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Astérisque, tome 118 (1984), p. 157-165
<http://www.numdam.org/item?id=AST_1984__118__157_0>

Société Mathématique de France
Astérisque 118 (1984) p.157 à 165.

THE INFLUENCE OF BOUNDARY GEOMETRY ON CAPILLARY SURFACES WITHOUT GRAVITY

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1. For given boundary geometry, the form of a capillary free surface can change strikingly, depending on whether or not an external force (gravity) field is present. We consider the historically important example of a cylinder Z of homogeneous material, closed at one end by a base of general section Ω and partly filled with liquid. We seek to characterize those configurations for which liquid can cover Ω and be in mechanical equilibrium. According to the Principle of Virtual Work, the associated energy functional

(1) $$E = \sigma(S - \beta S^{*} + 2HV)$$

will be stationary in equilibrium configuration. Here S is the (free) fluid surface area, σ the surface tension, S^{*} the area of wetted surface on Z, $\sigma\beta$ the adhesion coefficient of fluid to cylinder, $2\sigma H$ a Lagrange multiplier arising from the constraint that the volume V of fluid is prescribed. Formal variational procedures lead to a geometrical problem: to find those sections Ω such that there will exist a surface S of constant mean curvature H, which covers Ω and meets the cylinder walls Z in the constant angle $\gamma = \cos^{-1}\beta$ (measured between S and Ω). Our principal interest is directed toward configurations for which the energy will be minimized, and hence - in view of a theorem of Miranda [13] - we consider only surfaces that can be described non-parametrically by a function $z = u(x,y)$ over Ω. The variational condition applied to (1) then leads to an analytical formulation

$$(2) \qquad \operatorname{div} Tu \equiv 2H = \frac{\Sigma}{\Omega} \cos \gamma$$

in Ω , with

$$(3) \qquad Tu \equiv \frac{1}{\sqrt{1 + |Du|^2}} Du$$

and

$$(4) \qquad \nu \cdot Tu = \cos \gamma$$

on $\Sigma = \partial\Omega$. Here ν is outer directed normal on Σ. We use the symbols Σ , Ω , ... both to denote a set and to denote its measure.

We may always assume $0 \leq \gamma < \pi/2$. We assume Σ to be smooth except for a finite number of isolated corners P with interior angle 2α ($\leq 2\pi$). At P the condition (4) is not prescribed. It can be shown that a solution, whenever one exists, is nevertheless uniquely determined up to an additive constant; no growth condition at P need be imposed.

2. It was shown by Concus and Finn [2] that solutions of (2-4) may not exist, even for convex analytic Σ ; thus the structure of the solution set is quite different from what happens in a gravity field. The nonexistence is not an idiosyncrasy of the equations, it has been verified experimentally.

If a corner P appears, then no solution can exist if $\alpha + \gamma < \pi/2$; however, solutions for which $\alpha + \gamma = \pi/2$ are explicitly known [2].

3. The question of determining natural geometrical conditions on Ω for existence of a solution was addressed by Giusti and Weinberger [12], by Chen [1], by Finn [5] and by Finn and Giusti [10], with limited success. We describe here another approach to the problem, that has led to more inclusive results (Finn

[6, 7, 8], Concus and Finn [3, 4], Tam [17]). The underlying idea contacts on an observation of Concus and Finn [2], that whenever a solution exists, the functional

$$\Phi[\Gamma] \equiv \Gamma - \Sigma^* \cos\gamma + \left(\frac{\Sigma}{\Omega}\cos\gamma\right)\Omega^* \tag{5}$$

must be positive for any curve (or system of curves) Γ that cuts a subregion Ω^* from Ω and subarc Σ^* from Σ (see Figure 1).

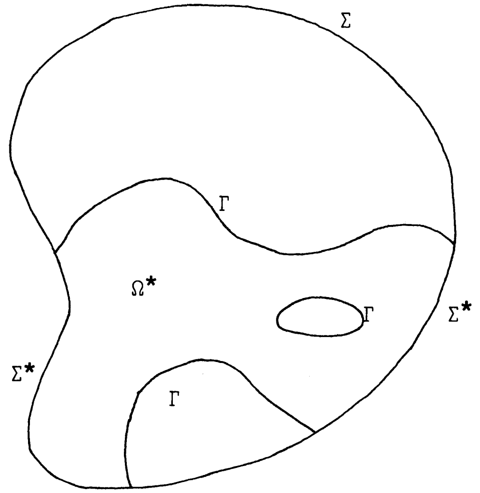

Figure 1

Giusti [11] showed that whenever there exists $\varepsilon > 0$ so that the modified functional

$$\Phi^{\varepsilon}[\Gamma] \equiv (1-\varepsilon)\Gamma - \Sigma^* \cos\gamma + \left(\frac{\Sigma}{\Omega}\cos\gamma\right)\Omega^* \tag{6}$$

is positive for all Γ as above, then a capillary surface must exist. In the special case $\gamma = 0$, he showed that $\Phi[\Gamma] > 0$ is already sufficient [12].

4. In order to see how these requirements relate to the geometry of Ω, it is

natural to seek those Γ that minimize the respective functionals. If we compare (5) with (1), we see a remarkable analogy, which shows that the problem of minimizing (5) is simply that of finding a capillary surface in one lower dimension. The only differences are a) the mean curvature is now prescribed in advance $\left(= \frac{\Sigma}{\Omega} \cos \gamma\right)$ and b) the bounding walls are no longer cylindrical, so that the problem must now be studied in a parametric formulation. The structure of the problem tells us, however, what to expect, and the result is proved in [7]: *A minimizing set* Γ *, whenever one exists, consists of circular arcs of radius* $R_\gamma = \frac{\Omega}{\Sigma \cos \gamma}$. *If* $\gamma \neq 0$ *, then* Γ *consists of a finite number of disjoint arcs, each of which either meets* Σ *at a point* P *, or else intersects* Σ *with angle* γ *, measured on the side of* Γ *opposite to that into which its curvature vector points. No arc of* Γ *can enter a point* P *at which* $2\alpha < \pi$ *, and no arc of* Γ *can include a semicircle.*

5. It can happen that no minimizing set exists. *Whenever that happens, there holds* $\Phi[\Gamma] > 0$ *, all* $\Gamma \subset \Omega$ [7]. Further, it is shown in [7] that *if at every corner* P *there holds* $\alpha + \gamma > \pi/2$ *, then there exists* $\varepsilon > 0$ *such that* $\Phi^{\varepsilon}[\Gamma] > 0$ *, all* $\Gamma \subset \Omega$. Under this condition, we obtain:

The nonexistence of a solution to the subsidiary variational problem for $\Phi[\Gamma]$ *is a sufficient condition for the existence of a solution to the original (capillary) variational problem for* $E[S]$.

The nonexistence of a minimizing set can be verified directly in many particular cases (e.g., it is easy to see that it holds in any parallelogram for which the smaller angle satisfies $\alpha + \gamma > \pi/2$). It also holds in any case for which the second variation on any extremal Γ can be made negative. In [7] this second variation is calculated explicitly. In polar coordinates r , θ referred to the center of an arc Γ , we find in terms of a variation $\dot{r} = \eta$,

$$I[\eta] \equiv \ddot{\Phi} = Q + \frac{\cot\gamma}{R_\gamma}\left[\left(1 - k_1\frac{R_\gamma}{\cos\gamma}\right)\eta_1^2 + \left(1 - k_2\frac{R_\gamma}{\cos\gamma}\right)\eta_2^2\right] \tag{7}$$

with

$$Q = \frac{1}{R_\gamma}\int_{\theta_1}^{\theta_2}\left(\eta'^2 - \eta^2\right)d\theta \quad . \tag{8}$$

Here the indices 1, 2 refer to the two points of contact with Σ, and k_1, k_2 are the curvatures of Σ at those points, considered as positive when the curvature vector points into Ω.

Under the constraint $\eta_1^2 + \eta_2^2 = 1$, the stationary points for $\ddot{\Phi}$ are the rigid motions

$$\eta = a\cos(\theta - \sigma) \tag{9}$$

where σ, a satisfy

$$\frac{2\sin 2\sigma}{\cos 2\sigma + \cos 2\delta} = (k_2 - k_1)\frac{R_\gamma}{\sin\gamma} \tag{10}$$

$$a^2 = \frac{1}{1 + \cos 2\sigma\cos 2\delta}\ , \quad \delta = \theta_2 - \theta_1 \quad . \tag{11}$$

These equations admit, in general, four distinct solutions.

For any choice of σ, a, we find from (7), (9)

$$\frac{1}{a^2}I[\eta] = -\sin 2\delta\cos 2\sigma + \cot\gamma\left\{\cos^2(\delta - \sigma)\left[1 - k_2\frac{R_\gamma}{\cos\gamma}\right] + \cos^2(\delta + \sigma)\left[1 - k_1\frac{R_\gamma}{\cos\gamma}\right]\right\} . \tag{12}$$

We find immediately the general result: *if* $\gamma > 0$ *and if for every strict subarc* Γ *of a semicircle of radius* R_γ *that meets* Σ *in equal angles* γ *(measured exterior to the semicircle) the relation* (12) *with one of the four* σ *from* (10) *yields a negative* I *, then there exists a solution of* (2-4).

Further sufficiency criteria appear in [7]. These criteria are useful in many particular cases, although for a general configuration they do require an investigation of the possible extremal configurations. The criterion can be made a priori in the case of a convex Ω , as then k_1 , $k_2 \geq 0$ and the angle δ can be estimated in terms of the maximal boundary curvature. We obtain the result [7]: *Suppose the boundary curvature* k *satisfies* $0 \leq k_m \leq k \leq k_M \leq \infty$. *Then a solution of* (2-4) *exists whenever either* $R_\gamma k_M \leq 1$ *or*

$$\min\left\{\frac{\sin^2\gamma}{R_\gamma k_M - \cos\gamma}, \cos\gamma\right\} + \left\{R_\gamma k_m - \cos\gamma\right\} > 0 \quad . \tag{13}$$

The particular case of a trapezoidal section is discussed in some detail in [6], where an anomalous behavior that had been observed for that section is clarified.

6. The extremals for the subsidiary problem have a curious property [4]: *Suppose there is a rigid displacement* η *of an extremal* Γ *, for which* $d\gamma = 0$ *at both points of contact with* Σ. *Then* $I[\eta] = 0$ *; further,* η *is an extremal for the functional* I *, in the sense that when* η *is expressed in the form* (9) *the parameters* σ *,* a *satisfy* (10) *and* (11).

7. It can be shown that to every section Ω there corresponds an angle γ_0 in $\left[0, \frac{\pi}{2}\right]$ such that if $\gamma_0 < \pi/2$, a solution of (2-4) exists for all $\gamma > \gamma_0$, while if $\gamma_0 > 0$, then no solution exists for $\gamma < \gamma_0$. Concus and Finn [3] studied the case $\gamma = \gamma_0$ and obtained the result:

Suppose $0 < \gamma_0 < \frac{\pi}{2}$. *If* Σ *is smooth, or if* $\alpha + \gamma_0 > \frac{\pi}{2}$ *at all corners, then no solution exists at* γ_0. *If one or more corners appear at which* $\alpha + \gamma_0 = \frac{\pi}{2}$ *, then a solution may or may not exist, depending on the geometry.*

It may at first seem surprising that a surface should exist when Σ is not smooth and fail to exist when Σ is smooth. However, the matter can be viewed from another point of view: If Σ is smooth, the capillary surface disappears in a continuous way as $\gamma \downarrow \gamma_0$, while if Σ has one or more corners, the surface can disappear in a discontinuous way.

In every case for which the surface fails to exist at γ_o, the surfaces corresponding to a sequence $\gamma \downarrow \gamma_o$ can be normalized so that at γ_o a limiting configuration is obtained, of a solution surface defined over a part Ω_o of Ω bounded in part by extremals Γ of the subsidiary problem, and which is asymptotic at Γ to vertical circular cylinders of radius R_γ over Γ. If $\Omega_o = \phi$ then the limiting surface consists of one or more vertical circular cylinders.

We note that in the above result, the case $\alpha + \gamma_o < \frac{\pi}{2}$ cannot occur. This follows from the general result (§2 above) that no solution can exist when $\alpha + \gamma < \frac{\pi}{2}$ holds at any corner. If $\alpha + \gamma = \frac{\pi}{2}$ at some corner, then the positivity of Φ^ε (§3 above) fails for every $\varepsilon > o$.

8. It is desirable to characterize those configurations with $\alpha + \gamma_o = \frac{\pi}{2}$, for which a solution will exist. A simple example is obtained by choosing Ω to be a regular polygon. A lower hemisphere whose equatorial circle circumscribes Ω then provides an explicit solution of (2-4), with $\alpha + \gamma = \frac{\pi}{2}$ at each corner. Larger values of γ are obtained by increasing the radius of the hemisphere, while for smaller γ there is no solution (discontinuous disappearance).

A general configuration does not seem to lend itself to a comparably simple discussion; however, Finn [8] proved the following result: *Suppose that at each corner* P *there is a lower hemisphere of radius* R_{γ_o} *which in some neighborhood of* P *meets the vertical cylinder walls over* Σ *in angles not larger than* γ_o. *Suppose also that* $\Phi[\Gamma] > 0$ *for all admissible* $\Gamma \subset \Omega$. *Then there exists a solution* $u(x)$ *of* (2-4) *in* Ω. *If* $u(x)$ *is normalized (e.g., so that* $\int_\Omega u\, dx = 0$ *), then* $u(x) < M < \infty$ *in* Ω, *depending only on the geometry and on the physical constants. The bounds are (in principle) explicit.*

This result was strengthened in important ways by Tam [17], who also extended it to any number of dimensions. Tam's proof proceeds along quite different lines, using an indirect argument based on the notion of generalized solution introduced by Miranda [14].

Siegel [15] studied the behavior of solutions $u(x;g)$ in a gravity field g, directed toward the base through the fluid, as $g \to 0$. He showed that *if* Σ *is smooth, if* $\gamma > 0$*, and if there exists a solution* $v(x)$ *of* (2-4) *in* Ω*, then (after normalization by additive constants)* $|u - v| = O(g)$ *in* Ω. Later Tam [17] weakened the restriction on Σ so as to allow corners, and showed that if $\gamma = 0$, then two different types of behavior can occur, depending on whether or not $v \in L^1(\Omega)$. Tam also provides new information on what happens in the case $\gamma = \gamma_0$.

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