex unipotent algebraic groups, so that* $\mathrm{St}(V,\nabla)$ *is in the category* $\mathbb{A}$ *and* $\mathfrak{st}(V,\nabla) = \mathrm{Lie}(\mathrm{St}(V,\nabla))$. *Moreover hte assignment* $k \longrightarrow 1 + k$ *is an isomorphism of affine varieties of* $\mathfrak{st}(W)$ *with* $\mathrm{St}(W)$ *for each open arc* $W \subset S^1$.

PROOF This is more or less an obvious consequence of the discussion in **I**, §3 and **II**, §2.3. Let E be the endomorphism bundle of V and ∇_E the connection on E associated to ∇. We know that for any open arc $W \subset S^1$,

$$\mathrm{St}(V,\nabla)(W) = 1 + \mathfrak{st}(V,\nabla)(W)$$

$$\mathfrak{st}(V,\nabla)(W) = \mathcal{H}_0(E)(W)$$

where $\mathcal{H}_0(E)(W)$ is the space of flat horizontal sections of (E,∇_E) on $\Gamma(W)$. Then, by **I**, Proposition 3.4.1, we have an isomorphism, depending only on the spectrum Σ of (V, ∇) and the partial ordering $<_u$ on it, of the sheaf $\mathrm{St}(V,\nabla)$ (resp. $\mathfrak{st}(V,\nabla)$) on W with the sheaf $\mathcal{S} = \mathcal{S}(B)$ (resp. $\mathfrak{s}(B)$) of subgroups of $GL(U)$ (resp. Lie subalgebras of $\mathfrak{gl}(U)$) whose stalk at any $w \in W$ is the group (Lie algebra) of all $g = (g_{\sigma\tau}) \in \mathrm{End}(U)$ such that

$$g_{\sigma\sigma} = 1 \quad (\text{resp. } g_{\sigma\sigma} = 0),$$

$$g_{\sigma\tau} = 0 \quad \text{unless } \sigma <_w \tau \quad (\sigma \neq \tau).$$

If we extend the partial ordering $<_w$ on Σ to a linear ordering $<<$ arbitrarily, then it is clear from the above that $g_{\sigma\tau} = 1$ or 0 if $\tau = \sigma$ and $g_{\sigma\tau} = 0$ if $\tau << \sigma$, so that if $g = 1 + h$, h is nilpotent and g is unipotent. ◆

We now introduce a natural filtration on $St(V,\nabla)$ indexed by real numbers. For any real number t, $St(V,\nabla)^{(t)}$ is the subsheaf of $St(V,\nabla)$ whose stalks at $u \in S^1$ are given by

$$St(V,\nabla)^{(t)}(u) = \{ g \in St(V,\nabla)(u) : (g-1)E(\omega) \sim 0\,(\Gamma(u)) \text{ if } \operatorname{ord}(\omega) > t \}$$

Here we recall (cf. I, §3) that for any differential form $\omega \in \mathcal{D}(u)$, $E(\omega)$ is $\exp(\int_u^z \omega^{\#}.\,dz)$. The fundamental result concerning the $St(V,\nabla)^{(t)}$ is the following.

PROPOSITION 3.1.2 *Let* $L = \{r_1,\ldots,r_m\}$ ($r_i \in \mathbf{Q}$, $r_1 < \ldots < r_m < -1$) *be the canonical levels of* (V,∇). *Then* $\{St(V,\nabla)^{(t)}\}$ *is a family of normal subsheaves of* $St(V,\nabla)$ *decreasing with* t. *Moreover, we have*

$$St(V,\nabla)^{(t)} = 0 \qquad (t < r_1 \text{ or } t \geq r_m),$$

$$St(V,\nabla)^{(t)} = St(V,\nabla)^{(r_k)} \quad (r_k \leq t < r_{k+1}).$$

Finally, writing $St^{(m-k)} = St(V,\nabla)^{(t)}$ *for* $t = r_k$, *one obtains the filtration*

$$St^{(0)} = St(V,\nabla) \supset St^{(1)} \supset \ldots St^{(m-1)} \supset St^{(m)} = 0$$

such that $St^{(i)}/St^{(j)} \in \mathbb{A}$ *whenever* $j > i$, *and* $St^{(i)}/St^{(i+1)}$ *is elementary with associated arc length* $\pi/(|r_{m-i}| - 1)$, $i = 0,1,\ldots,m-1$.

The proof of this proposition will be taken up in §§3.2- 3.4.

3.2 In this paragraph we shall suppose that (V,∇) is an unramified canonical form. Thus V is the trivial bundle $\mathbf{C} \times U$ where U is an n-dimensional vector space over $\mathbf{C}$, and $\nabla_{d/dz} = d/dz - B$ where

$$B = \sum_{r \in L} D_r z^r + z^{-1}C \qquad (L = \{r_1,\ldots,r_m\}, \quad r_i \in \mathbf{Z}, r_1 < \ldots < r_m < -1)$$

is an unramified canonical form. For any k, $1 \leq k \leq m$, we write $L(k) = \{ r_1,..,r_k \}$,

$$B_k = \Sigma_{r\in L(k)} D_r z^r + z^{-1}C,$$

and define ∇_k to be the connection on V whose connection matrix is the canonical form B_k which is being viewed as a section of the bundle End(V) on $\mathbf{C}^\times$. Let Σ (resp. $\Sigma(k)$) be the spectrum of (V, ∇) (resp. (V, ∇_k)). We identify Σ (resp.$\Sigma(k)$) with the joint spectrum of the D_r, $r \in L$ (resp. D_r, $r \in L(k)$); and for $\sigma \in \Sigma$ or $\Sigma(k)$ (depending on the context) let U_σ be the spectral subspace of U corresponding to σ and P_σ $(U \longrightarrow U_\sigma)$ the spectral projections. If $k < m$, let $B_{k,\sigma}$ $(\sigma \in \Sigma(k))$ be the restriction of

$$\Sigma_{r\in L\setminus L(k)} D_r z^r + z^{-1}C$$

to $V_\sigma = \mathbf{C} \times U_\sigma$, viewed as a section of End(V_σ) on $\mathbf{C}^\times$, and $\nabla_{k,\sigma}$ the connection on V_σ with $B_{k,\sigma}$ as the connection matrix. We write St, St_k, and $St_{k,\sigma}$ for the Stokes sheaves of (V,∇), (V,∇_k) and $(V_\sigma,\nabla_{k,\sigma})$ respectively. Finally, for $\alpha = \Sigma_{r\in L}\, a_r z^r . dz \in \Sigma$, let $\alpha_k = \Sigma_{r\in L(k)}\, a_r z^r . dz$.

It is easy to describe the Stokes sheaf of a canonical form explicitly. Indeed, it follows from **I**, §3.4 that for any open set $W \subset S^1$, $St(V, \nabla)(W)$ is the group of holomorphic maps g $(\Gamma(W) \longrightarrow GL(U))$ such that

(a) $dg/dz + [g, B] = 0$

(b) $g \sim 1$ $(\Gamma(W))$.

These differential equations can be solved at once to obtain the local structure of St. Select a branch $\log_W$ of log on $\Gamma(W)$ and define

$$\psi(z) = \exp(\Sigma_{r\in L} D_r\, z^{r+1}/r+1 + \log_W z \,.\, C) \qquad (z \in \Gamma(W)).$$

Then ψ $(\Gamma(W) \longrightarrow GL(U))$ is holomorphic and satisfies

$$(d\psi/dz)\, \psi^{-1} = B.$$

It is easy to see that if g $(\Gamma(W) \longrightarrow GL(U))$ is holomorphic, then

$$dg/dz + [g, B] = 0 \Leftrightarrow d/dz(\psi^{-1} g \psi) = 0.$$

In particular, if W is an open arc, the transformation

$$g \longrightarrow \psi^{-1} g \psi$$

is an isomorphism on W of $St(V, \nabla)$ with the subsheaf $\mathcal{S}(B)$ of the constant sheaf $GL(U)$ whose stalk at $w \in W$ is

$$\mathcal{S}(B)(W) = \{ h \in GL(U) : \psi h \psi^{-1} \sim 1 \ (\Gamma(w)) \}.$$

Thus, if $\sigma, \tau \in \Sigma$, and $\sigma \neq \tau$, then any $g \in St(W)$ can be represented as

$$(*) \qquad E(\sigma-\tau) \exp(\log_W z \,.\, C_\sigma)\, Y \exp(-\log_W z \,.\, C_\tau) = E(\sigma-\tau)\, h_{Y,\sigma\tau}$$

where $Y \in Hom(U_\tau, U_\sigma)$ is a constant. Note that

(a) $h_{Y,\sigma\tau}$ is a function of moderate growth

(b) $Y = 0$ unless $\rho_{\sigma-\tau}(u) < 0$ for all $u \in W$,

the latter being a consequence of **I**, Proposition 3.4.1.

PROPOSITION 3.2.1 *Fix* k, $1 \le k < m$. *Then there are maps* α $(St_k \to St)$ *and* β $(St \longrightarrow \bigoplus_{\sigma \in \Sigma(k)} St_{k,\sigma})$ *in the category* $\mathbb{A}$ *such that*

$$0 \longrightarrow St_k \longrightarrow St \longrightarrow \bigoplus_{\sigma \in \Sigma(k)} St_{k,\sigma} \longrightarrow 0$$

is an exact sequence. Moreover, this sequence splits , i.e., there is a map γ $(\bigoplus_{\sigma \in \Sigma(k)} St_{k,\sigma} \longrightarrow St)$ *such that* $\beta \circ \gamma = id$.

PROOF Let

$$\rho_k = \exp\left(\textstyle\sum_{r \in L \setminus L(k)} D_r\, z^{r+1}/r{+}1\right)$$

We define α, β, γ as follows : if $W \subset S^1$ is open and $g \in St_k(W)$, $h \in St\,(W)$, $t = (t_\sigma) \in \bigoplus_{\sigma \in \Sigma(k)} St_{k,\sigma}\,(W)$, then

$$\alpha(g) = \rho_k\, g\, \rho_k^{-1}, \quad \beta(h) = (P_\sigma h\, P_\sigma)_{\sigma \in \Sigma(k)}, \quad \gamma(t) = \textstyle\sum_{\sigma \in \Sigma(k)} P_\sigma t_\sigma P_\sigma$$

If we think of any $h \in St(W)$ as a matrix $(h_{\sigma\tau})_{\sigma,\tau \in \Sigma(k)}$, $\beta(h)$ is the matrix $(h_{\sigma\sigma}\delta_{\sigma\tau})$ and $\gamma(t_\sigma)$ is the matrix $(\delta_{\sigma\tau} t_\sigma)$. It is thus clear that $\beta \circ \gamma$ = identity, and that γ is multiplicative. The multiplicative nature of α is obvious. But for β it is not obvious and needs an argument. We consider the spectral decomposition $U = \bigoplus_{\varphi \in \Sigma} U_\varphi$ of W with respect to all the D_r, $r \in L$, and represent the $h \in St(W)$ as matrices $(h_{\varphi\psi})_{\varphi,\psi \in \Sigma}$. The multiplicativity of β then comes down to showing the following: for $\sigma \in \Sigma(k)$, let $S(\sigma)$ be the set of $\varphi \in \Sigma$ with $\varphi_k = \sigma$; then for $h,h' \in St(W)$ and $\varphi,\psi \in S(\sigma)$,

$$(hh')_{\varphi\psi} = \sum_{\lambda \in S(\sigma)} h_{\varphi\lambda} h'_{\lambda\psi}.$$

Now,

$$(hh')_{\varphi\psi} = \sum_{\lambda \in S(\sigma)} h_{\varphi\lambda} h'_{\lambda\psi} + \sum_{\tau \neq \sigma} \sum_{\mu \in S(\tau)} h_{\varphi\mu} h'_{\mu\psi}$$

where the τ in the second sum varies over $\Sigma(k)$. We claim that each summand of the inner sum in the second term (for fixed τ) is 0 . Indeed, let $u \in W$ be not on any Stokes line. Then we have either $\sigma <_u \tau$ or $\tau <_u \sigma$; in the first case, $\psi <_u \mu$ so that $h'_{\mu\psi} = 0$, while in the second case, $\mu <_u \varphi$ so that $h_{\varphi\mu} = 0$.

We must show next that α, β and γ map into the appropriate sheaves;

$\boxed{\alpha}$: For $g \in St_k$ we have,

$$d\alpha(g)/dz = [(d\rho_k/dz)\rho_k^{-1}, \rho_k g \rho_k^{-1}] + \rho_k[B_k, g] \rho_k^{-1}$$
$$= [B, \alpha(g)].$$

It remains to show that $\alpha(g) \sim 1$ $(\Gamma(W))$. If $\varphi, \psi \in \Sigma$ and $\varphi_k = \psi_k$, then $g_{\varphi\psi} = \delta_{\varphi\psi}$, so that we need only show that for any open arc W,

$$\alpha(g) \sim 0 \; (\Gamma(W) \qquad (\varphi_k \neq \psi_k).$$

Write $\sigma = \varphi_k$, $\tau = \psi_k$.Then, by the relation $(*)$,

$$\alpha(g)_{\varphi\psi} = E(\varphi - \psi) h_{Y,\sigma\tau}, \quad g_{\sigma\tau} = E(\sigma - \tau) h_{Y,\sigma\tau}.$$

Since $g \in St_k(W)$, it follows that g, and hence $\alpha(g)$, is zero unless $\rho_{\sigma-\tau}(u) < 0$ for all $u \in W$. On the other hand, as $\sigma \neq \tau$, $\text{ord}(\sigma-\tau) \leq r_k$, $\sigma-\tau$ and $\varphi-\psi$ have the same leading coefficient, so that $\rho_{\sigma-\tau}(u) = \rho_{\varphi-\psi}(u)$ $(u \in W)$. Hence, when $\rho_{\sigma-\tau}(u) < 0$ for all $u \in W$, we must have $E(\varphi-\psi) \sim 0\ (\Gamma(W))$. This implies that $\alpha(g)_{\varphi\psi} \sim 0\ (\Gamma(W))$ since $h_{Y,\sigma\tau}$ is of moderate growth.

$\boxed{\beta}$: It is trivial to check that β maps St (W) into $\bigoplus_{\sigma\in\Sigma(k)} St_{k,\sigma}(W)$.

$\boxed{\gamma}$: Similarly it is trivial that γ maps $\bigoplus_{\sigma\in\Sigma(k)} St_{k,\sigma}(W)$ into St (W).

$\boxed{\text{EXACTNESS}}$ For $g \in St_k(W)$, $g_{\sigma\sigma} = 1$ for $\sigma \in \Sigma(k)$, and so, $\beta(\alpha(g)) = 1$. We must now verify that if $h \in St\ (W)$ and $\beta(h) = 1$, then $g = \rho_k^{-1} h\, \rho_k \in St_k(W)$. A simple calculation shows that $dg/dz = [B_k, g]$ and so it remains to verify the flatness condition . Now $g_{\sigma\sigma} = 1$ for all σ and so we must verify that if $\varphi, \psi \in \Sigma$ and $\varphi_k = \sigma \neq \tau = \psi_k$, then $g_{\varphi\psi} \sim 0\ (\Gamma(W))$. But as before, in view of $(*)$,

$$h_{\varphi\psi} = E(\varphi-\psi)\, h_{Y,\sigma\tau}, \qquad g_{\varphi\psi} = E(\sigma-\tau)\, h_{Y,\sigma\tau};$$

and $h_{\varphi\psi}$, hence $g_{\varphi\psi}$ also, is zero unless $\rho_{\varphi-\psi}(u) < 0$ for all $u \in W$. In the latter case, as before, $\rho_{\sigma-\tau}(u) < 0$ for all $u \in W$, so that, as $h_{Y,\sigma\tau}$ is of moderate growth, $g_{\varphi\psi} \sim 0\ (\Gamma(W))$.

Finally, the morphic property of α, β, and γ is obvious. ◆

In view of this result we may identify St_k $(1 \le k \le m)$ as a normal subsheaf of St, with $St_m = St$. We put $St_0 = 0$; moreover, for $1 \le k < m$ and $\sigma \in \Sigma(k)$ we write $St_{k,k+1,\sigma}$ for the Stokes sheaf of $(V_\sigma, \nabla_{k,k+1,\sigma})$ where V_σ is the bundle $\mathbf{C} \times U_\sigma$ and $\nabla_{k,k+1,\sigma}$ is the connection whose matrix is the restriction to V_σ of

$$B_{k,k+1,\sigma} = \sum_{r\in L(k+1)\setminus L(k)} D_r z^r + z^{-1}C.$$

The following is then an immediate consequence of the above result.

COROLLARY3.2.2 *We have*

$$St_0 = 0 \subset St_1 \subset \dots \subset St_m = St$$

where St_k *is a normal subsheaf of* St. *Moreover*, *for* $0 \le k \le m$,

$$St_{k+1}/St_k \cong \bigoplus_{\sigma\in\Sigma(k)} St_{k,k+1,\sigma}$$

PROPOSITION 3.2.3 *Let* $r < -1$ *be an integer,* U' *a complex vector space of finite dimension , and* $B' = D' z^r + z^{-1}C'$ *a canonical form defining a connection* ∇' *on the bundle* $V' = \mathbf{C} \times U'$*. Then* $\mathrm{St}\,(V',\nabla')$ *is elementary with associated arc length* $\pi/(|r|-|)$.

PROOF If a,b are any two distinct eigenvalues of D', let $\Phi(a,b)$ be the set of rays through the points $u \in S^1$ such that $\mathrm{Re}((a-b)u^r) = 0$; then $\Phi = \bigcup_{a,b} \Phi(a,b)$ is the set of Stokes lines of $\mathrm{End}(V')$. In view of proposition 2.4.2, it is a question of proving two things:

(a) if K is an open arc $\subset S^1$ of length $\pi/(|r|-|)$ with end points not on Φ, then

$$H^0(K, \mathrm{St}\,(B')) = 0, \quad H^1(K,\mathrm{St}\,(B')) = 0$$

(b) if K,K' are arcs with $K' \subset K$ and $(K \setminus K') \cap \Phi = \varnothing$, then any section of $\mathrm{St}\,(B)$ on K' extends uniquely to a section on K.

The assertion (b) is a restatement of **I**,Proposition 3.4.2. Concerning the assertion (a), let $g \in H^0(K,\mathrm{St}\,(B'))$. If a,b are distinct eigenvalues of D', $\Phi(a,b)$ is a set of $2(|r|-1)$ rays with angle exactly $\pi/(|r|-|)$ between successive members and so one of them must meet Φ within K. Hence $g_{ab} = 0$, and so, as a, b are arbitrary, $g = 1$.

The vanishing of $H^1(K,\mathrm{St}\,(B'))$ is more delicate . We prove first that if K' is any open arc of length $\leq \pi/(|r|-|)$, the restriction map

$$H^1(S^1,\mathrm{St}\,(B')) \longrightarrow H^1(K',\mathrm{St}\,(B'))$$

is identically zero; this does not require B' to have only one level but depends rather on the fact that r is the principal level. If $\gamma \in H^1(S^1,\mathrm{St}\,(B'))$, we can find by the Malgrange-Sibuya theorem a meromorphic pair (V,∇) at $z = 0$ and an isomorphism ξ of its formalization with the formalization of $(V',\nabla_{B'})$, such that $((V,\nabla),\xi)$ is represented by γ. Thus we can find $\varepsilon > 0$, a covering (U_i) of S^1 by open arcs and isomorphisms x_i of $(V.\nabla)$ with $(V',\nabla_{B'})$ on $\Gamma(U_i)_\varepsilon$ such that $x_i^\wedge = \xi$ and the cocycle $(x_i\, x_j^{-1})$ represents γ. But $|K'| \leq \pi/(|r|-|)$, and so **I**,Theorem 2.2.4 allows us to find an isomorphism u of (V,∇) with $(V',\nabla_{B'})$ on

$\Gamma(K')_\varepsilon$ with $u^\wedge = \xi$; if $y_i = x_i u^{-1}$, it is immediate that $y_i \in St((V,\nabla))(U_i \cap K')$ and $c_i c_j^{-1} = y_i y_j^{-1}$, on $U_i \cap U_j \cap K'$. Thus $\mathfrak{T}$ trivializes on K'.

We now argue as in Proposition 2.4.2 to show that any torsor $\mathfrak{T}$ for St (B') on K is trivial. We claim first that if K' is an open arc and $K' \subset\subset K$, the restriction of $\mathfrak{T}$ on K' is actually the restriction of a torsor on S^1 for St(B'), to K'. Indeed, if we write $K' = (u',v')$ where $u',v' \in K$, there are small open arcs (v'',v') and (u',u'') with $u'' < v''$ on which $\mathfrak{T}$ is trivial ; we then take the trivial torsor on the arc $S^1 \setminus [u'',v'']$ and glue it to the restriction of $\mathfrak{T}$ on (u',v'). By the result established above, $\mathfrak{T}$ trivializes on K'. We now write K as $\bigcup_n K_n$ where $K_1 \subset\subset K_2 \subset\subset \ldots \subset\subset K_n \subset\subset \ldots\ldots K$, and $K \setminus K_1$ does not meet Φ. Then $\mathfrak{T}$ trivializes on all the K_n. If there are two sections t, t' of $\mathfrak{T}$ on K_n, then there is a section g of St (B') on K_n such that $g[t] = t'$. But as $(K \setminus K_n) \cap \Phi = \varnothing$, g extends uniquely to a section of St (B') on K, which must be the trivial section since $H^0(K, St(B')) = 0$. Thus $t = t'$, proving that $\mathfrak{T}$ is *uniquely trivializable* on K_n. The sections of $\mathfrak{T}$ on the K_n are thus coherent and build up to a section on K. This finishes the proof that $H^1(K, St(B')) = 0$. ◆

PROPOSITION 3.2.4 St *satisfies the conditions of Theorem* 2.4.1.

PROOF Let us observe that the connections $\nabla_{k,k+1,\sigma}$ all have a single level, and that this level is the same for all of them, namely r_{k+1}. The above proposition applies to each of them and shows immediately that

$$\bigoplus_{\sigma \in \Sigma(k)} St_{k,k+1,\sigma}$$

is elementary with arc length $\pi/(|r_{k+1}| - 1)$. The result now follows from Corollary 3.2.2. ◆

Let us write

$$St^{(t)} = St((V,\nabla))^{(t)}, \quad t \in \mathbf{R}$$

where $V = \mathbf{C} \times U$ and ∇ is the connection on V with connection matrix B .

PROPOSITION 3.2.5 *We have* ,

$$St(V,\nabla)^{(t)} = 0 \qquad (t < r_1 \textit{ or } \quad t \geq r_m),$$

$$St(V,\nabla)^{(t)} = St_k \quad (r_k \leq t < r_{k+1}).$$

PROOF Let W be an open arc and $g \in St^{(t)}(W)$, $t \in \mathbf{R}$. Then, when $\varphi,\psi \in \Sigma$, $(\varphi \neq \psi)$, we have, by $(*)$,

$$g_{\varphi\psi} = E(\varphi-\psi)\, h_{Y,\varphi\psi}.$$

Suppose first that $t < r_1$. Then $\mathrm{ord}(\varphi - \psi) \geq r_1 > t$ and so

$$\exp(-\log_W z.\, C_\varphi)\, E(-(\varphi-\psi))\exp(\log_W z.\, C_\psi)\, g_{\varphi\psi} \sim 0\ (\Gamma(W))$$

by the assumption on g. This shows that $Y = 0$, and hence that $g = 1$. Let us next suppose that $r_k \leq t < r_{k+1}$ where $k < m$, and write $f = \rho_k^{-1} g\, \rho_k$, $\varphi_k = \sigma$, $\psi_k = \tau$. If $\sigma = \tau$, then $\mathrm{ord}(\varphi-\psi) \geq r_{k+1} > t$, and so, we can argue as before that $g_{\varphi\psi} = 0$, so that $f_{\sigma\sigma} = 1$. If $\sigma \neq \tau$, we have

$$\varphi-\psi = (\sigma-\tau) + \theta, \quad \mathrm{ord}(\theta) \geq r_{k+1} > t,$$

and our assumption leads to the relation

$$f_{\sigma\tau} = E(-\theta)\, g_{\sigma\tau} \sim 0\ (\Gamma(W)).$$

In other words, we have verified that $\rho_k^{-1} g\, \rho_k \in St\,(B_k)(W)$, i.e., $St^{(t)}(W) \subset St_k$. To prove the reverse inclusion, let $f \in St_k$. Then $f_{\varphi\psi} = \delta_{\varphi\psi}$ for $\varphi,\psi \in \Sigma$ with $\sigma = \tau$; and if $\sigma \neq \tau$, we have

$$f_{\varphi\psi} = E(\varphi-\psi)\, h_{Y,\varphi\psi}, \qquad f_{\sigma\tau} = E(-\theta)\, f_{\varphi\psi},$$

and $\rho_{\varphi-\psi}(u) = \rho_{\sigma-\tau}(u)$ for all $u \in W$. Clearly we have to consider only the case when $\rho_{\varphi-\psi}(u) < 0$ for all $u \in W$, since otherwise $f_{\sigma\tau}$, and hence $f_{\varphi\psi}$ also, must vanish. Let $\rho_{\varphi-\psi}(u) < 0$ for all $u \in W$, and let η be any differential form in $\mathcal{D}(W)$ of order $> t$. Then

$$E(\eta)\, f_{\varphi\psi} = E(\varphi-\psi+\eta)\, h_{Y,\varphi\psi};$$

and for the leading term bz^q of $\varphi-\psi+\eta$, since it is the same as that of $\varphi-\psi$, we must have $\mathrm{Re}(bu^{q+1}/q+1) < 0$ for all $u \in W$. Hence, for any sector $\Gamma' \subset\subset \Gamma(W)$ one has the estimate

$$|E(\varphi-\psi+\eta)| \leq \exp(-a|z|^{q+1}) \quad (z \in \Gamma',\ z \longrightarrow 0),$$

$a > 0$ being some constant. But then, as $h_{Y,\varphi\psi}$ is of moderate growth, we must have

$$E(\eta)\, f_{\varphi\psi} \sim 0\ (\Gamma(W)).$$

This proves that $f \in St^{(t)}(W)$. Finally, let $t \geq r_m$, and let $f \in St\,(W)$. Then for any $\varphi, \psi \in \Sigma$, with $\varphi \neq \psi$, $\operatorname{ord}(\varphi - \psi) \leq r_m$ and so the last argument applies and shows that

$$E(\eta)\, f_{\varphi\psi} \sim 0\ (\Gamma(W))$$

for all $\varphi \neq \psi$. So $f \in St^{(t)}(W)$. ◆

3.3 In view of the results of § 3.2, it is clear that for completing the proof of Proposition 3.1.2 we must relate the Stokes sheaf of an arbitrary (V,∇) to that of an unramified canonical form. We shall do this in this paragraph and thus complete the proof of Proposition 3.1.2.

Let (V,∇) be a meromorphic pair at $z = 0$ and suppose that it is unramified. Then we can find an unramified canonical form B and an isomorphism η of the formalization of (V,∇) with that of (V_B, ∇_B). The Malgrange-Sibuya isomorphism described in **I**, Theorem 4.5.1 associates to the pair $((V,\nabla), \eta)$ a cohomology class $\alpha \in H^1(S^1, St)$ where $St = St\,(V_B, \nabla_B)$ as in § 3.2. We represent this cohomology class α by a suitable cocycle a coming from a good covering, for example, a finite covering (U_i) of S^1 by open arcs of length $< \pi/(|r_1|-1)$, r_1 being the principal level of ∇. Observe that if $((V', \nabla'), \xi)$ is a pair associated to (V,∇), then $((V', \nabla'), \eta\xi)$ is associated to (V_B, ∇_B), and this correspondence sets up a bijection

$$\varphi_a : H^1(S^1, St) \cong H^1(S^1, St(V,\nabla))$$

via the respective Malgrange-Sibuya maps.

PROPOSITION 3.3.1 *We have , for a suitable choice of the cocycle* a *representing* α,

$$St\,(V,\nabla) \cong St^{(a)},\ St\,(V,\nabla)^{(t)} \cong St^{(t)}(a), \qquad (t \in \mathbf{R})$$

where the superfix denotes twisting, and $\cong$ is an isomorphism of sheaves from the category $\mathbb{A}$. *Moreover, the bijection* φ_a *defined above coincides with twisting by* a *and is an isomorphism of affine spaces.*

PROOF There is an $\varepsilon > 0$ such that for each i we can find an isomorphism y_i of (V,∇) with (V_B,∇_B) on a sectorial domain $\Gamma(U_i)_\varepsilon$, the isomorphism being compatible with the asymptotic structures and inducing the formal isomorphism η. In particular, flatness of sections (of these as well as their associated bundles) is preserved under y_i. By the definition of the Malgrange-Sibuya map, the cocycle $a = (y_i y_j^{-1})$ determines the cohomology class α corresponding to $((V,\nabla),\eta)$. But y_i induces the isomorphism

$$g \longrightarrow y_i\, g\, y_i^{-1}$$

of St (V,∇) with the restriction $St_{(i)}$ of St to the arc U_i, so that we may now view St (V,∇) as obtained by glueing the $St_{(i)}$ via the isomorphisms

$$s \longrightarrow a_{ij} s\, a_{ij}^{-1}$$

of $St_{(j)}$ with $St_{(i)}$ on $U_i \cap U_j$. This proves that St $(V,\nabla) \cong St^{(a)}$; and as a_{ij} are multiplicatively flat, i. e., $\sim 1\ (\Gamma(U_i \cap U_j))$, it is also clear that St $(V,\nabla)^{(t)}$ gets identified with $St^{(t)(a)}$ in the same manner. Suppose now that $((V', \nabla'), \xi)$ is a pair associated to (V,∇) and x_i are isomorphisms of (V',∇') with (V_B,∇_B) on a sectorial domain $\Gamma(U_i)_\varepsilon$, the isomorphism being compatible with the asymptotic structures and inducing the formal isomorphism ξ. Write $g_{ij} = x_i x_j^{-1}$ and $h_{ij} = y_i x_i x_j^{-1} y_j^{-1}$. Then the cocycle (g_{ij}) represents $\Phi((V', \nabla'), \xi)) \in H^1(S^1, St(V,\nabla))$, while the cocycle (h_{ij}) represents $\Phi((V', \nabla'), \eta\xi)) \in H^1(S^1, St)$. The obvious relation

$$y_k\, g_{ij}\, y_k^{-1} = a_{ki}\, h_{ij}\, a_{ik} \qquad \text{on } U_i \cap U_j \cap U_k$$

shows, in view of the discussion in **II**, §1.3 (cf. relations (T) in the proof of Proposition 1.3.2, loc. cit) that $\Phi((V', \nabla'), \xi))$ is the twist by a of $\Phi((V', \nabla'), \eta\xi))$. Since twisting is functorial it is clear that it is an isomorphism of the affine schemes represented by $H^1(S^1, St)$. ♦

The next proposition relates the filtrations in the z-plane with those in the ζ-plane, $z = \zeta^d$. As usual $f = f_d$ is the map $\zeta \longrightarrow \zeta^d$. Let (V,∇) be a meromorphic pair (in the z-plane) at $z = 0$ and let $(V',\nabla') = f^*(V,\nabla)$. We do not suppose that (V,∇) is unramified, but only that (V',∇') is unramified ; this is certainly possible for a suitable d.

PROPOSITION 3.3.2 *For any* $t \in \mathbf{R}$ *let* $t' = d\,t + d - 1$. *Then*

$$St\,(V',\nabla')^{(t')} = f^*\,(St\,(V,\nabla)^{(t)} \qquad (t \in \mathbf{R})$$

PROOF Fix v on $S^{1,d}$, and let $u = f(v)$. We may work on a sufficiently small sector around v on which f is a diffeomorphism. The horizontal sections g and h of (V,∇) and (V',∇') correspond by $h(\zeta) = g(\zeta^d)$. Moreover for a differential form $\omega \in \mathcal{D}(u)$ one has $\mathrm{ord}(f^*\omega) = d\,\mathrm{ord}(\omega) + d - 1$. So

$$\begin{aligned} g \in St\,(V,\nabla)^{(t)} &\Leftrightarrow (g(z)-1)E(\omega) \sim 0\,(\Gamma(u)) \text{ for any } \omega \in \mathcal{D}(u) \\ &\qquad\qquad \text{with } \mathrm{ord}(\omega) > t \\ &\Leftrightarrow (h(\zeta)-1)E(\eta) \sim 0\,(\Gamma(v)) \text{ for all } \eta \in \mathcal{D}(v) \text{ with} \\ &\qquad \mathrm{ord}(\eta) > t' \\ &\Leftrightarrow h \in St(V',\nabla')^{(t')} \end{aligned}$$

which is what we wanted to prove. ◆

PROOF OF PROPOSITION 3.1.2 Fix a meromorphic pair (V,∇) at $z = 0$, with canonical levels $r_i \in (1/d)\,\mathbf{Z}$. Then $(V',\nabla') = f_d^*(V,\nabla)$ is unramified in the ζ-plane with canonical levels $r'_i = d\,r_i + d - 1 \in \mathbf{Z}$. Propositions 3.2.4, 3.2.5 and 3.3.1 show that Proposition 3.1.2 is true for (V',∇') with r_i replaced by r'_i and π by $d\pi$. But then Proposiiton 3.3.1 shows that Proposition 3.1.2 is true for (V,∇). ◆

3.4. Theorem 2.4.1 and Proposition 3.1.2 now lead to the following theorem.

THEOREM 3.4.1 *Let* (V,∇) *be a meromorphic pair at* $z = 0$. *Then* $St\,(V,\nabla)$ *is a sheaf from the category* $\mathbb{A}$. *The cohomology* $H^1(S^1, St\,(V,\nabla))$ *of the corresponding sheaf of unipotent group schemes in representable by an affine space of dimension equal to* $\mathrm{Irr}\,(E,\nabla_E)$, *the irregularity of the endomorphism bundle* (E,∇_E) *assocciated to* (V,∇).

PROOF Only the last assertion concerning the dimension of $H^1(S^1, \mathrm{St}\,(V,\nabla))$ needs a comment. Proposition 2.6.3 shows that this dimension is equal to $\dim H^1(S^1, \mathrm{st}\,(V,\nabla)) = \dim H^1(S^1, \mathcal{H}_0(\mathrm{End}\, V))$. But this is equal to $\mathrm{Irr}\,(E,\nabla_E)$ by definition (cf. §3.1). ♦

Let (V', ∇') be as in Proposition 3.3.2. From Theorem 2.7.10 we then obtain further the following theorem.

THEOREM 3.4.2 *We have*

$$H^1(S^1, \mathrm{St}\,(V,\nabla)) \cong H^1(S^{1,d}, \mathrm{St}\,(V',\nabla'))^{\mathrm{inv}}$$

where the superfix refers to the subspace of elements invariant under the natural action of μ_d.

Finally, it is clear from Proposition 3.2.1 that the conditions of Theorem 2.7.7 are satisfied by St, and so we have the following Theorem.

THEOREM 3.4.3 *Let notation and assumptions be as in Proposition 3.2.1. Then we have, canonically,*

$$H^1(S^1, \mathrm{St}) \cong \prod_{k \geq 0} \prod_{\sigma \in \Sigma(k)} H^1(S^1, \mathrm{St}_{k,k+1,\sigma})$$

where $\mathrm{St}_{k,k+1,\sigma}$ *are the Stokes sheaves of the elementary unramified pairs* $(V_\sigma, \nabla_{k,k+1,\sigma})$.

PART III : LOCAL MODULI

1 LOCAL MODULI SPACE FOR MARKED MEROMORPHIC PAIRS

1.1. The theorem of Malgrange-Sibuya (**I**, Theorem 4.5.1) says that the cohomology $H^1(S^1, St^0)$ of the Stokes sheaf St^0 of a meromorphic pair (V^0, ∇^0) parametrizes the set of isomorphism classes of *marked* meromorphic pairs formally isomorphic to (V^0, ∇^0), while Theorem 4.5.2 shows that for parametrizing the isomorphism classes of the *unmarked* pairs one has to go to the quotient of $H^1(S^1, St^0)$ by the group $G^\wedge(V^0, \nabla^0)$ of automorphisms of the formalization of (V^0, ∇^0). In Part **II** we have shown that $H^1(S^1, St^0)$ is an affine space over $\mathbf{C}$ in a natural manner. In this part we shall examine to what extent $H^1(S^1, St^0)$ and its quotients may be viewed as *local moduli spaces* for meromorphic pairs with fixed formal data. As is well known, this is really a question of studying the problem of classifying *analytic families* of meromorphic pairs upto meromorphic equivalence, by analytic maps into $H^1(S^1, St^0)$ and its quotients. We shall be interested only in the *local deformation theory*, so that only *germs* of families and their equivalence will be of concern to us. In this chapter we shall consider only the marked pairs, postponing to the next chapter the treatment of the unmarked case.

We begin with a brief discussion of analytic families of vector bundles and connections. A *family of vector bundles at* $z = 0$ is by definition a holomorphic vector bundle $\mathbf{V}$ on $\mathbf{\Delta} \times \Delta$ where $\mathbf{\Delta}$ (resp. Δ) is a polydisk (resp. disk) in $\mathbf{C}^d$ (resp. $\mathbf{C}$) centered at the origin. One may then identify $\mathbf{V}$ with the assignment $\lambda \longrightarrow V_\lambda$ $(\lambda \in \mathbf{\Delta})$, V_λ being the pull back bundle $i_\lambda{}^* \mathbf{V}$ on Δ corresponding to the map i_λ $(z \longrightarrow (\lambda, z))$. Holomorphic sections $\mathbf{s}$ of $\mathbf{V}$ may be identified with families of sections $\mathbf{s}(\lambda)$ of V_λ, $\mathbf{s}(\lambda)(z) = \mathbf{s}(\lambda : z)$. By a *meromorphic* section s (at $z = 0$) we mean a holomorphic section $\mathbf{s}$ of the restriction of $\mathbf{V}$ to $\mathbf{\Delta} \times \Delta^\times$ such that for some integer $q \geq 0$, $z^q \mathbf{s}$ extends to a holomorphic section of $\mathbf{V}$; here $\Delta^\times = \Delta \setminus (0)$. Let $\mathcal{O}$ be the algebra of germs of holomorphic functions at $(\mathbf{0}, 0) \in \mathbf{C}^d \times \mathbf{C}$ and let $\mathcal{Q} = \mathcal{O}[z^{-1}]$. It is then clear that the germs of meromorphic sections of $\mathbf{V}$ at $z = 0$ form a free $\mathcal{Q}$-

module, say **M**, of rank N = the rank of **V**. If **V**, **V'** are two families of vector bundles, a *meromorphic map* (at z = 0) $\mathbf{V} \longrightarrow \mathbf{V}'$ is by definition a holomorphic map s from the restriction of **V** to $\mathbf{\Delta} \times \Delta^{\times}$ to the corresponding restriction of **V'** such that for some integer $q \geq 0$, $z^q s$ extends to a holomorphic map of **V** to **V'**.

A *family of meromorphic pairs at* z = 0 is an assignment

$$(\mathbf{V}, \nabla) : \lambda \longrightarrow (V_\lambda, \nabla_\lambda)$$

such that

(a) **V** is a family of vector bundles at z = 0 and $V_\lambda = i_\lambda{}^*\mathbf{V}$

(b) for each $\lambda \in \mathbf{\Delta}$, $(V_\lambda, \nabla_\lambda)$ is a meromorphic pair at z = 0

(c) for any meromorphic section **s** of **V**,

$$(*) \qquad \nabla\mathbf{s} : \lambda, z \longrightarrow \nabla_{\lambda, d/dz}\, \mathbf{s}(\lambda)(z)$$

is again a meromorphic section of **V**.

Our concern is only with *germs* of families of such pairs at (**0**, 0). From the perspective of differential modules one views $\mathfrak{A}$ as a differential algebra with respect to the derivation $\partial/\partial z$ and considers the category of free differential modules **M** of finite rank over $\mathfrak{A}$. If $(\mathbf{V}, \nabla)$ is a family of meromorphic pairs at z = 0, one can associate to it in an obvious way a differential module $(\mathbf{M}, \nabla)$ over $\mathfrak{A}$ where **M** is the $\mathfrak{A}$-module of germs of meromorphic sections of **V** and ∇ is defined by $(*)$ above. This sets up an equivalence of categories and we shall not distinguish between these two categories. If we go to a concrete description using a trivialization of **V**, then $\mathbf{V} = \mathbf{\Delta} \times \Delta \times \mathbf{C}^N$, and the connections ∇_λ on $\Delta \times \mathbf{C}^N$ are given by

$$\nabla_{\lambda, d/dz} = d/dz - A(\lambda : z)$$

where A is an N×N matrix such that $z^q A$ is holomorphic on $\mathbf{\Delta} \times \Delta$ for some $q \geq 0$, i. e., $A \in \mathfrak{gl}(N, \mathfrak{A})$. In this way we are led to the context of a family of meromorphic differential equations that depend holomorphically on the parameter $\lambda \in \mathbf{\Delta}$.

Let $(\mathbf{V}, \nabla) = \{(V_\lambda, \nabla_\lambda)\}$ be a family of meromorphic pairs and let $(\mathbf{M}, \nabla)$ be the associated differential module over $\mathfrak{A}$. Let $\mathcal{O}_{d,1}$ be as in **I**, §1.5. Then we have a natural imbedding $\mathfrak{A} \hookrightarrow \mathcal{O}_{d,1}$ obtained by viewing any element of $\mathfrak{A}$ as a Laurent series in z all of whose coefficients are in $\mathcal{O}_d(\mathbf{\Delta})$ for some $\mathbf{\Delta}$. We thus have the differential module $(\mathbf{M}^\wedge, \nabla^\wedge)$ obtained from $(\mathbf{M}, \nabla)$ by extension of scalars $\mathfrak{A} \longrightarrow \mathcal{O}_{d,1}$. By a *marking of* $(\mathbf{V}, \nabla)$ *or* $(\mathbf{M}, \nabla)$ *by a differential module* $(M^{0\wedge}, \nabla^{0\wedge})$ *over* $\mathfrak{F}$ we mean an isomorphism

$$\xi : (\mathbf{M}^\wedge, \nabla^\wedge) \cong \mathcal{O}_{d,1} \otimes_{\mathfrak{F}} M^{0\wedge}$$

PROPOSITION 1.1.1 *In order that there is a marking of the family* $(\mathbf{V}, \nabla)$ *it is necessary and sufficient that it be isoformal, i. e., the formal isomorphism class of* $(V_\lambda, \nabla_\lambda)$ *does not depend on* λ.

PROOF This is immediate from **I**, Theorem 1.5.1.

Let (V^0, ∇^0) be a meromorphic pair at $z = 0$. An *isoformal family of marked pairs associated to or formally equivalent to* (V^0, ∇^0) may now be defined as a system $((\mathbf{V}, \nabla), \xi)$ where $(\mathbf{V}, \nabla)$ is a family of meromorphic pairs and ξ is a marking of it by $(M^{0\wedge}, \nabla^{0\wedge})$, the formalization of (V^0, ∇^0) ; we shall say that it is a *local isoformal deformation* of $((V_0, \nabla_0), \xi_0)$. By specializing ξ at the points λ we obtain a collection of isomorphisms

$$\xi_\lambda : (M_\lambda^\wedge, \nabla_\lambda^\wedge) \cong (M^{0\wedge}, \nabla^{0\wedge}),$$

where $(M_\lambda^\wedge, \nabla_\lambda^\wedge)$ is the formalization of $(V_\lambda, \nabla_\lambda)$, and we often identify ξ with the collection (ξ_λ). If $((\mathbf{V}, \nabla), \xi)$ and $((\mathbf{V}', \nabla'), \xi')$ are two isoformal families associated to (V^0, ∇^0). They are said to be *equivalent* if there is an isomorphism

$$\mathbf{a} : (\mathbf{M}, \nabla) \cong (\mathbf{M}', \nabla')$$

such that

$$\xi = \xi' \mathbf{a}^\wedge$$

where $\mathbf{a}^\wedge (\mathbf{M}^\wedge, \nabla^\wedge) \cong (\mathbf{M}'^\wedge, \nabla'^\wedge)$ is the natural extension of $\mathbf{a}$.

Let $St^0 = St(V^0, \nabla^0)$. Let $\mathcal{M}(V^0, \nabla^0)$ be, as in **I**, §1.4, the set of isomorphism classes of marked pairs formally equivalent to (V^0, ∇^0). After the results of **II**, Chapter 2, the space $H^1(S^1, St^0)$ may be viewed as an analytic manifold in a natural manner. Following the general principles of the theory of moduli one recognizes that in order to secure an interpretation of $H^1(S^1, St^0)$ as a moduli space for marked pairs one has to verify the following.

A (*Morphism property*). Fix a pair $((V_0, \nabla_0), \xi_0)$ whose isomorphism class is in $\mathcal{M}(V^0, \nabla^0)$ and let $\mathbf{f} = ((\mathbf{V}, \nabla), \xi)$ be a local isoformal deformation of it. Then for all λ near $\mathbf{0}$ the class of $((V_\lambda, \nabla_\lambda), \xi_\lambda)$ is in $\mathcal{M}(V^0, \nabla^0)$, and so by the Malgrange-Sibuya Theorem (**I**, Theorem 4.5.1) we obtain a map $\Phi_\mathbf{f}$ of neighbourhood of $\mathbf{0}$ into $H^1(S^1, St^0)$:

$$\Phi_\mathbf{f} : \lambda \longrightarrow \Phi((V_\lambda, \nabla_\lambda), \xi_\lambda).$$

It is obvious that the germ of this map depends only on the equivalence class of the family $((\mathbf{V}, \nabla), \xi)$. The *morphism property* is the assertion that this map is holomorphic.

B (*Criterion for equivalence*). This criterion says that two families

$$\mathbf{f} = ((\mathbf{V}, \nabla), \xi), \text{ and } \mathbf{f'} = ((\mathbf{V'}, \nabla'), \xi')$$

are equivalent if and only if $\Phi_\mathbf{f}$ and $\Phi_\mathbf{f'}$ define the same germ, i. e.,

$$\Phi_\mathbf{f} = \Phi_\mathbf{f'}$$

in a neighbourhood of $\mathbf{0}$.

C (*Existence of universal families*). Let $d = \dim H^1(S^1, St^0)$. Then, for any $((V_0, \nabla_0), \xi_0)$ a *universal local deformation* of it is a family $\mathbf{f} = ((\mathbf{V}, \nabla), \xi)$, defined over a polydisk Δ in $\mathbf{C}^d$, such that $V_0 = V_\mathbf{0}$, $\nabla_0 = \nabla_\mathbf{0}$, $\xi_\mathbf{0} = \xi_0$, and the map $\Phi_\mathbf{f}$ is a local analytic *isomorphism* of a neighbourhood of $\mathbf{0} \in \mathbf{C}^d$ into $H^1(S^1, St^0)$. The existence of such local universal deformations for arbitrary $((V_0, \nabla_0), \xi_0)$ is the third requirement for interpreting $H^1(S^1, St^0)$ as a moduli space for marked pairs.

THEOREM 1.1.2 $H^1(S^1, St^0)$ *is a local moduli space for marked meromorphic pairs whose formalizations are isomorphic to* (V^0, ∇^0).

We shall give the proof of this Theorem in the next four sections.

1.2. In this section we shall establish the morphism property. We begin with a simple observation. Let U, U' be open arcs of S^1 with $U \subset\subset U'$ and let $\Gamma = \Gamma(U)$, $\Gamma' = \Gamma(U')$ be the sectors on them. Let (M^0, ∇^0) be the differential module over $\mathfrak{F}_{cgt}$ of the germs of meromorphc sections of V^0 and $(M^{0\wedge}, \nabla^{0\wedge})$ its formalization. We then have the differential modules

$$\mathbf{M}^0(\Gamma') = A_{d,1}(\Gamma')\otimes_{\mathfrak{F}_{cgt}} M^0, \qquad \mathbf{M}^{0\wedge} = \mathcal{O}_{d,1}\otimes_{\mathfrak{F}_{cgt}} M^0$$

and the formalizing map $\mathbf{M}^0(\Gamma') \longrightarrow \mathbf{M}^{0\wedge}$. Suppose g is an automorphism of $\mathbf{M}^0(\Gamma')$ such that $g^\wedge = 1$, the identity automorphism of $\mathbf{M}^{0\wedge}$. It is then clear (for example, by choosing a trivialization of V^0) that g defines a family of automorphisms of (V^0, ∇^0) on the sectorial domain Γ_δ for some $\delta > 0$, preserving the asymptotic structure and satisfying

$$g(\lambda) \sim 1 \ (\Delta \times \Gamma_\delta).$$

Clearly each $g(\lambda)$ is an element of $St^0(U)$.

LEMMA 1.2.1 *The assignment*

$$\lambda \longrightarrow g(\lambda)$$

is an analytic map of a neighbourhood of **0** *into* $St^0(U)$. *Conversely, any such analytic map arises as above from some automorphism* g *of* $\mathbf{M}^0(\Gamma)$ *with* $g^\wedge = 1$.

PROOF We may assume that V^0 is the trivial bundle and that $\nabla^0{}_{d/dz} = d/dz - A^0$, where $A^0 \in \mathfrak{gl}(N, \mathfrak{F}_{cgt})$. Write $g(\lambda: z) = 1 + h(\lambda: z)$. Then h is an analytic map of $\Delta \times \Gamma_\delta$ into $\mathfrak{gl}(N, \mathbf{C})$ with $h \sim 0\ (\Delta \times \Gamma_\delta)$ satisfying

$$(*) \qquad dh/dz + hA^0 - A^0h = 0.$$

The solutions k of $(*)$ form a nilpotent Lie algebra, namely $\mathfrak{st}(U)$, and the map $1 \longrightarrow 1 + k$ is an isomorphism of affine varieties of $\mathfrak{st}(U)$ with $St^0(U)$ (cf.

I, Proposition 3.4.1, and **II**, Proposition 3.1.1), so that we need only to prove that $\lambda \longrightarrow h(\lambda)$ is analytic into $\mathfrak{st}(U)$. This is obvious since, for any $z_0 \in \Gamma_\delta$, the map $\lambda \longrightarrow h(\lambda : z_0)$ is analytic into $\mathfrak{gl}(N, \mathbf{C})$ while the map $k \longrightarrow k(z_0)$ is a *linear isomorphism* of $\mathfrak{st}(U)$ with a subspace of $\mathfrak{gl}(N, \mathbf{C})$. For the converse assertion it is enough to find for any $u \in U$ an open arc W, $u \in W \subset U$, such that $g(\lambda : z) \sim 1\ (\boldsymbol{\Delta} \times \Gamma(W)_\delta)$. By **I**, Proposition 3.4.1 we can find W and $y \in GL(N, A_b(\Gamma))$ such that $g(\lambda : z) = y(z)\, a(\lambda)\, y(z)^{-1}$ where $a(\lambda \longrightarrow a(\lambda))$ is an analytic map into $GL(N, \mathbf{C})$. It is then clear that $g(\lambda : z) \sim 1\ (\boldsymbol{\Delta} \times \Gamma(W)_\delta)$. ◆

Let us now consider a family $\mathbf{f} = ((\mathbf{V}, \nabla), \boldsymbol{\xi})$ of marked meromorphic pairs associated to (V^0, ∇^0). Let α be a positive number sufficiently small, say, $\alpha < \pi/(|r_1| - 1)$, r_1 being the principal level of the formalization of (V^0, ∇^0). Let $\mathfrak{U} = (U_i)$ be a finite covering of S^1 by open arcs of length $< \alpha$, and for each i let U_i' be an open arc of length $< \alpha$ with $U_i \subset\subset U_i'$; write $\Gamma_i = \Gamma(U_i)$, $\Gamma_i' = \Gamma(U_i')$. Since $\boldsymbol{\xi}$ is an isomorphism of $(\mathbf{M}^\wedge, \nabla^\wedge)$ with $\mathcal{O}_{d,1} \otimes_{\mathfrak{F}} M^{0\wedge}$, Theorem 2.2.3 allows us to choose for each i an isomorphism x_i,

$$x_i : \ A_{d,1}(\Gamma') \otimes_{\mathfrak{a}} \mathbf{M} \cong A_{d,1}(\Gamma') \otimes_{\mathfrak{F}_{cgt}} M^0, \qquad x_i^\wedge = \boldsymbol{\xi}.$$

If $g_{ij} = x_i x_j^{-1}$, g_{ij} is an automorphism of $A_{d,1}(\Gamma'_i \cap \Gamma_j') \otimes_{\mathfrak{F}_{cgt}} M^0$ and so defines a family of automorphisms $(g_{ij}(\lambda))_{\lambda \in \boldsymbol{\Delta}}$ of (V^0, ∇^0) on $(\Gamma_i \cap \Gamma_j)_\delta$ for sufficiently small $\boldsymbol{\Delta}$ and $\delta > 0$. Then $g(\lambda) = (g_{ij}(\lambda)) \in Z(\mathfrak{U} : St^0)$, the space of cocycles of St^0 attached to the covering $\mathfrak{U}$. Lemma 1.2.1 shows that $\lambda \longrightarrow g(\lambda)$ is an analytic map into $Z(\mathfrak{U} : St^0)$. But then it is clear from the discussion in **II**, §2.7 (Theorem 2.7.6) that $\lambda \longrightarrow [g(\lambda)]$ (= the cohomology class of $g(\lambda)$) is an analytic map into $H^1(S^1, St^0)$. This is of course the map $\Phi_{\mathbf{f}}$ defined by the family $\mathbf{f}$. This proves the morphism property. ◆

1.3. We shall now prove that $\Phi_{\mathbf{f}}$ determines the equivalence class of $\mathbf{f}$. We shall assume all the bundles to be trivial so that

$$V^0 = \Delta \times \mathbf{C}^N, \ \mathbf{V} = \mathbf{V}' = \boldsymbol{\Delta} \times \Delta \times \mathbf{C}^N.$$

Let $\mathbf{f}$ and $\mathbf{f}'$ be two families of marked meromorphic pairs that are isomorphic to (V^0, ∇^0) such that $\Phi_{\mathbf{f}} = \Phi_{\mathbf{f}'}$. We write A^0 for the connection matrix of ∇^0 and $A(\lambda : z)$ (resp. $A'(\lambda : z)$) for the connection matrix of ∇_λ (resp. ∇_λ'). In the notation of the previous section we now have a second family of cocycles $g'(\lambda) = (g_{ij}'(\lambda))_{\lambda \in \Delta}$, associated to ξ'(and choices of x_i' above ξ'), and the assumption is that $[g(\lambda)] = [g'(\lambda)]$. It is then immediate from **II**, Theorem 2.7.6 that there is a unique analytic map $\lambda \longrightarrow c(\lambda) = (c_i(\lambda))$ into $C(\mathfrak{A} : St^0)$ such that $g'(\lambda) = c(\lambda)[g(\lambda)]$ for all $\lambda \in \Delta$. Rewriting this in terms of the x_i and x_i' we find that

$$(*) \qquad x_i(\lambda)^{-1} c_i(\lambda)^{-1} x_i'(\lambda) = x_j(\lambda)^{-1} c_j(\lambda)^{-1} x_j'(\lambda) \qquad \text{on } \Gamma_i \cap \Gamma_j$$

for all i,j and $\lambda \in \Delta$. By Lemma 1.2.1 the collection $c_i(\lambda)_{\lambda \in \Delta}$ defines an automorphism c_i of $A_{d,1}(\Gamma_i \cap \Gamma_j) \otimes_{\mathfrak{F}_{cgt}} M^0$ with $c_i^\wedge = 1$; hence $x_i'^{-1} c_i^{-1} x_i = \mathbf{a}_i \in GL(N, A_{d,1}(\Gamma_i))$ with $\mathbf{a}_i[A] = A'$ and $\mathbf{a}_i^\wedge = \xi'^{-1}\xi$ for all i while $(*)$ shows that $\mathbf{a}_i$ and $\mathbf{a}_j$ coincide on $\Gamma_i \cap \Gamma_j$ for all i, j. We thus obtain an element $\mathbf{a} \in GL(N, \mathfrak{A})$ such that $\mathbf{a}[A] = A'$ and $\mathbf{a}^\wedge = \xi'^{-1}\xi$. It is now clear that $\mathbf{a}$ defines the equivalence of $\mathbf{f}$ and $\mathbf{f}'$. ◆

1.4. We shall now take up the construction of universal local deformations. This will be done in two stages. The first stage, to be carried out in this section, assumes that (V^0, ∇^0) is unramified; and the second stage, treated in the next section, completes the proof when this hypothesis is dropped. Throughout the proof we shall assume that all bundles are trivial with fiber $U = \mathbf{C}^N$. Thus to each element A of $\mathfrak{gl}(N, \mathfrak{F}_{cgt})$ we have a connection $\nabla(A)$ defined by

$$\nabla(A)_{d/dz} = d/dz - A,$$

on the trivial bundle $\Delta \times U$ for some disk Δ around $z = 0$, so that we may think of A or $\nabla(A)$ as defining the pair $(\Delta \times U, \nabla(A))$. We adopt a similar convention with regard to marked pairs and families, so that, for example, families are represented by elements A of $\mathfrak{gl}(N, \mathfrak{A})$ where $\mathfrak{A}$ is as in § 1.1. We write d for the irregularity of the endomorphism bundle of V^0 so that

$$d = \dim H^1(S^1, St^0).$$

Let $\nabla^0 = \nabla(A^0)$. We first suppose that $\nabla^0 \cong \nabla(B)$ under $GL(N, \mathfrak{F})$ where B is an unramified reduced canonical form. Let us consider the pair defined by $A_0 \in \mathfrak{gl}(N, \mathfrak{F}_{cgt})$ and an element ξ_0 of $GL(N, \mathfrak{F})$ such that $\xi_0[A_0] = A^0$. It is obvious that for the problem of constructing a universal family of local deformations of the marked pair (A_0, ξ_0) we may replace (A_0, ξ_0) by $(y[A_0], \xi_0 y^{-1})$ where $y \in GL(N, \mathfrak{F}_{cgt})$. Now there is an $\eta \in GL(N, \mathfrak{F})$ such that $\eta[A_0] = B$. If we truncate the Laurent series for η at a sufficiently high stage we shall obtain $y \in GL(N, \mathfrak{F}_{cgt})$ such that $y[A_0] = B + F$ where $F \in \mathfrak{gl}(N, \mathbf{C}\{z\})$. Thus there is no loss of generality in assuming that

$$(*) \qquad A_0 = B + F, \quad F \in \mathfrak{gl}(N, \mathbf{C}\{z\}).$$

We write $GL(N, \mathbf{C}\{z\})_1$ (resp. $GL(N, \mathbf{C}[[z]])_1$) for the subgroup of $GL(N, \mathbf{C}\{z\})$ (resp. $GL(N, \mathbf{C}[[z]])$) of elements whose leading term is 1. Also we use the usual notations and conventions regarding canonical forms (see **I**, §1.4). In particular,

$$B = \sum_{r \in L} z^r \otimes D_r + z^{-1} \otimes C,$$

where $L = \{r_1, r_2, \dots, r_m\}$ is the set of canonical levels of B, the r_i being integers with $r_1 < r_2 < \dots < r_m < -1$; C, D_r $(r \in L)$ are in $End(U)$ and commute with each other, and D_r is semisimple for all $r \in L$.

LEMMA 1.4.1 *If* $F_1 \in \mathfrak{gl}(N, \mathbf{C}[[z]])$, $\exists$ *a unique* $\xi \in GL(N, \mathbf{C}[[z]])_1$ *such that* $\xi[B + F_1] = B$.

PROOF Let $A_1 = B + F_1$. For the existence we may, in view of Lemma 6.2.2 of [BV 1], assume that $[A_1, D_r] = 0$ for all $r \in L$. By spectral splitting we come down to the case $A_1 = \sigma 1 + z^{-1}C + F_1$ where $\sigma \in \Sigma$, the spectrum of B. As C is reduced, Proposition 3.2 of [BV 1] applies to give $\xi \in GL(N, \mathbf{C}[[z]])_1$ such that $\xi[A_1] = \sigma 1 + z^{-1}C$. If $\xi' \in GL(N, \mathbf{C}[[z]])_1$ is such that $\xi'[A_1] = B$ also, then $t = \xi\xi'^{-1} \in GL(N, \mathbf{C}) \cap GL(N, \mathbf{C}[[z]])_1$, hence $t = 1$. ♦

Let $\mathfrak{gl}(N, \mathbf{C}[[z]])_+$ be the space of all $X \in \mathfrak{gl}(N, \mathbf{C}[[z]])$ with leading term 0.

LEMMA 1.4.2 *Let* $A_1 = B + F_1$ *be as in the preceding lemma. If* $Y^\wedge \in \mathfrak{gl}(N, \mathbf{C}[[z]])$, $\exists$ *a unique* $X^\wedge \in \mathfrak{gl}(N, \mathbf{C}[[z]])_+$ *such that* $\nabla(A_1)(X^\wedge) = Y^\wedge$.

PROOF By Lemma 1.4.1 it is enough to prove this with B in place of A_1. It is then a question of proving that for fixed $\sigma, \tau \in \Sigma$ and a given $Y^\wedge$,

$$Y^\wedge = Y_0 + zY_1 + \ldots. \qquad (Y_q \in \mathrm{Hom}_{\mathbf{C}}(U_\tau, U_\sigma)),$$

there is a unique $X^\wedge$,

$$X^\wedge = zX_1 + z^2X_2 + \ldots \qquad (X_q \in \mathrm{Hom}_{\mathbf{C}}(U_\tau, U_\sigma)),$$

with

$$dX^\wedge/dz + (\tau - \sigma)X^\wedge + z^{-1}(X^\wedge C_\tau - C_\sigma X^\wedge) = Y^\wedge.$$

Observe that the endomorphism $L(X' \longrightarrow X'C_\tau - C_\sigma X')$ of $\mathrm{Hom}_{\mathbf{C}}(U_\tau, U_\sigma)$ has as its only eigenvalues the numbers of the form $\alpha - \beta$ where α (resp. β) is an eigenvalue of C_τ (resp. C_σ); and $|\mathrm{Re}(\alpha - \beta)| < 1$ since C is reduced. The existence and uniqueness of the X_q follow from a simple calculation. Indeed, if $\sigma = \tau$, the X_q satisfy

$$(L + q + 1)(X_{q+1}) = Y_q \qquad (q \geq 0),$$

and since $L + q + 1$ is invertible, we are done. If $\sigma \neq \tau$, let

$$\tau - \sigma = c_{-r}z^{-r} + \ldots + c_{-2}z^{-2} \qquad (r \geq 2, c_{-r} \neq 0);$$

then the equations for the X_{q+r} $(q \geq -(r-1))$ become

$$X_{q+r} = c_{-r}^{-1}\left(Y_q - \Sigma_{2 \leq j < r}\, c_{-j}X_{q+j} - (L + q + 1)(X_{q+1})\right).$$

The existence and uniqueness of the X_q now follow by recursion. ♦

LEMMA 1.4.3 *Let* A_1, $X^\wedge$, $Y^\wedge$ *be as in Lemma* 1.4.2 *and suppose that* $F_1 \in \mathfrak{gl}(N, \mathbf{C}\{z\})$. *If* $W \subset S^1$ *is an open arc of length* $< \pi/(|r_1| - 1)$ *and* Y *is a holomorphic map* $\Gamma(W) \longrightarrow \mathrm{End}(\mathbf{C}^N)$ *such that* $Y \sim Y^\wedge$, *then we can find a holomorphic map* $X(\Gamma(W) \longrightarrow \mathrm{End}(\mathbf{C}^N))$ *such that on* $\Gamma(W)$,

$$\nabla(A_1)(X) = Y, \qquad X \sim X^\wedge.$$

PROOF The proof is by reduction to the case $F_1 = 0$. Suppose we have proved the lemma in this case. Let $\xi \in GL(N, \mathbf{C}[[z]])_1$ be such that $\xi[A_1] = B$. By **I**, Theorem 2.2.4 $\exists$ a holomorphic $x\ (\Gamma(W)_\delta \longrightarrow GL(N, \mathbf{C}))$ such that

$x[A_1] = B$ and $x \sim \xi$ on $\Gamma(W)$. If $Y_1{}^\wedge = \xi\, Y^\wedge \xi^{-1}$, $Y_1 = xY^\wedge x^{-1}$, $X_1{}^\wedge = \xi\, X^\wedge \xi^{-1}$, and $X_1\ (\Gamma(W)_\delta \longrightarrow \mathrm{End}(\mathbf{C}^N))$ is such that $\nabla(B)(X_1) = Y_1$ and $X_1 \sim X_1{}^\wedge$ on $\Gamma(W)_\delta$, then $X = x^{-1} X_1 x$ has the required properties. We shall now prove the lemma when $A_1 = B$. Select a holomorphic map $Z(\Gamma(W)_\delta \longrightarrow \mathrm{End}(\mathbf{C}^N))$ with $Z \sim X^\wedge\ (\Gamma(W))$; we shall seek X in the form $X = Z + P$, $P \sim 0(\Gamma(W))$. The equation for P is then seen to be

$$dP/dz + [P, B] = Q,$$

where

$$Q = -\,dZ/dz - [Z, B] + Y \sim 0\,(\Gamma(W)),$$

and this is equivalent to the system of equations

$$dP_{\sigma\tau}/dz + (\tau - \sigma)\, P_{\sigma\tau} + z^{-1}\,(P_{\sigma\tau}\, C_\tau - C_\sigma\, P_{\sigma\tau}) = Q_{\sigma\tau},\quad P_{\sigma\tau} \sim 0\,(\Gamma(W))$$

for all $\sigma, \tau \in \Sigma$. If $\sigma \neq \tau$ we use **I**, Theorem 2.3.1 to obtain the existence of $P_{\sigma\tau}$. If $\sigma = \tau$, then the equation becomes

$$dP_{\sigma\sigma}/dz + z^{-1}\,(P_{\sigma\sigma}\, C_\sigma - C_\sigma\, P_{\sigma\sigma}) = Q_{\sigma\sigma}, \qquad P_{\sigma\sigma} \sim 0\,(\Gamma(W)).$$

Write $h = \exp(\log z.\, C_{\sigma\sigma})$, $R = h^{-1} Q_{\sigma\sigma}\, h$; then, as $R \sim 0\,(\Gamma(W))$ we can find holomorphic $S\ (\Gamma(W)_\delta \longrightarrow \mathrm{End}(\mathbf{C}^N))$ such that $dS/dz = R$ on $\Gamma(W)_\delta$ and $S \sim 0\,(\Gamma(W))$; we may then take $P_{\sigma\sigma} = h\, S\, h^{-1}$. ♦

Let us write, for any $A \in \mathfrak{gl}(N, \mathbf{C}\{z\})$, $\mathfrak{st}(A)$ for the infinitesimal Stokes sheaf of the pair $(\Delta \times U, \nabla(A))$ and $St(A)$ for the corresponding Stokes sheaf.

LEMMA 1.4.4 *Let* $A = B + F$, $F \in \mathfrak{gl}(N, \mathbf{C}\{z\})$. *Define*

$$R(A) = \left\{ Y \in \mathfrak{gl}(N, \mathbf{C}\{z\}) : Y = \nabla(A)\,(X) \ \textit{for some}\ X \in \mathfrak{gl}(N, \mathbf{C}\{z\})_+ \right\}.$$

Then there is a natural isomorphism of complex vector spaces

$$h : \mathfrak{gl}(N, \mathbf{C}\{z\}) \,/\, R(A) \cong H^1(S^1, \mathfrak{st}(A)).$$

In particular,

$$\dim_{\mathbf{C}}\,(\mathfrak{gl}(N, \mathbf{C}\{z\}) \,/\, R(A)) = d.$$

PROOF Let $\mathcal{S}$ be the sheaf on S^1 of germs of holomorphic maps $\Gamma(U)_\delta \longrightarrow \mathrm{End}(\mathbf{C}^N)$ whose entries have asymptotic expansions in $\mathbf{C}[[z]]$ and let $\mathcal{S}_+$ be the subsheaf of those elements whose asymptotic expansions lie in $\mathfrak{gl}(N, \mathbf{C}[[z]])_+$. Then $\nabla(A)$ is a sheaf map of $\mathcal{S}$ into itself and the preceding lemma implies that $\nabla\mathcal{S}_+$ is precisely the sheaf $\mathcal{S}$. Let us next introduce the sheaf $\mathfrak{M}$ which is the subsheaf of $\mathcal{S}_+$ of all elements that are in the kernel of $\nabla(A_0)$. We then have the following exact sequence

$$0 \longrightarrow \mathfrak{M} \longrightarrow \mathcal{S}_+ \longrightarrow \mathcal{S} \longrightarrow 0$$

where the map $\mathcal{S}_+ \longrightarrow \mathcal{S}$ is the one defined by $\nabla(A)$. Let us now look at the corresponding long exact sequence. It is easy to verify that

$$H^0(S^1, \mathcal{S}_+) = \mathfrak{gl}(N, \mathbf{C}\{z\})_+, \qquad H^0(S^1, \mathcal{S}) = \mathfrak{gl}(N, \mathbf{C}\{z\}).$$

Moreover, as the kernel of $\nabla(A)$ on $\mathfrak{gl}(N, \mathbf{C}\{z\})_+$ is zero, we have

$$H^0(S^1, \mathfrak{M}) = 0.$$

Hence we obtain the exact sequence

$$0 \longrightarrow \mathfrak{gl}(N, \mathbf{C}\{z\})_+ \longrightarrow \mathfrak{gl}(N, \mathbf{C}\{z\}) \longrightarrow H^1(S^1, \mathfrak{M}) \longrightarrow$$

$$\longrightarrow H^1(S^1, \mathcal{S}_+) \longrightarrow H^1(S^1, \mathcal{S}) \longrightarrow 0$$

where the map going from $\mathfrak{gl}(N, \mathbf{C}\{z\})_+$ to $\mathfrak{gl}(N, \mathbf{C}\{z\})$ is $\nabla(A)$. Thus we get the exact sequence

$$0 \longrightarrow \mathfrak{gl}(N, \mathbf{C}\{z\})/R(A) \longrightarrow H^1(S^1, \mathfrak{M}) \longrightarrow H^1(S^1, \mathcal{S}_+)$$

and it is a question of proving that

$$\ker\,(H^1(S^1, \mathfrak{M}) \longrightarrow H^1(S^1, \mathcal{S}_+)) = H^1(S^1, \mathfrak{st}(A)).$$

Suppose we take an element of this kernel and represent it by a cocycle (g_{ij}) with respect to a finite covering (U_i) of S^1 by open arcs of sufficiently small length. If $g_{ij} \sim g_{ij}{}^\wedge$, then $\nabla(A)(g_{ij}) = 0 \Rightarrow \nabla(A)(g_{ij}{}^\wedge) = 0$; lemma 1.4.2 now allows us to conclude that $g_{ij}{}^\wedge = 0$, as $g_{ij}{}^\wedge$ lies in $\mathfrak{gl}(N, \mathbf{C}[[z]])_+$. Thus the cocycle belongs to $\mathfrak{st}(A)$. In the other direction, suppose that we start with a cocycle (g_{ij}) associated to $\mathfrak{st}(A)$. By the Mlagrange-Sibuya Theorem (**I**, Theorem 4.4.1) we can find an element $c^\wedge \in \mathfrak{gl}(N, \mathbf{C}[[z]])$ and $c_i \in \mathcal{A}(U_i)$ so that $c_i \sim c_i^\wedge$ and $g_{ij} = c_i - c_j$. As $c^\wedge$ is determined only mod $\mathfrak{gl}(N, \mathbf{C}\{z\})_+$, we may assume that $c^\wedge \in \mathfrak{gl}(N, \mathbf{C}[[z]])_+$. But this is just the assertion that the cocycle trivializes in $\mathcal{A}_+$. ◆

Let us continue with $A_0 = B + F$ where $F \in \mathfrak{gl}(N, \mathbf{C}\{z\})$. Consider a family of connections $\nabla(L(\lambda))$ $(\lambda \in \Delta)$, Δ a polydisk in $\mathbf{C}^d$, such that

(a) $L(\mathbf{0}) = A_0$

(b) $L(\lambda) = B + M(\lambda)$ where $M(\lambda) \in \mathfrak{gl}(N, \mathbf{C}\{z\})$, and for some $\delta > 0$, and some analytic map $M\ (\Delta \times \Delta_\delta \longrightarrow \mathfrak{gl}(N, \mathbf{C}))$,

$$M(\lambda)(z) = M(\lambda : z)$$

DEEFINITION $\nabla(L(\lambda))$ is an *infinitesimally versal family* if the linear span $\mathfrak{L}$ of $(\partial L/\partial\lambda_i)_{\lambda=\mathbf{0}}$ $(1 \le i \le d)$ is linearly independent of $R(A)$, $R(A)$ being as in Lemma 1.4.4. Note that $(\Delta \times U, \nabla(L(\lambda))$ is an analytic family of meromorphic pairs. It is clear from Lemma 1.4.4 that

$$\mathfrak{gl}(N, \mathbf{C}\{z\}) = R(A) \oplus \mathfrak{L}.$$

It is also easy to see that we can always find such an infinitesimally versal family of connections through A. Indeed, if $\mathfrak{L}$ is a linear subspace of dimension d complementary to $R(A)$ in $\mathfrak{gl}(N, \mathbf{C}\{z\})$ and $\{B_j\}_{1 \le j \le d}$ is a basis for $\mathfrak{L}$, and if we set

$$L(\lambda) = A + \textstyle\sum_{1 \le j \le d} \lambda_j B_j,$$

then it is obvious that $\{L(\lambda)\}$ is an infinitesimally versal family of connections through A.

Our aim now is to show that an infinitesimally versal family of connections through A defines a universal family of local deformations of a pair (A, ξ). We need a preparatory lemma.

LEMMA 1.4.5 *Let* $q \ge 1$ *be an integer,* $\Delta \subset \mathbf{C}^q$ *a polydisk, and* $L = B + M$ *where* $M\ (\Delta \times \Delta \longrightarrow \mathfrak{gl}(N, \mathbf{C}))$ *is analytic ; let* $L(\lambda)(z) = L(\lambda : z)$. *Let* η *be an element of* $GL(N, \mathfrak{F})$ *such that* $\eta[B] = A^0$. *For each* $\lambda \in \Delta$ *let*

$\alpha(\lambda)$ *be the unique element of* $GL(N, \mathbf{C}[[z]])_1$ *such that* $\alpha(\lambda)[L(\lambda)] = B$. *Then* $\xi(\lambda \longrightarrow \xi(\lambda) = \eta\alpha(\lambda))$ *defines an element of* $GL(N, \mathcal{O}_{q,1}(\Delta_1))$ *for some* $\Delta_1 \subset \Delta$, *and* $(\nabla(L(\lambda)), \xi(\lambda))$ *is a family of marked meromorphic pairs formally isomorphic to* $\nabla(B)$. *In particular, if* L *is infinitesimally versal, there is* $\Delta_1 \subset \Delta$ *and* $\xi \in GL(N, \mathcal{O}_{d,1}(\Delta_1))$ *such that* $(\nabla(L), \xi)$ *is a family of marked meromorphic pairs associated to* $\nabla(A^0)$.

PROOF It is enough to prove that α $(\lambda \longrightarrow \alpha(\lambda))$ defines an element of $GL(N, \mathcal{O}_{q,1}(\Delta_1))$. In view of the uniqueness of $\xi(\lambda)$ it suffices to exhibit a Δ_1 and an element ζ of $GL(N, \mathcal{O}_{q,1}(\Delta_1))$ such that $\zeta[L] = B$. But this is precisely what is done in Corollary 6.7.6 of [BV 2]. The second assertion is clear from Lemma 1.4.1. ♦

PROPOSITION 1.4.6 *Suppose* $\{L(\lambda)\}$ *is an infinitesimally versal family of connections through* A_0 *and* $(\nabla(L), \xi)$ *the corresponding family of marked pairs. Then* $(\nabla(L), \xi)$ *is universal for* $(\nabla(A_0), \xi(\mathbf{0}))$.

PROOF We fix a finite covering $\mathfrak{a} = (U_i)$ of S^1 by open arcs of length $< \pi/(|r_1| - 1)$. Then, as in § 1.2 we obtain a cocycle g which is an analytic map of a (sufficiently small) polydisk Δ into $Z = Z(\mathfrak{a} : St(A^0))$ such that the Malgrange-Sibuya class $\Phi(L(\lambda), \xi(\lambda))$ is the cohomology class of $g(\lambda)$. Now, by **II**, Theorem 2.7.6, the map

$$\varphi : c \longrightarrow c[g(\mathbf{0})]$$

ia an analytic diffeomorphism into Z. Let S denote the range of the differential of this map. We must prove that the map $\lambda \longrightarrow [g(\lambda)]$ has an injective differential at $\lambda = \mathbf{0}$, and for this it is a question of proving that if w_p $(1 \le p \le d)$ are complex numbers,

$$\textstyle\sum_{1 \le p \le d} w_p \left(\partial g/\partial \lambda_p\right)_{\lambda = \mathbf{0}} \in S \implies w_1 = w_2 = \dots = w_d = 0.$$

Let us fix the w_p and assume that

$$(*) \qquad \textstyle\sum_{1 \le p \le d} w_p \left(\partial g/\partial \lambda_p\right)_{\lambda = \mathbf{0}} \in S.$$

Define

$$M = \Sigma_{1 \le p \le d}\, w_p \left(\partial L/\partial \lambda_p\right)_{\lambda = 0} \in \mathfrak{gl}(N, \mathbf{C}\{z\}).$$

It is then sufficient to show that if $(*)$ is true, then $M \in R(A)$; for, as L is infinitesimally versal, the map

$$(u_1, u_2, \ldots, u_d) \longrightarrow \Sigma_{1 \le p \le d}\, u_p \left(\partial L/\partial \lambda_p\right)_{\lambda = 0}$$

is a linear isomorphism of $\mathbf{C}^d$ onto a subspace of $\mathfrak{gl}(N, \mathbf{C}\{z\})$ that is complementary to $R(A)$.

For brevity write

$$\partial = \Sigma_{1 \le p \le d}\, w_p \left(\partial/\partial \lambda_p\right)_{\lambda = 0}.$$

Then $(*)$ means that for some tangent vector τ to $C(\mathfrak{A} : St(A^0))$ at 1 we have $\partial g = (d\varphi)_{c = 1}(\tau)$. Let $t \longrightarrow c(t)$ be an analytic map of a neighbourhood of 0 in $\mathbf{C}$ into $C(\mathfrak{A} : St(A^0))$ such that $c(0) = 1$ and $(dc/dt)_{t=0} = \tau$. Then $\tau \in C(\mathfrak{A} : \mathfrak{st}(A^0))$, say $\tau = (\tau_i)$. If we write $c(t) = (c_i(t))$, we have

$$\partial g = \left(d/dt\right)_{t=0} \left(c_i(t)\, g_{ij}(\mathbf{0})\, c_j(t)^{-1}\right)$$

and hence

$$\partial g_{ij} = \tau_i\, g_{ij}(\mathbf{0}) - g_{ij}(\mathbf{0})\, \tau_j$$

on $\Gamma(U_i \cap U_j)_\delta$. But as $g_{ij} = x_i x_j^{-1}$, we have, using the above formulae for ∂g_{ij},

$$x_i(\mathbf{0})^{-1}\, \partial x_i - x_i(\mathbf{0})^{-1}\, \tau_i\, x_i(\mathbf{0}) = x_j(\mathbf{0})^{-1}\, \partial x_j - x_j(\mathbf{0})^{-1}\, \tau_j\, x_j(\mathbf{0})$$

on $\Gamma(U_i \cap U_j)_\delta$ for all i, j. These relations show that there is an analytic map ζ of the punctured disk $\Delta_\delta{}^\times$ into $\mathrm{End}(\mathbf{C}^N)$ such that

(#) $$x_i(\mathbf{0})^{-1}\, \partial x_i - x_i(\mathbf{0})^{-1}\, \tau_i\, x_i(\mathbf{0}) = \zeta$$

on $\Gamma(U_i)_\delta$ for all i. On the other hand, as $\tau_i \in \mathfrak{st}(A^0)(U_i)$, we have $\tau_i \sim 0$ on $\Gamma(U_i)$ while $x_i(\mathbf{0}) \sim \xi(\mathbf{0})$ and $\partial x_i \sim \partial \xi$ on the same sector. Hence,

$$\zeta \sim \xi(\mathbf{0})^{-1} \partial \xi = \alpha(\mathbf{0})^{-1}\, \partial \alpha \quad (\Gamma(U_i))$$

for all i. As usual we conclude that ζ is meromorphic on the full disk. As $\alpha(\lambda)$ is in $GL(N, \mathbf{C}[[z]])_1$ for all λ, $\partial\alpha \in \mathfrak{gl}(N, \mathbf{C}[[z]])_+$, so that $\zeta \in \mathfrak{gl}(N, \mathbf{C}\{z\})_+$.

To prove that $M \in R(A)$ it is enough to prove that

$$(**) \qquad d\zeta/dz + [\zeta, A_0] = -M.$$

First, as $x_i[L] = A^0$ on $\Delta_1 \times \Gamma(U_i)_\delta$, we have,

$$dx_i(\lambda)/dz + x_i(\lambda)\, L(\lambda) - A^0 x_i(\lambda) = 0.$$

We take $\lambda = \mathbf{0}$ in this and also apply ∂ to it to get the following two relations

$$dx_i(\mathbf{0})/dz + x_i(\mathbf{0})\, A_0 - A^0 x_i(\mathbf{0}) = 0,$$

(##)

$$d(\partial x_i)/dz + (\partial x_i)\, A_0 + x_i(\mathbf{0})\, M - A^0(\partial x_i) = 0.$$

A not too difficult calculation based on the relations (#) and (##) shows that (**) follows from them. ◆

Essentially the same calculations lead to the following condition for deciding when a given family is universal. Let $A_0 \in \mathfrak{gl}(N, \mathfrak{F}_{cgt})$, $\xi_0[A_0] = A^0$.

PROPOSITION 1.4.7 *Let* $\{L(\lambda), \xi(\lambda)\}$ *be a family of marked meromorphic pairs defined for* λ *in a polydisk* $\Delta \subset \mathbf{C}^q$ *with* $L(\mathbf{0}) = A_0$, $\xi(\mathbf{0}) = \xi_0$. *Write* $\mathfrak{X} \subset \mathfrak{gl}(N, \mathfrak{F}_{cgt}) \times \mathfrak{gl}(N, \mathfrak{F})$ *for the linear span of* $(\partial_i L, \partial_i \xi)$, *where* $\partial_i = (\partial/\partial\lambda_i)_{\lambda=\mathbf{0}}$ $(1 \le i \le d)$. *Let us now suppose that* $\mathfrak{X}$ *satisfies the following conditions*:

(a) $\dim_{\mathbf{C}} \mathfrak{X} = q$

(b) $\mathfrak{X}$ *is linearly independent of the range of the map*

$$\eta \longrightarrow (\nabla(A_0)(\eta), -\xi_0\eta)$$

of $\mathfrak{gl}(N, \mathfrak{F}_{cgt})$ *into* $\mathfrak{gl}(N, \mathfrak{F}_{cgt}) \times \mathfrak{gl}(N, \mathfrak{F})$.

Then $\lambda \longrightarrow \Phi((L(\lambda), \xi(\lambda))$ *has injective differential at* $\lambda = \mathbf{0}$. *Suppose* $q = d$.

Then $\{(L(\lambda), \xi(\lambda)\}$ *is a universal family of local deformations of* (A_0, ξ_0).

PROOF We proceed exactly as before and construct $\zeta \in \mathfrak{gl}(N, \mathcal{F}_{cgt})$ such that $\nabla(A_0)(-\zeta) = M = \partial L$, with $\zeta \sim \xi(\mathbf{0})^{-1}\,\partial\xi$; as ζ is a convergent element, $\zeta = \xi(\mathbf{0})^{-1}\,\partial\xi$. So, $(\nabla(A_0)(-\zeta), \xi_0\,\zeta) = (\partial L, \partial\xi)$, which implies, by (b), that $\partial L = 0, \partial\xi = 0$, so that $\partial = 0$. ♦

1.5. In this section we shall drop the assumption that (V^0, ∇^0) is unramified. Let b be an integer divisible by its ramification index and let us introduce the complex plane $\mathbf{C}_\zeta$ of $\zeta = z^{1/b}$. We shall generally use the symbol ~ to indicate the lift to $\mathbf{C}_\zeta$ of an object in $\mathbf{C}_z$. We shall also identify $H^1(S^1, St(A^0))$ with $H^1(S^1, St(A^{0\sim}))^{inv}$, the subspace of cohomology classes that are invariant under the Galois group μ_b, as we are allowed to do so by **II**, Theorem 3.4.2. Fix a pair (A_0, ξ_0) associated to A^0 so that $\xi_0[A_0] = A^0$. Let h_0 be the class in $H^1(S^1, St(A^0))$ defined by (A_0, ξ_0). We write Φ (resp. $\Phi^\sim$) for the Malgrange-Sibuya map associated to (A_0, ξ_0) (resp. $(A_0^\sim, \xi_0^\sim)$), so that

$$\Phi(A_0, \xi_0) = \Phi^\sim(A_0^\sim, \xi_0^\sim) = h_0.$$

By what we have established so far, we can find a universal local deformation $\{A'(\mu), \xi'(\mu)\}$ of $(A_0^\sim, \xi_0^\sim)$). By taking the preimage of the invariant classes under the map $\mu \longrightarrow \Phi^\sim(A'(\mu), \xi'(\mu))$ we see that there is a family $\{A(\lambda), \xi(\lambda)\}$ defined for λ in a polydisk Δ of dimension $d = \dim H^1(S^1, St(A^0))$ with the following properties:

(a) $A(\mathbf{0}) = A_0^\sim, \xi(\mathbf{0}) = \xi_0^\sim$

(b) $\lambda \longrightarrow \Phi^\sim(A(\lambda), \xi(\lambda))$ is an analytic diffeomorphism of Δ with an open neighbourhood of h_0 in $H^1(S^1, St(A^{0\sim}))^{inv}$.

We shall show that this family is equivalent to a family coming from the *z*-plane. More precisely we shall prove the following Proposition.

PROPOSITION 1.5.1 *There exists an* $y \in GL(N, \mathcal{Q}_\zeta)$ *and a family* $(B(\lambda), \eta(\lambda))$ *with* $B(\lambda) \in \mathfrak{gl}(N, \mathbf{C}\{z\})$, $\eta(\lambda) \in GL(N, \mathcal{F})$, *and* $\eta(\lambda)[B(\lambda)] = A^0$, *such that* $y(\lambda)[A(\lambda)] = B(\lambda)^\sim$ *and* $\xi(\lambda)\,y^{-1} = \eta(\lambda)^\sim$ *for all* λ.

PROOF The proof requires three lemmas.

LEMMA 1.5.2 *There is a finite covering* $\mathfrak{B} = (U_i)$ *of* $S^{1,b}$ *by open arcs and gauge transformations* $x_i\ (\Delta \times \Gamma(U_i)_\delta \longrightarrow GL(N, \mathbf{C}))$ *such that*

(a) $x_i \sim \xi\ (\Delta \times \Gamma(U_i)_\delta)$

(b) $x_i[A] = A^{0\sim}$

(c) $\mathfrak{B}$ *is a good covering and the set of indices* i *admits a free action by* μ_b *such that* $\omega U_i = U_{\omega(i)}\ (\omega \in \mu_b)$

(d) *the cocycles* $g(\lambda) = (x_i(\lambda)\, x_j(\lambda)^{-1})$ *are invariant under* μ_b, *i. e.,*

$$x_{\omega(i)}(\lambda : \omega\zeta)\, x_{\omega(j)}(\lambda : \omega\zeta)^{-1} = x_i(\lambda : \zeta)\, x_j(\lambda : \zeta)^{-1}$$

PROOF Let N be an integer $>> 1$. We shall take $U_i = (t_i, t_{i+2})$ where the $t_i\ (i \in \mathbf{Z},\ t_i = t_{i+Nb})$ are points of division of $S^{1,b}$ into Nb arcs of equal length, ordered counter clockwise; if $\omega = e^{2i\pi/b}$, then $\omega U_i = U_{i+N}$. We first find gauge transformations u_i having the properties (a) and (b); the condition (d) need not be satisfied, and the point of the proof is to show that the u_i can be adjusted so that this can be assured. The argument is a variation of the one used in **II**, Proposition 1.3.3.

Let $h(\lambda)$ be the cocycle $(u_i(\lambda)\, u_j(\lambda)^{-1})$. Then for any $\omega \in \mu_b$ the transform of $h(\lambda)$ by ω^{-1} is given by

$$\omega^{-1}.\, h(\lambda) = \big((\omega^{-1}.\, u_{\omega(i)}(\lambda))\, (\omega^{-1}.\, u_{\omega(i)}(\lambda))\big) \quad (\omega^{-1}.u_k(\lambda : \zeta) = u_k(\lambda : \omega\zeta)).$$

The assumption is that $\omega^{-1}.\, h(\lambda)$ and $h(\lambda)$ define the same cohomology class for all λ. So, by **II**, Theorem 2.7.6 we can find unique $c(\omega : \lambda) \in C(\mathfrak{B} : St(A^{0\sim}))$ such that $\omega.\, h(\lambda) = c(\omega : \lambda)[h(\lambda)]$ for all λ and $\lambda \longrightarrow c(\omega : \lambda)$ is analytic for all ω. We have

$$(*) \qquad c(1 : \lambda) = 1, \quad c(\sigma\omega : \lambda) = (\sigma.\, c(\omega : \lambda))\, c(\sigma : \lambda),$$

where the action of ω on the coboundary $c = (c(i))$ is given by

$$(\omega.\, c)(i) = \omega.\, c(\omega^{-1}(i)) \qquad ((\omega.f)(\zeta) = f(\omega^{-1}\zeta))$$

We shall now construct an analytic map $\lambda \longrightarrow d(\lambda)$ into $C(\mathfrak{B} : St(A^{0\sim}))$ such that

$$(**) \qquad c(\sigma : \lambda) = (\sigma . d(\lambda))\, d(\lambda)^{-1}$$

for all λ. Supposing that we have done this, we shall now complete the proof of the Lemma. By Lemma 1.2.1 we can view the map $\lambda \longrightarrow d(\lambda)$ as a collection of analytic maps d_i $(\lambda, \zeta \longrightarrow d(\lambda : \zeta)(i))$ into $GL(N, \mathbf{C})$ such that $d_i^{-1} \sim 1$ and $d_i[A^{0\sim}] = A^{0\sim}$ on $\Delta \times \Gamma(U_i)_\delta$. So if $x_i = d_i^{-1} u_i$, then (a) and (b) are satisfied obviously. Furthermore (d) is also true; for $g(\lambda) = d(\lambda)^{-1}[h(\lambda)]$, and a simple calculation shows that $g(\lambda)$ is invariant under μ_b.

It remains to construct d. We write each of the indices i $(0 \le i < N b)$ uniquely as $\omega(j)$ for some $\omega \in \mu_b$ and some j, $0 \le j < N$, and the sections of $St(A^{0\sim})$ on U_i uniquely as $\omega . s$ where s is a section on U_j. Then we can identify the collection $(c(\sigma : \lambda))$ with the collection of analytic maps $(d(\sigma, \omega, j))$ where

$$d(\sigma, \omega, j) : \Delta \longrightarrow St(A^{0\sim})$$

$$c(\sigma : \lambda)(\omega(j)) = \omega . d(\sigma, \omega, j)(\lambda).$$

The relations satisfied by the $c(\sigma : \lambda)$ then translate into

$$d(1, \omega, j) = 1, \qquad d(\sigma\tau, \omega, j) = d(\tau, \sigma^{-1}\omega, j)\, d(\sigma, \omega, j).$$

It follows immediately from these that

$$d(\tau, \eta, j) = d(\eta^{-1}\tau, 1, j)\, d(\eta^{-1}, 1, j)^{-1}.$$

If we define

$$d(\lambda)(\omega(j)) = \omega . d(\omega^{-1}, 1, j)(\lambda),$$

the preceding relations retranslate into

$$c(\tau : \lambda)(\eta(j)) = \tau . d(\lambda)(\tau^{-1}\eta(j))\, d(\lambda)(\eta(j))^{-1},$$

which is just a restatement of $(**)$. ♦

LEMMA 1.5.3 *There is* $t_\omega \in GL(N, \mathfrak{A}_\zeta)$ $(\omega \in \mu_b)$ *such that*

$$t_\omega = \xi^{-1}\,\xi^\omega \qquad (\omega \in \mu_b)$$

where $\xi^\omega(\lambda:\zeta) = \xi(\lambda:\omega^{-1}\zeta)$. *In particular*

$$t_{\omega\sigma} = t_\omega\,(t_\sigma)^\omega.$$

PROOF From the relation (d) of the preceding Lemma we find that

$$x_i(\lambda:\zeta)^{-1}\,x_{\omega(i)}(\lambda:\omega\zeta) = x_j(\lambda:\zeta)^{-1}\,x_{\omega(j)}(\lambda:\omega\zeta)$$

on $\Gamma(U_i \cap U_j)_\delta$ for all i, j, while the left side is asymptotic to $\xi(\lambda:\zeta)^{-1}\,\xi(\lambda:\omega\zeta)$. So we conclude in the usual manner that there is an element $t'(\omega) \in GL(N, \mathfrak{a}_\zeta)$ such that $t'(\omega)(\lambda:\zeta)$ coincides with $x_i(\lambda:\zeta)^{-1}\,x_{\omega(i)}(\lambda:\omega\zeta)$ on $\Delta \times \Gamma(U_i)_\delta$ for all i. If we write $t_\omega = t'(\omega^{-1})$, we get, on replacing the x_i by their asymptotic expansions, the required relation between the ξ's and t's. ◆

LEMMA 1.5.4 *There is an* $y \in GL(N, \mathfrak{a}_\zeta)$ *such that*

$$t_\omega = y^{-1}\,y^\omega \qquad (\omega \in \mu_b).$$

PROOF The argument is a minor variation of the usual proof of the vanishing of $H^1(G, GL(N, K))$ for a *field* K/k with Galois group G ([Se], p. 159). For any γ in the ring $\mathrm{End}(\mathcal{O}_{d,1}{}^N)$ of N×N matrices over $\mathcal{O}_{d,1}$, we form the sum over μ_b,

$$D(\gamma) = \textstyle\sum_\omega t_\omega\,\gamma^\omega.$$

It is now a question of showing that for some $\gamma \in \mathrm{End}(\mathfrak{a}_\zeta{}^N)$, $D(\gamma) \in GL(N, \mathfrak{a}_\zeta)$. For, suppose this is true for $\gamma = \gamma_0$. Then Lemma 1.5.3 shows that $D(\gamma_0)$ satisfies the relation $D(\gamma_0)^\omega = t_\omega^{-1}\,D(\gamma_0)$, so that the proof is completed on taking $y = D(\gamma_0)^{-1}$.

To prove that $D(\gamma) \in GL(N, \mathfrak{a}_\zeta)$ for some $\gamma \in \mathrm{End}(\mathfrak{a}_\zeta{}^N)$ we argue as follows. Since $t_\omega = \xi^{-1}\,\xi^\omega$ we have $D(\xi^{-1}) = b\,\xi^{-1} \in GL(N, \mathcal{O}_{d,1})$. So, if ξ_m is the Laurent polynomial obtained by dropping from ξ all terms z^r with $r > m$, we have, $\lim_{m\to\infty} \xi_m = \xi$ in the adic topology, so that $\xi_m \in GL(N, \mathfrak{a}_\zeta)$ if m is sufficiently large. Hence,

$$\lim_{m\to\infty} D(\xi_m{}^{-1}) = D(\xi^{-1}) \in GL(N, \mathcal{O}_{d,1}),$$

showing that $D(\xi_m^{-1}) \in GL(N, \mathcal{O}_\zeta)$ for sufficiently large m. ◆

The proof of Proposition 1.5.1 is now immediate. For, by Lemmas 1.5.3 and 1.5.4, we have,

$$\xi\, y^{-1} = (\xi\, y^{-1})^\omega \qquad (\omega \in \mu_b),$$

and so there is $\eta \in GL(N, \mathcal{O}_{d,1})$ such that $\xi\, y^{-1} = \eta^\sim$. Then $\eta^\sim y[A] = A^{0\sim}$ or $y[A] = \eta^{\sim -1}[A^{0\sim}]$, so that $y[A]$ is invariant. We then define $B(\lambda)$ by $y[A(\lambda)] = B(\lambda)^\sim$. ◆

With this the proof of Theorem 1.1.2 is complete. ◆

2 LOCAL MODULI SPACE FOR MEROMORPHIC PAIRS

2.1 We shall now treat the moduli problem for the meromorphic pairs themselves without any markings. We fix a pair (V^0, ∇^0) as before and consider the set $\mathfrak{M}(V^0, \nabla^0)$ (resp. $\mathfrak{I}(V^0, \nabla^0)$) of asomorphism classes of marked (resp. unmarked) pairs associated to (V^0, ∇^0). Let $G^{0\wedge}$ be the group of automorphisms of the formalization $(M^{0\wedge}, \nabla^{0\wedge})$ of (V^0, ∇^0). By Theorem 4.5.2 $G^{0\wedge}$ operates on $\mathfrak{M}(V^0, \nabla^0)$ and we have a natural bijection

$$\mathfrak{I}(V^0, \nabla^0)) \cong G^{0\wedge} \backslash \mathfrak{M}(V^0, \nabla^0);$$

the map

$$\mathfrak{M}(V^0, \nabla^0) \longrightarrow \mathfrak{I}(V^0, \nabla^0))$$

is the one that takes the isomorphism class of $((V, \nabla), \xi)$ to the isomorphism class of (V, ∇). Our aim now is to prove the following proposition.

PROPOSITION 2.1.1 $G^{0\wedge}$ *may be viewed in a natural manner.as a complex affine algebraic group acting morphically on* $H^1(S^1, St^0)$, *and the actions on* $H^1(S^1, St^0)$ *and* $\mathfrak{M}(V^0, \nabla^0)$ *commute with the Malgrange-Sibuya isomorphism* Φ.

The proof requires a little preparation. We begin with a few remarks on sheaves of complex associative algebras. Let X be a topological space and $\mathcal{A}$ be a sheaf of complex associative algebras (always with units). Then for any **C**-algebra R we have a sheaf $\mathcal{A}(R)$ of associative R-algebras defined by $\mathcal{A}(R)(U) = R \otimes_{\mathbf{C}} \mathcal{A}(U)$, $U \subset X$ being open; the assignment $R \longrightarrow \mathcal{A}(R)$ is a covariant functor. Hence, denoting the group of units of any ring M by $M^\times$, we see that

$$\mathcal{A}(R)^\times : U \longrightarrow \mathcal{A}(R)(U)^\times$$

is a sheaf of groups on X and that $R \longrightarrow \mathcal{A}(R)^\times$ is a covariant functor. We thus obtain a sheaf of group schemes over **C** (see **II**, Chapter 2); if for an open set

U, $\mathcal{A}(U)$ is finite dimensional, then $R \longrightarrow \mathcal{A}(R)^\times(U)$ is an affine group scheme over **C**. We shall be interested in the case when $X = S^1$ and $\mathcal{A}(U)$ is finite dimensional for all open arcs U. If $\mathcal{H}^0$ is the $\mathcal{D}$-filtered local system of germs of sectorial sections of (V^0, ∇^0), then taking $\mathcal{A} = \mathrm{End}(\mathcal{H}^0)$ or $\mathcal{A} = \mathrm{End}(\mathrm{Gr}\,\mathcal{H}^0)$ we obtain examples of the above situation. We write Aut^0 and $\mathrm{Aut}^{0\wedge}$ for the corresponding sheaves of group schemes of the units. It is clear that St^0 is a normal subsheaf of Aut^0. Moreover, as the $\mathcal{D}$-filtered structure of $\mathcal{H}^0$ arises from $\mathrm{Gr}\,\mathcal{H}^0$ on sufficiently small open arcs it follows that for such arcs U,

$$0 \longrightarrow \mathrm{St}^0(R)(U) \longrightarrow \mathrm{Aut}^0(R)(U) \longrightarrow \mathrm{Aut}^{0\wedge}(R)(U) \longrightarrow 0$$

is exact. Hence

$$0 \longrightarrow \mathrm{St}^0 \longrightarrow \mathrm{Aut}^0 \longrightarrow \mathrm{Aut}^{0\wedge} \longrightarrow 0$$

is an exact sequence of sheaves of group schemes.

PROOF By **I**, Corollary 4.8.3, we have a natural isomorphism

$$G^{0\wedge} = \mathrm{Aut}(M^{0\wedge}, \nabla^{0\wedge}) \cong \mathrm{Aut}(\mathrm{Gr}\,\mathcal{H}^0).$$

Now the right side is an affine algebraic group in a natural manner; indeed, we may identify $\mathrm{Aut}(\mathrm{Gr}(\mathcal{H}^0))$ with the group of graded automorphisms of the complex vector space $W = \mathrm{Gr}\,\mathcal{H}^0(1)$ that commute with the monodromy action of **Z**. So the first assertion is proved. For the second assertion it is a question of defining a natural action of $\mathrm{Aut}^{0\wedge}(R)$ on $H^1(S^1, \mathrm{St}^0(R))$ for each **C**-algebra R that is functorial in R. Let $\gamma \in H^1(S^1, \mathrm{St}^0(R))$ and $\eta \in \mathrm{Aut}^{0\wedge}(R)$. Let (U_i) be a finite covering of S^1 by sufficiently small arcs and let (g_{ij}) be a representative of γ associated to this covering. We choose $y_i \in \mathrm{Aut}^0(R)(U_i)$ such that the image of y_i is η, and define

$$\eta.\gamma = [(y_i\, g_{ij}\, y_i^{-1})]$$

where [a] is the cohomology class of the cocycle a. It is a routine matter to verify that this class is independent of the choices made in the construction. If we specialize R to be **C**, it is simple to verify that for any pair $((V, \nabla), \xi)$ for which $\Phi((V, \nabla), \xi) = \gamma$, $\Phi((V, \nabla), \eta\xi) = \eta.\gamma$. ♦

REMARK It is not difficult to see that a map $\lambda \longrightarrow \eta(\lambda)$ of a polydisk Δ into Morph$((M^{0\wedge}, \nabla^{0\wedge}), (M^{0\wedge}, \nabla^{0\wedge}))$ is analytic if and only if the $\eta(\lambda)$ are defined by an element $\eta \in \mathrm{End}(\mathcal{O}_{d,1}{}^N)$. In particular, maps of Δ into $G^{0\wedge}$ are analytic if and only if they are defined by elements of $GL(N, \mathcal{O}_{d,1}{}^N)$.

PROPOSITION 2.1.2 *We have a natural bijection*

$$\Psi : \mathcal{J}(V^0, \nabla^0) \cong G^{0\wedge}\backslash H^1(S^1, St^0).$$

PROOF This is immediate from Proposition 2.1.1. ◆

PROPOITION 2.1.3 $G^{0\wedge}$ *is connected.*

PROOF Let $W = \mathrm{Gr}\,\mathcal{H}^0(1) = \bigoplus_{\omega \in \mathcal{S}(1)} W_\omega$ and let L be the monodromy action of $1 \in \mathbf{Z}$ on W. $G_{0^\wedge}$ may then be identified with the group of all collections $\{g(\omega)\}$ where $g(\omega) \in GL(W_\omega)$ and $g(1.\,\omega) = L\,g(\omega)\,L^{-1}$ for al ω. If $\{\omega_i\}$ is a system of representatives for the monodromy action of $\mathbf{Z}$ on $\mathcal{S}(1)$ and $b(i)$ is the order of the stabilizer of ω_i, then $G^{0\wedge} \cong \prod G_i$ where G_i is the centralizer of $L^{b(i)}$ for all i. But, if Z_i is the subspace of $\mathrm{End}(W_{\omega(i)})$ that centralizes $L^{b(i)}$, G_i is the subset of elements of Z_i with nonzero determinant; it is therefore connected because the complex line joining any two points in it meets the complement in a finite set at most. ◆

2.2 The work of the preceding section shows that we are in a familiar paradigm, namely that of algebraic actions of affine algebraic groups on affine varieties. We want to look at the quotient

$$G^{0\wedge}\backslash H^1(S^1, St^0),$$

and recall the well known fact that to get a "good" quotient certain "bad" orbits may have to be removed. We shall now discuss briefly what should be done in the present context.

Let G be a connected complex Lie group acting on a connected complex manifold M, the action being written $g, x \longrightarrow g.\,x$. By a *quotient* of M by G we mean a pair (Y, f) where Y is a complex manifold and $f\,(M \longrightarrow Y)$ is an analytic map which is everywhere submersive and whose fibres are precisely the G-orbits in M. It is easy to see that such a quotient, if it exists, is essentially unique. For arbitrary M, G, let us form the orbit space

$$M^* = G\backslash M, \qquad \pi : M \longrightarrow M^*,$$

equipped with the quotient topology and the sheaf $\mathcal{O}$ such that for any open $U^* \subset M^*$ $\mathcal{O}(U^*)$ is the algebra of G-invariant analytic functions on $\pi^{-1}(U^*)$. A point $m \in M$ (or its image $m^* \in M^*$) is called *smooth* if M^* looks like a complex manifold at m^*, and for some open U^* containing m^*, (U^*, π) is a quotient of $\pi^{-1}(U^*)$. If M^{sm} is the open set of smooth points and M^{*sm} is its image in M^*, it is then clear that (M^*, π) is the quotient of M^{sm}. *However,* M^{*sm} *need not be Hausdorff.* If $\delta(m)$ is the dimension of the orbit $G.m$ $(m \in M)$ and $\delta = \max_{m \in M} \delta(m)$, it is easy to see that $\{m \in M : \delta(m) = \delta\}$ is a dense G-invariant open set and that M^{sm} is contained in that set; in particular,

$$\dim M^* = \dim M - \delta.$$

These remarks apply to the action of $G^{0\wedge}$ on $H^1(S^1, St^0)$. We shall call a pair *smooth* if its isomorphism class is in $H^1(S^1, St^0)^{sm}$. We then have

THEOREM 2.2.1 $H^1(S^1, St^0)^{*sm}$ *is a local moduli space for the meromorphic pairs that are formally equivalent to* (V^0, ∇^0) *and its dimension is* $d^* = d - \delta$ *where* δ *is the maximum of the dimensions of the orbits of* $G^{0\wedge}$ *in* $H^1(S^1, St^0)$.

PROOF This is just a formal consequence of Theorem 1.1.2. We must remember that in view of **I**, Theorem 1.5.1, given any isoformal family $\{(V_\lambda, \nabla_\lambda)\}$ of meromorphic pairs formally equivalent to (V^0, ∇^0), we can always find an analytic family of isomorphisms ξ_λ of the formalizations of $(V_\lambda, \nabla_\lambda)$ with that of (V^0, ∇^0) so that $\{((V_\lambda, \nabla_\lambda), \xi_\lambda\}$ becomes a family of marked pairs associated to (V^0, ∇^0). Then

$$\Psi(V_\lambda, \nabla_\lambda) = \pi \circ \Phi(((V_\lambda, \nabla_\lambda), \xi_\lambda)$$

The morphism property is now clear. Let $\{(V'_\lambda, \nabla'_\lambda)\}$ be another family with $\Psi(V'_\lambda, \nabla'_\lambda) = \Psi(V_\lambda, \nabla_\lambda) \in H^1(S^1, St^0)^{*sm}$, formally equivalent to (V^0, ∇^0), and $\{((V'_\lambda, \nabla'_\lambda), \xi'_\lambda\}$ a marked family associated to it. Then an elementary Lie group theoretic argument shows that there is an anlytic map $\lambda \longrightarrow \eta_\lambda$ into $G^{0\wedge}$

such that $\Phi((V'_\lambda, \nabla'_\lambda), \eta_\lambda\xi'_\lambda) = \Phi((V_\lambda, \nabla_\lambda), \xi_\lambda)$. This implies the equivalence of the families $((V'_\lambda, \nabla'_\lambda), \eta_\lambda\xi'_\lambda)$ and $((V'_\lambda, \nabla'_\lambda), \xi_\lambda)$ and hence of $\{(V'_\lambda, \nabla'_\lambda)\}$ and $\{(V_\lambda, \nabla_\lambda)\}$. Finally let $\{((V_\lambda, \nabla_\lambda), \xi_\lambda)\}$ be universal for $((V_0, \nabla_0), \xi_0)$. Let N be a submanifold through $\Phi((V_0, \nabla_0), \xi_0)$ of dimension d^* such that π is a diffeomorphism on N and N' the preimage of N via the map

$$\lambda \longrightarrow \Phi((V_\lambda, \nabla_\lambda), \xi_\lambda),$$

then the restriction to a neighbourhood of **0** of $\lambda \longrightarrow \Psi((V_\lambda, \nabla_\lambda))$ is a universal local deformation of $\Psi((V_0, \nabla_0))$. ♦

We shall now obtain a criterion for a family to be universal at a given point of the moduli space. We shall suppose that all the bundles are of the form $V = \Delta \times C^N$ with the connections $\nabla(A)$, $A \in \mathfrak{gl}(N, \mathfrak{F}_{cgt})$. For any subset $E \subset \mathfrak{gl}(N, \mathfrak{F})$ let $E_{cgt} = E \cap \mathfrak{gl}(N, \mathfrak{F}_{cgt})$. Define, for any $A \in \mathfrak{gl}(N, \mathfrak{F}_{cgt})$,

$$M^\wedge(A) = (\nabla(A)(\mathfrak{gl}(N, \mathfrak{F}))_{cgt},$$

$$M(A) = \nabla(A)(\mathfrak{gl}(N, \mathfrak{F}_{cgt}),$$

$$N(A) = \{X^\wedge : X^\wedge \in \mathfrak{gl}(N, \mathfrak{F}),\ \nabla(A)(X^\wedge) = 0\}.$$

PROPOSITION 2.2.1 *Fix* $A_0 \in \mathfrak{gl}(N, \mathfrak{F}_{cgt})$, $\xi_0 \in GL(N, \mathfrak{F})$ *such that* $\xi_0[A_0] = A^0$. *If* δ_0 *is the dimension of the orbit of* $\varphi = \Phi((V, \nabla(A_0), \xi_0)$ *under* $G^{0\wedge}$, *then*

$$\dim (M^\wedge(A_0)/M(A_0)) = d - \delta_0.$$

PROOF The key to the proof is contained in the following two lemmas.

LEMMA 2.2.2 *We have*

$$\xi_0 N(A_0)\xi_0^{-1} = \mathrm{Lie}(G^{0\wedge}), \quad \xi_0 N(A_0)_{cgt}\, \xi_0^{-1} = \mathrm{Lie}(G^{0\wedge}{}_\varphi),$$

$G^{0\wedge}{}_\varphi$ *being the stabilizer of* φ *in* $G^{0\wedge}$.

PROOF $\xi_0 N(A_0)\xi_0^{-1}$ is easily seen to be $N(A^0)$ which may obviously be identified naturally with $\mathrm{Lie}(G^{0\wedge})$. We now take up the second formula. We choose and fix a finite covering $\mathfrak{U} = (U_i)$ of S^1 by open arcs of sufficiently small length and calculate all cohomologies using it. Let $x_i \in GL(N, A_1(\Gamma(U_i)))$

be such that $x_i \sim \xi_0$ and $x_i[A_0] = A^0$ on $\Gamma(U_i)$ for all i, and let $g_{ij} = x_i x_j^{-1}$ so that $g = (g_{ij})$ represents φ. Let $\eta \in \mathrm{Lie}(G^{0\wedge}) = N(A^0)$; by **I**, Theorerm 2.2.4 we can find $\sigma_i \in \mathrm{End}(A_1(\Gamma(U_i))^N)$ with $\sigma_i \sim \eta$ and $\nabla(A^0)(\sigma_i) = 0$ on $\Gamma(U_i))_\delta$ for all i. Then it is not difficult to see that $\eta \in \mathrm{Lie}(G^{0\wedge}{}_\varphi)$ if and only if there are $\alpha_i \in \mathfrak{st}^0(U_i)$ such that

$$\sigma_i g_{ij} - g_{ij} \sigma_j = \alpha_i g_{ij} - g_{ij} \alpha_j \qquad \text{on } \Gamma(U_i \cap U_j)_\delta$$

for all i, j. Indeed, if $y_i(t) = 1 + t\sigma_i$ $(t \in \mathbf{C})$, then $t \longrightarrow y_i(t)$ is a curve through 1 in $G^{0\wedge}$ with η as its tangent vector at $t = 0$ and so the left side of the above equation is just $(d/dt)_{t=0}(y_i(t)\, g_{ij}\, y_j(t)^{-1})$; so the condition that $\eta \in \mathrm{Lie}(G^{0\wedge}{}_\varphi)$ is that this derivative should be the same as the result of applying to g a tangent vector to the orbit of g under $C(\mathfrak{A}: St^0)$, i. e., to an element of the form given by the right side of the above equation. Substituting $g_{ij} = x_i x_j^{-1}$ we can rewrite this as

$$x_i^{-1} \sigma_i x_i - x_j^{-1} \sigma_j x_j = x_i^{-1} \alpha_i x_i - x_j^{-1} \alpha_j x_j$$

or

$$(*) \qquad x_i^{-1} \sigma_i x_i - x_i^{-1} \alpha_i x_i = x_j^{-1} \sigma_j x_j - x_j^{-1} \alpha_j x_j$$

on $\Gamma(U_i \cap U_j)_\delta$ for all i, j. The expressions above are $\sim \xi_0^{-1}\eta\xi_0$ and are solutions of the equation $\nabla(A_0)(u) = 0$ $(u \in \mathrm{End}(A_1(\Gamma(U_i))^N)$. The proof is now completed quickly. If $\eta \in \mathrm{Lie}(G^{0\wedge}{}_\varphi)$, $(*)$ is true and so there is $\zeta \in \mathrm{End}(\mathfrak{F}_{cgt}{}^N)$ such that

$$\zeta = x_i^{-1} \sigma_i x_i - x_i^{-1} \alpha_i x_i$$

on $\Gamma(U_i)_\delta$ for all i. But $\zeta = \xi_0^{-1}\, \eta\, \xi_0$, so that $\xi_0 \zeta \xi_0^{-1} = \eta$ and $\nabla(A_0)(\zeta) = 0$, showing that $\zeta \in N(A_0)_{cgt}$. In the other direction if we start with $\zeta \in N(A_0)_{cgt}$ and define η to be $\xi_0\zeta\xi_0^{-1}$, we have $\eta \in \mathrm{Lie}(G^{0\wedge})$; we choose σ_i as before and define the α_i by the last displayed equation; then $\alpha_i \in \mathfrak{st}^0(U_i)$ and $(*)$ is satisfied, showing that $\eta \in \mathrm{Lie}(G^{0\wedge}{}_\varphi)$. ♦

LEMMA 2.2.3 *We can find a natural linear isomorphism*

$$N(A_0)/N(A_0)_{cgt} \cong \mathcal{S} \subset H^1(S^1, \mathfrak{st}(A_0))$$

and a natural linear isomorphism

$$M^\wedge(A_0)/M(A_0) \cong H^1(S^1, \mathcal{st}(A_0))/\mathcal{S}.$$

PROOF We begin with a simple observation. Suppose $\nabla(A_0)(X^\wedge) = Y^\wedge$ where $X^\wedge \in \mathfrak{gl}(N, \mathfrak{F})$ and $U \subset S^1$ is an open arc of sufficiently small length. If $Y(\Gamma(U)_\delta \longrightarrow \mathrm{End}(\mathbf{C}^N))$ is holomorphic and is such that $Y \sim Y^\wedge(\Gamma(U))$, we can find holomorphic $X(\Gamma(U)_\delta \longrightarrow \mathrm{End}(\mathbf{C}^N))$ such that $X \sim X^\wedge$ and $\nabla(A_0)(X) = Y$. For proving this we may go over to a covering complex plane so that there is no loss of generality in assuming that A_0 is unramified. Further, by using **I**, Theorem 2.2.4, we may come down to the case when A_0 is a canonical form. The argument is now completed as in Lemma 1.4.3.

We shall now go on with the proof. Let $\mathfrak{R}$ be the sheaf on S^1 of germs of elements of $\mathrm{End}(A_1(\Gamma)^N)$, i. e., of holomorphic maps $\Gamma(U)_\delta \longrightarrow \mathrm{End}(\mathbf{C}^N)$ whose entries have asymptotic expansions in $\mathfrak{F}$. Then $\nabla(A_0)$ is a sheaf map of $\mathfrak{R}$ into itself and we write $\nabla\mathfrak{R}$ for its range. The remark made in the preceding paragraph implies that $\nabla\mathfrak{R}$ is precisely the subsheaf of $\mathfrak{R}$ of those elements whose asymptotic expansions belong to $\nabla(A_0)(\mathrm{End}(\mathfrak{F}^N))$. Let us also introduce the sheaves $\mathfrak{N}$ and $N(A_0)$, where $\mathfrak{N}$ is the subsheaf of $\mathfrak{R}$ of all elements that are in the kernel of $\nabla(A_0)$, and $N(A_0)$ is the constant sheaf with coefficients in $N(A_0)$. We then have the following two exact sequences

$$0 \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{R} \xrightarrow{\nabla(A_0)} \nabla\mathfrak{R} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{st}(A_0) \longrightarrow \mathfrak{N} \xrightarrow{\wedge} N(A_0) \longrightarrow 0$$

where $\wedge$ in the second sequence is the map that takes any element to its asymptotic expansion and is surjective in view of the remark above. Let us look at the corresponding long exact sequences. Since

$$H^0(S^1, \mathfrak{N}) = N(A_0)_{cgt}, \quad H^0(S^1, \mathfrak{R}) = \mathfrak{gl}(N, \mathfrak{F}_{cgt}), \quad H^0(S^1, \nabla\mathfrak{R}) = M^\wedge(A_0),$$

we obtain the exact sequences

$$\mathfrak{gl}(N, \mathfrak{F}_{cgt}) \xrightarrow{\nabla(A_0)} M^\wedge(A_0) \longrightarrow H^1(S^1, \mathfrak{N}) \longrightarrow$$

$$\longrightarrow H^1(S^1, \mathfrak{R}) \longrightarrow H^1(S^1, \nabla\mathfrak{R}) \longrightarrow 0$$

and

$$0 \longrightarrow N(A_0)_{cgt} \longrightarrow N(A_0) \longrightarrow H^1(S^1, \mathfrak{st}(A_0)) \longrightarrow$$
$$\longrightarrow H^1(S^1, \mathfrak{N}) \longrightarrow H^1(S^1, N(A_0)) \longrightarrow 0.$$

Let

$$K_1 = \mathrm{kernel}\big(H^1(S^1, \mathfrak{N}) \longrightarrow H^1(S^1, \mathfrak{R})\big),$$
$$K_2 = \mathrm{kernel}\big(H^1(S^1, \mathfrak{N}) \longrightarrow H^1(S^1, N(A_0))\big).$$

It is enough to prove that $K_1 = K_2$; for then the first of the above exact sequences gives $M^\wedge(A_0)/M(A_0) \cong K_1$ and the second gives an imbedding $N(A_0)/N(A_0)_{cgt} \cong \mathcal{B} \subset H^1(S^1, \mathfrak{st}(A_0))$ such that $H^1(S^1, \mathfrak{st}(A_0))/\mathcal{B} \cong K_2$. If $g \in K_1$ and is represented by the cocycle (g_{ij}) associated to a finite covering (U_i) of S^1 by open arcs of sufficiently small length, then $g_{ij} = c_i - c_j$, $c_i \in \mathrm{End}(A_1((\Gamma(U_i))^N)$, and $c_i \sim c_i^\wedge$ for all i. As $\nabla(A_0)(g_{ij}) = 0$, we have $\nabla(A_0)(g_{ij}^\wedge) = 0$, so that $\nabla(A_0)(c_i^\wedge) = \nabla(A_0)(c_j^\wedge)$; if we then choose and fix an index k and write $d_i^\wedge = c_i^\wedge - c_k^\wedge$, then $g_{ij}^\wedge = d_i^\wedge - d_j^\wedge$ and $\nabla(A_0)(d_i^\wedge) = 0$. Thus $g \in K_2$. If $g \in K_2$ then we have the relations $g_{ij}^\wedge = c_i^\wedge - c_j^\wedge$ with $\nabla(A_0)(c_i^\wedge) = 0$; choose $c_i \sim c_i^\wedge$ and write $h_{ij} = g_{ij} - (c_i - c_j)$. Then (h_{ij}) is a cocycle for the subsheaf of flat sections of $\mathfrak{R}$. This is however the sheaf that occurs in the additive Malgrange-Sibuya Theorem, and so by **I**, Theorem 4.4.1, we can find $a^\wedge \in \mathrm{End}(\mathfrak{F}^N)$ and $a_i \in \mathrm{End}(A_1(\Gamma(U_i))^N)$ such that $a_i \sim a^\wedge$ for all i and $h_{ij} = a_i - a_j$ for all i, j. But then $g_{ij} = c_i' - c_j'$ where $c_i' = c_i + a_i \sim c_i^\wedge + a^\wedge$, proving that g maps to zero in $H^1(S^1, \mathfrak{R})$. ♦

The proof of Proposition 2.2.1 may now be completed easily. By the two preceding lemmas

$$\dim M^\wedge(A_0)/M(A_0) = d - (\dim \mathrm{Lie}(G^{0\wedge}) - \dim \mathrm{Lie}(G^{0\wedge}{}_\varphi) = d - \delta_0.$$

This proves the Proposition. ♦

PROPOSITION 2.2.4 *Let* $\Delta \subset \mathbf{C}^q$, $\{(V, \nabla(A(\lambda))\}$ *an isoformal family of meromorphic pairs and* $A(\mathbf{0}) = A_0$. *If* $\mathfrak{X}$ *is the* **C**-*linear span of* $\partial_i A$ $(\partial_i = (\partial/\partial\lambda_i)_{\lambda=\mathbf{0}}, 1 \le i \le d)$, *then* $\mathfrak{X} \subset M^\wedge(A_0)$. *If we assume that* $\Psi(V, \nabla(A_0))$ *is a smooth point*, $\dim \mathfrak{X} = q$, *and* $\mathfrak{X} \cap M(A_0) = 0$, *then the map*

$$\lambda \longrightarrow \Psi(V, \nabla(A(\lambda))$$

has an injective differential at $\lambda = \mathbf{0}$. *In particular, when these assumptions are made and* $q = d^*$, $\{(V, \nabla(A(\lambda))\}$ *is universal at* $\lambda = \mathbf{0}$.

PROOF We begin by showing that $\partial_i A \in \nabla(A_0)(\mathfrak{gl}(N, \mathcal{F}))$; this will prove the first assertion. Now, by **I**, Theorem 1.5.1 there is a $\xi \in GL(N, \mathcal{O}_{q,1})$ such that $\xi(\lambda)[A(\lambda)] = A_0$ for all λ. If we apply ∂_i to the relation

$$(d\xi(\lambda)/dz)\, \xi(\lambda)^{-1} + \xi(\lambda)\, A(\lambda)\xi(\lambda)^{-1} = A_0$$

we get, after an easy calculation,

$$\partial_i A = -\nabla(A_0)(\xi_0^{-1}\partial_i\xi),$$

which proves what we want. To prove the injectivity of the differential of the map $\gamma\ (\lambda \longrightarrow \Psi(V, \nabla(A(\lambda)))$ under the additional conditions assumed above we proceed as in Propositions 1.4.6 and 1.4.7. Let α be the map $y \longrightarrow y.\,[g(\mathbf{0})]$ of $G^{0\wedge}$ into $H^1(\mathfrak{a} : St^0))$ and let $d\alpha_1$ be its differential at the identity element of this group. If ∂ is as in loc. cit., we must then prove the following implication : if $\eta \in \mathrm{Lie}(G^{0\wedge})$, then

$$(*) \qquad d\alpha_1(\eta) - \partial g = 0 \implies \partial = 0.$$

Now $\mathrm{Lie}(G^{0\wedge})$ is the Lie algebra of all $P \in \mathrm{End}(\mathcal{F}^N)$ such that $\nabla(A^0)(P) = 0$ and so we can choose by **I**, Theorem 2.2.4 $y_i \in \mathrm{End}(A_1(\Gamma(U_i))^N)$ such that $y_i \sim \eta$ and $\nabla(A^0)(y_i) = 0$ for all i. Going over to $Z(\mathfrak{a} : St^0)$ the condition $d\alpha_1(\eta) = \partial g$ then becomes

$$\partial g_{ij} = \tau_i\, g_{ij}(\mathbf{0}) - g_{ij}(\mathbf{0})\, \tau_j + y_i\, g_{ij}(\mathbf{0}) - g_{ij}(\mathbf{0})\, y_j$$

for a suitable element $(\tau_i) \in C(\mathfrak{a} : st^0)$. If we now substitute for g_{ij} the expression $x_i\, x_j^{-1}$ we get after some calculation the relations

$$x_i(\mathbf{0})^{-1}\, \partial x_i - x_i(\mathbf{0})^{-1}\, \tau_i\, x_i(\mathbf{0}) - x_i(\mathbf{0})^{-1}\, y_i\, x_i(\mathbf{0}) =$$

$$x_j(\mathbf{0})^{-1}\, \partial x_j - x_j(\mathbf{0})^{-1}\, \tau_j\, x_j(\mathbf{0}) - x_j(\mathbf{0})^{-1}\, y_j\, x_j(\mathbf{0})$$

on $\Gamma(U_i \cap U_j)_\delta)$ for all i, j. As these expressions are $\sim \xi(\mathbf{0})^{-1}\partial\xi - \xi(\mathbf{0})^{-1}\, \eta\, \xi(\mathbf{0})$ we conclude as usual that there is $\zeta \in \mathrm{End}(\mathcal{F}_{cgt}{}^N)$ such that

$$\zeta = x_i(0)^{-1}\,\partial x_i - x_i(0)^{-1}\,\tau_i\,x_i(0) - x_i(0)^{-1}\,y_i\,x_i(0) \qquad \text{on } \Gamma(U_i)_\delta$$

for all i. Exactly as in Propositions 1.4.6 and 1.4.7 we now find that

$$\nabla(A_0)(\zeta) = -\partial A.$$

In other words, $\partial A \in M(A_0)$. This forces $\partial A = 0$ by our hypotheses and hence we get $\partial = 0$. ◆

2.3 The obvious shortcoming of the above results is that there seems to be no simple way to determine when $\Psi((V, \nabla(A_0)))$ is smooth. We have however not yet used the fact that we are in the *algebraic* cadre, a fact that becomes significant especially when $G^{0\wedge}$ is *reductive*. Let X be an irreducible affine variety over **C** and let G be a connected reductive group over **C** operating morphically on X. Following Mumford [MF] we call a point $x \in X$ *stable* if G. x, the orbit of x, has maximal dimension and is closed in the Zariski topology (this is equivalent to its being closed in the usual complex topology). Let X^{st} be the set of stable points; it may be empty(look at the action $\lambda, t \longrightarrow \lambda. t$ of $\mathbf{C}^{\times}$ on **C**), but if it is nonempty, then it is a Zariski open G-invariant set. Let us now suppose that $X^{st} \neq \emptyset$. As the subring of G-invariants of the coordinate ring **C**[X] of X is finitely generated, it is an affine algebra which is an integral domain, and so we can introduce the irreducible affine variety X^* which is its maximal spectrum, together with the natural map $\pi : X \longrightarrow X^*$. It is now a consequence of the geometric invariant theory of reductive groups (cf. [MF], pp. 27-30, [Ses], pp. 283-288) that $\pi(X^{st}) := X^{st*}$ is open in X^*, that $X^{st} = \pi^{-1}(X^{st*})$, and that (X^{st*}, π) is a *good quotient* of X^{st} by G. This last property means the following :

(a) $\pi : X^{st} \longrightarrow X^{st*}$ is open and surjective

(b) the fibers of π above X^{st*} are precisely the G-orbits in X^{st}

(c) if $U \subset X^{st}$ is a G-invariant open set and $\pi(U) = U^*$, then a function f on U^* is regular if and only if $f \circ \pi$ is regular on $U = \pi^{-1}(U^*)$.

Let $X^{\#}$ be the preimage in X^{st} of the set $X^{st\#}$ of simple points in X^{st*}. It is then clear that $X^{\#} \subset X^{sm}$, and that $(X^{st\#}, \pi)$ is a quotient of $X^{\#}$ by G in the complex analytic category. We note that $X^{st\#}$ is *automatically Hausdorff.* Applied to our special situation these remarks lead to the following Theorem. Write for brevity $H^1 = H^1(S^1, St^0)$. A meromorphic pair is called *stable simple* if there is a marking for it so that the corresponding element of H^1 is stable and simple for $G^{0\wedge}$; this is obviously independent of the choice of the marking.

THEOREM 2.3.1 *Suppose that* $G^{0\wedge}$ *is reducive and* $H^{1,st}$ *is not empty. Then it is Zariski open and* $G^{0\wedge}$*-invariant. If* $\pi : H^{1,st} \longrightarrow H^{1,st*}$ *is the natural map, then the open set of simple points of the quasi affine variety* $H^{1,st*}$ *is a local moduli space for the isomorphism classes of stable simple pairs.*

The obvious questions that arise now are the following :

1. Are there stable points ?

2. How does one recognize that a given pair defines a stable point in the moduli space ?

We do not have definitive answers to these questions and so we shall devote the remainder of this chapter to a few remarks and examples that illustrate the notion of stability for meromorophic pairs.

Bessel and Whittaker connections(I,§5) The space H^1 is $\mathbf{C}^2$ while $G^{0\wedge}$ is $\mathbf{C}^\times$ acting on H^1 by λ, $(a, b) \longrightarrow (\lambda a, \lambda^{-1}b)$. It is then obvious that $H^{1,st}$ is the set $\{(a, b) : ab \neq 0\}$ and $G^{0\wedge}/H^{1,st} \cong \mathbf{C}^\times$. In particular all stable points are simple. Further $H^{1,sm} = \mathbf{C}^2 \setminus \{(0, 0)\}$, and it is easy to see that $G^{0\wedge}/H^{1,sm}$ is not separated; indeed, it is the classical example of a nonseparated analytic space where two copies of $\mathbf{C}$ with coordinates t_1 and t_2 are glued along $\mathbf{C}^\times$ via the identification $t_1 = t_2$. To get a Hausdorff quotient we have to omit not only $(0, 0)$ but at least one of the two coordinate axes. The reader should go back to the discussion in **I**,§5 for the criterion for the stability of a Bessel or Whittaker connection.

Existence of stable points We assume that $A^0 = B$ where B is an unramified reduced canonical form,

$$B = \sum\nolimits_{r \in L} D_r z^r + z^{-1}C.$$

The stabilizer of $\nabla(B)$ in $GL(N, \mathfrak{F})$ is then known to be G_B, the stabilizer of B in $GL(N, \mathbf{C})$ ([BV 1], Theorem 7.2). Theorem 3.4.3 of **II** gives a description of $H^1(S^1, St)$ and it is not difficult to see that the isomorphism there is G_B-equivariant. We thus have

PROPOSITION 2.3.2 *The isomorphism*

$$H^1(S^1, St) \cong \prod_{k \geq 0} \prod_{\sigma \in \Sigma(k)} H^1(S^1, St_{k,k+1,\sigma})$$

is G_B*-equivariant, the action of* G_B *on the right side being obtained via the natural maps* $G_B \longrightarrow$ *stabilizer of* $B_{k,k+1,\sigma}$.

Let us now assume that the restriction of C to the spectral subspaces of $\{D_r\}_{r \in L}$ is simple, namely, has no repeated eigenvalues. We shall therefore assume that the D_r and C are all diagonal. Then G_B is the diagonal subgroup $\mathbf{D}$ of $GL(N, \mathbf{C})$. We then have the following Theorem (cf. [BV 3]). We identify $\mathbf{C}^\times$ with the subgroup of scalar multiples of the identity in $\mathbf{D}$.

PROPOSITION 2.3.3 *Under the above assumptions* $H^{1,st}$ *is nonempty,* $\mathbf{D}/\mathbf{C}^\times$ *acts freely on* $H^{1,st}$, *and* $H^{1,st*}$ *is a quasiaffine variety of dimension equal to* $d - N + 1$.

PROOF We shall begin by considering the special case when B has a single level,

$$B = D_r z^r + z^{-1} C,$$

where C is reduced. We do not assume that C has a simple spectrum on each eigenspace of D_r but shall consider the action of $\mathbf{D}$ on the cohomology. Let $\{\sigma_j\}_{1 \leq j \leq M}$ be the spectrum of D_r and for $i \neq j$ let S_{ij} be the set of $2q$ Stokes lines associated to $(\sigma_i - \sigma_j) z^r$ where $q = |r_1| - 1$. We work with the special good covering $(U_\alpha)_{\alpha \in \mathbf{Z}}$ of S^1 obtained by dividing S^1 into $4q$ arcs of equal length by the points t_i $(i \in \mathbf{Z}, t_i = t_{i+4q})$ ordered counter clockwise and taking $U_\alpha = (t_{\alpha-1}, t_{\alpha+1})$; we shall suppose also that the t_i are not on any Stokes line. Since the length of the U_α is π/q it is clear that *each* S_{ij} *has exactly one member meeting each* U_α. So the sections of the Stokes sheaf St on each U_α are trivial and hence

$$H^1(S^1, St) = Z((U_\alpha) : St) = \prod_{1 \leq \alpha \leq 4q} St(W_\alpha), \qquad W_\alpha = U_\alpha \cap U_{\alpha+1}.$$

We denote the elements of $H^1(S^1, St)$ by $(g(\alpha))_{1 \le \alpha \le 4q}$. We write the elements of $End(\mathbf{C}^N)$ as block matrices $(a_{ij})_{1 \le i,\, j \le M}$ defined by the spectral decomposition of D_r, the elements of $\mathbf{D}$ as $(u_j)_{1 \le j \le M}$, and for fixed i, j, denote the entries of a_{ij} by $a_{ij,rs}$. We choose branches of the logarithm on the W_α and identify (cf. **I**, Proposition 3.4.1) $St(W_\alpha)$ with the subgroup of $GL(N, \mathbf{C})$ of block matrices (a_{ij}) with

$$a_{ii} = 1, \quad a_{ij} = 0 \text{ unless } Re(-q^{-1}(\sigma_i - \sigma_j)\, z^{-q}) < 0 \text{ on } W_\alpha \ (i \neq j).$$

The action of $(u_j) \in \mathbf{D}$ on $Z((U_\alpha) : St)$ is given by

$$(u_j),\ (g(\alpha)_{ij}) \longrightarrow (h(\alpha)_{ij}), \qquad h(\alpha)_{ij,rs} = u_{i,r}\, g(\alpha)_{ij,rs}\, u_{j,s}^{-1}.$$

We now introduce $A(g)$, the associative algebra with unit generated by the matrices $g(\alpha)$ and consider the condition

(∗) for each (i, j) with $i \neq j$ and (r, s), $\exists\ h \in A(g)$ such that $h_{ij,rs} \neq 0$.

We shall now prove that if g satisfies the condition (∗) then the stabilzer of g in $\mathbf{D}$ is $\mathbf{C}^\times$ and that the $\mathbf{D}$-orbit of g is closed. Let $u = (u_j) \in \mathbf{D}$ stabilize g, so that it centralizes $A(g)$. Then by (∗) we have $u_{i,r} = u_{j,s}$ so that $u \in \mathbf{C}^\times$. To prove that the $\mathbf{D}$-orbit of g is closed we must show that the map $u \longrightarrow u.\, g$ from $\mathbf{D}/\mathbf{C}^\times$ into $End(\mathbf{C}^N)^{4q}$ is proper in the complex topology. If L is a compact set in $End(\mathbf{C}^N)^{4q}$, then for each $h \in A(g)$ there is a compact set $K(h)$ in $End(\mathbf{C}^N)$ such that the subset L_g of all u in $\mathbf{D}$ for which $u.\, g \in L$ will satisfy $u\, h\, u^{-1} \in K(h)$. Let us fix (i, j) with $i \neq j$ and (r, s) and select $h, h' \in A(g)$ such that $h_{ij,rs} \neq 0$ and $h'_{ji,sr} \neq 0$. Then there is a compact set $K \subset \mathbf{C}$ such that

$$u \in L_g \Longrightarrow u_{ir}\, h_{ij,rs}\, u_{js}^{-1} \in K, \qquad u_{js} h'_{ji,sr}\, u_{ir}^{-1} \in K.$$

It is easy to conclude from these relations that u belongs to a compact set mod $\mathbf{C}^\times$.

In order to complete the proof of the existence of stable $\mathbf{D}$-closed orbits of dimension $N - 1$ in this special case we must show that there exist g such that $A(g)$ satisfies (∗). Actually we shall construct cocycles g such that

(∗∗) for each (i, j) with $i \neq j$ and (r, s), $\exists\ \alpha$ such that $g(\alpha)_{ij,rs} \neq 0$.

In fact, for any α, we take $g(\alpha)$ to be a matrix such that $g(\alpha)_{ij} = 0$ except when $\mathrm{Re}(-q^{-1}(\sigma_i - \sigma_j)\, z^{-q}) < 0$ on W_α in which case we take it to be a matrix with all entries nonzero. To see that g satisfies (**) consider *four successive values of* α, say $\alpha = \beta, \beta + 1, \beta + 2, \beta + 3$ and fix i, j, with $i \neq j$. Then exactly two of the arcs W_β meet a Stokes line from S_{ij}, and these two arcs *cannot be adjacent* ; further $\mathrm{Re}(-q^{-1}(\sigma_i - \sigma_j)\, z^{-q})$ keeps the same sign on each of the other two arcs *and the two signs are opposite to each other.* So g satisfies (**) when α is one of these four indices. This finishes the proof in the special case.

For the general case it is now enough to observe, using Proposition 2.3.2, that $H^1(S^1, St_1)$ is a closed **D**-stable subspace of $H^1(S^1, St')$; here St' is the Stokes sheaf of $D_{r'} z^{r'} + z^{-1}C$, r' being the principal level of $\nabla(B)$. This proves the Proposition. ◆

From the preceding proof the following result is immediate.

PROPOSITION 2.3.4 *Suppose* $B = D_r z^r + z^{-1} C$ *where* C *is reduced and has a simple spectrum on each eigenspace of* D_r. *Let* $\mathcal{D}$ *be the space matrices with zero off-diagonal blocks and* g *a cocycle such that* $A(g) + \mathcal{D} = \mathrm{End}(\mathbf{C}^N)$. *Then* g *defines a stable cohomology class.*

The above Proposition gives a useful criterion for stability in the special case discussed because of recent results of Ramis on the structure of the Galois differential group [Ra 2,3,4]. In the special situation considered by us, the results of Ramis imply that the representing cocycle of a pair whose Galois differential group has an irreducible action satisfies the condition of the above Proposition. It is not difficult to give examples where the above condition is satisfied. For instance, suppose that $2q \geq \Sigma_{i \neq j}\, m_i m_j$ where m_i is the multiplicity of the eigenvalue σ_i of D_r. Then essentially the same argument as in the proof of Proposition 2.3.3 leads to the construction of a cocycle satisfying the condition of Proposition 2.3.4. We shall not pursue these matters any further here.

APPENDIX

SOME HISTORICAL REMARKS

1. The aim of this appendix is to provide a brief historical supplement to the paper. Our intention is not to give an exhaustive historical survey but to give some additional perspective to some of the themes treated here. The reader may consult with profit the book of Majima [Maj] where another historical account is available as well as the article of Bertrand [Be].

D-modules, systems of meromorphic differential equations and the categorical language

The classical theory of differential equations was entirely concerned with solutions of differential equations of arbitrary degree N and the associated $N \times N$ systems of degree 1. The concept of D-modules or differential modules over differential rings and its use in the classification of meromorphic systems was initiated by Manin [Ma] . Among other things he characterized the D-modules which arise from meromorphic systems with regular singularities. In a pioneering and influential work [De 1] Deligne developed the theory of meromorphic connections with regular singularities on bundles defined on smooth algebraic varieties of arbitrary dimension. Further, in his letters to Malgrange [De2], Deligne outlined in the categorical language a complete description of the category of germs of meromorphic pairs in the neighborhood of an irregular singular point (**I**, §4). Deligne's treatment used the formal classification of Hukuhara-Turrittin-Levelt and the Malgrange-Sibuya isomorphism of the set of isomorphism classes of marked meromorphic pairs at a point formally isomorphic with a given pair with the first cohomology of the Stokes sheaf of that pair(see below).

Formal structure and reduction theory

It was Fabry who first constructed in his 1885 thesis [Fa] a full set of N linearly independent solutions of a scalar meromorphic differential equation of degree N,

$$(*) \qquad D_N u = 0, \qquad D_N = (d/dz)^N + a_{N-1}(z)(d/dz)^{N-1} + \ldots + a_0(z),$$

in the neighborhood of a singular point, say $z = 0$. The solutions were constructed over the extension $\mathfrak{F}_q = \mathfrak{F}[t]$, $t = z^{1/q}$, and were of the form

$$f_{jh} = e^{Q_j(t)} t^{\mu_j} \sum_{0 \le j \le h} (\log t)^j g_{ihj} \qquad (0 \le h \le m_j - 1);$$

here $Q_j(t)$ are distinct Laurent polynomials in t containing only negative powers of t, the g_{ihj} are in $\mathbf{C}[[t]]$, and m_j are integers ≥ 1 with $m_1 + m_2 + \ldots = N$. Then in the 1930's Cope took up this theme in two fundamental papers [Co]. He proved that any formal differential operator D_N of degree N defined over $\mathfrak{F}$ can be factorized over a suitable extension $\mathfrak{F}_q$ as a product of N differential operators of degree 1, and showed further the equivalence of linear $N \times N$ systems of degree 1 with linear scalar differential equations of degree N. In the language of differential modules this equivalence may be formulated as follows. Let K be a nontrivial differential field of characteristic 0, $\mathcal{D}$ the algebra of differential operators over K, and U any differential module of dimension N over K; then U is a cyclic module, and $U \cong \mathcal{D}/\mathcal{D}D_N$ for a suitable D_N, the set of possible D_N being in canonical bijection with the set of cyclic vectors. The factorization theorem was later reproved by Malgrange [Mal 2] using Newton polygons and by Robba [Ro] using Henselian techniques. It is not difficult to show that the results of Fabry and Cope are completely equivalent.

The main question in the formal theory of first order meromorphic systems

$$(*) \qquad dF/dz = A(z) F \qquad (A \in \mathfrak{gl}(N, \mathfrak{F}))$$

is their reduction to a canonical form under $GL(N, \mathfrak{F})$ or $GL(N, \mathfrak{F}^{cl})$. This was resolved by the combined efforts of Hukuhara [Hu], Turrittin [Tu] and Levelt [Le]. See also [BJL 1] and [BV 1] for some refinements of this work. The principal level or Katz invariant for a meromorphic system was explicitly discussed by Katz [Ka]; Poincaré [Po] had treated this formal invariant in the context of scalar equations of degree N.

The formal reduction theory for systems when the coefficients come from a general differential ring was the main theme of [BV 2]. The complete description of the category of differential modules over $R[[z]][z^{-1}]$ where R is a local ring is still not available, although [BV 2] contains a theory for the so-called well-behaved modules, which includes the reduction theory of (*) when A(z) depends analytically on an arbitrary number d of complex parameters $\lambda = (\lambda_1, \ldots, \lambda_d)$. These results are essential for the discussion of moduli problems for the systems (*).

Analytic theory and asymptotic structures on sectors

It was Poincaré [Po] who first discovered the analytic significance of the formal solutions of (*) when he showed that in generic cases of (*) there exist analytic solutions which were asymptotic to Fabry's solutions in sufficiently small sectors. Later Trjitzinsky [Tr] showed that there exist analytic solutions asmptotic to fixed formal solutions of an arbitrary system (*) in sectors bounded by certain "spectral curves" coming from the spectrum of the associated canonical form. It was however Hukuhara [Hu] who obtained the definitive version of the asymptotic existence theorem, and in doing this also discovered the correct way to define the Stokes lines. Hukuhara proved that the asymptotic existence theorem holds for sectors containing in their interior atmost one Stokes line. As a corollary Hukuhara showed that given a formal reduction of (*) to a "weak" canonical form and any sector as above there was an analytic reduction asymptotic to this formal reduction in this sector. This was completed by Malmquist [Malm] who was able to replace "weak" canonical form by canonical form. Turrittin [Tu] gave another independent proof of Hukuhara's asymptotic result.

Sibuya [Si 1] showed that the spectral splitting part of the asymptotic analytic reduction process could be carried out in suitable sectors and also treated spectral splitting for the generic case when (*) depends analytically on one parameter.

Already in Turrittin's work the problem of refining the asymptotic theory by introducing the setting of Gevrey classes of functions appeared in a natural manner. This aspect, which we have not touched at all in this paper, has been pursued in great depth by Ramis. The reader should consult the papers of Ramis [Ra] as well as that of Ramis and Sibuya [Ra-Si]. In addition much of the theory discussed above is treated in Wasow's classic treatise [Wa]. For a more detailed survey of the history of the asmptotic existence theorems see the excellent discussion in [Maj].

The Stokes sheaf and cohomological methods

A very important step in the modern approach to these classical problems was taken by Malgrange and Sibuya when they introduced sheaves of functions with asymptotic expansions and introduced for the first time cohomological methods into the theory [Mal 3] [Si 2].Their work highlighted the importance of the Stokes sheaf and its cohomology for the classification of meromorphic systems and led to what we have labelled the Malgrange-Sibuya isomorphism theorem (cf. **I**, §4). The work of Deligne [De 2] that we mentioned earlier uses the Malgrange-Sibuya isomorphism theorem to obtain a natural equivalence of the category of all meromorphic pairs at $z = 0$ with the category of $\mathcal{D}$-filtered local systems on S^1 (**I**, Theorem 4.7.3). For another but shorter exposition of Deligne's theorem see [Mal 4]. Majima [Maj] has extended much of this theory to the case of integrable connections in several variables.

The affine nature of $H^1(S^1, St)$

Deligne had already observed in [De2] that $H^1(S^1,St)$ was intrinsically a smooth variety. Balser, Jurkat and Lutz [Bal] [BJL] [J] showed (translating into our language) that for a certain canonical good covering $\mathfrak{A}$ of S^1 (or more precisely its universal covering space) $H^1(\mathfrak{A} :St) = H^1(S^1, St)$ is an affine space whose dimension is the irregularity of the endomorphism bundle of the formal pair chosen as the formal model. In an earlier version of the present paper [BV4] a proof was given in the unramified case that $H^1(S^1, St)$ had an intrinsic affine space structure and that its dimension was the irregularity of the of the endomorphism bundle. In [De3] Deligne outlined the proof given here. Of course the essence of this proof is a much stronger statement giving conditions when the H^1 of a sheaf of unipotent group schemes on S^1 is representable by an affine space. The freeness of the action of the coboundary group on the space of cocycles, obtained in [BJL] [J] [BV 4] appears in our present version as the vanishing of the H^0 of all the twists of the Stokes sheaf (see **II**, §§1,2). Earlier Malgrange [Mal 4] had proved the intrinsic affine nature of $H^1(S^1,St)$ when the leading coefficient of the canonical form of the formal model has distinct eigenvalues.

Local moduli

The precise notion of local moduli for isoformal unmarked pairs in the unramified case seems to have first appeared in [BV3]. An exposition of this is given in [BV5]. It should be pointed out that the formal reduction theory with parameters as given in [BV2] is essential for the treatment of local moduli. The idea of treating the moduli problem for marked pairs so that $H^1(S^1, St)$ itself (and not a quotient of it as would be the case if only unmarked pairs are considered) is the moduli space goes back to Deligne [De 2] as mentioned in [Mal 4]. The moduli problem for certain *nonlinear* systems is considered in the paper of Martinet and Ramis [MR].

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RÉSUMÉ

La présente monographie est consacrée à l'étude locale des solutions des systèmes d'équations différentielles méromorphes linéaires au voisinage d'un point singulier irrégulier. Si le point singulier est régulier, le groupe de monodromie contient l'essentiel de l'information. En revanche, le cas d'un point singulier irrégulier est bien plus compliqué. Cela est dû au fait que les solutions formelles de l'équation au voisinage d'un tel point sont d'ordinaire divergentes. Néanmoins, les solutions formelles sont séries asymptotiques pour les solutions analytiques dans tous les secteurs ayant la singularité pour sommet, pourvu que l'angle soit suffisamment petit. En général, si l'on fixe une solution formelle Φ, les solutions analytiques pour lesquelles Φ est une série asymptotique ne sont pas uniques. En fait, si l'on fait tourner le secteur, les solutions analytiques pour lesquelles Φ est une série asymptotique changeront en général : c'est le ≪ phénomène de Stokes ≫. Le but de cette monographie est de fournir une analyse systématique de ce phénomène et d'étudier comment il est affecté par des variations isoformelles des équations.

Le langage naturel pour exposer les principaux théorèmes est celui de germes de fibrés vectoriels holomorphes munis de connexions méromorphes. Si on fixe une telle paire (V_0, ∇_0), le théorème de Malgrange–Sibuya affirme qu'il y a une équivalence naturelle entre l'ensemble de triplets :

$$\left\{((V, \nabla), \xi) \,\middle|\, \xi : (V, \nabla) \cong (V_0, \nabla_0) \text{ est un isomorphisme formel} \right\}$$

défini à un isomorphisme analytique près, et la première cohomologie d'un certain faiseau $St(V, \nabla)$ de groupes sur S^1. Ce faisceau, qui s'appele le faisceau de Stokes de (V, ∇), est un faisceau de schémas en groupes algébriques unipotents sur **C**. Le résultat fondamental de la Partie **II** affirme que le foncteur de première cohomologie est représentable par un espace affine sur **C** dont la dimension est **l'irrégularité** du fibré $End(V, \nabla)$. Dans la Partie **III** on démontre que l'espace analytique complexe sousjacent à cet espace affine est un espace de modules locaux pour les déformations locales isoformelles de la paire (V, ∇). Dans la Partie **I** on développe le langage modernedes équations différentielles méromorphes linéaires. En particulier, on expose les théorèmes de Malgrange–Sibuya et Deligne qui traitent de la catégorie de germes de fibrés vectoriels holomorphes munis de connexions méromorphes.