

Astérisque

MELVYN B. NATHANSON

Thin bases in additive number theory

Astérisque, tome 147-148 (1987), p. 315-317

http://www.numdam.org/item?id=AST_1987__147-148_315_0

© Société mathématique de France, 1987, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

THIN BASES IN ADDITIVE NUMBER THEORY

Melvyn B. Nathanson

Let B be a set of nonnegative integers, and let hB denote the set of all sums of h elements of B . The set B is a basis (resp. asymptotic basis) of order h if hB contains all (resp. all sufficiently large) natural numbers. The squares, the k -th powers, and the primes are the classical examples of asymptotic bases in additive number theory.

Let $B(x)$ denote the number of positive integers in the set B that do not exceed x . If B is an asymptotic basis of order h , then it is easy to show that $B(x) > c_1 x^{1/h}$ for some constant $c_1 > 0$ and all $x > x_1$. An asymptotic basis B of order h is called thin if $B(x) < c_2 x^{1/h}$ for some constant $c_2 > 0$ and all $x > x_2$. Thin bases exist. Indeed, for each $h \geq 2$, Cassels [1] constructed a family of bases B of order h such that $B(x) \sim c x^{1/h}$ as $x \rightarrow \infty$. It is not known if the classical sequences in additive number theory contain subsequences that are thin bases.

Let A be a finite set of nonnegative integers, and let $|A|$ denote the cardinality of A . If $\{0, 1, \dots, n\} \subseteq hA$, then A is called a basis of order h for n . Clearly, if A is a basis of order h for n , then $|A| > n^{1/h}$.

In this report I state some recent results on the additive basis properties of subsets of the squares, k -th powers, and primes.

THEOREM (Choi-Erdős-Nathanson[2]). For every $n > 1$ there exists a finite set A of squares such that A is a basis of order 4 for n and

$$|A| < cn^{1/3} \log n$$

where $c = 4/\log 2$.

This is proved by means of an explicit construction. Note that the set of all squares up to n contains $[n^{1/2}] + 1$ elements.

THEOREM (Erdős-Nathanson[3]). For every $\varepsilon > 0$ there exists a set B of squares such that

- (i) B is a basis of order 4,
- (ii) If $n \neq 4^T(8k+7)$, then $n \in 3B$,
- (iii) $B(x) \sim cx^{(1/3)+\varepsilon}$ for some $c > 0$ as $x \rightarrow \infty$

The proof uses the probability method of Erdős and Rényi. The Theorem is best possible except for the ε in the exponent in (iii).

Zöllner combined the two results above to obtain the following.

THEOREM (Zöllner[7]). For every $\varepsilon > 0$ there exists n_0 such that if $n > n_0$ there is a finite set A of squares such that A is a basis of order 4 for n and

$$|A| < n^{(\frac{1}{4})+\varepsilon}$$

This result is best possible except for the ε in the exponent.

THEOREM (Zöllner[8]). Let $h \geq 4$. For every $\varepsilon > 0$ there exists a set B of squares such that B is a basis of order h and

$$B(x) < x^{(1/h)+\varepsilon}$$

for all $x > x_0$.

THEOREM (Wirsing[6]). Let $h \geq 4$. There exists a set B of squares such that B is a basis of order h and

$$B(x) < c(x \log x)^{1/h}$$

for some constant $c=c(h) > 0$ and all $x > x_0$.

Both Wirsing and Zöllner use probability methods to obtain their results, and, consequently, it is not yet possible to describe explicitly a sparse sequence of squares that is a basis of order 4.

There are some results on thin versions of Waring's problem.

THEOREM (Nathanson[4]). Let $k \geq 3$ and $s > s_0(k)$. Let $0 < \varepsilon < 1/s$. There exists a set B of nonnegative k -th powers such that B is a basis of order s and

$$B(x) \sim cx^{1-(1/s)+\varepsilon}$$

for some constant $c > 0$ as $x \rightarrow \infty$.

The proof requires the Hardy-Littlewood asymptotic formula for the number of representations of an integer as the sum of s k -th powers, as well as the Erdős-Rényi probability method.

There is a finite version of the preceding theorem. Let $f(n,k,s)$ denote the cardinality of the smallest finite set A of k -th powers such that A is a basis of order s for n . Clearly, $f(n,k,s) > n^{1/s}$.

Define

$$\beta(k,s) = \limsup_{n \rightarrow \infty} \frac{\log f(n,k,s)}{\log n}$$

Let $g(k)$ denote the smallest integer h such that the set of all non-negative k -th powers is a basis of order h .

THEOREM (Nathanson[5]). For $k \geq 3$ and $s \geq g(k)$,

$$f(n,k,s) < 2(s-g(k)+1) n^{1/(s-g(k)+k)}$$

In particular, $\beta(k,s) \sim 1/s$ as $s \rightarrow \infty$.

Finally, there is the following beautiful result on sums of primes.

THEOREM (Wirsing[6]). For $h \geq 3$, there is a set P of primes such that

- (i) $n \in hP$ for all $n > n_0$ such that $n \equiv h \pmod{2}$,
- (ii) $P(x) < c(x \log x)^{1/h}$ for some constant $c > 0$ and all $x > x_0$.

REFERENCES

1. J.W.S. Cassels, "Über Basen der natürlichen Zahlenreihe, Abh. math. Semin. Univ. Hamburg 21 (1957), 247-257.
2. S.L.G. Choi, P. Erdős, and M.B. Nathanson, Lagrange's theorem with $N^{1/3}$ squares, Proc. Amer. Math. Soc. 79 (1980), 203-205.
3. P. Erdős and M. B. Nathanson, Lagrange's theorem and thin subsequences of squares, in J. Gani and V. K. Rohatgi(eds.), Contributions to Probability, Academic Press, New York, 1981, pp.3-9.
4. M.B. Nathanson, Waring's problem for sets of density zero, in M.I. Knopp(ed.), Number Theory, Philadelphia 1980, Lecture Notes in Mathematics, Springer-Verlag, Vol. 899,1981, pp. 301-310.
5. M. B. Nathanson, Waring's problem for finite intervals, Proc. Amer. Math. Soc., to appear.
6. E. Wirsing, Thin subbases, preprint.
7. J. Zöllner, Über eine Vermutung von Choi, Erdős und Nathanson, Acta Arith. 45 (1985).
8. J. Zöllner, Der Vier-Quadrate-Satz und ein Problem von Erdős und Nathanson, Dissertation, Johannes Gutenberg-Universität, Mainz, 1984.

Melvyn B. Nathanson
 Department of Mathematics
 Rutgers-The State University
 Newark, New Jersey 07102
 U.S.A.