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EXISTENCE AND UNIQUENESS OF NONLINEAR REALIZATIONS

by

Bronislaw JAKUBCZYK

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1.- INTRODUCTION.

The realization problem can be formulated as follows. Given an input-output system ("black box") described by an input-output map, find an internal description of the system (called realization of the system) and show that a "minimal internal description" is, in a sense, unique. In the case of discrete time and a general (set theoretical) input-output map, this is a problem of the automata theory (cf.[6]) and is solved by introducing the concept of "state space" as a "minimal memory" of the system.

We are concerned with the case of continuous-time systems with the output having finite number of real valued components. The problem has a satisfactory solution for the case of linear systems (cf. KALMAN [6] and the bibliography cited there) and bilinear systems (cf. e.g.[3]). In the case of the input-output map given by a finite Voltera series direct constructions of linear-analytic realizations were given by BROCKETT [1] and CROUCH [2] (see these proceedings).

In the general, nonlinear case a basic result was obtained by SUSSMANN [8], [10] (for related topics see SUSSMANN [9], [11] and HERMANN, KERNER [4]), who proved that if an input-output map has a realization which is either analytical or smooth symmetric, then it has a minimal realization which is unique up to a diffeomorphism.

Here, we give general necessary and sufficient conditions for existence of realizations of nonlinear input-output maps. We show that two minimal realizations are diffeomorphic (our definition of minimality is slightly modified with respect to [10]). We outline the construction of a realization in the general case. The detailed construction and proofs are contained in [5] .

2.- INPUT-OUTPUT MAP OF A GIVEN CONTROL SYSTEM.

Consider a control system of the form

$$(1) \qquad \dot{x} = f(x,u) \, , \quad x(0) = x_o$$
$$y = h(x) \, ,$$

where $x(t) \in X$ n-dimensional, differentiable manifold, $u(t) \in \Omega$ and $y(t) \in R^r$. Let U be a class of admissible control functions u defined on finite subintervals $[0, t_u)$ of $R_+ = [0, \infty)$. We assume that for any $u \in U$ the equation $\dot{x} = f(x,u)$ has a well defined unique solution on $[0, t_u]$. Let Φ^f_u denote the diffeomorphism $\Phi^f_u : X \longrightarrow X$ which maps initial points of the trajectories into their finite points. For a given quadruple $\Sigma = (X, f, h, x_o)$ we define the input-output map $p_\Sigma : U \longrightarrow R^r$ of system (1) by

$$(2) \qquad p_\Sigma(u) = h(\Phi^f_u(x_o)) \, .$$

3.- CONTROL SEMIGROUP AND CONTROL GROUP.

For the sake of simplicity, we shall consider here the class of piecewise constant controls only (see [5] for a general class of controls). Let Ω denote a set of admissible values of controls (its elements will be denoted by α, β). Denote by

$$(3) \qquad a = (t_k \, a_k) \dots (t_2 \, \alpha_2)(t_1 \, \alpha_1)$$

the piecewise constant function $[0, \sigma_k) \longrightarrow \Omega$, $a(\tau) = \alpha_i$ for $\tau \in [\sigma_{i-1}, \sigma_i)$, $\sigma_i = \sum_{j=1}^{i} t_j$ $(\sigma_o = 0)$, where $t_i \in R_+ = [0, \infty)$ and $k \geq 0$. The set of all such functions will be denoted by S and its elements by a, b, c. There is a natural semigroup structure in S with multiplication defined by concatenation

$$(4) \qquad ba = (\tau_m \beta_m) \dots (\tau_1 \beta_1)(t_k \alpha_k) \dots (t_1 \alpha_1)$$

where $b = (\tau_m \beta_m) \dots (\tau_1 \beta_1)$. The identity in S is the empty sequence (3).

There is a natural action of R_+ on S

(5) $$ta = ((t\,t_k)\,\alpha_k)\dots((t\,t_1)\,\alpha_1)$$

(expansion). We identify $\alpha = (1\alpha)$.

The semigroup S can be extended to a group G called control group (see LOBRY [7]). The elements of G are formal sequences of the form (3) with $t_i \in R$ and multiplication defined by (4) , where we identify $(t_1\alpha)(t_2\alpha) = (t_1 + t_2)\alpha$ and $(0\alpha) = e$. The element ta is defined by (5) for $t > 0$ and by $ta = ((t\,t_1)\alpha_1)\dots((t\,t_k)\alpha_k)$ for $t < 0$.

4.- INPUT-OUTPUT SYSTEMS.

Assume that R^r is our output space. Any mapping $p : S \longrightarrow R^r$ will be called an input-output map. By an input-output system we shall mean the triple (S, p, R^r) . To have existence of realizations we shall impose two basic assumptions on the input-output map p (they have parallel versions if p is defined on the group G) .

Denote $\underline{b} = (b_1, \dots, b_m)$, $b_i \in S$ $(b_i \in G)$, $m \geq 1$, $\underline{a} = (a_1, \dots, a_q)$ $a_i \in S$ $(a_i \in G)$, $q \geq 1$, $\underline{t} = (t_1, \dots, t_q)$, $t_i \in R_+$ $(t_i \in R)$ and define the functions : $\psi_{\underline{a}}^{\underline{b}} : R_+^q \longrightarrow R^{rm}$ $(R^q \longrightarrow R^{rm})$ by $\psi_{\underline{a}}^{\underline{b}} = (\psi_{\underline{a}}^{b_1}, \dots, \psi_{\underline{a}}^{b_m})$, where

$$\psi_{\underline{a}}^{b_i}(\underline{t}) = p(b_i\,(t_q\,a_q)\dots(t_1\,a_1)) .$$

It may be useful to imagine $(t_q\,a_q)\dots(t_1\,a_1)$ as a basic control and b_i , $i = 1, \dots, m$, as measure experiments.

Let $k = 2, 3, \dots, \infty, \omega$. The regularity assumption on p takes the form

(A.1) The functions $\psi_{\underline{a}}^{\underline{b}}$ belong to the class C^k for any $\underline{a}, \underline{b}, m \geq 1, q \geq 1$. In the case of $k = \omega$ and p defined on the semigroup S we shall also need a stronger version of (A.1).

(A.1)' The functions have real analytic extensions onto R^q for any $\underline{a}, \underline{b}, m \geq 1, q \geq 1$.

Define

$$\operatorname{rank} p = \sup_{\underline{a}, \underline{b}, \underline{t}} \operatorname{rank} D\psi \frac{\underline{b}}{\underline{a}} (\underline{t})$$

where $D\psi$ denotes the differential of ψ. We shall also assume that

(A.2) $\operatorname{rank} p < \infty$.

In the nonanalytical case the following additional assumption will be used

(A.3) $\forall a \in \Omega \ \exists \beta \in \Omega \quad \forall a, b \in S \quad \forall t > 0 \quad p(b(t\beta)(t\alpha)a) = p(ba) = p(b(t\alpha)(t\beta)a)$.

5.- REALIZATIONS.

Now we shall precise what we mean by realizations of the input-output system (S, p, R^r). The quadruple $\Sigma = (X, f, h, x_o)$ will be called a C^k realization of the input-output system (S, p, R^r), $k = 2, 3, \ldots, \infty, \omega$, if

(i) X is a C^k manifold (Hausdorff, without boundary) and $x_o \in X$,

(ii) $f : X \times \Omega \longrightarrow TX$ is a function such that the vector fields $f(.,\alpha)$ are complete and generate C^k flows $\Phi^f_{(t\alpha)}$,

(iii) $h : X \longrightarrow R^r$ is a function of the class C^k ,

(iv) the input-output map p_Σ is equal to p i.e.

$$p(a) = h(\Phi^f_a(x_o)), \ a \in S .$$

The realization is called reachable (weakly reachable) if $\forall x \in X \ \exists a \in S \ (a \in G)$ $\Phi^f_a(x_o) = x$ (for $a = (t_k\alpha_k) \ldots (t_1\alpha_1) \in G$ we define $\Phi^f_a = \Phi^f_{(t_k\alpha_k)} \circ \ldots \circ \Phi^f_{(t_1\alpha_1)}$. It is called observable if $\forall x_1, x_2 \in X$, $x_1 \neq x_2$ $\exists b \in S$ $h(\Phi^f_b(x_1)) \neq h(\Phi^f_b(x_2))$. A reachable and observable realization is called minimal. Weakly reachable and observable realization is called C^ω-minimal (minimal in the class C^ω). The realization is called symmetric if $\forall \alpha \in \Omega \ \exists \beta \in \Omega \ \forall x \in X \ f(x,\alpha) = -f(x,\beta)$.

Two C^k realizations Σ and Σ' are said to be C^k-diffeomorphic if there is a C^k diffeomorphism $\chi : X \longrightarrow X'$ which carries Σ to Σ' i.e.

$$(D\chi f) \circ \chi^{-1} = f' , \ h \circ \chi^{-1} = h' , \ \chi(x_o) = x'_o .$$

6. - THE MAIN RESULT.

The following theorem gives general conditions for existence and uniqueness of realizations of the system (S, p, R^r).

Theorem 1. - Let $k = 2, 3, \ldots, \infty, \omega$. The input-output system (S, p, R^r) has a C^k realization if and only if the map p can be extended to a map $\bar{p} : G \longrightarrow R^r$ which satisfies (A.1) and (A.2).

Any two C^k, $k = 2, \ldots, \infty$, minimal (C^ω-minimal) realizations of (S, p, R^r) are C^k (C^ω) diffeomorphic.

The existence criterion of the above theorem is somewhat implicit. However the criterion can be transformed to an explicit form for two important classes of realizations.

Theorem 2. - a) $k = 2, \ldots, \infty$. Any input-output system, which satisfies (A.1), (A.2) and (A.3) has a minimal, symmetric, C^k realization Σ such that $\dim X = \operatorname{rank} p$.

b) $k = \omega$. Any input-output system which satisfies (A.1)' and (A.2) has a C^ω-minimal realization Σ such that $\dim X = \operatorname{rank} p$.

Theorems 1, 2 are reformulations of the results of [5] (extended version). Namely, the existence part of Theorem 1 is contained in Theorem 4 of [5] and the uniqueness part of Theorem 1 in [5]. The full proofs are contained in [5]. Below we shall outline the proof of the existence part of Theorem 1.

7. - NECESSITY.

If there exists a realization Σ, then the extension $\bar{p} : G \longrightarrow R^r$ can be defined by

$$\bar{p}(a) = h(\Phi^f_a(x_o)), \quad a \in G .$$

We define the following maps $\psi_{\underline{a}} : R^q \longrightarrow X$, $\psi^{\underline{b}} : X \longrightarrow R^{rm}$,

$$\psi_{\underline{a}}(\underline{t}) = \Phi^f_{(t_q a_q)\ldots(t_1 a_1)}(x_o)$$

$$\psi^{\underline{b}}(x) = (h(\Phi^f_{b_1}(x)), \ldots, h(\Phi^f_{b_m}(x))).$$

We have that $\psi_{\underline{a}}^{\underline{b}} = \psi^{\underline{b}} \circ \psi_{\underline{a}}$, thus it is easy to see that (A.1) and (A.2) are satisfied and rank $\leqslant$ dim X .

8.- CONSTRUCTION OF A REALIZATION (OUTLINE).

For a given map $\overline{p} : G \longrightarrow R^r$ we introduce an equivalence relation in G

$$a \sim b \Leftrightarrow \forall c \ \ \overline{p}(ca) = \overline{p}(cb) .$$

We define

$$X = G/\sim$$

and $[a]$ denotes the equivalence class of a . Define the maps $\Phi_a : X \longrightarrow X$, $a \in G$, and $h : X \longrightarrow R^r$ by

$$\Phi_a([b]) = [ab], \ h([b]) = \overline{p}(b)$$

and let $x_o = [e]$.

The topology in X is defined as the strongest topology such that the maps $\psi_{\underline{a}} : R^q \longrightarrow X$ are continuous for all $\underline{a} = (a_1, \ldots, a_q)$, $q \geqslant 1$, where

$$\psi_{\underline{a}}(\underline{t}) = \Phi_{(t_q a_q) \ldots (t_1 a_1)}(x_o) .$$

The C^k differential structure on X is introduced by defining the class of real valued functions of the class C^k on X :

$$\varphi \in C^k(X, R) \overset{df}{\Leftrightarrow} \varphi \circ \psi_{\underline{a}} \in C^k(R^q, R) \quad \forall \underline{a} .$$

Using (A.1) and (A.2) it can be proved that X is C^k , finite dimensional manifold and Φ_a , h are functions of the class C^k . The vector fields $f(.,\alpha)$ are defined as infinitesimal vector fields of the flows $\Phi_{(t\alpha)}$.

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