Astérisque

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Astérisque, tome 118 (1984), p. 33-53
<http://www.numdam.org/item?id=AST_1984__118__33_0>

Société Mathématique de France
Astérisque 118 (1984) p.33 à 53

LEAST AREA PROBLEMS WITH A VOLUME CONSTRAINT

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1. INTRODUCTION.

Let Ω be an open subset of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$ and let Γ be a (Borel) subset of $\partial\Omega$. In this lecture we wish to consider the following problem:

$$(P_V)\quad \begin{cases} \text{to minimize } \mathcal{F}(E) = \int_\Omega |D\phi_E| + \int_{\partial\Omega} |\phi_E - \phi_\Gamma| dH_{n-1} \text{ among the Borel subsets } E \text{ of} \\ \Omega \text{ satisfying } |E| = \text{measure of } E = V \quad (0 < V < |\Omega|) \end{cases}$$

Here ϕ_E denotes the characteristic function of the set E, $\int_\Omega |D\phi_E|$ is the perimeter of E in Ω, i.e.

$$\int_\Omega |D\phi_E| = \sup\left\{ \int_E \operatorname{div}\Phi(x)dx,\ \Phi \in C_o^1(\Omega,\mathbb{R}^n),\ \|\Phi\|_\infty \leq 1 \right\} .$$

It is well-known that every set E with $\int_\Omega |D\phi_E| < +\infty$ has a trace integrable on $\partial\Omega$, which we still denote by ϕ_E.
Roughly speaking, we are looking for a solution set E taking on the prescribed boundary value Γ, such that the area of its free surface $\partial E \cap \Omega$ yields a minimum when compared to the area of the free boundary of any admissible set having the same volume in Ω and the same trace on $\partial\Omega$. In our formulation, the "Dirichlet condition" Γ is actually retained in the functional itself, as a "penalty term".

The existence problem.

If Ω is a bounded set, it is very easy to prove the existence of a solution to problem (P_V). In fact, from Ascoli-Arzelà theorem and a standard regularization of the characteristic function ϕ_E , we get easily the following

Compactness theorem.

If $\int_\Omega |D\phi_{E_h}| \leq$ constant (independent of h), then there exist an increasing sequence $j(h)$ and a set E such that

$$\phi_{E_{j(h)}} \xrightarrow{h\to+\infty} \phi_E \quad \text{in} \quad L^1(\Omega) \quad .$$

On the other hand, the functional $\mathcal{F}$ is lower semicontinuous in the $L^1(\Omega)$-topology. These two results enable us to get the existence of a solution to problem (P_V) in the case $|\Omega| < +\infty$. But in the general case, if Ω is not bounded, we can assure only $L^1_{loc}(\Omega)$-convergence to ϕ_E and therefore, it is not guaranteed that the limit set E verifies the volume constraint. Our first problem is then:

Problem I. To prove an existence theorem for problem (P_V) in the case $|\Omega| = +\infty$.

The regularity problem.

Of course, although our definition of area spans very strange boundaries, we expect that a minimum to problem (P_V) has a regular boundary (except perhaps for a small singular set).

In 1960 E. De Giorgi proved the analiticity of the reduced boundary $\partial^* E$ of a set E , minimizing surface area in an open set $\Omega \subset \mathbb{R}^n$. (See [7]).

Then, M. Miranda obtained the following estimate on the measure of the singular points: $H_{n-1}(\partial E - \partial^* E) = 0$ (see [14]). Later, H. Federer improved this result, showing that actually $H_s(\partial E - \partial^* E) = 0 \quad \forall\, s > n-8$. This is the best possible result, in view of the well-known "Simon's cone" in $\mathbb{R}^8$. (See [9], [3], [13]).

We want to consider the new problem arising when a volume constraint is imposed to the solution:

Problem II. To prove a regularity theorem for the boundaries of the minima to problem (P_V).

Lagrange multipliers.

It is very natural to ask for the possibility to drop the volume constraint by adding a suitable "penalty term" to the functional. This consideration leads us to the following:

Problem III. Find, for every $V \in (0,|\Omega|)$, a suitable "regular" function $g : (0,|\Omega|) \to \mathbb{R}$ such that problem (P_V) is equivalent to the problem of minimizing $\mathcal{F}(E) + g(|E|)$ among the Borel subsets of Ω (that is, without any volume constraint).

We answer now to these three questions. It is a very remarkable fact that the same idea permits us to solve problems I, II and III. The key argument in our method is the isoperimetric property of the sphere combined with an iterative argument.

2. PROBLEM I: EXISTENCE IN THE CASE $|\Omega| = +\infty$.

There is a particular situation with $|\Omega| = +\infty$ in which the solution to problem (P_V) is well-known. It is the case $\Omega = \mathbb{R}^n$. The solutions in this case are spheres of measure V. However, the proof in this situation relies on a symmetrization process which is not applicable to more general situations. We recall here the

Isoperimetric property of the sphere.

For every Borel subset $E \subset \mathbb{R}^n$ it holds

$$(1) \qquad |E| \wedge |\mathbb{R}^n - E| \le c(n)\left(\int_{\mathbb{R}^n} |D\phi_E|\right)^N$$

where $N = \dfrac{n}{n-1}$ and $c(n)$ is a constant such that (1) becomes an equality if E (or $\mathbb{R}^n - E$) is a sphere. (See [6], [11]).

We pass now to prove an existence result in a more general situation. The following result is proved in [2], where the following assumptions are made:

H1) Γ is a bounded subset of $\partial\Omega$.

H2) There exists a ball $B \subset \Omega$ with $|B| = V$.

The following example shows that H2) is necessary:

Example. Let $\Omega = \bigcup_{j=1}^{\infty} B_j \subset \mathbb{R}^n$, where $\{B_j\}_j$ is a sequence of disjoint balls, $|B_j| = V - \frac{V}{2^j}$, let $\Gamma = \emptyset$.

Put $E_j = B_j \cup F_j \, (j \geq 2)$, where F_j is a sphere, $F_j \subset B_1$, $|F_j| = \frac{V}{2^j}$. We have

$$\mathcal{F}(E_j) \downarrow n\omega_n^{\frac{1}{n}} V^{\frac{n-1}{n}} = \text{perimeter of a sphere of measure } V ,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

On the other hand, from the isoperimetric property of the sphere, it must be

$$\mathcal{F}(E) > n\,\omega_n^{\frac{1}{n}} V^{\frac{n-1}{n}} \qquad \forall\, E \subset \Omega \quad |E| = V$$

and therefore in this case the solution to problem (P_V) does not exist.

Put

$$\mathcal{E}_V = \{E \subset \Omega : |E| = V\}$$

the class of admissible sets. From conditions H1), H2) we immediately obtain the existence of a set $E \in \mathcal{E}_V$ with $\mathcal{F}(E) < +\infty$. (In fact, it suffices to take $E = B$). The proof of the existence theorem under the hypotheses H1), H2) will be given in several steps. Put

$$\Omega_j = \{x \in \Omega : |x| < j\} \quad ;$$

let $j_o \in \mathbb{N}$ be such that $B \subset \Omega_j$, $\Gamma \subset \partial\Omega_j$ $\forall\, j \geq j_o$. For $j \geq j_o$ let E_j be a minimum of the functional

$$(2) \qquad E \to \int_{\Omega_j} |D\phi_E| + \int_{\partial\Omega_j} |\phi_E - \phi_j| \, dH_{n-1}$$

in the class

(3) $\{E \subset \Omega_j : |E| = V\}$.

Since $|\Omega_j| < +\infty$, we already know the existence of such sets E_j.

Step 1. There exists $j_1 > j_o$ such that, for $j \geq j_1$, there exists r_j, $j_o \leq r_j \leq j_1$ with

$$\int_{|x| = r_j} \phi_{E_j} dH_{n-1} = 0 \quad .$$

Step 2. From Step 1 we can deduce, for $j \geq j_1$, the existence of a minimum $\tilde{E}_j \subset \Omega_{r_j}$ for problem (2)-(3).

Step 3. The minimum property of the sets $\tilde{E}_j$ and the compactness theorem imply the existence of a subsequence (which we still label $\{\tilde{E}_j\}$) and of a set $E_o \in \mathcal{E}_V$ such that

$$\tilde{E}_j \xrightarrow{j \to +\infty} E_o \quad \text{in} \quad L^1(\Omega) \quad .$$

Step 4. From the lower semicontinuity of $\mathcal{F}$ and the minimum property of the approximating sets $\tilde{E}_j$ we can easily infer that E_o is a minimum for $\mathcal{F}$ among the bounded sets in $\mathcal{E}_V$. This in turn implies easily that E_o is a minimum for problem (P_V).

As the proofs of steps 2, 3 and 4 are somewhat standard, we concentrate our attention to the proof of step 1 (for details see [2]). To this aim, we need an isoperimetric-type lemma. Introduce the following notations:

For $j_o \leq t_1 < t_2 < t_3 \leq j_1$, put

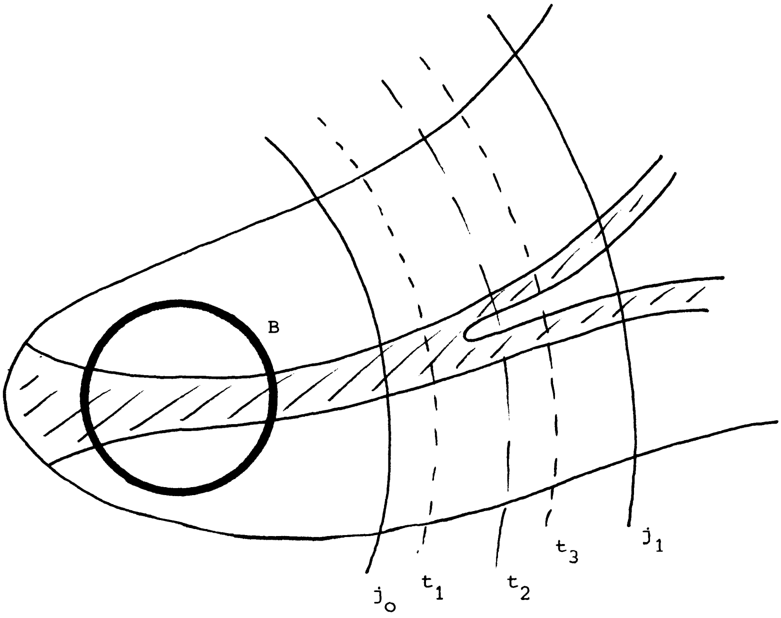
$$(t_1, t_2) = \Omega_{t_2} - \overline{\Omega_{t_1}} \quad , \quad (t_2, t_3) = \Omega_{t_3} - \overline{\Omega_{t_1}}$$

and for $j \geq j_1$ let

$$v_1 = |E_j \cap (t_1, t_2)| \quad , \quad v_2 = |E_j \cap (t_2, t_3)| \quad , \quad v = v_1 + v_2$$

and

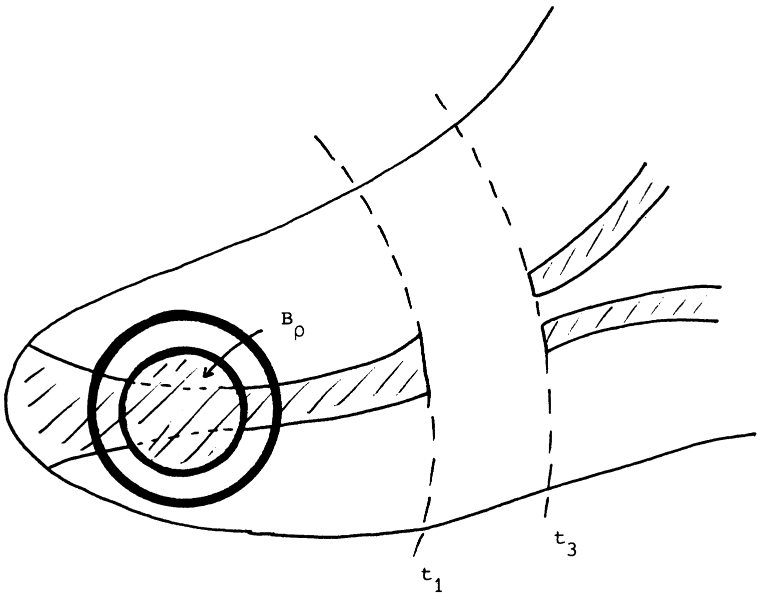
$$m = \max_{i=1,2,3} \int_{|x|=t_i} \phi_{E_j} \, dH_{n-1} \ .$$



For simplicity, assume the trace of E_j on $|x| = t_i$ $(i = 1,2,3)$ is continuous (i.e., $\phi^+_{E_j} = \phi^-_{E_j}$ on $|x| = t_i$).

Define a set $F \in \mathcal{E}_V$ in the following way:

$$F = \begin{cases} E_j & \text{in } \Omega_j - (t_1,t_3) - B \\ \emptyset & \text{in } (t_1,t_3) \\ E_j \cup B_\rho & \text{in } B \end{cases}$$



Here $B_\rho \subset B$ is a suitable ball such that $|F| = V$.

Since $\mathcal{F}(E_j) \leq \mathcal{F}(F)$, the isoperimetric property of the sphere and inequality (1.10) in [15] get us the following:

Isoperimetric-type inequality.

With the notations introduced above, we have

(4) $$v_1 \wedge v_2 \leq c_1 \cdot m^N \quad ,$$

where $c_1 = c_1(n)$ *only depends on the dimension* n *and* $N = \frac{n}{n-1}$ *(for details see* [2]*).*

We are ready to prove step 1: Fix $j \geq j_1$; later we will choose j_1 in a suitable way.

We will construct r_j as the limit of two sequences $\{a_k\}$, $\{b_k\}$, with a_k increasing and b_k decreasing towards r_j , and such that

$$(5) \qquad \lim_k \int_{|x|=a_k} \phi_{E_j} dH_{n-1} = \lim_k \int_{|x|=b_k} \phi_{E_j} dH_{n-1} = 0 \quad .$$

From (5) we easily get

$$\int_{|x|=r_j} \phi_{E_j} dH_{n-1} = 0 \quad ,$$

since E has finite perimeter in Ω.

The sequences $\{a_k\}$ and $\{b_k\}$ will be constructed by an iterative process, starting with

$$(6) \qquad a_o = j_o \ , \quad b_o = j_1 \quad .$$

Given a_k and b_k $(k \geq 0)$, with $a_k < b_k$, we define

$$(7) \qquad \ell_k = b_k - a_k \ , \quad v_k = |E_j \cap (a_k, b_k)| \quad .$$

Clearly, for any $h_k \in \left(0, \frac{\ell_k}{3}\right)$, it is possible to find three points t_i^k $(i = 1,2,3)$, such that the trace of E on $|x| = t_i^k$ is continuous for $i = 1,2,3$, and moreover

$$(8) \qquad \begin{cases} t_1^k \in (a_k, a_k + h_k) \\ t_2^k \in \left(a_k + \frac{\ell_k}{2} - \frac{h_k}{2}, \ a_k + \frac{\ell_k}{2} + \frac{h_k}{2}\right) \\ t_3^k \in (b_k - h_k, b_k) \end{cases}$$

while

$$(9) \qquad \int_{|x|=t_i^k} \phi_{E_j} dH_{n-1} \leq \frac{v_k}{h_k} \qquad (i = 1,2,3) \quad .$$

Now, setting

$$(10)\qquad \begin{cases} v_i^k = |E_j \cap (t_i^k, t_{i+1}^k)| \\ v_{k+1} = v_1^k \wedge v_2^k \end{cases}$$

we are in position to construct the new points a_{k+1} , b_{k+1} , choosing the end-points of the interval corresponding to v_{k+1} : that is

$$(11)\qquad (a_{k+1}, b_{k+1}) = \begin{cases} (t_1^k, t_2^k) & \text{if } v_1^k \le v_2^k \\ (t_2^k, t_3^k) & \text{otherwise.} \end{cases}$$

We construct in this way the required sequence $\{a_k\}$, $\{b_k\}$. Let

$$(12)\qquad m_k = \max_{i=1,2,3} \int_{|x|=t_i^k} \phi_{E_j} dH_{n-1} \quad .$$

Using (9) it follows that

$$(13)\qquad m_k \le \frac{v_k}{h_k} \quad \forall\, k \quad .$$

Now the problem is to pick out a sequence $h_k \in \left(0, \frac{\ell_k}{3}\right)$ such that

$$(14)\qquad \lim_k m_k = 0 \quad .$$

Take $h_k = \frac{\ell_o}{9 \cdot 4^k}$. Using the isoperimetric-type inequality and (13) we obtain:

$$m_o \le \frac{v_o}{h_o}$$

$$v_1 \le c_1 m_o^N \le c_1 \left(\frac{v_o}{h_o}\right)^N$$

$$m_1 \le \frac{v_1}{h_1} \le c_1 \frac{v_o^N}{h_o^N \cdot h_1}$$

$$v_2 \le c_1 m_1^N \le c_1^{1+N} \frac{v_o^{N^2}}{h_o^{N^2} \cdot h_1^N}$$

$$m_2 \le \frac{v_2}{h_2} \le c_1^{1+N} \frac{v_o^{N^2}}{h_o^{N^2} \cdot h_1^N \cdot h_2}$$

$$v_3 \le c_1 m_2^N \le c_1^{1+N+N^2} \frac{v_o^{N^3}}{h_o^{N^3} \cdot h_1^{N^2} \cdot h_2^N}$$

$$m_3 \le \frac{v_3}{h_3} \le c_1^{1+N+N^2} \cdot \frac{v_o^{N^3}}{h_o^{N^3} \cdot h_1^{N^2} \cdot h_2^N \cdot h_3}$$

$$\vdots$$

$$m_k \le c_1^{1+N+\ldots+N^{k-1}} \cdot \frac{v_o^{N^k}}{h_o^{N^k} \cdot h_1^{N^{k-1}} \cdot h_{k_1}^N \cdot h_k} = \left[c_1^{\sum_{i=1}^{k-1} N^{-i}} \cdot \frac{v_o}{\prod_{i=o}^{k} h_i^{N^{-i}}} \right]^{N^k} \le$$

$$\le \left(\text{const} \cdot \frac{v_o}{\rho_o}\right)^{N^k} \quad \text{i.e.,}$$

$$m_k \le \left(\text{const} \cdot \frac{v_o}{\rho_o}\right)^{N^k} . \tag{14}$$

It suffices then to take j_1 so large that

$$\text{const} \cdot \frac{v_o}{\rho_o} = \text{const} \cdot \frac{v_o}{j_1 - j_o} \le \text{const} \cdot \frac{V}{j_1 - j_o} < 1$$

and Step 1 is thus proved.

3. PROBLEM II: REGULARITY OF THE MINIMAL BOUNDARIES ENCLOSING A GIVEN VOLUME.

(The results in this paragraph were obtained in [12]).

Let E be a solution to problem (P_V). Define

$$\psi(E,x,\rho) = \int_{\overline{B_\rho}(x)} |D\phi_E| - \inf\left\{\int_{\overline{B_\rho}(x)} |D\phi_F| \,,\; F - \overline{B_\rho}(x) = E - \overline{B_\rho}(x)\right\} \,,\; (x \in \Omega,\; 0 < \rho < \text{dist}(x,\partial\Omega)) \,.$$

The classical regularity result for minimal boundaries relies mainly on two basic facts:

<u>first</u>, an estimate of the type

$$\psi(E,x,\rho) \leq \text{constant} \cdot \rho^n \quad ;$$

<u>second</u>, a procedure which allows to get minimal cones starting from arbitrary minimal sets, by "blowing up" at boundary points. The estimate on the function ψ can be obtained easily if one can prove the existence of an interior and an exterior point to the minimal set E (see [12], Proposition 1). We want to say something about this last point, which we consider the most interesting part of our method.

It suffices to prove only the existence of exterior points of E in Ω, since the statement about interior points follows arguing in the same way on $\Omega - E$.

Let $B_\rho(x) \subset \Omega$, denote by t_1, t_2, t_3 three points such that $0 < t_1 < t_2 < t_3 < \rho$ and assume the trace of E on ∂B_{t_i} (same centre as B_ρ) is continuous for $i = 1,2,3$.

Define

$$m = \max_{i=1,2,3} \int_{\partial B_{t_i}} \phi_E \, dH_{n-1} \quad ,$$

$$v_1 = |E \cap (t_1,t_2)| \qquad (t_1,t_2) = B_{t_2} - \overline{B}_{t_1}$$

$$v_2 = |E \cap (t_2,t_3)| \qquad (t_3,t_2) = B_{t_3} - \overline{B}_{t_2}$$

$$v = v_1 + v_2$$

We have then the:

<u>Isoperimetric-type inequality</u>.

With the notations given above, there is a constant c_1 *(only depending on the dimension* n *) such that, if*

(15) $$v \leq \omega_n \left(\frac{t_3 - t_1}{2}\right)^n$$

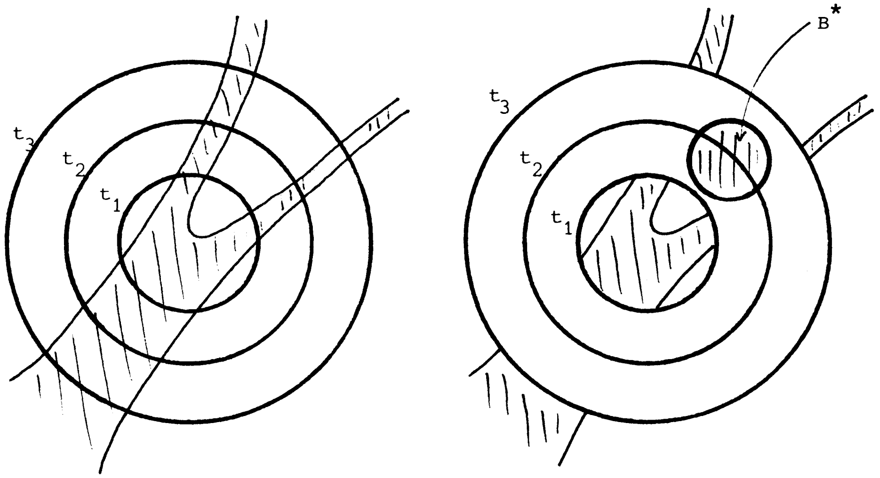
then

(16) $$v_1 \wedge v_2 \leq c_1 m^N$$

where again $N = \frac{n}{n-1}$.

The condition (15) is essential here. This condition allows us to replace the portion of E contained in the annulus (t_1, t_3) by a ball, say B^* , of the same measure, that is:

$$|B^*| = v \quad \text{with} \quad B^* \subset (t_1, t_3) \quad .$$



Let $F = (E - E \cap (t_1, t_3)) \cup B^*$; then, owing to the minimality of E , we have

$$\int_{B_\rho} |D\phi_E| \leq \int_{B_\rho} |D\phi_F|$$

and (16) follows (see [12], Lemma 2).

The existence of an exterior point relies on the existence of a $t \in (0, \rho)$ such that the trace of E on ∂B_t (same centre as B_ρ) vanishes here:

(17) $$\int_{\partial B_t} \phi_E \, dH_{n-1} = 0 \quad .$$

From (17) it is easy to prove that actually $|E \cap B_t| = 0$, as claimed.
The proof of (17) is essentially the same as that of Step 1 in the existence theorem; we use an analogous iterative process beginning with $a_o = 0$, $b_o = \rho$.
The new difficulty relies on the fact that, at each step in the iterative process, condition (15) must be verified. So we need $|E \cap B_\rho|$ to be "very small". In other words, we must be able to pick the sphere $B_\rho \subset \Omega$ such that $|E \cap B_\rho|$ is "very small". This is a consequence of a general property for sets of finite perimeter.

<u>Lemma</u>. *Let* Q *be an hypercube in* $\mathbb{R}^n$ *with side* $\ell > 0$, *and let* E *be a set of finite perimeter in* Q *such that*

(18) $$0 < |E \cap Q| < |Q| \quad .$$

For every $p \in \left(0, \frac{n^2}{n-1}\right)$ *and for every* $A > 0$ *we can find two balls* B_{ρ_1}, $B_{\rho_2} \subset Q$, *with arbitrarily small radii* ρ_1, ρ_2, *satisfying:*

(19) $$|E \cap B_{\rho_1}| < A\rho_1^p$$
$$|B_2 - E| < A\rho_2^p$$

<u>Sketch of the proof</u>. Clearly, for $0 \le p \le n$, statement (19) holds with B_{ρ_1}, B_{ρ_2} centred at points of density 0 or 1 respectively; assume that $p > n$. For a fixed natural number k, denote by

$$\left\{ Q_i^{(k)} \, , \, i = 1,2,\ldots,2^{kn} \right\}$$

the family of open disjoint subcubes with side $2^{-k}\ell$ into which Q can be divided. Then consider the following partition of the subcubes $Q_i^{(k)}$:

$$A_k = \left\{ Q_i^{(k)} : \varepsilon_k \le |E \cap Q_i^{(k)}| \le \left(\frac{\ell}{2^k}\right)^n - \varepsilon_k \right\}$$

$$\mathcal{B}_k = \left\{ Q_i^{(k)} \ : \ |E \cap Q_i^k| < \varepsilon_k \right\}$$

$$\mathcal{C}_k = \left\{ Q_i^{(k)} \ : \ |E \cap Q_i^k| > \left(\frac{\ell}{2^k}\right)^n - \varepsilon_k \right\} \quad ,$$

where $\varepsilon_k > 0$ is a suitable sequence going to 0. We claim that there is a k_o such that

(20) $\quad \mathrm{card}\,\mathcal{B}_k > 0 \ , \ \mathrm{card}\,\mathcal{C}_k > 0 \quad \forall\, k \geq k_o \quad .$

To see this, consider an element $Q_i^{(k)} \in \mathcal{A}_k$; it follows from the definition of $\mathcal{A}_k$ that

$$|E \cap Q_i^{(k)}| \wedge |Q_i^{(k)} - E| \geq \varepsilon_k \quad .$$

In view of the Sobolev-Poincarè inequality (see [10]) this implies:

$$\varepsilon_k^{\frac{n-1}{n}} \leq c_2 \int_{Q_i^{(k)}} |D\phi_E| \quad ,$$

with c_2 only depending on the dimension n.

Thus we obtain the following estimate on $\mathrm{card}\,\mathcal{A}_k$:

$$\frac{\mathrm{card}\,\mathcal{A}_k \cdot \varepsilon_k^{\frac{n-1}{n}}}{c_2} \leq \int_Q |D\phi_E| \quad .$$

This enables us to get (20) by taking the sequence ε_k in a suitable way. The choice

$$\varepsilon_k = \frac{A\ell^p}{2^{p(k+1)}}$$

contributes to this aim.

Now, for a given $k \geq k_o$ we can pick out a subcube $Q_i^{(k)} \in \mathcal{B}_k$; denoting by B_{ρ_1} the ball inscribed in $Q_i^{(k)}$ (with radius $\rho_1 = 2^{-(k+1)} \cdot \ell$) we obtain from the definition of $\mathcal{B}_k$ that

$$|E \cap B_{\rho_1}| \leq |E \cap Q_i^{(k)}| < 2^{-p(k+1)} \cdot A \cdot \ell^p = A\rho_1^p \quad .$$

In a similar way, if B_{ρ_2} denotes the ball inscribed in a subcube $Q_i^{(k)} \in \mathfrak{C}_k$, we have

$$|B_{\rho_2} - E| \leq |Q_i^{(k)} - E| < 2^{-p(k+1)} \cdot A \cdot \ell^p = A\rho_2^p \quad .$$

We state now the regularity result:

Regularity theorem.

If E *minimizes perimeter with a volume constraint in an open subset* Ω *of* $\mathbb{R}^n (n \geq 2)$ *, then* $\partial^* E \cap \Omega$ *is an analytic* $(n-1)$*-manifold, and moreover* $H_s[(\partial E - \partial^* E) \cap \Omega] = 0 \quad \forall\, s > n-8$.

4. PROBLEM III: LAGRANGE MULTIPLIERS.

For the sake of semplicity we confine ourselves to the case $|\Omega| < +\infty$, $V \in (0, |\Omega|)$.

Define

$$(21) \qquad g_{\lambda,V}(t) = \lambda (t-V)^2 \ , \qquad \lambda > 0 \ , \quad V \in (0,|\Omega|) \ , \quad t \in \mathbb{R}$$

and

$$(22) \qquad \mathfrak{G} = \{g_{\lambda,V} : \lambda > 0,\ V \in (0,|\Omega|)\} \quad .$$

For every function $g \in \mathfrak{G}$, let E_g be a minimum for the functional

$$\mathcal{F}_g(E) = \mathcal{F}(E) + g(|E|)$$

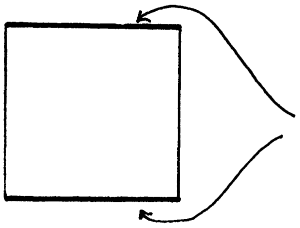
among the Borel subsets $E \subset \Omega$. It is very easy to prove that the set

$$v(\mathfrak{G}) = \{|E_g| : g \in \mathfrak{G}\}$$

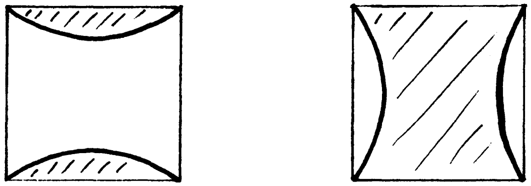
is dense in $(0,|\Omega|)$.

The following example shows that $v(\mathfrak{G})$ doesn't necessarily coincide with $(0,|\Omega|)$.

Example $(n = 2)$. Let $\Omega = (0,1) \times (0,1)$ and Γ as in the picture.



The isoperimetric property of the sphere implies that the minima of the functionals $\mathcal{F}_g$ can be only of the types indicated in the picture, where, in both cases, $\Omega \cap \partial Eg$ is constituted by two arcs of circle with the same radions.



A B

Moreover, for every $g \in \mathcal{G}$ there is at the most a minimum of the type A) and there is at most a minimum of the type B).

In situation A) we have

$$\frac{d(\text{length})}{d(\text{area})} > 0$$

while in the situation B) it is

$$\frac{d(\text{length})}{d(\text{area})} < 0 \quad .$$

Since $g'_{\lambda,V}(V) = 0$, from the preceding remarks it follows that

(i) If $Eg_{\lambda,V}$ is of the type A), then $|Eg_{\lambda,V}| < V$

while

(ii) If $Eg_{\lambda,V}$ is of the type B), then $|Eg_{\lambda,V}| > V$.

We will prove now that $1/2 \notin V(\mathcal{G})$, that is,

$$|Eg_{\lambda,V}| \neq 1/2 \quad \forall \lambda > 0 \quad \forall V \in (0, |\Omega|) \quad .$$

First case. $V \leq 1/2$.

Then, if $Eg_{\lambda,V}$ is of the type A), from (i) we have $|Eg_{\lambda,V}| < 1/2$. Suppose the set E of the type B) and measure 1/2 is a minimum for $\mathcal{F}g_{\lambda,V}$. Let E^* be the set of type A) and measure 1/2. Because $\mathcal{F}g_{\lambda,V}(E) = \mathcal{F}g_{\lambda,V}(E^*)$, the set E^* should be a minimum for $\mathcal{F}g_{\lambda,V}$, and this is not possible.

Second case. $V > 1/2$.

We argue in the same way.

We consider now the functions

$$\text{(23)} \qquad \phi_{\lambda,V}(t) = \lambda|V - t| \ , \quad \lambda > 0 \ , \quad V \in (0, |\Omega|) \ , \quad t \in \mathbb{R} \ .$$

Noting that the estimates in the proof of the lemma on page 45 only depend on ℓ , $|E \cap Q|$, $\int_Q |D\phi_E|$, A and p , we obtain the following uniform version of that lemma:

Lemma. *Let* Q *be a hypercube in* $\mathbb{R}^n$ *with side* $\ell > 0$, *and let* E *be a set of finite perimeter in* Q *such that*

$$0 < a < |E \cap Q| < b < |Q|$$

$$\int_Q |D\phi_E| < c$$

*(*a,b,c *are positive constants).*

For every $p \in \left(0, \frac{n^2}{n-1}\right)$ *and for every* $A > 0$ *there exist*

$$r = r(a,b,c,\ell,p,A)$$

and two balls $B_1, B_2 \subset Q$ *of radius* r, *such that*

$$(24) \qquad \begin{cases} |E \cap B_1| < A \cdot r^p \\ |B_2 - E| < A \cdot r^p \end{cases}$$

Then we have

<u>Lemma</u>. *Let* $V \in (0, |\Omega|)$. *Then there exist* $\lambda_o = \lambda_o(V)$ *and* $r_o = r_o(V)$ *such that for every* $\lambda \geq \lambda_o$, *there are two balls* B_λ , $\tilde{B}_\lambda \subset \Omega$ *with radius* r_o *and*

$$|E\phi_{\lambda,V} \cap B_\lambda| = 0$$

$$|E\phi_{\lambda,V} \cap \tilde{B}_\lambda| = |\tilde{B}_\lambda| \quad .$$

<u>Proof</u>. Noting that $\lim_{\lambda \to +\infty} |E\phi_{\lambda,V}| = V$ and taking $\tilde{a}$, $\tilde{b}$, $0 < \tilde{a} < V < \tilde{b} < |\Omega|$, we obtain that there is a λ_o such that

$$(25) \qquad \tilde{a} \leq |E_{\phi_{\lambda,V}}| \leq \tilde{b} \quad \forall \lambda \geq \lambda_o \quad .$$

From (25) it follows the existence of $\ell > 0$, a, b, $0 < a < b < \ell^n$, such that for every $\lambda \geq \lambda_o$ there exists a cube $Q_\lambda \subset \Omega$ with side ℓ satisfying

$$a \leq |E_{\phi_{\lambda,V}} \cap Q_\lambda| \leq b \quad \forall \quad \lambda \geq \lambda_o \quad .$$

Let's now choose A and p as in the proof of (17). Using the preceding lemma, take two balls $B_1 = B_1(x_\lambda)$, $B_2 = B_2(x_\lambda) \subset \Omega$ with radius $r = r(a,b,c,\ell,p,A)$ and verifying (24). Then we proceed as in the proof of (17) beginning the iterative process with

$$a_o = r/2 \ , \quad b_o = r \quad .$$

We obtain the lemma with $r_o = r/2$.

From the lemma above we obtain the following

<u>Theorem</u>. *Let* $V \in (0, |\Omega|)$, λ_o, r_o *as above. Then, for every* $\lambda > \max\left\{\lambda_o, \frac{n}{r_o}\right\}$, *we have*

$$|E_{\phi_{\lambda,V}}| = V \quad .$$

Proof. By contradiction suppose $|E_{\phi_{\lambda,V}}| < V$. We can then move $\tilde{B}_\lambda$, while remaining in Ω, until it reaches a new position $\tilde{\tilde{B}}_\lambda$ such that

$$|E_{\phi_{\lambda,V}}| < |E_{\phi_{\lambda,V}} \cup \tilde{\tilde{B}}_\lambda| \leq V \quad .$$

From the inequality (1.10) in [15] we obtain

$$\mathcal{F}_{\phi_{\lambda,V}}\left(E_{\phi_{\lambda,V}} \cup \tilde{\tilde{B}}_\lambda\right) = \mathcal{F}\left(E_{\phi_{\lambda,V}} \cup \tilde{\tilde{B}}_\lambda\right) + \lambda\left(V - |E_{\phi_{\lambda,V}} \cup \tilde{\tilde{B}}_\lambda|\right) \leq \int_\Omega |\phi_{E_{\phi_{\lambda,V}}}| +$$

$$+ \int_{\partial\Omega} |\phi_{E_{\phi_{\lambda,V}}} - \phi_\Gamma| + \frac{n}{r_o}|\tilde{\tilde{B}}_\lambda - E_{\phi_{\lambda,V}}| + \lambda\left(V - |E_{\phi_{\lambda,V}} \cup \tilde{\tilde{B}}_\lambda|\right) = \mathcal{F}\left(E_{\phi_{\lambda,V}}\right) +$$

$$+ \frac{n}{r_o}|\tilde{\tilde{B}}_\lambda - E_{\phi_{\lambda,V}}| + \lambda\left(V - |E_{\phi_{\lambda,V}}|\right) - \lambda|\tilde{\tilde{B}}_\lambda - E_{\phi_{\lambda,V}}| = \mathcal{F}_{\phi_{\lambda,V}}\left(E_{\phi_{\lambda,V}}\right) +$$

$$+ \left(\frac{n}{r_o} - \lambda\right)|\tilde{\tilde{B}}_\lambda - E_{\phi_{\lambda,V}}| < \mathcal{F}_{\phi_{\lambda,V}}\left(E_{\phi_{\lambda,V}}\right)$$

absurd.

Therefore $|E_{\phi_{\lambda,V}}| \geq V$.

In the same way we can prove that

$$|E_{\phi_{\lambda,V}}| \leq V \quad .$$

5. CONCLUDING REMARKS.

The techniques we have presented here have been used in the study of some questions in capillarity and in problems concerning rotating masses of fluids. See [1], [4], [5], [8]. See also Congedo - Emmer's lecture in these Proceedings.

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