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Inverse local times, positive sojourns, and maxima for Brownian motion

by Frank B. KNIGHT

Let $B(t)$, $B(0) = 0$, be a standard Brownian motion on R^1, and let $\ell(t,x) = \frac{d}{dx}\int_0^t I_{(-\infty,x)}(B(s))ds$ be its local time, continuous in (t,x) with probability 1 by Trotter's Theorem. We denote $\ell(t,0)$ by $\ell_0(t)$, and for $\alpha > 0$ let $T(\alpha) = \ell_0^{(-1)}(2\alpha) = \inf(t : \frac{1}{2}\ell_0(t) > \alpha)$ (the factor $\frac{1}{2}$ is introduced for notational convenience later). In the present work we are concerned with finding the joint distribution of the three random variables $T(\alpha)$, $S^+(\alpha) = \int_0^{T(\alpha)} I_{(0,\infty)}(B(s))ds$, and $M^+(\alpha) = \max_{t\leq T(\alpha)} B(t)$. To indicate how one arrives at this problem, we recall that $\ell(T(\alpha),x)$ is a tractable process in parameter x which can be used to define the three variables by the formulae

$$T(\alpha) = \int_{-\infty}^{\infty} \ell(T(\alpha),x)dx,\ S^+(\alpha) = \int_0^{\infty} \ell(T(\alpha),x)dx, \text{ and}$$
$$M^+(\alpha) = \inf\{x > 0 : \ell(T(\alpha),x) = 0\},\ P - a.s.$$

Indeed, $\frac{1}{2}\ell(T(\alpha),x)$ defines two independent diffusions with initial value α and infinitesimal generator $y\frac{d^2}{dy^2}$, namely the processes $\frac{1}{2}\ell(T(\alpha),x)$, $x \geq 0$, and $\frac{1}{2}\ell(T(\alpha),-x)$, $x \geq 0$. For reference to these facts, we can cite [7, Theorem 5.3.20]. Therefore, the above joint distribution easily extends to cover that of the 5 variables obtained by including $S^-(\alpha)$ and $M^-(\alpha)$, defined in the obvious way. Moreover one can hope for a reasonably simple

answer. Unfortunately, however, even this seemingly natural problem is by no means trivial, and the answer (insofar as we can give one) is not entirely simple.

A second reason for giving attention to this particular problem is its connection with the Brownian bridge. We recall that, as shown by P. Lévy, the process $b(t) = B(t) - tB(1)$, $0 \leq t \leq 1$, is independent of $B(1)$, and thus can be expressed as $B(t)$ conditional on $B(1) = 0$. Hence the name "Brownian bridge" (although, to be sure, the "bridge" is clearly in an extremely bad state of repair). Now letting $s(x)$ denote the local time at x of $b(t)$, it is well known (and easy to see) that we have the conditional equivalence in law

$$(1) \qquad (s(x) \,|\, s(0) = 2\alpha) \equiv (\ell(T(\alpha), x) \,|\, T(\alpha) = 1).$$

The densities of $s(0)$ and $T(\alpha)$ being known from Lévy's work, it follows that with the 5 variables introduced above one can also express the joint density of the 5 corresponding variables

$$(s(0), \int_0^\infty s(x)dx, \int_{-\infty}^0 s(x)dx, \max_{0\leq t\leq 1} b(t), \min_{0\leq t\leq 1} b(t)).$$

Besides, this also follows by a recent result of Biane, LeGall, and Yor [1, Theorem 1], from which it is easy to deduce the following identity of Laplace transforms

$$(2) \qquad E \exp - (\lambda_1 M^+(\alpha) T^{-\frac{1}{2}}(\alpha) + \lambda_2 S^+(\alpha) T^{-1}(\alpha) + \lambda_3 \ln(\alpha T^{-\frac{1}{2}}(\alpha)))$$

$$= \sqrt{\frac{2}{\pi}}\, E \exp - (\lambda_1 \max_t b + \lambda_2 \int_0^\infty s(x)dx + (\lambda_3 - 1)\ln s(0)), \quad \lambda_i > 0.$$

Remark. The choice of $\alpha = 1$ in the above paper was evidently only for

convenience, and in any case the left side of (2) does not depend on α, as seen by a scaling argument. We do not know whether (2) is the most expeditious way to transfer the results for $B(t)$ to $b(t)$ in the general setting. In the special case $\lambda_1 = 0$ it is easier to proceed directly, as done below (Corollary 1).

Let us now begin by giving the three marginal densities of $T(\alpha)$, $S^+(\alpha)$, and $M^+(\alpha)$.

Theorem 1. (a) The density of $T(\alpha)$ is $\alpha\sqrt{\frac{2}{\pi}}\, x^{-\frac{3}{2}} \exp(-2\alpha^2/x)$, $x > 0$. (b) The density of $S^+(\alpha)$ is that of $T(\alpha/2)$. (c) The density of $M^+(\alpha)$ is $\alpha x^{-2} \exp(-\alpha/x)$, $x > 0$.

Proof. (a) It follows by Lévy's equivalence $(B(t) - M(t), M(t)) \equiv (|B(t)|, \ell_0(t))$, where $M(t) = \max_{s \le t} B(s)$, that $T(\alpha) \equiv M^{(-1)}(2\alpha)$ [7, Theorem 2.5]. (b) As noted above, $T(\alpha) = S^+(\alpha) + S^-(\alpha)$ is a sum of independent, identically distributed random variables. Consequently, $S^+(\alpha) \equiv T(\alpha/2)$ as asserted. (c) Again as observed above, the diffusion $\frac{1}{2}\ell(T(\alpha),x)$, $x \ge 0$, has $M^+(\alpha)$ as passage time to 0, i.e. $M^+(\alpha) = \inf\{x > 0 : \frac{1}{2}\ell(T(\alpha),x) = 0\}$. Its initial value being α, we can refer for example to [7, Theorem 4.3.6] for the distribution of the passage time (the present process is equivalent to twice the process treated there).

Remark. This last result is not to be confused with the density of $\max_{x>0} \ell(T(\alpha),x)$, which is elementary. Indeed, since $\ell(T(\alpha),x)$, is a diffusion in natural scale, $P\{\max_{x>0} \ell(T(\alpha),x) < y\} = (y - 2\alpha)y^{-1}$, $y > 2\alpha$. Therefore, its density is simply $2\alpha y^{-2}$, $y > 2\alpha$.

The corresponding results for the Brownian bridge $b(t)$ are due to P. Lévy, with the possible exception of that for $\max b(t)$. Of course $T(\alpha)$ corresponds simply to 1, by definition of $b(t)$. On the other hand, the

density of $s(0)$ is the same as that of $M(1)$ given that $M(1) = B(1)$, by Lévy's equivalence cited above (because $|B(t)|$ conditioned by $|B(1)| = 0$ is equivalent to $|b(t)|$). This density is $x \exp(-\frac{x^2}{2})$, according to [9, §42, (10)]. The fact that $\int_0^1 I_{(0,\infty)}(b(t))dt$ has a uniform distribution on $(0,1)$ (constant density 1) is known as P. Lévy's Law [5, 2.6, Problem 1]. Finally, the density of $\max_{0\le t\le 1} b(t)$ can be expressed by convoluting the passage time density of $B(t)$ to x with the return time density to 0, and dividing by the density of $B(1)$ at 0, which is just $(2\pi)^{-\frac{1}{2}}$. In this way we obtain the probability that the maximum exceeds x. The derivative then expresses the density, but in the form of an untractable integral. Actually, however, it can be simplified, since we know from Doob's paper [4, §6] that the distribution of the maximum is just $1 - \exp(-2x^2)$, $0 < x$.

We also easily obtain a joint density for $b(t)$. Stated in conditional form, it is

Corollary 1. The conditional density of $\int_0^1 I_{(0,\infty)}(b(t))dt$ given that $s(0) = 2\alpha$ is

$$(2\sqrt{2\pi})^{-1}\alpha(t(1-t))^{-\frac{3}{2}} \exp\left[\frac{-\alpha^2(2t-1)^2)}{2t(1-t)}\right],\quad 0 < t < 1.$$

Proof. In view of the equivalence (1), the required density is also that of $(S^+ | T(\alpha) = 1)$, where $T(\alpha) = S^+ + S^-$ as before. By Theorem 1 this is the convolution of two densities as for $T(\alpha/2)$, divided by $\alpha\sqrt{\frac{2}{\pi}}\exp(-2\alpha^2)$, which reduces easily to the expression stated.

A worthwhile check of this result is to derive from it P. Lévy's Law for $\int_0^1 I_{(0,\infty)}(b(t))dt$. We multiply by the density of $\frac{1}{2}s(0)$, which in terms of α is $2(2\alpha \exp(-2\alpha^2))$ by Lévy's result quoted before, and integrate. The

change of variables $\beta = \alpha(t(1-t))^{-\frac{1}{2}}$ reduces the integral to

$\sqrt{\frac{2}{\pi}} \int_0^\infty \beta^2 e^{-\frac{1}{2}\beta^2} d\beta = 1$ for all $t \in (0,1)$, i.e. the uniform density.

Returning now to our original problem of $(T(\alpha), S^+, M^+)$ (we drop α in S^+, M^+ since it is fixed throughout) it is clear that the foregoing proof contains also the joint density of $(T(\alpha), S^+)$, or more precisely that of $(S^+|T(\alpha))$. We will not pause to write the exact expression, but go on to consider densities involving M^+. Since S^- and M^+ are conditionally independent given S^+, it suffices to consider the joint density of S^+ and M^+, from which that of the triple follows immediately by adding S^-. The approach which seems most workable here is to derive the conditional density of S^+ given $M^+ = m > 0$, and this will be done by first deriving its Laplace transform $E(\exp - \lambda S^+ | M^+ = m)$.

To this end, we can apply the representation of D. Ray [13] for the reversed (conditional) process $\frac{1}{2}\ell(T(\alpha), m - x)$, $0 \leq x \leq m$. Letting $\tau(y)$ be a diffusion equivalent to $|B_4|^2(y)$, where $|B_4|$ is a 4-dimensional Bessel process, the representation of Ray shows that $\ell(T(\alpha),x)$, $0 \leq x \leq m$, conditional on $M^+ = m$ is equivalent in law to $(1+x)^2\tau((1+x)^{-1} - (1+m)^{-1})$, conditional on $\tau(0) = 0$ and $\tau(1-(1+m)^{-1}) = 2\alpha$. This argument is given in [7, Corollary 5.1.8], where the process involved there need only be conditioned by the requirement of 2α as inital value at $x = 0$. Let us remark, before going on, that the transform we derive will be checked later by the entirely different methods of excursion theory, but the present method has interesting biproducts. Accordingly, we have the representation

$$(4) \qquad (S^+|M^+ = m) \equiv \int_0^m (1+x)^2 \tau((1+x)^{-1} - (1+m)^{-1})dx$$

$$= \int_{(1+m)^{-1}}^1 u^{-4}\tau(u - (1+m)^{-1})du$$

$$= (1 + m)^4 \int_0^{m(1+m)^{-1}} ((1 + m)^4 v^4 + 1)^{-1} \tau(v)dv,$$

by elementary changes of variables. Now let us represent $\tau(v) \equiv \sum_{k=1}^{4} B_k^2(v)$, where $B_1, \ldots, B_4$ are independent Brownian motions. The condition $\tau(m(1 + m)^{-1}) = 2\alpha$ can be divided arbitrarily among $B_k^2(m(1 + m)^{-1})$, $1 \leq k \leq 4$, since $(B_k(m(1 + m)^{-1}), 1 \leq k \leq 4)$ is uniformly distributed over the sphere of radius $\sqrt{2\alpha}$ under the conditioning. It will be convenient to assume that $B_k(m(1 + m)^{-1}) = \sqrt{\frac{\alpha}{2}}$, $1 \leq k \leq 4$, so that our 4 terms in the integral (4) are both independent and identically distributed. Next, we apply a scaling argument to change the upper limit $m(1 + m)^{-1}$ to 1. Namely, letting $W_k(t) = \left[\frac{m + 1}{m}\right]^{\frac{1}{2}} B_k\left[\frac{m}{m + 1} t\right]$, the integral (4) can be written in the form

$$(5) \qquad \sum_{k=1}^{4} (m + 1)^2 m^2 \int_0^1 (m^4 s^4 + 1)^{-1} W_k^2(s)ds \quad \left[s = \frac{m + 1}{m} v\right]$$

in which $W_k(s)$ are independent Brownian motions starting at 0 and conditioned by $W_k(1) = \left[\frac{(m + 1)\alpha}{2m}\right]^{\frac{1}{2}}$, $1 \leq k \leq 4$.

Now we are ready to introduce the Laplace transform of the distribution of expression (5). At first thought this appears to depend on the three parameters (λ, α, m). However, it is plausible that the scaling property of B_k should introduce an invariance in the transform under an appropriate scaling relation among these parameters. Routine scaling argument (which we shall omit) indeed shows that the transform actually depends only on the pair of parameters $m\sqrt{\lambda}$ and αm^{-1}.

Granting this invariance, we can now derive the transform by appeal to some of the oldest known Wiener integrals.

Theorem 2. $E \exp(-\lambda s^+ | M^+ = m) = 2m^2\lambda(\sinh m\sqrt{2\lambda})^{-2}\exp(\alpha m^{-1}(1 - m\sqrt{2\lambda} \coth m\sqrt{2\lambda}))$.

Proof. Set $m^2\lambda = \mu$, and let $m \longrightarrow 0+$ with μ fixed and $\alpha m^{-1} = r$ fixed. The condition $W_k(1) = \left[\frac{(m+1)\alpha}{2m}\right]^{\frac{1}{2}}$ in (5) becomes $W_k(1) = \left[\frac{r}{2}\right]^{\frac{1}{2}}$, and the transform of (5) becomes $(E \exp(-\mu \int_0^1 W_k^2(s)ds))^4$. Evidently there is no difficulty with this limit procedure since the law of W_k depends continuously on the given value of $W_k(1)$ in view of Lévy's representation $W(t) = b(t) + tW(1)$. Now this transform is well-known to be $(2\mu)(\sinh\sqrt{2\mu})^{-2} \exp r(1 - \sqrt{2\mu}\coth\sqrt{2\mu})$, as required.

The history of this last result is tied to P. Lévy in an interesting way. In [8, 10° (1940)], after introducing the "stochastic area" of a plane Brownian motion given its value at $t = 1$, Lévy showed that this area is Gaussian with mean 0 and variance $\sigma^2 = \frac{1}{4}\int_0^t (W_1^2(s) + W_2^2(s))ds$, where (W_1, W_2) denote the two Brownian components. At this point, he did not succeed in obtaining the exact Laplace transform. However, by 1951 in [10, (1.3.4)] he had succeeded in obtaining the Fourier transform of the area by an entirely different method (by Fourier series expansion of the path functions). In view of his earlier observation, this is just the Laplace transform of σ^2 with λ^2 in place of λ and a factor $\frac{1}{2}$ in the exponent. However, at least in [10] Lévy does not mention this fact.

Meanwhile, building in part on the work of Cameron and Martin, who had by 1944 obtained the Laplace transform of $\int_0^1 B^2(s)ds$ when $B(1)$ is not given in advance, E. W. Montroll in 1952 published explicit expressions including that for $E \exp(-\lambda \int_0^1 B^2(s)ds)$, conditional on arbitrary $B(0)$ and $B(1)$ [12, (3.41)]. This of course contains the case we need, as does also previous work of Lévy. One can argue that our case is even implicit in the papers of Cameron and Martin, but we were not able to extract it from their general formulas.

The present application has an interesting biproduct, as follows.

Corollary 2. For the one-dimensional Brownian motion starting at 0, we have

$$E(\exp - \lambda \int_0^1 B^2(t)dt \,|\, B^2(1) = c) = E(\exp - \lambda \int_0^1 (1 + t^4)^{-1} B^2(t)dt \,|\, B^2(1) = 2c).$$

Proof. Instead of letting $m \longrightarrow 0$ as in the above proof, we just set $m = 1$ in 5) and use the invariance of the transform.

Remark. The result of Theorem 2 can also be obtained by suitably differentiating the transform from Theorem 2.1 of [6, (1969)]. Indeed, this result is easily equivalent to it, and we first obtained it by this method (which in turn was based on a random walk approximation). Thus the present result apparently dates from 1969. However, the present method, based on Ray's representation, seems to give a new insight as well as yielding Corollary 2. Below we treat a third method which leads, besides, to an inversion of the transform.

There is also a second consequence which seems quite surprising, so we state it as

Theorem 3.(NB) For all $\alpha > 0$, $E \exp(-\lambda S^+ M^{-2}) = 2\sqrt{2\lambda}\,(\sinh 2\sqrt{2\lambda})^{-1}$. In other words, S^+M^{-2} has the same distribution as $4 \int_0^1 (W_1^2(s) + W_2^2(s))ds$ given that $W_1^2(1) + W_2^2(1) = 0$, i.e. 4 times the integral of a squared Bessel bridge of dimension 2.

(NB) A probabilistic explanation for this identity, together with various related developments obtained by P. Biane and M. Yor during the Colloque, is to appear subsequently.

Proof. By Theorem 1, (c) and Theorem 2 we need only calculate

$$E\exp(-\lambda S^{+}M^{-2}) = \int_0^\infty E\exp\left[-\frac{\lambda}{m^2}S^{+}\right]\alpha\, m^{-2}\exp\left[-\frac{\alpha}{m}\right]dm$$

$$= 2\lambda(\sinh\sqrt{2\lambda}\,)^{-2}\int_0^\infty \exp\left[-\frac{\alpha}{m}\sqrt{2\lambda}\coth\sqrt{2\lambda}\right]\frac{\lambda}{m^2}\,dm$$

$$= 2\lambda(\sinh\sqrt{2\lambda}\,)^{-2}\left[\sqrt{2\lambda}\coth\sqrt{2\lambda}\right]^{-1}$$

$$= 2\sqrt{2\lambda}\,(\sinh 2\sqrt{2\lambda}\,)^{-1}.$$

We now consider the meaning and inversion of the transform of Theorem 2. The factor $2m^2\lambda(\sinh m\sqrt{2\lambda}\,)^{-2}$ is the transform of m^2X where the distribution of X depends neither on m nor on α, and has transform $2\lambda(\sinh\sqrt{2\lambda})^{-2}$. This transform is familiar from "excursion theory." It is the square of the transform of 1st passage time to 0 of a 3- dimensional Bessel process, starting at 1. According to D. Williams' well-known characterization of the Brownian excursion law (see for example [14, p. 233]) this is the transform of the duration of a Brownian excursion of maximum 1. Then it is apparent what meaning is to be ascribed to the factor under consideration. It is the transform of the duration of the unique excursion by time $T(\alpha)$ which reaches the maximum m.

The inversion of this transform is therefore a convolution of two identically distributed first-passage-time distributions. These distributions also go back to P. Lévy. The detailed inversion is found in [3] which noted the simplification which occurs in the present (3-dimensional) case. Subsequently it occurred in many papers and contexts, of which we will cite only [2, (3.17)] and [11]. In terms of Chung's distribution $F_1(x) = \sum_{n=-\infty}^{\infty}(-1)^n\exp(-n^2x)$, $0 < x < \infty$, the passage time distribution is $F_1\left[\frac{\pi^2 x}{2}\right]$.

Louchard [11] gives the representation $F_1(x) = 2\theta\left[\frac{4x}{\pi}\right] - \theta\left[\frac{x}{\pi}\right]$ in terms of the 3rd Jacobi θ-function, as well as the relationship to the Kolmogorov-Smirnov distribution, and other applications. The convolution of two terms with distribution $F_1(x)$ has distribution $F_2(x) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2x}(1 - 2n^2x)$ [2, (4.9)], and consequently we have the

First Inversion. The factor $2m^2\lambda(\sinh m\sqrt{\lambda}\,)^{-2}$ is the transform of $\frac{d}{dx}F_2\left[\frac{\pi^2}{2}m^{-2}x\right]$.

Remarks. By contrast, the maximum of a Brownian excursion of duration ℓ has distribution $F_2\left[\frac{2}{\ell}x^2\right]$, $0 < x < \infty$, [2, Theorem 7]. Why the same $F_2(x)$ is involved in both of these apparently reverse situations would seem to us to deserve a probabilistic explanation of some sort.

Turning now to the second factor

(6) $\qquad \exp \alpha m^{-1}(1 - m\sqrt{2\lambda}\ \operatorname{cotanh} m\sqrt{2\lambda}\,),$

we need only recall a little of the Ito excursion theory to understand its signification. With local time as time parameter, the excursions of $B(t)$ above 0 form a Poisson point process, and the expected frequency of excursion maxima has density $x^{-2}dx$. Therefore, $m^{-1} = \int_m^{\infty} x^{-2}dx$ is the expected number of excursions by $T(1)$ whose maxima are at least m, and $\exp(-\alpha m^{-1})$ is the probability of no such excursions by $T(\alpha)$. Now it is easy to guess the meaning of (6), which is the transform of the total duration of positive excursions by time $T(\alpha)$ conditional on the assumption that all have maxima less than m. But then the factor $\exp(-\alpha\sqrt{2\lambda}\ \operatorname{cotanh} m\sqrt{2\lambda}\,)$ must be the transform of the duration of positive excursions by time $T(\alpha)$ for the process $B(t)$, killed (or absorbed) upon reaching $[m,\infty)$.

This last can be represented in the form $\exp\left[-\alpha \int_0^\infty (1 - e^{-\lambda y}) n_+(dy)\right]$, where $n_+(dy)$ is the Lévy measure of $T(\alpha)$ for the above killed process (actually, $n_+(dy)$ also has a point mass m^{-1} at $+\infty$ to allow for the case $T(\alpha) = \infty$, which must be included in our representation). Now to calculate $n_+(dy)$ we can use the Lévy formula [5, 6.2] $n_+(dy) = \lim_{\epsilon \to 0+} \epsilon^{-1} P^\epsilon(T_0 \in dy)$, where T_0 denotes the passage time to 0 of the process absorbed at m. For $\epsilon > 0$, $P^\epsilon(T_0 \in dy)$ is the probability of absorption at 0 for the process with two absorbing barriers at 0 and m. It is a quantity whose density has a well-known trigonometric series expansion [7, Theorem 4.1.1], and we obtain easily

$$(7) \qquad n_+(dy) = \lim_{\epsilon \to 0+} \left[\epsilon^{-1} \frac{\pi}{m^2} \sum_{n=1}^{\infty} (-1)^n n \cos n\pi \sin \frac{n\pi\epsilon}{m} \exp\left[-\frac{1}{2}\left(\frac{n\pi}{m}\right)^2 y\right]\right] dy$$

$$= \pi^2 m^{-3} \sum_{n=1}^{\infty} n^2 \exp\left[-\frac{1}{2}\left(\frac{n\pi}{m}\right)^2 y\right] dy.$$

To apply this to $T(\alpha)$, however, we have to remember that the local time of [5] is $\frac{1}{2}$ of our $\ell_0(t)$, so that actually we are using the appropriate definition of $T(\alpha)$. Therefore, it follows that

$$(8) \qquad \exp\left(-\alpha\sqrt{2\lambda} \operatorname{cotanh} m\sqrt{2\lambda}\right)$$

$$= \exp - \alpha\left[m^{-1} + \int_0^\infty (1 - e^{-\lambda y}) \pi^2 m^{-3} \sum_{n=1}^{\infty} n^2 \exp\left[-\frac{1}{2}\left(\frac{n\pi}{m}\right)^2 y\right] dy\right]$$

$$= \exp - \alpha\left[m^{-1} + \lambda m^{-1} \sum_{n=1}^{\infty} \int_0^\infty \exp\left[-\lambda y - \frac{1}{2}\left(\frac{n\pi}{m}\right)^2 y\right] dy\right]$$

$$= \exp - \alpha\left[m^{-1} + 2\lambda m^{-1} \sum_{n=1}^{\infty} \left[\lambda + \frac{1}{2}\left(\frac{n\pi}{m}\right)^2\right]^{-1}\right].$$

where we integrated by parts for the second equality (after checking that $\lim_{y\to 0+} y \sum_{1}^{\infty} e^{-n^2 y} = 0$).

Remarks. One can also calculate the Lévy formula $E \exp(-\lambda T(\alpha)) = \exp(-\alpha G_\lambda^{-1}(0,0))$ where $G_\lambda(0,0)$ is the Green density of $B(t)$ with absorbing barriers at both $\pm m$. This gives the result $\exp(-2\alpha\sqrt{2\lambda}\ \operatorname{cotanh}\ m\sqrt{2\lambda}\ )$ for the transform of $T(\alpha)$ for the process with two barriers, as expected, but it requires using an interesting identity from complex variables, namely

$$(9) \qquad a^{-1} \sum_{n=1}^{\infty} \left[\lambda + \frac{1}{2}\left[\frac{(2n-1)\pi}{2m}\right]^2\right] = (2\lambda)^{-\frac{1}{2}} \tanh\, m\sqrt{2\lambda}.$$

Another curious observation is that we can express the density $n_+(dy)(dy)^{-1}$ from (7) by using the same F_2 as in the first inversion:

$$(10) \qquad n_+(dy)(dy)^{-1} = \sqrt{2\pi}\, y^{-3/2} F_2(2m^2 y^{-1}),$$

where we applied the identity $\theta(x) = x^{-\frac{1}{2}}\theta(x^{-1})$ as in [2, (4.10)].

Returning to (8), we observe that it has a form which can be inverted as an infinite convolution of distributions with transforms

$$(11) \qquad \exp\left[-2\alpha\lambda m^{-1}\left[\lambda + \frac{1}{2}\left[\frac{n\pi}{m}\right]^2\right]^{-1}\right],$$

provided that these inversions can be identified and that the convolution series converges. It is clear from (8) that these are the transforms of compound Poisson random variables with intensity parameter $\lambda_n = \alpha\pi^2 m^{-3} n^2 \int_0^\infty \exp\left[-\frac{1}{2}\left[\frac{n\pi}{m}\right]^2 y\right] dy = 2\alpha m^{-1}$, not depending on n. Letting X_1

have the density $\frac{1}{2}\left[\frac{\pi}{m}\right]^2 \exp\left[-\frac{1}{2}\left[\frac{\pi}{m}\right]^2 y\right]$, the n^{th} compounding random variable is equivalent to $n^{-2}X_1$. Since the convolution corresponds to a sum of independent random variables, with expectations $\mu_n = n^{-2}(4\alpha m\pi^{-3})$ and variances $\sigma_n^2 = n^{-4}(16\alpha m^{-3}\pi^{-5})$, it follows that the series converges with probability 1. Hence the inversions of $\exp\left[-2\alpha\lambda m^{-1} \sum_{n=1}^{N} \left[\lambda + \frac{1}{2}\left[\frac{n\pi}{m}\right]^2\right]^{-1}\right]$ converge in law as $N \longrightarrow \infty$. Also, since the summands are positive, and have a density apart from the probability mass at 0, it follows that the limit law (which has no point mass at 0) has a density. To see this, one need only remark that, conditional on any of the first N terms being non-zero, the sum has a density, and as $N \longrightarrow \infty$ this given event tends to probability 1.

Turning to the inversion of the terms (11), we observe that this is related once more to the process $\frac{1}{2}\,\ell\,(T(\alpha),t)$, $t > 0$, with generator $y\,\frac{d^2}{dy^2}$. For convenience here we let t denote the parameter (instead of x) and let 0 be introduced as an absorbing boundary. Then the semigroup $T_t f(x)$ satisfies $T_t(e^{-\lambda x}) = \exp(-\lambda x(1 + \lambda t)^{-1})$, in accordance with [7, (4.3.12)] (which is for $\frac{1}{2}$ the present process, i.e. $\frac{1}{4}\,\ell(T(\alpha),t)$). It is easy to see that, for each n, the present transform (11) is the special case with $x_n = 4\alpha m(n\pi)^{-2}$, $t_n = 2m^2(n\pi)^{-2}$. The inversion is therefore the transition density of this process with x_n as starting point and t_n as time parameter. Referring to [7, (4.3.13)], it is (in terms of variable y)

$$(12) \qquad t_n^{-1}\sqrt{x_n y^{-1}}\left[\exp - \left[\frac{x_n + y}{t_n}\right]\right] I_1\left[2t_n^{-1}\sqrt{x_n y}\right]dy,\ y > 0,$$

where I_1 is the modified Bessel function, and we also have the point mass $\exp(-2\alpha m^{-1})$ at 0 corresponding to absorption by time t_n.

Unfortunately, in spite of the convolution properties of the semigroup, we do not see any easy way to convolute these distributions explicitly, nor do we

know why the ubiquitous generator $y\frac{d^2}{dy^2}$ is again implicated. Let us simply state, in conclusion, the

Second Inversion. The factor (6) has as inversion the limit as $N \longrightarrow \infty$ of the convolution from $n = 1$ to N of the probability distributions (12).

Final Remark. Noting that $\frac{x_n}{t_n} = \frac{2\alpha}{m}$ does not depend on n, it is easy to see that the distributions (12) are all derived from a single distribution by changes of scale. Indeed, for $n = 1$ the density (12) reduces to

$$(13) \qquad \pi m^{-\frac{3}{2}}\left[\frac{\alpha}{y}\right]^{\frac{1}{2}}\left[\exp - \left[\frac{2\alpha}{m} + \frac{\pi^2}{2m^2}y\right]\right]I_1\left[2\pi m^{-\frac{3}{2}}(\alpha y)^{\frac{1}{2}}\right],$$

and if X_1 denotes a random variable with this density on $(0,\infty)$, and $P\{X_1 = 0\} = \exp(-2\alpha m^{-1})$, then for every n the random variable $X_n = n^{-2}X_1$ has the distribution (12).

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