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their higher symmetries**

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## REPRESENTATION MODELS FOR CLASSICAL GROUPS AND THEIR HIGHER SYMMETRIES

BY

I.M. GELFAND and A.V. ZELEVINSKY

The main results presented in this talk are published with complete proofs in [1]. We give also some new results obtained by the authors jointly with V.V. SERGANOVA. The conversations with V.V. SERGANOVA enable us also to clarify the formulations of [1] related to supermanifolds. We are very grateful to her.

Let  $G$  be a reductive algebraic group over  $\mathbf{C}$ . A representation of  $G$  which decomposes into the direct sum of all its (finite dimensional) irreducible algebraic representations each occurring exactly once is called a *representation model for  $G$* . The H. Weyl's unitary trick shows that the construction of such a model is equivalent to the construction of a representation model for the compact form of  $G$ ; the language of complex groups is more convenient for us here. The classical example of a representation model for  $SO_3$  is the natural representation in the space of functions on the two dimensional sphere.

We study two approaches to representation models which are in some sense dual to each other. The first one is to realize a representation model of  $G$  as induced representation  $\text{Ind}_M^G \tau$ . When  $\tau = 1$ , we say that this is a *geometric realization* of a model; in this case a model is realized in the space of regular functions on the homogeneous space  $G/M$ .

The second approach is to realize a model as the restriction to  $G$  of a certain special representation of an overgroup of  $G$ . More precisely, we shall construct an overgroup  $L \supset G$  and a representation of the Lie algebra  $\mathfrak{l}$  of  $L$  which is an extension of the action of the Lie algebra  $\mathfrak{g}$  of  $G$  on the representation model of  $G$ . The action of  $\mathfrak{l}$  on a representation model for  $G$  will be called *higher symmetry* of the model (in [1] we used the term "hidden

symmetry” but the present terminology looks more natural). An important example of higher symmetry was recently discovered by L.C. BIEDENHARN and D. FLATH [2] : they construct an action of the Lie algebra  $\mathfrak{so}_8$  on the representation model for  $SL_3$ .

We shall construct geometric realizations and higher symmetries of representation models for all classical groups  $G$ . Thus,  $G$  will be one of the groups  $GL(n, \mathbf{C})$  ( $n \geq 2$ ),  $O(n, \mathbf{C})$  ( $n \geq 3$ ) or  $Sp(2n, \mathbf{C})$  ( $n \geq 1$ ) (the letter  $\mathbf{C}$  will be omitted for brevity). All our constructions exhibit the remarkable duality between the orthogonal and symplectic groups, and between symmetric and exterior algebras. As a consequence supergroups and supermanifolds will naturally appear in a “purely even” situation.

The systematic study of representation models was initiated in [3]. For classical groups the result of [3] may be formulated as follows :

**THEOREM 1** [3]. — *For any classical group  $G$  define the subgroup  $M$  and the representation  $\tau$  of  $M$  according to the following table (the embedding of  $M$  into  $G$  is standard, and  $\lambda(n)$  denotes the natural representation of a subgroup of  $GL(n)$  in the exterior algebra  $\bigwedge^*(\mathbf{C}^n)$ ).*

|        |              |                          |                        |              |
|--------|--------------|--------------------------|------------------------|--------------|
| $G$    | $GL(n)$      | $O(2n+1)$                | $O(2n)$                | $Sp(2n)$     |
| $M$    | $O(n)$       | $O(n+1) \times O(n)$     | $O(n) \times O(n)$     | $GL(n)$      |
| $\tau$ | $\lambda(n)$ | $\lambda(n+1) \otimes 1$ | $\lambda(n) \otimes 1$ | $\lambda(n)$ |

*Then in each case the induced representation  $\text{Ind}_M^G \tau$  is a representation model for  $G$ .*

In fact, in [3] the model  $\text{Ind}_M^G \tau$  was constructed for any connected reductive group  $G$ . The homogeneous space  $G/M$  is the complexification of the symmetric space of maximal rank corresponding to  $G$ .

The first main result of [1] is the construction of a new series of models which are in some sense dual to models from THEOREM 1.

**THEOREM 2** [1]. — *For any classical group  $G$  different from  $GL(2n+1)$  and  $O(2n+1)$  define the subgroup  $M$  and the representation  $\tau$  of  $M$  according to the following table (the embedding of  $M$  into  $G$  is standard, and  $\sigma(n)$  denotes the natural representation of a subgroup of  $GL(n)$  in the symmetric algebra  $S^*(\mathbf{C}^n)$ ).*

|        |              |             |                          |                        |
|--------|--------------|-------------|--------------------------|------------------------|
| $G$    | $GL(2n)$     | $O(2n)$     | $Sp(4n+2)$               | $Sp(4n)$               |
| $M$    | $Sp(2n)$     | $GL(n)$     | $Sp(2n+2) \times Sp(2n)$ | $Sp(2n) \times Sp(2n)$ |
| $\tau$ | $\sigma(2n)$ | $\sigma(n)$ | $\sigma(2n+2) \otimes 1$ | $\sigma(2n) \otimes 1$ |

Then in each case the induced representation  $\text{Ind}_M^G \tau$  is a representation model for  $G$ .

The spaces  $G/M$  in THEOREM 2 also are complexifications of symmetric spaces but not of maximal rank.

There is a specific correspondence between models from THEOREMS 1 and 2. Namely, each of the models from THEOREM 1 transforms into a model from THEOREM 2 if all groups and representations are changed as follows :  $GL(n)$  is everywhere replaced by  $GL(2n)$ ,  $O(n)$  by  $Sp(2n)$ ,  $Sp(2n)$  by  $O(4n)$ , and  $\lambda(n)$  by  $\sigma(2n)$ . The meaning of this procedure is not clear.

For the groups  $GL(2n+1)$  and  $O(2n+1)$  not covered by THEOREM 2 there are also important representation models constructed in [4].

THEOREM 3 [4]. — Let  $G = GL(2n+1)$  or  $O(2n+1)$ . Define the subgroup  $M_0$  in  $G$  as follows. For  $G = GL(2n+1)$  put  $M_0 = Sp(2n)$ , the embedding of  $M_0$  into  $G$  being the composition of standard embeddings  $Sp(2n) \subset GL(2n) \subset GL(2n+1)$ . For  $G = O(2n+1)$  put  $M_0 = GL(n)$ , the embedding of  $M_0$  into  $G$  being the composition of standard embeddings  $GL(n) \subset O(2n) \subset O(2n+1)$ . Then in each case  $\text{Ind}_{M_0}^G 1$  is a representation model for  $G$ .

It turns out that each of the models from THEOREM 2 also may be realized as  $\text{Ind}_{M_0}^G 1$ ; for this we choose a subgroup  $M_0$  in  $M$  such that  $\tau = \text{Ind}_{M_0}^M 1$ . To describe  $M_0$  we need some terminology. Let  $GL(n-1/2)$  denote the affine subgroup in  $GL(n)$ , and  $Sp(2n-1)$  the affine subgroup in  $Sp(2n)$ , i.e., in each case the stabilizer of a nonzero linear functional on the space of the standard representation. Therefore, in the series of groups  $GL(n)$  the index  $n$  now may be half-integer, and in the series of groups  $Sp(n)$   $n$  may be any positive integer. This terminology enables us to unify THEOREMS 2 and 3 :

THEOREM 4 [1]. — For any classical group  $G$  define the subgroup  $M_0$  in  $G$  according to the following table (for  $G = GL(2n+1)$  or  $O(2n+1)$  the embedding of  $M_0$  into  $G$  is described in THEOREM 3, and in other cases  $M_0$  is embedded in a standard way into the subgroup  $M$  from THEOREM 2).

| $G$    | $GL(n)$   | $O(n)$        | $Sp(2n)$               |
|--------|-----------|---------------|------------------------|
| $M_0$  | $Sp(n-1)$ | $GL((n-1)/2)$ | $Sp(n-1) \times Sp(n)$ |
| $\tau$ | 1         | 1             | 1                      |

Then in each case  $\text{Ind}_{M_0}^G 1$  is a representation model for  $G$ .

THEOREM 4 gives geometric realizations of representation models for all classical groups. In particular, each model from THEOREM 2 is naturally

interpreted as a geometric realization. The symmetry between THEOREMS 1 and 2 suggests that such an interpretation must exist for models from THEOREM 1. We shall show that each representation model from THEOREM 1 is naturally realized as the space of functions on a certain supermanifold (THEOREMS 6 and 7).

Our next goal is to show that each of the representation models from THEOREM 4 admits higher symmetry, i.e., the natural action of the Lie algebra  $\mathfrak{l}$  of an overgroup  $L \supset G$ . The overgroup  $L$  will be always classical or the product of two classical groups (instead of  $O(n)$  it will be sometimes more convenient to take its connected component of identity  $SO(n)$ ).

For any classical group  $L$  we define the homogeneous space  $\Gamma(L)$  as follows :  $\Gamma(GL(2n))$  is the Grassmannian of  $n$ -dimensional subspaces in  $\mathbb{C}^{2n}$ ,  $\Gamma(GL(2n+1))$  is the Grassmannian of  $(n+1)$ -dimensional subspaces in  $\mathbb{C}^{2n+1}$ ,  $\Gamma(O(2n+1))$  is the manifold of all isotropic  $n$ -dimensional subspaces in  $\mathbb{C}^{2n}$ ,  $\Gamma(O(2n))$  is the manifold of all isotropic  $n$ -dimensional subspaces in  $\mathbb{C}^{2n+1}$ ,  $\Gamma(Sp(2n))$  is the manifold of all Lagrangian ( $n$ -dimensional) subspaces in  $\mathbb{C}^{2n}$ .  $\Gamma(L)$  is connected when  $L \neq O(2n)$ , and  $\Gamma(O(2n))$  has two connected components (two isotropic subspaces  $X$  and  $Y$  belong to the same component of  $\Gamma(O(2n))$  if dimensions of  $X \cap Z$  and  $Y \cap Z$  have the same parity for some (and hence for any)  $Z \in \Gamma(O(2n))$ ). Finally, for  $L_1$  and  $L_2$  classical we put  $\Gamma(L_1 \times L_2) = \Gamma(L_1) \times \Gamma(L_2)$ .

The homogeneous space  $\Gamma(L)$  will be called the *basic Grassmannian* of  $L$ . It is a compact Hermitian symmetric space. We think that basic Grassmannians and their superanalogues defined below are important geometric objects whose significance reaches far beyond their applications to representation models.

For any classical group  $G$  define the overgroup  $L \supset G$  according to the following table (embeddings  $G \subset L$  will be specified below) :

|     |            |                           |                         |            |
|-----|------------|---------------------------|-------------------------|------------|
| $L$ | $SO(2n+2)$ | $O(2n+2) \times SO(2n+2)$ | $SO(2n+2) \times O(2n)$ | $GL(2n+1)$ |
| $G$ | $GL(n)$    | $O(2n+1)$                 | $O(2n)$                 | $Sp(2n)$   |

THEOREM 5 [1]. — *Let  $G$  be a classical group and  $L \supset G$  be the overgroup just defined.*

(a) *The action of  $G$  on the basic Grassmannian  $\Gamma(L)$  has the unique open orbit  $\Omega_0$ . This orbit is isomorphic as a  $G$ -space to the space  $G/M_0$  from THEOREM 4.*

(b) *The representation of  $G$  on the space  $C[\Omega_0]$  of regular functions on  $\Omega_0$  is a representation model for  $G$ .*

(c) *The natural action of the Lie algebra  $\mathfrak{l}$  of  $L$  on  $\mathbf{C}[\Omega_0]$  is a higher symmetry of the model.*

Note that for  $G = GL(3)$  we have  $L = SO(8)$ , so we have constructed a natural action of the Lie algebra  $\mathfrak{so}_8$  on the representation model for  $GL(3)$ . This gives a geometric explanation of the construction of [2].

Let us specify the embedding of  $G$  into  $L$  and give an explicit description of  $\Omega_0$  in THEOREM 5.

1)  $G = GL(n)$ ,  $L = SO(2n+2)$ . Let  $L$  be realized in  $(2n+2)$ -dimensional vector space  $W$  (over  $\mathbf{C}$ ) with a nondegenerate symmetric bilinear form. Choose two complementary isotropic subspaces  $\bar{V}$  and  $\bar{V}'$  in  $W$  and a decomposition  $\bar{V} = V + \mathbf{C}v$  into the sum of  $n$ -dimensional and 1-dimensional subspaces. The embedding of  $G$  into  $L$  is a composition of standard embeddings  $GL(n) \subset GL(n+1) \subset SO(2n+2)$ . Thus, the subgroup  $G$  of  $L$  consists of transformations preserving  $V$ ,  $v$  and  $\bar{V}'$ .

Let  $\Gamma = \Gamma(L)$  be realized as the set of  $(n+1)$ -dimensional isotropic subspaces  $X$  in  $W$  such that  $\dim(X \cap \bar{V}')$  is even. Then  $\Omega_0$  consists of the subspaces  $X \in \Gamma$ , which are transversal to each of the subspaces  $V$ ,  $\mathbf{C}v$ , and  $\bar{V}'$ .

2)  $G = O(2n+1)$ ,  $L = O(2n+2) \times SO(2n+2)$ . Let  $L_1 = O(2n+2)$  and  $L_2 = SO(2n+2)$  so that  $L = L_1 \times L_2$ . Let  $L_1$  and  $L_2$  be realized in the same  $(2n+2)$ -dimensional vector space  $\bar{W}$ . We choose a nondegenerate subspace  $W$  in  $\bar{W}$  of codimension 1, and realize  $G$  as the group of orthogonal transformations of  $W$ . Define the embeddings  $i_1 : G \rightarrow L_1$  and  $i_2 : G \rightarrow L_2$  as follows : the action of both  $i_1(g)$  and  $i_2(g)$  on  $W$  coincides with that of  $g$ , and on the orthogonal complement of  $W$  in  $\bar{W}$   $i_1(g)$  is the identity and  $i_2(g)$  is the multiplication by  $\det(g)$ . The embedding  $G \rightarrow L$  is defined by  $g \mapsto (i_1(g), i_2(g))$ .

The basic Grassmannian  $\Gamma = \Gamma(L)$  consists of the pairs  $(X, Y)$  where  $X$  runs over the set of all  $(n+1)$ -dimensional isotropic subspaces in  $\bar{W}$ , and  $Y$  runs over any one of two connected components of this set. We have

$$\Omega_0 = \{(X, Y) \in \Gamma : X \cap Y \cap W = 0\}.$$

3)  $G = O(2n)$ ,  $L = SO(2n+2) \times O(2n)$ . Let  $L_1 = SO(2n+2)$  so that  $L = L_1 \times G$ . Let  $L_1$  be realized in a  $(2n+2)$ -dimensional vector space  $\bar{W}$ , and  $G$  in its nondegenerate subspace  $W$  of codimension 2. Define the embedding  $i : G \rightarrow L_1$  as the composition of the standard embedding  $O(2n) \rightarrow O(2n+1)$  and the embedding  $i_2 : O(2n+1) \rightarrow SO(2n+2)$  just defined. The embedding  $G \rightarrow L$  sends  $g$  to  $(i(g), g)$ .

The basic Grassmannian  $\Gamma = \Gamma(L)$  consists of pairs  $(X, Y)$  where  $X$  runs over a component of the the set of  $(n+1)$ -dimensional isotropic subspaces

in  $\overline{W}$ , and  $Y$  runs over the set of  $n$ -dimensional isotropic subspaces in  $W$ . We have

$$\Omega_0 = \{(X, Y) \in \Gamma : X \cap Y = 0, \dim(X \cap W) = n - 1\}.$$

4)  $G = Sp(2n)$ ,  $L = GL(2n + 1)$ . The embedding  $G \subset L$  is a composition of standard embeddings  $Sp(2n) \subset GL(2n) \subset GL(2n + 1)$ . In other words, let  $L$  be realized in a  $(2n + 1)$ -dimensional vector space  $\overline{W}$ . Choose a subspace  $W$  in  $\overline{W}$  of codimension 1, a (nondegenerate) symplectic form  $B$  on  $W$ , and a vector  $z \in \overline{W} \setminus W$ . Then  $G$  is embedded in  $L$  as the subgroup preserving  $W$ ,  $B$  and  $z$ .

The basic Grassmannian  $\Gamma = \Gamma(L)$  is the set of all  $(n + 1)$ -dimensional subspaces in  $\overline{W}$ . Let  $p_W$  denote the projection of  $\overline{W}$  onto  $W$  along  $\mathbf{C}z$ . Then  $\Omega_0$  consists of  $X \in \Gamma$  transversal to  $W$  and  $\mathbf{C}z$ , and such that the restriction of  $B$  on each of the subspaces  $X \cap W$  and  $p_W(X)$  has maximal possible rank.

Now we shall obtain a geometric realization and higher symmetry for models from THEOREM 1. For this let us look at the geometric realization of a model from another point of view. Let  $E$  be an “abstract” representation model for  $G$ , i.e., the direct sum of all irreducible algebraic representations of  $G$ . A geometric realization of the model, i.e., the realization of  $E$  as the space of regular functions on an algebraic variety with an action of  $G$ , induces on  $E$  the additional structure of a commutative algebra such that  $G$  acts on  $E$  by algebra automorphisms. Conversely, such a structure on  $E$  allows us to identify  $E$  with the space of regular functions on an algebraic  $G$ -variety. Now observe that the realization of  $E$  from THEOREM 2 naturally induces the structure of a commutative algebra on  $E$  even if the subgroup  $M_0$  from THEOREM 4 is not yet introduced. Indeed, the induced representation  $\text{Ind}_M^G \tau$  is naturally realized in the space of (regular) sections of the vector bundle on  $G/M$  whose fibres are isomorphic to  $\tau$ ; but  $\tau$  is by definition a commutative algebra.

Consider again models from THEOREM 1. Since  $\tau$  acts now not on the symmetric but on the exterior algebra, it follows that this realization of a model induces on  $E$  the structure of a commutative superalgebra such that  $G$  acts on  $E$  by its automorphisms. The arguments above show that it is natural to regard such a realization also as a geometrical one but acting now in the space of functions on a certain supermanifold.

We assume the reader to be familiar with basic facts about supermanifolds and supergroups (see, e.g., [5]). Recall that a supermanifold  $\Omega$  is defined as a pair consisting of the underlying manifold  $\Omega^{(0)}$  and the structural sheaf on  $\Omega^{(0)}$ ; a (regular) function on  $\Omega$  is a section of the structural sheaf. Using this definition we see that the representation model  $\text{Ind}_M^G \tau$  from THEOREM 1 may indeed be regarded as the space of regular functions on the supermanifold

$\Omega$  whose underlying manifold is  $G/M$ , and the stalks of the structural sheaf are isomorphic to  $\tau$ .

It is quite natural that the notion of a higher symmetry for these models must also be modified : now the “overgroup”  $L$  will be a supergroup, and it will be its Lie superalgebra  $\mathfrak{l}$  which will act on the representation model for  $G$ . To construct them we need some terminology related to classical supergroups.

Let  $GL(p|q)$  denote the supergroup of all (even) automorphisms of a  $p|q$ -dimensional vector superspace. (To be more formal, this means that for any commutative superalgebra  $\Lambda$  the set of  $\Lambda$ -points of the supermanifold  $GL(p|q)$  is the set of automorphisms of the  $\Lambda$ -module  $\Lambda^{p|q}$ , see [5].) All verbal descriptions of various supergroups and supermanifolds given below will have similar meaning. The change of parity gives an isomorphism  $GL(p|q) \cong GL(q|p)$ . We denote by  $O(p|q)$  the subsupergroup of  $GL(p|q)$  preserving a nondegenerate symmetric (in the supersense [5]) even bilinear form on a  $p|q$ -dimensional vector superspace. The usual sign convention shows that such a form exists only for even  $q$ , and that the underlying manifold of  $O(p|q)$  is  $O(p) \times Sp(q)$ . The supergroup  $O(p|q)$  is commonly denoted by  $OSp(p|q)$  but our notation makes symmetry with purely even case more clear. The supergroup  $Sp(p|q)$  is defined in a similar way; the change of parity leads to an isomorphism  $Sp(p|q) \cong O(q|p)$ . The supergroup  $O(p|q)$  is not connected when  $p \geq 2$ ; let  $SO(p|q)$  denote its component of identity.

We extend the definition of the basic Grassmannian  $\Gamma(L)$  to the case when  $L$  is one of the supergroups  $GL(1|n)$  or  $SO(2|2n)$  : define  $\Gamma(GL(1|2n))$  to be the supermanifold of  $0|n$ -dimensional subspaces in a  $1|2n$ -dimensional vector superspace,  $\Gamma(GL(1|2n+1))$  to be the supermanifold of  $0|(n+1)$ -dimensional subspaces in a  $1|(2n+1)$ -dimensional vector superspace, and  $\Gamma(SO(2|2n))$  to be one of the connected components of the supermanifold of isotropic  $1|n$ -dimensional subspaces in a  $2|2n$ -dimensional vector superspace equipped with a nondegenerate symmetric bilinear form. These supermanifolds are Hermitian symmetric superspaces in the sense of [6].

Let  $G$  be a classical group. Since the representation model for  $G$  is now realized in the space of functions on a supermanifold it is only natural that  $G$  itself should be regarded as a supergroup. We shall use the identifications  $GL(n) = GL(0|n)$ ,  $O(n) = Sp(0|n)$ ,  $Sp(2n) = O(0|2n)$  thus realizing  $G$  by transformations of a purely odd vector superspace. Define the oversupergroup  $L \supset G$  according to the following table (the embeddings  $G \rightarrow L$  are quite similar to those from THEOREM 5) :

|     |                   |                  |                           |
|-----|-------------------|------------------|---------------------------|
| $L$ | $SO(2 2n)$        | $GL(1 n)$        | $SO(2 2n) \times O(0 2n)$ |
| $G$ | $GL(n) = GL(0 n)$ | $O(n) = Sp(0 n)$ | $Sp(2n) = O(0 2n)$        |

THEOREM 6 [1]. — *Let  $G$  be a classical group realized as a supergroup as above, and  $L \supset G$  the oversupergroup just defined.*

(a) *The basic Grassmannian  $\Gamma(L)$  contains the unique minimal  $G$ -invariant open subsupermanifold  $\Omega$ .*

(b) *The representation of  $G$  on the space  $\mathbf{C}[\Omega]$  of regular functions on  $\Omega$  is naturally isomorphic to the representation  $\text{Ind}_M^G \tau$  from THEOREM 1 and so is a representation model for  $G$ .*

(c) *The natural action of the Lie superalgebra  $\mathfrak{l}$  of  $L$  on  $\mathbf{C}[\Omega]$  is the higher symmetry of the model.*

This is essentially THEOREM 3' from [1]. The present notation makes the result entirely similar to that of THEOREM 5. The explicit description of  $\Omega$  is left to the reader; it is quite similar to that given after THEOREM 5.

Our next goal is to prove for supermanifolds  $\Omega$  from THEOREM 6 an analogue of THEOREM 4, i.e., to represent  $\Omega$  as a homogeneous supermanifold. In contrast with THEOREM 4  $\Omega$  is not a homogeneous space under the action of  $G$  in the sense of supermanifolds (see [5]) but will be represented in the form  $\overline{G}/\overline{M}$  where  $\overline{G}$  is a certain oversupergroup of  $G$  and  $\overline{M}$  a subsupergroup of  $\overline{G}$ . To define  $\overline{G}$  and  $\overline{M}$  we need some terminology. Let  $GL(1/2|n)$  denote the supergroup of (even) transformations of a  $1|n$ -dimensional vector space preserving a nonzero even linear functional,  $Sp'(1|n)$  the supergroup of transformations of a  $1|n$ -dimensional vector space preserving a skew-symmetric even bilinear form  $B$  of rank  $0|n$  and acting trivially on  $\text{Ker } B$ , and  $O'(1|2n)$  the supergroup of transformations of a  $1|2n$ -dimensional vector space preserving a symmetric even bilinear form of rank  $0|2n$  and acting trivially on its kernel. The following theorem is obtained jointly with V.V. SERGANOVA.

THEOREM 7. — *For each classical group  $G$  realized as a supergroup as above define the oversupergroup  $\overline{G}$  and the subsupergroup  $\overline{M}$  of  $\overline{G}$  according to the following table (the embeddings of  $G$  and  $\overline{M}$  into  $\overline{G}$  are natural).*

|                |            |                           |                             |             |
|----------------|------------|---------------------------|-----------------------------|-------------|
| $\overline{G}$ | $GL(1 n)$  | $Sp'(1 2n)$               | $Sp'(1 2n+1)$               | $O'(1 2n)$  |
| $G$            | $GL(0 n)$  | $Sp(0 2n)$                | $Sp(0 2n+1)$                | $O(0 2n)$   |
| $\overline{M}$ | $Sp'(1 n)$ | $Sp(0 n) \times Sp'(1 n)$ | $Sp(0 n+1) \times Sp'(1 n)$ | $GL(1/2 n)$ |

*Then in each case the homogeneous space  $\overline{G}/\overline{M}$  is isomorphic as a supermanifold with action of  $G$  to the supermanifold  $\Omega$  from THEOREM 6.*

In particular, we obtain the action of  $\overline{G}$  on the space  $\mathbf{C}[\Omega]$  which is by THEOREM 6 a representation model for  $G$ . This may be regarded as an “integrable” version of higher symmetry. Our usual parallelism with even case suggests that such symmetry must in turn exist for models from

THEOREM 4. To formulate the result we need some more terminology. Let  $Sp'(2n+1)$  (resp.  $O'(2n+1)$ ) denote the group of linear transformations of  $\mathbb{C}^{2n+1}$  preserving a skew-symmetric (resp. symmetric) bilinear form of rank  $2n$  and acting trivially on its kernel. For each classical group  $G$  define the overgroup  $\overline{G}$  and the subgroup  $\overline{M}$  of  $\overline{G}$  according to the following table (the embeddings of  $G$  and  $\overline{M}$  into  $\overline{G}$  are natural).

|                |              |            |             |           |
|----------------|--------------|------------|-------------|-----------|
| $\overline{G}$ | $GL(2n+1/2)$ | $GL(2n+2)$ | $O'(2n+1)$  | $O(2n+2)$ |
| $G$            | $GL(2n)$     | $GL(2n+1)$ | $O(2n)$     | $O(2n+1)$ |
| $\overline{M}$ | $Sp(2n)$     | $Sp(2n+2)$ | $GL(n+1/2)$ | $GL(n+1)$ |

  

|                |                           |                             |
|----------------|---------------------------|-----------------------------|
| $\overline{G}$ | $Sp'(4n+1)$               | $Sp'(4n+3)$                 |
| $G$            | $Sp(4n)$                  | $Sp(4n+2)$                  |
| $\overline{M}$ | $Sp(2n) \times Sp'(2n+1)$ | $Sp(2n+2) \times Sp'(2n+1)$ |

The next result is an even counterpart of THEOREM 7; it is also obtained with V.V. SERGANOVA.

THEOREM 8. — For any classical group  $G$  let the overgroup  $\overline{G} \supset G$  and the subgroup  $\overline{M}$  of  $\overline{G}$  be defined as above.

(a) The action of  $G$  on the homogeneous space  $\Omega = \overline{G}/\overline{M}$  has the unique open orbit  $\Omega_0$ , and it is isomorphic to the space  $G/M_0$  from THEOREM 4.

(b) The complement of  $\Omega_0$  in  $\Omega$  has codimension  $\geq 2$ . Therefore  $\mathbb{C}[\Omega_0] = \mathbb{C}[\Omega]$ .

(c) The action of  $G$  on the representation model  $\mathbb{C}[\Omega_0]$  extends naturally to the action of  $\overline{G}$ .

*Remarks.*

1) The groups  $\overline{G}$  from THEOREM 8 and the supergroups  $\overline{G}$  from THEOREM 7 are of independent interest. The group  $Sp'(2n+2)$  was recently introduced in [7], where its representations are studied, and some interesting combinatorial applications are obtained. Note that this group is different from  $Sp(2n+1)$  introduced above: the unipotent radical  $Sp'(2n+1)$  is abelian while that of  $Sp(2n+1)$  is isomorphic to the Heisenberg group.

2) For any classical group  $G$  consider the overgroup  $\overline{G}$  from THEOREM 8 and the overgroup  $L$  from THEOREM 5 (or their super-analogues from THEOREMS 7 and 6). It turns out that sometimes (but not in all cases) there are natural embeddings of  $\overline{G}$  into  $L$  and of  $\Omega$  from THEOREM 8 into the basic Grassmannian  $\Gamma(L)$ . The interaction of  $\overline{G}$  and  $L$  will be studied elsewhere.

Our construction of higher symmetries is of independent interest besides applications to representation models. Recall that the spaces  $G/M$  from THEOREMS 1 and 2 are complexifications of Riemannian symmetric spaces; in THEOREMS 5 and 6 we have constructed in particular an open embedding of each  $G/M$  into the special compact Hermitian symmetric space, viz., one of the basic Grassmannians defined above. This construction is analogous to the twistor construction of R. PENROSE : recall that a twistor is a two-dimensional plane in  $\mathbf{C}^4$ , and the corresponding Grassmannian is a compactification of the complexification of the Minkowsky space  $\mathbf{R}^4$ . Our last result is such a construction for all classical symmetric spaces : it is obtained jointly with A.B. GONCHAROV.

By a classical compact group we shall mean one of the groups  $U(n, \mathbf{R})$ ,  $U(n, \mathbf{C})$  or  $U(n, \mathbf{H})$  (i.e., the orthogonal, unitary or symplectic compact group, respectively), or the product of several such groups. The complexifications of these groups are respectively  $O(n)$ ,  $GL(n)$  and  $Sp(2n)$ . By a classical compact symmetric space we mean the space of the form  $X = U_0/U_0^\theta$ , where  $U_0$  is the identity component of a classical compact group  $U$ , and  $U_0^\theta$  is the subgroup of fixed points of an involutive automorphism  $\theta$  of  $U_0$ . The complexification  $X_C$  of  $X$  is realized as  $G_0/G_0^\theta$  where  $G_0$  and  $G_0^\theta$  are complexifications of  $U_0$  and  $U_0^\theta$ . The list of irreducible spaces  $X$  and  $X_C$  is given in THEOREM 9 below; note that it is different from the usual list (see [8]) since our  $X$  is not necessarily simply connected.

THEOREM 9 [1]. — *Let  $X$  be an irreducible classical compact symmetric space, and  $X_C = G_0/G_0^\theta$  be its complexification. Define the overgroup  $L_0 \supset G_0$  and the homogeneous  $L_0$ -space  $\Gamma$  according to the following table (the basic Grassmannians  $\Gamma(L_0)$  are defined above, and  $\Gamma_k^n$  denotes the Grassmannian of  $k$ -dimensional subspaces in  $\mathbf{C}^n$ ).*

|          |                                     |                                     |
|----------|-------------------------------------|-------------------------------------|
| $X$      | $U(n, \mathbf{C})/U(n, \mathbf{R})$ | $U(n, \mathbf{H})/U(n, \mathbf{C})$ |
| $X_C$    | $GL(n)/O(n)$                        | $Sp(2n)/GL(n)$                      |
| $L_0$    | $Sp(2n)$                            | $Sp(2n) \times Sp(2n)$              |
| $\Gamma$ | $\Gamma(L_0)$                       | $\Gamma(L_0)$                       |

|          |                                        |                                      |
|----------|----------------------------------------|--------------------------------------|
| $X$      | $U_0(2n, \mathbf{R})/U(n, \mathbf{C})$ | $U(2n, \mathbf{C})/U(n, \mathbf{H})$ |
| $X_C$    | $SO(2n)/GL(n)$                         | $GL(2n)/Sp(2n)$                      |
| $L_0$    | $SO(2n) \times SO(2n)$                 | $SO(4n)$                             |
| $\Gamma$ | $\Gamma(L_0)$                          | $\Gamma(L_0)$                        |

|          |                                                                       |                                                                 |
|----------|-----------------------------------------------------------------------|-----------------------------------------------------------------|
| $X$      | $U_0(n, \mathbf{R}) / (U(k, \mathbf{R}) \times U(n-k, \mathbf{R}))_0$ | $U(n, \mathbf{C}) / U(k, \mathbf{C}) \times U(n-k, \mathbf{C})$ |
| $X_C$    | $SO(n) / S(O(k) \times O(n-k))$                                       | $GL(n) / GL(k) \times GL(n-k)$                                  |
| $L_0$    | $GL(n)$                                                               | $GL(n) \times GL(n)$                                            |
| $\Gamma$ | $\Gamma_k^n$                                                          | $\Gamma_k^n \times \Gamma_{n-k}^n$                              |

|          |                                      |                                                                       |
|----------|--------------------------------------|-----------------------------------------------------------------------|
| $X$      | $U(n, H) / U(k, H) \times U(n-k, H)$ | $(U_0(n, \mathbf{R}) \times U_0(n, \mathbf{R})) / U_0(n, \mathbf{R})$ |
| $X_C$    | $Sp(2n) / Sp(2k) \times Sp(2(n-k))$  | $(SO(n) \times SO(n)) / SO(n)$                                        |
| $L_0$    | $GL(2n)$                             | $SO(2n)$                                                              |
| $\Gamma$ | $\Gamma_{2k}^{2n}$                   | $\Gamma(L_0)$                                                         |

|          |                                                                 |                                      |
|----------|-----------------------------------------------------------------|--------------------------------------|
| $X$      | $(U(n, \mathbf{C}) \times U(n, \mathbf{C})) / U(n, \mathbf{C})$ | $(U(n, H) \times U(n, H)) / U(n, H)$ |
| $X_C$    | $(GL(n) \times GL(n)) / GL(n)$                                  | $(Sp(2n) \times Sp(2n)) / Sp(2n)$    |
| $L_0$    | $GL(2n)$                                                        | $Sp(4n)$                             |
| $\Gamma$ | $\Gamma(L_0)$                                                   | $\Gamma(L_0)$                        |

Then in each case the action of  $G_0$  on  $\Gamma$  has the unique open orbit, and it is isomorphic to  $X_C$  as a  $G_0$ -space.

*The concluding remark.* — We have already mentioned the construction of a  $\mathfrak{so}_8$ -action on the representation model for  $SL(3)$  given in [2]. This construction appears in [2] as a tool for the problem of canonical description of tensor operators for  $\mathfrak{g} = \mathfrak{sl}_3$ . The latter problem is equivalent to that of canonical decomposition of the tensor product of two irreducible  $\mathfrak{g}$ -modules into irreducible components. A different new approach to this problem based on the study of canonical bases in irreducible  $\mathfrak{g}$ -modules is developed in the recent authors' note [9].

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