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A SURVEY ON THE SINGULARITIES AND STABILITY OF DIFFERENTIAL FORMS

by

Martin Golubitsky* and David Tischler*

Our purpose in these lectures is to describe the status of the following:

Problem: Find generic classifications for the spaces of germs of C^∞ differential p-forms and closed C^∞ differential p-forms.

Work on this problem began with the thesis of Jean Martinet [13] published in 1970. In the intervening eight years both examples and counterexamples have broadened our knowledge indicating, in particular, that a new method for handling infinite dimensional unfolding spaces must be developed before significant progress on the general problem can be achieved. Our focus will be twofold. First, we shall describe the essentially complete enumeration of stable forms which gives but one small part of the stated problem and second, we shall describe the total moduli space for certain generic singularities giving substance to our statement above.

In particular, we give a complete generic classification for germs of non-zero analytic (n-1)-forms on $\mathbf{R}^n$ for all n as well as the C^∞ classification when n = 2 or 3. Aside from the simple classification of n-forms this is the first such result. As this is the only new result given in this survey we present a complete proof in §2. (See 2.14, 2.16, and 2.17.)

The structure of this paper is as follows: The first section contains a

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general description of what is meant by a generic classification. A particular class of singularities - called algebraic - are defined along with the notion of stability. A sharp dimension count shows for which stable p-forms exist. In section 2 we describe the stable forms along with part of the generic classification. Aside from the results mentioned above on (n-1)-forms we outline some results about moduli for 1-forms. The corresponding results known for closed forms are given in section 3. Global results (of which there are few) and integrable forms (with singularities) are described in sections 4 and 5 respectively.

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§1. Introduction.

In this Introduction we describe more precisely what we mean by a generic classification and offer as a candidate a generic set of forms as the basis for this classification.

Let D^p (resp. $\mathcal{D}^p$) denote the space of germs of C^∞ (closed) p-forms on $\mathbf{R}^n$ at 0.

Definition 1.1: Two forms w and w' in D^p are equivalent (by pull-back) if there is a germ of a C^∞ diffeomorphism $\varphi : (\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^n,0)$ such that $\varphi^* w' = w$.

We denote by $\mathrm{Diff}_0(\mathbf{R}^n)$ the group of all such diffeomorphism germs. Also we let $J^k(D^p)$ (resp. $J^k(\mathcal{D}^p)$) denote the k-jets of p-forms (resp. closed p-forms) and for $w \in D^p$ let $j^k w : \mathbf{R}^n \longrightarrow J^k(D^p)$ be defined by $x \longmapsto$ the k-jet of w at x pulled-back by translation to the origin in $\mathbf{R}^n$.

Definition 1.2: A k-th order singularity for p-forms is a submanifold Σ of $J^k(D^p)$ (resp. $J^k(\mathcal{D}^p)$) which is invariant under the action of $\mathrm{Diff}_0(\mathbf{R}^n)$ induced by pull-back.

Let

(1.3) $$\Sigma(w) = (j^k w)^{-1}(\Sigma) .$$

Definition 1.4: Let Σ be a k-th order singularity. Then w has a generic Σ-singularity at x if $x \in \Sigma(w)$ and $j^k w \pitchfork \Sigma$ at x.

It is immediate that if w has a generic Σ-singularity at 0, then $\Sigma(w)$ is a submanifold germ of $\mathbb{R}^n$ at 0 and

(1.5) $$\operatorname{codim} \Sigma(w) = \operatorname{codim} \Sigma .$$

Martinet [13] observes that the Thom Transversality Theorem holds for differential forms. In particular, let $D^p(M)$ (resp. $\mathcal{B}^p(M)$) denote the space of globally defined p-forms (closed p-forms) on the n-manifold M. Also for $w \in \mathcal{B}^p(M)$ let $[w]$ denote its de Rham cohomology class in $H^p(M)$.

Lemma 1.6: Let Σ be a k-th order singularity. Let

$$T_\Sigma = \{w \in D^p(M) \mid j^k w \pitchfork \Sigma\} .$$

Then T_Σ is a residual set. Note that the corresponding statement for closed forms is also true.

Moreover if w is closed and $\Sigma \subset J^k(\mathcal{B}^p)$ then there exists a w' arbitrarily close to w with $[w'] = [w]$ and w' in T_Σ.

The topology used in this lemma is the Whitney C^∞ (or strong) topology. This space is a Baire space so that a residual set - which is the countable intersection of open, dense subsets - is dense. The "moreover" part of the lemma follows as the perturbations necessary to move w to w' in T_Σ may be assumed to be exact.

Definition 1.7: (a) A generic set of p-forms on M is a set G which is both residual (preferably open and dense) and invariant under the action of Diff(M) on $D^p(M)$.

(b) A generic set of germs in D^p is a set

$$\tilde{G} = \{\text{germs of } w \text{ at } 0 \mid w \in G\}$$

where G is a generic set of p-forms on $\mathbb{R}^n$.

Our basic problem is, in reality, two problems. First one must find a generic set $\tilde{G}$ and then enumerate the equivalence classes in $\tilde{G}$. There is a natural candidate for $\tilde{G}$ defined in terms of the usual algebraic operations on the differential algebra of forms which we now describe. The following is an inductive definition.

Definition 1.8: A first order algebraic singularity is a manifold $\Sigma \subset J^1(D^p)$ (resp. $J^1(\mathcal{B}^p)$) which is described by a finite number of expressions like

(a) $w^k \wedge (dw)^{\ell}(0) = 0$ and

(b) $w^k \wedge (dw)^{\ell}(0) \neq 0$.

To iterate this definition let w have a generic first order algebraic singularity at 0 Note that $\Sigma(w)$ is a submanifold of $\mathbb{R}^n$ and thus one may either pull-back or restrict w to $\Sigma(w)$. The first order algebraic singularities for the pull-back and restriction will be the second order algebraic singularity for W.

Finally, in certain cases, invariant vector fields or line fields appear and contraction by or Lie differentiation with respect to these fields will generate algebraic singularities. Also, the decomposability (factoring into one forms) of w is an algebraic invariant.

This iterative definition is, of course, inspired by the Thom-Boardman singularities for mappings.

In general, it seems difficult to prove that the algebraic singularities define a stratification of $J^k(D^p)$ for all k and p. However if this process does give a stratification, then it is well-behaved enough so that the forms transverse to it will be a generic set. This follows as the algebraic singularities - however complicated - are countable in number.

Specifically Pelletier [21] and [20] has essentially shown that for 1-forms, 2-forms, and closed 2-forms, the algebraic singularities do form a stratification of J^k. We shall describe his stratification for closed 2-forms in more detail in section 3.

We have found that for the examples we have considered this process works well and as the remainder of these lectures will be based on specific examples no harm will come by assuming that algebraic singularities do yield a stratification.

Our method of classification will be to put the various geometric data, that is, the singular sets $\Sigma_i(w)$ obtained from generic algebraic singularities, into some fixed position in $\mathbb{R}^n$ and then determine what are the normal forms for this situation.

The simplest situation is when there is exactly one normal form; i.e., the algebraic singularity determines the form. These turn out to be the stable forms and are the ones that we shall classify first.

Definition 1.9: w in D^p is *stable* if for every nearby w' to w there is a diffeomorphism germ $\varphi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, x)$ for x near 0 such that $\varphi^* w' = w$. In the corresponding definition for closed forms one must assume that w' is closed.

See [13] or [5] for a more precise technical definition.

Lemma 1.10: *If a stable form* w *has a* Σ-*singularity at* 0 *then it must have a generic* Σ-*singularity at* 0.

Proof: We assume $j^k w(0) \in \Sigma$. By the transversality theorem (Lemma 1.6) there is a w' near w such that $j^k w' \pitchfork \Sigma$. As w is stable we may assume that w is equivalent to w'. Since Σ is invariant under the action of $\mathrm{Diff}_0(\mathbb{R}^n)$ $j^k w \pitchfork \Sigma$ at 0.

It follows that stable forms take on all algebraic singularities generically and the outline given above for finding stable forms does lead to an essentially complete enumeration.

We call non-algebraic singularities *analytic singularities* as they usually appear as modal parameters in the determination of normal forms for a given algebraic singularity.

We close the Introduction with one specific result which will give a focus for the examples in the subsequent sections.

__Proposition 1.11:__ (a) __If $2 \leq p \leq n-2$ there are no stable forms,__ and (b) __if $3 \leq p \leq n-2$ there are no stable closed forms.__

__Proof:__ This is just a combinatorial problem. Let $\mathcal{O}_w^k$ = orbit in $J^k(D^p)$ of w under the action of $\mathrm{Diff}_0^{k+1}(\mathbf{R}^n)$ - the group of invertible $(k+1)$-jets. $\mathcal{O}_w^k$ is a submanifold as $\mathrm{Diff}_0^{k+1}(\mathbf{R}^n)$ is a finite dimensional Lie group. If we can show that $\operatorname{codim} \mathcal{O}_w^k > n$ for some k, then Lemma 1.10 will show that w cannot be stable. Note that

$$
\begin{aligned}
\text{(1.12)} \qquad \operatorname{codim} \mathcal{O}_w^k &= \dim J^k(D^p) - \dim \mathcal{O}_w^k \\
&\geq \dim J^k(D^p) - \dim \mathrm{Diff}_0^{k+1}(\mathbf{R}^n) \quad .
\end{aligned}
$$

Recall that the dimension of the space of polynomials of degree k on $\mathbf{R}^n$ is given by the combinatorial symbol $\binom{n+k}{n}$. Thus

$$
\text{(1.13)} \qquad \operatorname{codim} \mathcal{O}_w^k \geq \binom{n}{p}\binom{n+k}{n} - n\binom{n+k+1}{n} \; .
$$

The expression on the RHS of (1.13) is a polynomial of degree n in k and the coefficient of the term k^n is given by

$$
\text{(1.14)} \qquad \binom{n}{p} - n \; .
$$

This is easily seen to be positive for $2 \leq p \leq n-2$. Thus for k large enough $\operatorname{codim} \mathcal{O}_w^k$ is greater than n and stability fails.

The number count for closed forms is similar. Here we assume that $k > n$. The Poincaré lemma shows that the sequence

$$
\text{(1.15)} \qquad \cdots \longrightarrow J^k(D^p) \xrightarrow{\ d\ } J^{k-1}(D^{p+1}) \xrightarrow{\ d\ } J^{k-2}(D^{p+2}) \longrightarrow \cdots
$$

is exact. As $J^k(\mathcal{Z}^p) = \operatorname{Ker} d$ we have that

$$\dim J^k(\mathcal{D}^p) = \sum_{i=0}^{n-p} (-1)^i \dim J^{k-i}(D^{p+i}) \tag{1.16}$$
$$= \sum_{i=0}^{n-p} (-1)^i \binom{n}{p+i} \binom{n+k-i}{n} \ .$$

The RHS of (1.16) is also a polynomial of degree n in k whose top order coefficient is

$$\sum_{i=0}^{n-p} (-1)^i \binom{n}{p+i} \ . \tag{1.17}$$

As $\binom{n}{s} = \binom{n-1}{s-1} + \binom{n-1}{s}$ for $s \leq n-1$ we have that

$$\dim J^k(\mathcal{D}^p) = \binom{n-1}{p-1} k^n + \cdots \tag{1.18}$$

where $\cdots$ indicates lower order terms in k. Letting $\mathcal{O}_w^k$ now stand for the orbit of w in $J^k(\mathcal{D}^p)$ under the action of $\mathrm{Diff}_0^{k+1}(\mathbf{R}^n)$ we have using (1.12) that

$$\operatorname{codim} \mathcal{O}_w^k \geq \left(\binom{n-1}{p-1} - n \right) k^n + \cdots \ . \tag{1.19}$$

One now checks that for $3 \leq p \leq n-2$ the first coefficient is positive.

Note: As $k \longrightarrow \infty$ in both situations we see that in fact $\operatorname{codim} \mathcal{O}_w^k$ also approaches $+\infty$.

A reasonable question - given this proposition - is whether there are any stable forms at all. The answer is classical as volume forms, contact forms, and symplectic forms are stable.

§2. Classification and stability for forms.

We consider two separate problems in this section which, in reality, have the same spirit. The first is the classification of stable p-forms which by Proposition 1.11 restricts us to $p = 0,1,n-1$, and n. The second is the general classification problem which we consider only in the range of stable forms. Clearly $p = 1$ and $n - 1$ are the interesting cases; in fact, it is surprising to us how rich in structure these two cases actually are. As the extremes $p = 0$ and n are easy to describe we dispense with them first.

(A) $p = 0$.

A 0-form is just a function $w : (\mathbf{R}^n,0) \longrightarrow \mathbf{R}$. There are two possible generic algebraic singularities; namely

$$(2.1) \qquad \text{(a)} \quad dw(0) \neq 0 \qquad \text{and} \qquad \text{(b)} \quad dw(0) = 0.$$

For (b) to occur generically w must have a Morse singularity at 0. The associated normal forms are well-known:

$$(2.2) \qquad \text{(a)} \quad w(x) = x_1 + c$$

$$\text{(b)} \quad w(x) = -(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + \cdots + x_n^2 + c\,.$$

Note that the value $w(0) = c$ is an invariant of pull-back thus giving a simple example of an analytic singularity. Only (a) yields a stable form as the critical value is an obstruction to stability in (b).

(B) $p = n$.

Again there are only two possible algebraic singularities given by

$$(2.3) \qquad \text{(a)} \quad w(0) \neq 0 \qquad \text{and} \qquad \text{(b)} \quad w(0) = 0.$$

The first case is just that of a volume form. If (b) holds generically then the singular set $\Sigma(w)$ is a hypersurface in $\mathbf{R}^n$ which may be assumed to be $x_1 = 0$. Normal forms for these cases are:

$$(2.4) \qquad \text{(a)} \quad w(x) = dx_1\wedge\ldots\wedge dx_n \quad \text{and} \quad \text{(b)} \quad w(x) = x_1 dx_1\wedge\ldots\wedge dx_n\,.$$

As a result of these normal forms both singularities are stable, thus providing an example - the only example - where stable p-forms give a complete generic classification of p-forms.

(C) $p = n-1$.

At this moment $(n-1)$-forms provide the most satisfactory example for the classification of stable forms as well as for the generic classification problem. There are n different examples of stable $(n-1)$-forms, none of which were known classically, as well as an almost complete generic classification for non-zero (n-1)-forms. We shall discuss, in order, the algebraic singularities, the stable forms, and the generic classification.

There are four first order algebraic singularities:

$$\text{(2.5)}\qquad \begin{array}{ll} \text{(a)} & w(0) = 0,\ dw(0) = 0 \\ \text{(b)} & w(0) \neq 0,\ dw(0) \neq 0 \\ \text{(c)} & w(0) = 0,\ dw(0) \neq 0 \\ \text{(d)} & w(0) \neq 0,\ dw(0) = 0 . \end{array}$$

The rich structure lies in case (d); for completeness we dispense with (a), (b), and (c) first. It is clear that (a) cannot occur generically as the associated submanifold Σ has codimension equal to $n+1$. For both (b) and (c) there is a uniquely defined vector field Y defined by

$$Y \lrcorner\, dw = w . \tag{2.6}$$

In (b) this vector field is non-zero so one may choose coordinates $x_1,\ldots,x_{n-1},y$ with $Y = \frac{\partial}{\partial y}$. It follows [13] that w has the normal form

$$w(x,y) = (1+y)dx_1 \wedge \ldots \wedge dx_{n-1} \tag{2.7}$$

and is thus stable. When (c) holds generically the vector field Y has an isolated zero at the origin. It is not hard to see that the eigenvalues of the linear part of Y at 0 are invariants of w. Thus we get at least n modal parameters, and see that forms satisfying (c) are not stable. The complete classification is equivalent to classifying the vector fields Y with divergence $\equiv 1$ (since $\mathcal{L}_Y\, dw = dw$) up to equivalence given by volume preserving diffeomorphisms.

We now concentrate on the last - and most interesting case - (d). First we define some higher order algebraic singularities. Note that $\operatorname{Ker} w$ is a well-defined line field as $w(0) \neq 0$. Choose Y to be a non-zero vector field in $\operatorname{Ker} w$ and define

$$\Sigma_k = \{w \text{ satisfying (d)} \,|\, \mathcal{L}_Y^i\, w(0) = 0 \ (i < k) \text{ and } \mathcal{L}_Y^k\, w(0) \neq 0\} \tag{2.8}$$

where $\mathcal{L}_Y^i$ indicates i Lie differentiations with respect to Y. Next choose coordinates $x_1,\ldots,x_{n-1},y$ on $\mathbb{R}^n$ so that in coordinates one has

$$w(x,y) = g(x,y)dx \tag{2.9}$$

where $dx = dx_1 \wedge \ldots \wedge dx_{n-1}$, and

$$\Sigma_k = \left\{ \frac{\partial^i g}{\partial y^i}(0) = 0 (1 \leq i \leq k) \text{ and } \frac{\partial^{k+1} g}{\partial y^{k+1}}(0) \neq 0 \right\} . \tag{2.10}$$

One now observes that for w to have a generic Σ_k-singularity at 0 implies that $k \leq n$ and that $g(x,y)$ is a universal unfolding of $g(0,y)$ in the sense of catastrophe theory. The universal unfolding theorem implies that w is equivalent to

$$w_\mu = \mu(x)f(x,y)dx \quad \text{with} \quad \mu(0) \neq 0 \tag{2.11}$$

where

$$f(x,y) = \pm y^{k+1} + x_{k-1}y^{k-1} + \cdots + x_1 y + 1 \tag{2.12}$$

and $\mu(x)$ is the determinant of the change of coordinates on the x-variables necessary to put g in the normal form f. Observe that if $k < n$, then a change of coordinates given by $\overline{x_{n-1}} = \int_0 \mu(x)dx_{n-1}$ can put (2.11) in the same form with $\mu(x) \equiv 1$. To summarize cases (b) and (d) we have

<u>Theorem</u> 2.13 [5]: <u>An</u> (n-1)-<u>form</u> w <u>is stable iff</u> w <u>has a generic</u> Σ_k <u>singularity for</u> $k < n$ <u>in which case</u> w <u>is equivalent to the following</u>:

$$(1 + x_1 y + \cdots + x_{k-1}y^{k-1} \pm y^{k+1})dx . \tag{2.14}$$

In order to complete the proof of Theorem 2.13 we must show that the case $k = n$ yields an unstable situation. In fact we shall show more:

<u>Theorem</u> 2.14: <u>Let</u> μ <u>and</u> ν <u>be analytic functions of</u> x. <u>Then</u> w_μ <u>is equivalent to</u> w_ν <u>iff</u> $\mu \equiv \nu$.

This theorem shows that at least for non-zero analytic (n-1)-forms one can give a complete generic description. In fact we can make some statements about the C^∞ classification which are slightly different. Define

$$(2.15) \qquad U = \{x \in \mathbb{R}^{n-1} \mid f_x(y) = \sigma \quad \text{has } n \text{ distinct real roots}\} .$$

<u>Theorem</u> 2.16: <u>Let</u> μ <u>and</u> ν <u>be</u> C^∞ <u>germs defined on</u> x. <u>If</u> w_μ <u>is equivalent to</u> w_ν <u>then</u> $\mu \equiv \nu$ <u>on</u> U.

Since U is an open set in $\mathbb{R}^{n-1}$ with 0 in $\overline{U}$ we see that (2.16) implies (2.14). Also Theorem 2.16 completes the proof of (2.13). Not only are the forms w_μ not stable when $k = n$ but this generic algebraic singularity type has an infinite dimensional moduli space. The note after Proposition 1.11 shows that for $2 \leq p \leq n-2$ this is true for any singularity type. Thus the example given here is the rule not the exception. However, for $2 \leq p \leq n-2$ the moduli have not been explicitly identified. Finally we observe:

<u>Proposition</u> 2.17: <u>For</u> $n = 2$ <u>and</u> 3 <u>one has that</u> w_μ <u>is equivalent to</u> w_ν <u>iff</u> $\mu = \nu$ <u>on</u> U.

Thus in these cases a complete description for the generic C^∞ singularities can also be given. We conjecture that this proposition is true for all n but have not yet been able to obtain a proof. We now give a proof of Theorem 2.16. First we need a lemma.

<u>Lemma</u> 2.18: <u>Let</u> $f(y)$ <u>be a polynomial such that</u>

(i) $f(y) = y^{n+1} + a_{n-1}y^{n-1} + \cdots + a_1 y + 1$,

(ii) $f'(y) = 0$ <u>has</u> n <u>distinct roots</u> $y_1, \ldots, y_n$,

(iii) $f(y_n) \neq 0$

(iv) $b_i = f(y_i)/f(y_n)$ <u>is given for</u> $1 \leq i \leq n-1$.

<u>Then</u> f <u>is uniquely determined</u>.

<u>Proof</u>: The assumptions imply that $f(y_i) = b_i c$ where $c = f(y_n)$ and $b_n = 1$. Observe that (i) implies that given the y_i's and b_i's one obtains a system of

$n \times n$ linear equations in the variables a_j and c as follows:

$$(2.19) \qquad \begin{pmatrix} -1 - y_1^{n+1} = -b_1 c + a_1 y_1 + \cdots + a_{n-1} y_1^{n-1} \\ \vdots \qquad\qquad \vdots \\ -1 - y_n^{n+1} = -b_n c + a_1 y_n + \cdots + a_{n-1} y_n^{n-1} \end{pmatrix}$$

The system (2.19) has a unique solution if

$$(2.20) \qquad D = \det \begin{pmatrix} -b_1 & y_1 & \cdots & y_1^{n-1} \\ & \vdots & & \\ -b_n & y_n & \cdots & y_n^{n-1} \end{pmatrix} \neq 0 .$$

But

$$(2.21) \qquad D = -\frac{1}{f(y_n)} \det \begin{pmatrix} f(y_1) & y_1 & \cdots & y_1^{n-1} \\ \vdots & & & \\ f(y_n) & y_n & \cdots & y_n^{n-1} \end{pmatrix} .$$

Thus by (i) we have

$$(2.22) \qquad -f(y_n)\, D = \det \begin{pmatrix} 1 & y_1 & \cdots & y_1^{n-1} \\ \vdots & & & \\ 1 & y_n & \cdots & y_n^{n-1} \end{pmatrix} + \det \begin{pmatrix} y_1^{n+1} & y_1 & \cdots & y_1^{n-1} \\ \vdots & & & \\ y_n^{n+1} & y_n & \cdots & y_n^{n-1} \end{pmatrix}$$

$$= V + (-1)^{n-1} y_1 \cdots y_n\, A$$

where V and A have the obvious meaning.

Observe that V is a Vandermondian determinant which is non-zero as long as the y_i's are distinct. Thus by (ii) and (iii) the lemma is proved if we can show that $A = 0$. Consider

$$(2.23) \qquad \det \begin{pmatrix} 1 & y_1 & \cdots & y_1^n \\ \vdots & & & \\ 1 & y_n & \cdots & y_n^n \\ 1 & t & \cdots & t^n \end{pmatrix} = \pm \prod_{i=1}^{n} (t - y_i) \prod_{j<i} (y_j - y_i)$$

$$= \pm\, f_y(t)\, D$$

since this matrix is also Vandermondian. Note that the coefficient of t^{n-1} on the left hand side of (2.23) is just $\pm A$ whereas the form of $f_y(t)$ given by differentiation of (i) shows that this coefficient must equal zero.

<u>Proof</u> <u>of</u> <u>Theorem</u> 2.16: Suppose that $\varphi : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ is a diffeomorphism such that $\varphi^* w_\mu = w_\nu$. Note that 0 must be preserved as this singularity type occurs at isolated points. Moreover φ must preserve Ker $w_\mu = \left\{\frac{\partial}{\partial y}\right\}$ so φ has the form

$$\varphi(x,y) = (X(x), Y(x,y)) \quad . \tag{2.24}$$

Moreover expanding the equation $\varphi^* w_\mu = w_\nu$ in coordinates yields the relation

$$f(X,Y) = B(x) f(x,y) \quad \text{with} \quad B(0) \neq 0 \ . \tag{2.25}$$

Define $S = \{f_y = 0\}$ and let $\pi(x,y) = x$ be the projection of $\mathbf{R}^n \longrightarrow \mathbf{R}^{n-1}$. By definition (2.15) there are exactly n points $(x, y_1(x)), \ldots, (x, y_n(x))$ in $\pi^{-1}(x) \cap S$ for x in U. Moreover, since $\pi|(S \cap \pi^{-1}(U))$ is a submersion we may assume that each $y_i : U \longrightarrow \mathbf{R}$ is smooth. Next define

$$Q(x) = (f(x,y_1(x))/f(x,y_n(x)), \ldots, f(x,y_{n-1}(x))/f(x,y_n(x))) \ . \tag{2.26}$$

Note that $Q : U \longrightarrow \mathbf{R}^{n-1}$ is well-defined as we may assume that $(x, y_n(x))$ is near $(0,0)$ and $f(0) = 1$.

Note that the set $\{dw_\mu = 0\}$ is just S so $\varphi : S \longrightarrow S$ is guaranteed. As a result, (2.25) shows that

$$Q(X(x)) = Q(x) \ . \tag{2.27}$$

Finally observe that the conclusion of Lemma 2.18 is equivalent to stating that Q is injective. Hence we have that $X(x) = x$. One now concludes that $Y(x,y) = y$ on $S \cap \pi^{-1}(U)$ by continuity. (Note that Ker w is oriented so that $\frac{\partial Y}{\partial y}(0,0) > 0$.) Thus $\varphi|S \cap \pi^{-1}(U) =$ identity and $\mu = \nu$ on U. Q.E.D.

The proof of Proposition 2.17 will be divided into two parts. First we conjugate w_μ to w_ν on S and then off S. Both halves rely on Moser's Method [16] for conjugating forms. As this method is both simple and useful we isolate it here.

Moser's Method 2.28: Suppose that one wants to conjugate two p-forms w and w'. Then let $w_t = w + t(w'-w)$. Assume that there is a diffeomorphism φ_t such that

$$\varphi_t^* w_t = w \quad \text{for} \quad 0 \leq t \leq 1 . \tag{2.29}$$

Then differentiation with respect to t indicated by $\cdot$ yields

$$\mathcal{L}_{V_t} w_t + \dot{w}_t = 0 \quad \text{for} \quad 0 \leq t \leq 1 \tag{2.30}$$

where $V_t = \dot{\varphi}_t$. Now note that the diffeomorphisms φ_t solving (2.29) can be found by integration if the linear problem

$$V_t \lrcorner\, dw_t + d(V_t \lrcorner\, w_t) = w - w' \tag{2.31}$$

can be solved for vector fields V_t. One technicality is that $V_t(0) = 0$ so that V_t can be integrated to $t = 1$ on a neighborhood of 0. This observation we shall call Moser's Method; it is most often applied when one of the two terms of the LHS of (2.31) can be assumed to be zero.

Lemma 2.32: *Suppose there is a diffeomorphism* $\varphi : (\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^n,0)$ *of the form* (2.24) *satisfying*:

(a) $\varphi(S) = S$

(b) $\varphi^* w_\mu = w_\nu$ on S

(c) $\varphi(0,y) = (0,y)$.

Then there is a diffeomorphism $\psi : (\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^n,0)$ *such that* $\psi^* w_\mu = w_\nu$.

Proof: The assumption that φ has the form (2.24) implies that

$$\varphi^* w_\mu = \gamma(x,y) f(x,y) dx \equiv w_\gamma \tag{2.33}$$

where

$$\gamma(x,y) = \mu(X)f(X,Y)\ \det(dX)/f\ . \tag{2.34}$$

Therefore

$$\gamma_y = \mu(X)\ \det(dX)(f_y(X,Y)fY_y - f_y f(X,Y))/f^2\ . \tag{2.35}$$

Since $\varphi : S \longrightarrow S$ by (a) we see that

$$f_y(X,Y) = \tau(x,y)f_y(x,y) \tag{2.36}$$

for some smooth function τ. Hence

$$\gamma_y = \sigma f_y \tag{2.37}$$

for some smooth function σ. Moreover we claim that

$$\sigma(0) = 0\ . \tag{2.38}$$

To evaluate $\sigma(0)$ is to compute γ_y/f_y at 0 by (2.37). This computation yields

$$\sigma(0) = \mu(0)\ \det(dX)_0(\tau(0)-1)\ . \tag{2.39}$$

Note that assumption (c) implies that $Y_y(0) = 1$. Next evaluate (2.36) at $x = 0$ and use (c) to obtain

$$f_y(0,y) = \tau(0,y)f_y(0,y)\ . \tag{2.40}$$

Thus $\tau(0) = 1$ and the claim is proved from (2.39).

We now show - using Moser's Method - that (2.37) and (2.38) suffice to prove the lemma. We wish to show that w_γ and w_ν are equivalent. Assume that the vector field V_t has the form

$$V_t = a(x,y,t)\frac{\partial}{\partial y}\ . \tag{2.41}$$

Then the linear problem (2.31) becomes in this case in coordinates

$$a(t\gamma_y f+(\nu+t(\gamma-\nu))f_y) = (\nu-\gamma)f\ . \tag{2.42}$$

Now substitute for γ_y using (2.37) to obtain

$$a = (\nu-\gamma)f/f_y(t\sigma f+\nu+t(\gamma-\nu)) \tag{2.43}$$

assuming the function on the RHS of (2.43) is actually C^∞. Note that (2.38) implies that the second factor in the denominator when evaluated at 0 is $\nu(0) \neq 0$. Here one must observe that (b) and (c) imply that $\gamma(0) = \mu(0)$ while $\mu \equiv \nu$ on U implies that $\gamma(0) = \nu(0)$. Thus to show that a is C^∞ we only need show that $(\gamma-\nu)/f_y$ is C^∞. Since $\gamma = \nu$ on S by (b), $\{f_y = 0\} = S$, and f_y is non-singular we see that a is smooth. Moreover $a(0,0,t) \equiv 0$. Hence this a solves the linear problem (2.31) and Moser's Method proves the lemma.

<u>Proof of Proposition</u> 2.17 <u>when</u> $n = 2$: The proof is immediate from Lemma 2.32. For $n = 2$ we have that $f(x,y) = y^3 + xy + 1$, $S = \{x = -3y^2\}$, and U is the set $x < 0$. Observe that

$$w_\mu|S = \mu(-3y^2)(1-2y^3)dx = \nu(-3y^2)(1-2y^3)dx = w_\nu|S \tag{2.44}$$

since $\mu \equiv \nu$ on U. Now apply the lemma.

The case $n = 3$ requires a special argument which we isolate in the following:

<u>Lemma</u> 2.45: <u>Let</u> $w = \mu(x,y)dxdy$ <u>and</u> $w' = \nu(x,y)dxdy$ <u>be</u> 2-<u>forms on</u> $\mathbb{R}^2$ <u>near</u> 0 <u>satisfying</u>:

(a) $\mu = \nu$ <u>for</u> $x \leq 0$,

(b) $\nu = g\mu$ <u>where</u> $g > 0$, <u>and</u>

(c) $\mu = 0$ <u>is contained in the half-plane</u> $x \leq 0$ <u>with</u> $d\mu \neq 0$ <u>along</u> $\mu = 0$.

<u>Then there is a diffeomorphism</u> $\varphi : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R}^2,0)$ <u>such that</u> $\varphi^* w = w'$ <u>and</u> $\varphi = id_{\mathbb{R}^2}$ <u>on</u> $x \leq 0$.

<u>Proof</u>: We again apply Moser's Method. This time assuming that V_t has the form

$$V_t = A(x,y,t) \frac{\partial}{\partial x}. \tag{2.46}$$

In coordinates the linearized problem (2.31) is

$$\frac{\partial}{\partial x}[A(\nu+t(\mu-\nu)] = \nu - \mu. \tag{2.47}$$

Integration yields

$$A = \int_0^x (\nu-\mu)dx/(\nu+t(\mu-\nu)) \tag{2.48}$$

as long as A is C^∞. Note that

$$\nu + t(\mu-\nu) = \mu(t+(1-t)g) \tag{2.49}$$

by (b) so that the denominator in (2.48) vanishes precisely when $\mu = 0$. As $\nu - \mu \equiv 0$ for $x \leq 0$ by (a) we see that (c) implies that A is C^∞. Also $A(0,0,t) \equiv 0$. This solves the linear problem (2.31) and the lemma is proved since $V_t \equiv 0$ for $x \leq 0$.

<u>Proof of Proposition</u> 2.17 <u>when</u> $n = 3$: For this case we have

$$f(x,y) = y^4 + x_2y^2 + x_1y + 1 \tag{2.50}$$

and

$$f_y(x,y) = 4y^3 + 2x_2y + x_1 . \tag{2.51}$$

The surface S and the set U are pictured in Figure 2.1. In particular U is the interior of the cusp curve $x_1^2/8 + x_2^3/27 = 0$.

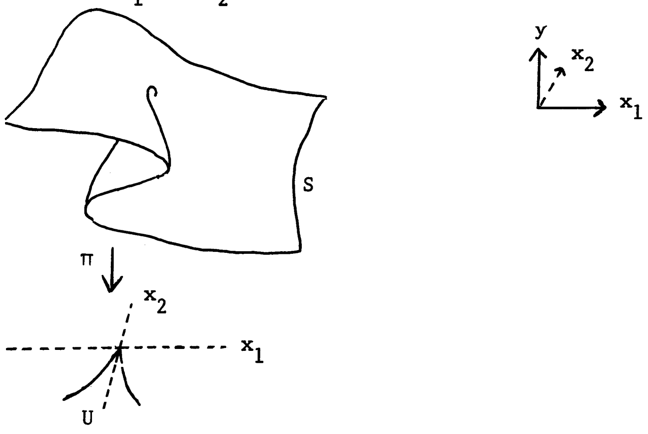

Figure 2.1

Lemma 2.32 implies that it is sufficient to conjugate w_μ to w_ν on S by a diffeomorphism which is the identity on the pleated part of the surface S. Note that one may parametrize S by x_2 and y as $x_1 = -2x_2y - 4y^3$. Thus w_μ pulled back to S has the form

$$(2.52) \qquad -\mu(-2x_2y-4y^3,x_2)(1-x_2y^2-3y^4)(2x_2+12y^2)dydx_2 \equiv \tilde{\mu}dydx_2 \;.$$

Similarly for w_ν. A computation shows that the pleated part of the surface S, i.e. $\pi^{-1}(U) \cap S$, is bounded by the parabola

$$(2.53) \qquad x_2 = -\frac{3}{2}y^2 \;.$$

The fact that $\mu \equiv \nu$ on U implies that $\tilde{\mu} \equiv \tilde{\nu}$ on the interior of the parabola (2.53). Also $\tilde{\mu} = 0$ is the parabola $x_2 = -6y^2$ and $d\tilde{\mu} \neq 0$ on $\tilde{\mu} = 0$. Finally $\tilde{\nu} = g\tilde{\mu}$ for $g > 0$ since $\tilde{\nu} = \tilde{\mu}$ on U. Hence Lemma 2.45 implies that w_μ is conjugate to w_ν on S by a diffeomorphism which is the identity on $\pi^{-1}(U) \cap S$. This is sufficient to show that φ may be extended to a diffeomorphism $\tilde{\varphi}$ on $\mathbb{R}^3$ of the form (2.24) with $\tilde{\varphi} = \mathrm{id}$ on $\pi^{-1}(U)$. So Lemma 2.32 is applicable and the proposition is proved.

Our discussion of non-zero (n-1)-forms is now complete.

(D) p = 1.

In [13] Martinet shows that there are at least two stable 1-forms on $\mathbb{R}^n$ for every n though the character of these forms depends on whether n is odd or even. We show in [5] that on $\mathbb{R}^3$ and $\mathbb{R}^4$ Martinet's examples are in fact the only ones; we also conjecture that this statement is true for all n thus giving a complete classification of all stable forms.

We first describe the stable 1-forms and then show that for the simplest non-stable singularity on $\mathbb{R}^4$ an infinite dimensional moduli space appears. As will become clear this moduli space - as constructed - has a different character from the moduli space for (n-1)-forms. It is also somewhat less satisfying than the example of (n-1)-forms in that we have not shown that we have the total space of invariants.

Let $n = 2k + 1$, then the stable singularities for a 1-form w are given by

(2.54) $$w \wedge (dw)^k(0) \neq 0$$

(2.55) $w \wedge (dw)^k(0) = 0$ generically with $S = \{w \wedge (dw)^k = 0\}$, $w(0) \neq 0$, and $i_S^*(dw)^k(0) \neq 0$ where $i_S : S \longrightarrow \mathbb{R}^{2k+1}$ is inclusion.

The singularity (2.54) is the classical contact form and is well-known to have the normal form

(2.56) $$w = dz + x_1 dy_1 + \cdots + x_k dy_k \ .$$

The normal form obtained by Martinet for (2.55) is

(2.57) $$w = \pm\, z dz + (1+x_1)dy_1 + x_2 dy_2 + \cdots + x_k dy_k \ .$$

When $n = 2k$ the stable singularities may be described as follows:

(2.58) (a) $w(0) \neq 0$ and $(dw)^k(0) \neq 0$

(b) $w(0) \neq 0$, $(dw)^k(0) = 0$, and $w \wedge (dw)^{k-1}(0) \neq 0$.

The associated normal forms are

(2.59) (a) $w = (1+x_1)dy_1 + x_2 dy_2 + \cdots + x_k dy_k$

(b) $w = (1 \pm x_1^2)dy_1 + x_2 dy_2 + \cdots + x_k dy_k$.

Now to describe the moduli space for 1-forms on $\mathbb{R}^4$ alluded to above. Given the stability results it is natural to assume that the following algebraic singularity is present:

(2.60) (i) $w(0) \neq 0$

(ii) $(dw)^2(0) = 0$

(iii) $w \wedge (dw)(0) = 0$.

As we assume that this singularity is generic we have

(2.61) $$S = \{dw^2 = 0\}$$

is a 3-dimensional submanifold of $\mathbb{R}^4$. This follows as $(dw)^2$ is a 4-form so that its zeros are given by a single function. Let $i_S : S \longrightarrow \mathbb{R}^4$ denote inclusion. Then define

$$(2.62) \qquad T = \{i_S^*(w \wedge dw) = 0\} .$$

Generically T is a 2-dimensional submanifold of S as $i_S^*(w \wedge dw)$ is a 3-form on S whose zeros are also given by a single function.

We can now make two further non-degeneracy assumptions:

(2.60) (iv) $i_T^* \, dw(0) \neq 0$

(v) $i_T^* \, w(0) \neq 0$.

Note that assumptions (iv) and (v) imply the existence of a non-zero vector field U on T satisfying

$$(2.63) \qquad U \lrcorner \, i_T^* \, dw = i_T^* \, w .$$

For future reference we let σ_t denote the one-parameter group generated by U on T.

We now state our theorem. Let $\mathcal{E}_1$ denote the space of germs of C^∞ functions from $\mathbb{R} \longrightarrow \mathbb{R}$ at 0. Then

<u>Theorem</u> 2.64: (a) <u>There is a map</u> $H_w : T \longrightarrow \mathcal{E}_1$ <u>such that if</u> $w' = \varphi^* w$ <u>for some diffeomorphism</u> $\varphi : (\mathbb{R}^4,0) \longrightarrow (\mathbb{R}^4,0)$ <u>then</u> $H_{w'} = H_w \circ (\varphi | \varphi^{-1}(T))$.

(b) <u>For any</u> w <u>satisfying</u> (2.60) w <u>may be perturbed to</u> w' <u>also satisfying</u> (2.60) <u>so that</u> $H_{w'}(0)$ <u>is an arbitrary germ near</u> $H_w(0)$.

Although the actual moduli space is not identified by this theorem it is shown to be infinite dimensional. Part (a) shows that the image of H_w is an at most 2-dimensional subspace of the infinite dimensional vector space $\mathcal{E}_1$ which is invariantly defined up to pull-back. On the other hand part (b) shows that this image may be changed in an infinite dimensional number of ways, by small perturbations of w.

We sketch the proof; the details are given in [5].

Lemma 2.65: *There exists a unique vector field* V *on* S *such that*

(a) $V \in \operatorname{Ker} dw$ *and*

(b) $V(i_S^* w(V)) = \pm 1$.

The definition of T (2.62) implies that on T, $\operatorname{Ker} i_S^* w \supset \operatorname{Ker} i_S^* dw$ so that $i_S^* w(V) = 0$ on T. Moreover (2.60) (iv) shows that $\operatorname{Ker} i_S^* dw$ is one-dimensional and transverse to T. The existence of a V satisfying (b) is essentially the statement that the zero of $i_S^* w(V)$ on T for $V(0) \neq 0$ is a simple one in the transverse direction to T. The $\pm$ sign depends on various orientations established by w and dw and only one sign obtains for a given w. We may think of Lemma 2.65 as giving a parametrization to the direction transverse to T in S.

Lemma 2.66: *Let* $\tilde{V}$ *be any smooth extension of* V *to* $\mathbb{R}^4$. *Then there is a vector field* Z *on* $\mathbb{R}^4$ *such that* (a) Z *in* $\operatorname{Ker} dw$ *on* S

(b) $Z \pitchfork S$

(c) $Z\ dw(Z,\tilde{V}) = 1$.

This lemma is similar to Lemma 2.65 as now we try to give a parametrization to a direction transverse to S in $\mathbb{R}^4$. Note that $dw(\cdot,\tilde{V}) \equiv 0$ on S. Moreover $\operatorname{Ker} dw$ is two-dimensional on S and transverse to S by (2.60) (ii) and (iv). Thus it is possible to choose a line field in $\operatorname{Ker} dw$ on S and transverse to S. Condition (c) just gives a way of parametrizing this line field. Clearly the choice of Z is not unique as it depends on the choice of line field. However, one has

Lemma 2.67: *Let* Z *and* Z' *be chosen to satisfy Lemma* 2.66. *Then* $Z - Z'$ *is a multiple of* V *on* S.

We now have all the information necessary to define our invariant.

Let $g : T \longrightarrow \mathbb{R}$ be given by

$$g = Z \lrcorner\, w \quad . \tag{2.68}$$

Note that Lemma 2.67 shows that g is well-defined on T as $V \lrcorner w = 0$ on T as noted above.

Now recall U defined by (2.63) and define

$$H_w(q) = g(\sigma_t(q)) . \tag{2.69}$$

More precisely $H_w(q)$ is the germ of $g(\sigma_t(q))$ in the real-variable t at $t = 0$. As everything was defined functorially Theorem 2.64 (a) holds.

To sketch part (b) of the Theorem let $\ell : \mathbf{R}^4 \longrightarrow \mathbf{R}$ be a function such that

$$\ell(S) \equiv 0 \quad \text{and} \quad d\ell(Z) \equiv 1 . \tag{2.70}$$

Let $\psi : \mathbf{R}^4 \longrightarrow \mathbf{R}$ be arbitrary and consider

$$w_\psi = w + d(\ell\psi) . \tag{2.71}$$

The following are obvious:

$$dw_\psi = dw, \ S_\psi = S, \ i_S^* w_\psi = i_S^* w, \ T_\psi = T . \tag{2.72}$$

Therefore w_ψ also satisfies (2.60) (i) - (v).

Next observe that

$$U_\psi = U, \ V_\psi = V, \quad \text{and} \quad Z_\psi = Z . \tag{2.73}$$

Therefore

$$g_\psi = Z \lrcorner w_\psi = Z \lrcorner w + \psi = g + \psi \tag{2.74}$$

and

$$H_{w_\psi}(q) = H_w(q) + \psi(\sigma_t(q)) . \tag{2.75}$$

As ψ was arbitrary on T, part (b) is proved.

§3. Singularities of closed forms.

Most of this section will be devoted to the discussion of stability of germs of closed 2-forms on $\mathbb{R}^4$. Before beginning this discussion we observe that the complexity of possible algebraic singularities for closed forms is greatly reduced; in particular, algebraic singularities are generated only by equations of the form

$$w^k = 0 . \tag{3.1}$$

As a result, it is conceivable to classify the algebraic singularities for closed forms. For closed 2-forms this program has been carried out by Pelletier and will be described later.

As indicated in Proposition 1.11 the search for stable closed forms restricts one to looking at p-forms where $p = 1,2,n-1$, and n. For the general classification problem one wants to look at more general p. Little is known in general, but there are some results by Turiel [23] for closed 3-forms on $\mathbb{R}^5$ which will be described at the end of this section.

In fact the only interesting case of stable closed forms is that of $p = 2$. For $p = n$ was covered in (2.4) as every n-form is automatically closed and stable closed 1-forms are described by Morse Theory. In particular, a closed 1-form $w = df$ is stable if either f is non-singular or f has a non-degenerate singularity at 0. The normal forms are

(3.2)

(a) $w(x) = dx_1$

(b) $w(x) = \sum_{i=1}^{n} \pm x_i dx_i$.

The discussion of stable (n-1)-forms is subsumed in the discussion of Darboux Theorem below.

We first describe some generalities about singularities of closed 2-forms which are taken mainly from [13].

Let w be in $\mathcal{B}^2$, then the rank of w at x is $2p$ if $w(x)^p \neq 0$ while $w(x)^{p+1} = 0$. The kernel of w at x is defined by

(3.3) $$\operatorname{Ker} w(x) = \{X \in T_x\mathbf{R}^n \mid X \lrcorner\, w(x) = 0\} .$$

The following are equivalent:

(3.4) (a) rank $w(0) = 2p$

(b) $\dim \operatorname{Ker} w(0) = n - 2p$

(c) $w(0) = dx_1 \wedge dy_1 + \cdots + dx_p \wedge dy_p$ for an appropriate basis of $T_0\mathbf{R}^n$.

It is clear that rank is an invariant of diffeomorphisms, so we have the following algebraic singularity.

(3.5) $$\Sigma_c = \{w \mid \operatorname{corank} w = c\} \subset J^0(\mathcal{B}^2)$$

where corank = n-rank. Martinet [13, p.107] has shown that Σ_c is a submanifold of $J^0(\mathcal{B}^2)$ of codimension $c(c-1)/2$. Moreover, the stratification of $J^0(\mathcal{B}^2)$ given by the various Σ_c is precisely the one given by the action of $\operatorname{Diff}^1_0(M)$ on $J^0(\mathcal{B}^2)$ and is thus coherent. So the set of closed 2-forms on M whose 0-jet is transverse to this stratification is both open and dense.

<u>Theorem</u> 3.6 (Darboux): <u>If</u> $n = 2p$ <u>and</u> $w(0) \in \Sigma_0$ <u>then there exist coordinates on a neighborhood of</u> 0 <u>such that</u>

(3.7) $$w = dx_1 \wedge dy_1 + \cdots + dx_p \wedge dy_p .$$

A very nice proof of Theorem 3.6 is given by Moser [16] using what we have entitled Moser's Method 2.28. This theorem has been generalized in several ways.

<u>Theorem</u> 3.8: (a) <u>If</u> $n = 2p + 1$ <u>and</u> $w(0) \in \Sigma_1$ <u>then there exist coordinates near</u> 0 <u>such that</u> (3.7) <u>holds.</u>

(b) <u>Suppose</u> $w \in \mathcal{B}^2$ <u>has constant rank</u> $2p$ <u>near</u> 0 <u>where</u> $2p < n$. <u>Then there exist coordinates near</u> 0 <u>such that</u> (3.7) <u>holds.</u>

(c) <u>Suppose</u> w <u>is a decomposable closed</u> ℓ-<u>form with</u> $\ell < n$. <u>Then there exist coordinates near</u> 0 <u>such that</u> $w = dx_1 \wedge \cdots \wedge dx_\ell$.

It is trivial to see that (a) follows from (b) and that (3.7) describes a stable closed 2-form when $p = 2n$ or $p = 2n+1$ as both Σ_0 and Σ_1 are open sets. Theorem 3.8 (b) and (c) have essentially the same proof; we sketch the proof of (b).

By assumption the plane field given by $\operatorname{Ker} w(x)$ has constant dimension $n - 2\ell$ and is integrable because $dw = 0$ and w^{ℓ} is decomposable at each point (using (3.4) (c)). Locally this foliation defines a fibration $\pi : \mathbf{R}^n \longrightarrow \mathbf{R}^{2\ell}$. It follows from $dw = 0$ that w is π-basic; i.e., there exists $\bar{w}$ in $\mathbf{R}^{2\ell}$ such that $w = \pi^*\bar{w}$. On $\mathbf{R}^{2\ell}$, $\bar{w}$ is closed with corank 0, thus Darboux Theorem yields the result.

<u>Corollary</u> 3.9: <u>Suppose</u> $w \in \mathcal{D}^{n-1}$ <u>and</u> $w(0) \neq 0$. <u>Then there exist coordinates near</u> 0 <u>such that</u>

$$w = dx_1 \wedge \cdots \wedge dx_{n-1} \, . \tag{3.10}$$

<u>As a result</u> w <u>is stable</u>.

Observe that (3.10) gives the only stable $(n-1)$-form. Genericity implies that if $w(0) = 0$ then zero is an isolated zero. Let Ω be a volume form, then $w = X \lrcorner \, \Omega$ for some vector field X. The ratios of the eigenvalues of the linear part of X at 0 are invariants of w (independent of Ω). These ratios may be changed by perturbing X by a volume preserving vector field so stability of w fails.

The question of stability for closed 2-forms is best understood when $n = 4$. We restrict our description of higher order singularities to that case. Observe from (3.5) that generically only Σ_0 and Σ_2 singularities may occur as $\operatorname{codim} \Sigma_4 = 6$. Moreover Σ_0 singularities are described by Darboux Theorem.

Generically

$$\Sigma_2(w) = \{x \in \mathbf{R}^4 \,|\, w(x)^2 = 0\} \tag{3.11}$$

is a submanifold of $\mathbf{R}^4$ of codimension 1. The higher order singularities are contained in Σ_2; as we shall see there are three possibilities.

First, note that at each point $x \in \Sigma_2(w)$ there is a well-defined 2-plane Ker $w(x)$, and either x is in

(3.12) $$\text{(a)} \quad \Sigma_{20}(w) = \{x \mid \operatorname{Ker} w(x) \pitchfork T_x \Sigma_2\}$$

$$\text{or} \quad \text{(b)} \quad \Sigma_{22}(w) = \{x \mid \operatorname{Ker} w(x) \subset T_x \Sigma_2\} \ .$$

These sets correspond to 1st order singularities. Note that $\Sigma_{20}(w)$ is open in $\Sigma_2(w)$ and that generically $\Sigma_{22}(w)$ is a submanifold of $\Sigma_2(w)$ of codimension 2; i.e. a curve. To see that, let N be the normal vector to $\Sigma_2(w)$ at 0 and observe that (b) holds iff $N \cdot \operatorname{Ker} w(x) = 0$ which yields two independent conditions. Also note that for $\Sigma_{22}(w)$ to occur generically is a 2-jet condition; we call such generic singularities Σ_{22} .

Proposition 3.13 [13, p.157]: *If w has a Σ_{20} singularity at 0 then there exist coordinates near 0 such that*

(3.14) $$w = x\,dx \wedge dy + dz \wedge dt \ .$$

Thus Σ_{20} singularities are stable.

We now show that there are several distinct types of Σ_{22} singularities. Let

(3.15) $$\text{(a)} \quad \Sigma_{220}(w) = \{x \in \Sigma_{22}(w) \mid \operatorname{Ker} w(x) \pitchfork T_x \Sigma_{22}(w)\}$$

$$\text{(b)} \quad \Sigma_{221}(w) = \{x \in \Sigma_{22}(w) \mid \operatorname{Ker} w(x) \supset T_x \Sigma_{22}(w)\} \ .$$

Note that $\Sigma_{220}(w)$ is an open set in $\Sigma_{22}(w)$ and thus Σ_{220} points are 2nd order singularities. Generically $\Sigma_{221}(w)$ is a submanifold of codimension 1 in $\Sigma_{22}(w)$ and thus occurs at isolated points. The generic Σ_{221} singularity is a 3rd order singularity.

Martinet has shown [13, p.127] that there is a residual set in $\mathcal{D}^2(\mathbb{R}^4)$ which at each point takes on one of the four generic singularity types described above: Σ_0, Σ_{20}, Σ_{220}, Σ_{221} of codimension 0, 1, 3, 4 respectively.

Theorem 3.16: Let $w \in \mathcal{D}^2(\mathbb{R}^4)$. Then w is stable (at 0) iff 0 is a Σ_0, Σ_{20}, or Σ_{220} singularity of w.

What remains to be shown are the stability of Σ_{220} and the instability of Σ_{221}. Specifically

Theorem 3.17 [22]: Let w have a Σ_{220} singularity at 0. Then there exist coordinates near 0 such that w is one of

(3.18)
$$\text{(a)}\quad w_e = (-zdz-tdt)\wedge\nu + dx\wedge\nu + xdz\wedge dt$$
$$\text{(b)}\quad w_h = (-zdz+tdt)\wedge\nu + dx\wedge\nu + xdz\wedge dt$$

where $\nu = dy + zdt$.

In (3.18) e and h stand for elliptic and hyperbolic respectively, the nomenclature being described below. In these coordinates $\Sigma_2(w) = \{x = 0\}$ and $\Sigma_{22}(w) = \{x = z = t = 0\}$. Letting $i : \Sigma_2 \longrightarrow \mathbb{R}^4$ be inclusion we see that the first two terms give $w|\Sigma_2(w)$ and the first term is i^*w.

Proposition 3.19 [6]: Let w have a Σ_{221} singularity at 0. Then there exists a path $w_s \in \mathcal{D}^2(\mathbb{R}^4)$ such that

(a) $w_0 = w$

(b) for each s, w_s has a Σ_{221} singularity at 0

(c) w_s is not equivalent to $w_{s'}$, if $s \neq s'$.

As an example, let

(3.20)
$$w_s = ((s-1)y-z+t)dt + (t-z)dz)\wedge\nu + dx\wedge\nu + xdz\wedge dt + R$$

where $\nu = dy + zdt$ and R is a closed 2-form whose coefficients are homogeneous polynomials of degree 3. The term R is necessary to guarantee that w_s has a generic Σ_{221} singularity, although the modal parameter s can be observed on the 2-jet level. As in (3.18) the first two terms of (3.20) gives $w|\Sigma_2(w)$ and the first term is i^*w.

We shall give a sketch of these two results, the details being found in the references. It is clear from the way (3.18) and (3.20) were given that there is a similarity between the results; this similarity stems from the following:

Lemma 3.21: Let $w \in \mathcal{D}^2(\mathbb{R}^4)$ have a generic Σ_{22} singularity. Then there exist 1-forms α and ν in $\mathcal{D}^1(\Sigma_2(w))$ such that

(i) $w|\Sigma_2 = \pi^* i^* w + dx \wedge \pi^* \nu$

where $\pi : \mathbb{R}^4 \longrightarrow \Sigma_2$ is a projection with $\pi|\Sigma_2 = id_{\Sigma_2}$ and $\Sigma_2 = \{x = 0\}$.

(ii) $\nu \wedge d\nu(0) \neq 0$ (ν is a contact form)

(iii) $i^* w = \alpha \wedge \nu$

(iv) $\alpha \wedge d\nu \equiv 0$.

Proof: [6, p. 220] We sketch the details here. The existence of ν satisfying (i) follows from the genericity of Σ_2. Let $\bar{w} = i^* w$. Since $w^2 \equiv 0$ on Σ_2 one finds that $\bar{w} \wedge \nu \equiv 0$. Since $\nu(0) \neq 0$ there exists an α satisfying (iii). Of course $\alpha + f\nu$ would do just as well; with that leeway one can choose α satisfying (iv) as well. Finally, $\nu \wedge d\nu(0) \neq 0$ follows from the facts that Σ_2 is generic, $d\bar{w} \equiv 0$, and $\bar{w}(0) = 0$, this last fact being equivalent to $0 \in \Sigma_{22}(w)$.

Remarks: (a) As ν is a contact form there exist coordinates on Σ_2 such that $\nu = dy + zdt$. The particular form of α may be read from (3.18) and (3.20).

(b) We can now define elliptic and hyperbolic Σ_{220} singularities. Let X be the vector field on Σ_2 given by $X \lrcorner\, \nu \wedge d\nu = i^* w$. Then Σ_{220} may be defined by $\{i^* w = 0\}$ or $\{X = 0\}$. Let L be the linear part of X at 0. Then one of the three eigenvalues of L must be zero as X is zero on the curve Σ_{22}. Moreover trace $L = 0$ as $\mathcal{L}_X \nu \wedge d\nu \equiv 0$. In the Σ_{220} case the other eigenvalues are non-zero. The Σ_{220} singularity is elliptic (hyperbolic) if these eigenvalues are both imaginary (real). For a Σ_{221} singularity all the eigenvalues of L are zero.

Of course the vector field X above depends on the choice of ν. However X is defined up to a non-zero function multiple independent of ν so that the nature of the eigenvalues of L is well-defined.

Assuming the normal form for ν one may assume that

(3.22) (a) $\Sigma_{22}(w) = \{z = t = 0\}$ for Σ_{220} singularities
and (b) $\Sigma_{22}(w) = \{y = 0,\ z = t\}$ for Σ_{221} singularities,

i.e. $\Sigma_{22}(w)$ may be put into the normal form (3.22) by a diffeomorphism preserving ν.

The vector field $\frac{\partial}{\partial y}$ is the characteristic vector field of ν; i.e. $\frac{\partial}{\partial y} \lrcorner\, \nu \equiv 1$ and $\frac{\partial}{\partial y} \lrcorner\, d\nu \equiv 0$. Condition (iv) of Lemma 3.21 implies that $\alpha = adt + bdz$ where $a,b : \Sigma_2 \longrightarrow \mathbf{R}$. Note that the vector field X is in $\operatorname{Ker} \alpha \cap \operatorname{Ker} \nu$. This vector field is crucial to Roussarie's proof of Theorem 3.17; it also gives one a way of seeing why that proof fails for Σ_{221} singularities. The following is the key result. Let w_t be a curve of closed 2-forms with Σ_{220} singularities at 0 and X_t be the associated vector fields. (We may of course assume - after an initial conjugacy - that $\Sigma_2(w_t)$, $\Sigma_{22}(w_t)$, and ν_t are independent of t.)

<u>Proposition</u> 3.23 [22]: <u>Let</u> $h_t : \Sigma_2 \longrightarrow \mathbf{R}$ <u>satisfy</u> $h_t = 0$ <u>on</u> $\Sigma_{22}(w)$. <u>Then the differential equation</u>

$$X_t\, g_t = h_t \tag{3.24}$$

<u>may be solved for</u> g_t.

The proof of Proposition 3.23 uses the facts that the eigenvalues of the linear part of X are non-zero and that X_t is in $\operatorname{Ker} \nu$, as well as much of the machinery set up by Roussarie in [22] to solve such O.D.E.'s. The reduction of the stability question to Proposition 3.23 was first observed by Martinet [13, p.163] to show that Σ_{220} singularities are formally infinitesimally stable. We now outline the reduction process, leaving the details of the proof of Proposition 3.23 to [22].

The reduction is obtained in three steps. The general idea is to use Moser's Method (2.28). Let w' be near w. We may assume that the following data are the same for both w and w': $\Sigma_2(w)$, $\Sigma_{22}(w)$, $w(0)$, and ν. Let $w_t = w + t(w'-w)$.

By Moser's Method it is sufficient to solve

$$\alpha \wedge d(V_t \lrcorner w_t) = -\dot{w}_t \tag{3.25}$$

for V_t. Since w_t is exact in a neighborhood of 0, it suffices to find V_t and $f_t : \mathbb{R}^4 \longrightarrow \mathbb{R}$ such that

$$V_t \lrcorner w_t = f_t + \dot{\alpha}_t \tag{3.26}$$

where $\dot{\alpha}_t + f_t$ is the general antiderivative of $-\dot{w}_t$.

<u>Lemma</u> 3.27 [22, Lemma 19, p. 166]: <u>To solve</u> (3.26) <u>smoothly, it suffices to find</u> f_t <u>and a pointwise solution</u> V_t <u>to</u> (3.26).

The proof of Lemma 3.27 uses the genericity of Σ_2 and the fact that $w|\Sigma_2$ has constant rank. This pointwise condition is automatically satisfied off Σ_2 as w_t is symplectic there.

<u>Lemma</u> 3.28 [22, Lemma 21, p. 170 and Lemma 20, p. 168]: <u>To solve</u> (3.26), <u>it suffices to solve</u> (3.26) <u>for the pull-back of</u> w <u>to</u> Σ_2. <u>In particular, one must find a</u> V_t <u>at each point on</u> $\Sigma_2(w)$ <u>and</u> $g_t : \Sigma_2(w) \longrightarrow \mathbb{R}$ <u>solving</u>

$$V_t \lrcorner i^* w_t = i^*(\dot{\alpha}_t) + dg_t \; . \tag{3.29}$$

The proof of Lemma 3.28 uses the genericity of Σ_2 and Σ_{22}. A necessary condition for solving (3.29) is obtained by contracting (3.29) with X_t yielding

$$X_t \lrcorner (i^*(\dot{\alpha}_t)+dg_t) \equiv 0 \quad \text{on} \quad \Sigma_2(w) \; . \tag{3.30}$$

<u>Lemma</u> 3.31 [22, Lemma 20, p. 168]: <u>If a function</u> g_t <u>exists satisfying</u> (3.30) <u>then</u> (3.26) <u>may be solved.</u>

Here again the genericity of Σ_{22} is crucial. Setting $h_t = -X_t \lrcorner i^*(\dot{\alpha}_t)$ shows - through (3.30) - that Proposition 3.23 is sufficient to prove the stability result for Σ_{220} singularities.

We now outline the proof that there always exists at least 1 modal parameter for Σ_{221} singularities. Let

(3.32) $$S_w = \{\rho \in D^1(\Sigma_2(w)) \mid \rho \wedge i^* w \equiv 0\} .$$

Using the genericity of Σ_{22} one can show that S_w is generated by α and ν as a module over C^∞ functions on $\Sigma_2(w)$. Let w' be near w; we look for obstructions to the existence of a diffeomorphism $\Phi : \mathbf{R}^4 \longrightarrow \mathbf{R}^4$ such that $\Phi^* w' = w$. It is clear that Φ preserves $\Sigma_2(w)$, $\Sigma_{22}(w)$, and $\Sigma_{221}(w) = \{0\}$. Moreover

(3.33) $$\Phi^* \nu = f\alpha + g\nu$$

for some functions f and g since pulling back $i^*(w') \wedge \nu = 0$ by Φ shows that $\Phi^* \nu \in S_w$.

Let $D\Phi(0)$ denote the Jacobean of Φ at 0.

<u>Lemma</u> 3.34 [6, p. 223]: <u>Assume</u> $\Phi^* \nu = f\alpha + g\nu$ <u>and that</u> Φ <u>preserves</u> $\Sigma_{22}(w)$. <u>Then, in an appropriate basis,</u>

(3.35) $$D\Phi(0) = \begin{pmatrix} \lambda^2 & * & * \\ 0 & \lambda & * \\ 0 & 0 & \lambda \end{pmatrix} .$$

Let $\alpha = a\,dt + b\,dz$. Then using (3.22) (b) write

(3.36) $$\begin{aligned} a &= a_1 y + a_2(z-t) + \cdots \\ b &= b_1 y + b_2(z-t) + \cdots \end{aligned} .$$

Note that $b_2 = -a_2$ as $d(\alpha \wedge \nu) = 0$. Similarly we may write $\alpha' = a'dt + b'dz$. Then

<u>Lemma</u> 3.37 [6, p. 223]: <u>If</u> $\Phi^* w' = w$, <u>then</u>

$$\frac{(a_1+b_1)^4}{a_2^5} = \frac{(a_1'+b_1')^4}{a_2'^5} .$$

The proof follows from the restriction on $D\Phi(0)$ given in Lemma 3.34. In general letting $w_s = w + s y dy \wedge dt$ varies the invariant described in Lemma 3.37.

Remark: If one tries to use the Moser Method for Σ_{221} singularities one has to solve an equation of the type $X_s(g_s) = h_s$ where, in the example (3.20)

$$X = (-y-z+t)\frac{\partial}{\partial z} + (t-z)\frac{\partial}{\partial t} + (z^2-zt)\frac{\partial}{\partial y}\ . \tag{3.38}$$

For this X that equation is not even formally solvable when $h = yt^2$.

For higher order singularities of closed 2-forms on $\mathbf{R}^n$, $n > 4$ there are some results. For Σ_{220} singularities Roussarie [22] has shown that in most cases they are 3-determined.

Pelletier [20] has generalized the type of higher order singularity discussed here. We will briefly indicate his results.

These singularities define a stratification where each strata is indexed by a finite triangular array of subscripts $\{a_{ij}\}$, $i \leq j$. The entries $\{a_{ii}\}$ on the diagonal define a nested family of submanifolds $\Sigma_{a_{11}}(w_1) \supset \Sigma_{a_{22}}(w_2) \supset \dots \supset \Sigma_{a_{kk}}(w_k)$, where w_{i+1} denotes the pull-back of w to $\Sigma_{a_{ii}}(w_i)$, $w_1 = w$, and as above $\Sigma_{a_{ii}}(w_i)$ denotes the points where w_i has corank a_{ii}. A jet of a closed 2-form w belongs to $\Sigma_{\{a_{ij}\}}$ provided $\dim \operatorname{Ker} w_i \cap T\Sigma_{a_{j-1,j-1}} = a_{ij}$, where $\Sigma_{a_{00}} = R^n$. For example, the singularity Σ_{220} above would be denoted $\Sigma_{\begin{smallmatrix}220\\31\\1\end{smallmatrix}}$.

Theorem 3.39 [20]: The sets $\Sigma_{\{a_{ij}\}}$ define a stratification of the jets of closed 2-forms for which the codimension of each strata is independent of the dimension of the manifold $\mathbf{R}^n$. A similar stratification for jets of all 2-forms is also given in [20].

We have seen that there are no stable closed p-forms for $3 \leq p \leq n-2$. There are simplifying assumptions that allow some classifications in this range. For example, one can consider those p-forms for which the germ (jet) of w at a point x is equivalent to the germ (jet) of w at all points in some neighborhood of x. This situation has been studied by Turiel [23] in the case of 3-forms on $\mathbf{R}^5$. We will briefly describe one of his results.

At a point x, a 3-form w can be written in one of three ways:

(3.40) (a) $w(x) = 0$

(b) $w(x) = dx_1 \wedge dx_2 \wedge dx_3$

(c) $w(x) = dt \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$

since any element of $\Lambda^3(\mathbf{R}^5)$ is factorable. The case where w is decomposable at each point is covered by Darboux Theorem 3.6. We restrict to the case where (3.40) (c) holds for each x.

Let $S_w = \{\rho \in D^1 | \rho \wedge w = 0\}$. S_w is generated as a module over the functions on $\mathbf{R}^5$ by a single one-form ρ, $\rho \neq 0$.

Theorem 3.41 [23]: (a) If there is a $\rho \in S_w$ such that $d\rho = 0$ then in a neighborhood of 0 there are coordinates in which $w = dt \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$.

(b) If there is a $\rho \in S_w$ such that $\rho d\rho \neq 0$, $\rho(d\rho)^2 = 0$ then in a neighborhood of 0, $w = (dt + x_2 dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$.

There are other normal forms for 3-forms on $\mathbf{R}^5$ in [23] using the assumption that w is left invariant by a transitive pseudogroup of local diffeomorphisms. Some of these models depend on parameters.

§4. Global stability.

Until this point we have only discussed the stability of germs of p-forms. One can also ask whether there are any globally stable p-forms on an n-manifold M. As we shall see, there are several possible questions the most naive of which follows from

Definition 4.1: A form $w \in D^p(M)$ is (globally) stable if for every w' near w there is a diffeomorphism φ on M near the identity such that $\varphi^* w' = w$.

This is one stituation where the appellage "naive" can actually be proved as

Proposition 4.2 [4]: There are no globally stable forms on a compact manifold.

Proof: The local theory implies that $p = 0,1,n-1$, and n are the only possibilities. If w is in $D^1(M^{2k+1})$ then $\int_M |w \wedge dw^k|$ is invariant under diffeomorphisms of M and can easily be perturbed. There are similar invariants in the other cases.

The corresponding definition of stability of closed forms is of more interest with one added hypothesis. The de Rham cohomology class $[w] \in H^p(M)$ is clearly an invariant of w. We ask whether closed p-forms can be stable with respect to perturbations by exact forms. Again the local theory demands that we restrict our attention to $p = 1,2,n-1$, and n. Here there are positive results.

<u>Theorem</u> 4.3 [16]: <u>Let</u> M^n <u>be a compact manifold. The following forms are stable in their cohomology classes</u>:

(a) <u>volume forms</u>

(b) <u>symplectic forms</u>

(c) <u>nowhere vanishing</u> 1-<u>forms</u>.

The proof is given by (2.28). Part (c) is not contained in [16], but its proof is similar [8]. Part (a) may be strengthened since the set of volume forms in the same cohomology class is connected. Thus for volume forms w and w' we see that w' is equivalent to w iff $[w] = [w']$. It is an open question whether the symplectic forms in a given cohomology class are all equivalent. It is not true, in general, that all non-zero closed 1-forms in the same cohomology class are equivalent [9]. However for compact 3-manifolds M if $[w] = [w']$ where w and w' are nowhere zero then w' is equivalent to w by an isotopy of M^3 [10].

The closest situation to symplectic on odd dimensional manifolds is a form $w \in \mathcal{D}^2(M^{2k+1})$ for which $w^k \neq 0$. Such w have Σ_1 singularities at each point which implies that w has a stable germ at each point. Unlike symplectic forms, these forms are not in general stable in their cohomology class as can be seen from the following example.

Let (x,y,z) be the standard coordinates on $T^2 \times \mathbf{R}$, where T^2 is the 2-dimensional torus. Let φ be a diffeomorphism of T^2 near the identity which preserves the closed 2-form $dx \wedge dy$ and has support in a small neighborhood of some point. The diffeomorphism of $T^2 \times \mathbf{R}$ given by $(x,y,z) \longrightarrow (\varphi(x,y), z+1)$ leaves $dx \wedge dy$ invariant and induces a closed 2-form on the quotient manifold T^3. By varying φ one can vary the kernel field of the induced 2-form on T^3. These forms are all cohomologous but the orbit structure of the kernel fields can differ. Therefore any of these 2-forms on T^3 will not be stable because there are inequivalent forms arbitrarily nearby.

This example also explains why there is no version of Theorem 4.3 for closed n-1 forms on M^n even though there are stable germs of closed n-1 forms.

One can extend Theorem 4.3 (a) and (c) to the case of forms with zeros. For $\mathcal{D}^n(M^n)$ and $\mathcal{D}^1(M^n)$ there is a generic set of forms which have stable germs at each point. For n-forms the models are given by (2.4) and for closed 1-forms the models are given by (3.2). It is clear that for a generic $w \in \mathcal{D}^1(M)$ there is a normal form in the neighborhood of the set $\Sigma = \{x \in M^n | w(x) = 0\}$ since Σ consists of a finite number of points. Therefore w is stable in a neighborhood of Σ. The same result is true for generic n-forms. For if $w \in \mathcal{D}^n(M^n)$, then there is a function $f : M \longrightarrow \mathbf{R}$, such that $\Sigma = \{f = 0\}$, $df \neq 0$ along Σ. Furthermore, near Σ $w = f \wedge df \wedge \pi^* \eta$, where η is a non-zero (n-1)-form on Σ and π is a given projection of the neighborhood of Σ onto Σ. The form η may be chosen in advance up to sign (Σ is oriented since it can be thought of as the boundary of an oriented manifold). Therefore w is stable in a neighborhood of Σ.

Using a proof similar to that of Theorem 4.3, one obtains

<u>Proposition</u> 4.4: <u>Let</u> M^n <u>be compact. Let</u> w <u>and</u> w' <u>be nearby generic forms in either</u> $\mathcal{D}^n(M^n)$ <u>or</u> $\mathcal{D}^1(M^n)$ <u>and suppose that</u> $[w-w'] = 0$ <u>in</u> $H^n_c(M-\Sigma)$ <u>where</u> H^n_c <u>denotes de Rham cohomology with compact supports. Then</u> w <u>and</u> w' <u>are equivalent.</u>

<u>Notes</u>: (1) This cohomology condition is sensible as w is stable along Σ so one may assume that $w = w'$ near Σ.

(2) The corresponding generalization for closed 2-forms will not work as $\Sigma = \{x \in M^{2k} | w^k(x) = 0\}$ is odd dimensional. In general the pull-back of w to Σ is not stable as seen by the example after Theorem 4.3.

§5. Integrable 1-forms.

As we have seen in §2 there is not yet a satisfactory classification for germs of 1-forms. In the case of integrable 1-forms there is to some extent a classification. A 1-form w is said to be integrable if $w \wedge dw = 0$. Let w be a germ of an integrable 1-form at 0.

Definition 5.1. w has a first integral if there exist real-valued functions f, g, $g(0) \neq 0$ such that $w = gdf$.

If w has a first integral, then the foliation defined by $\operatorname{Ker} w$ is given by the level surfaces of f. Therefore those 1-forms with first integrals can be understood by using the classification of function germs under right-left equivalence as in [14], [3]. Using [17] one shows that 1-forms gdf and $\bar{g}d\bar{f}$ are equivalent as foliations if and only if f and $\bar{f}$ are right-left equivalent as function germs.

Of course, integrable 1-forms with first integrals exclude those integrable 1-forms whose corresponding foliation (with singularities) has holonomy. Suppose $w(0) = 0$, $w = \sum_{i=1}^{n} a_i dx_i$. If the ideal generated by the $\{a_i\}$ in the ring of functions has finite codimension one says that w has an algebraically isolated zero.

Conjecture (Moussu): If w is a C^∞ integrable 1-form which has an algebraically isolated zero, then w has a C^∞ first integral if the corresponding foliation has no holonomy.

What is known is the following:

Theorem 5.2 [17]: *If w is the germ of an integrable 1-form on $\mathbb{R}^n$, $n \geq 3$ then w possesses a formal first integral. That is, there are functions f, g, $g(0) \neq 0$ such that $j^\infty w = j^\infty(gdf)$.*

In the case that w is analytic, one has

Theorem 5.3 [12]: *If w is an analytic 1-form, $n \geq 3$, with an algebraically isolated zero, then there exist analytic functions f, g, $g(0) \neq 0$ such that $w = g\,df$.*

There are also some results on the existence of first integrals in the C^∞ case. If $w(0) = 0$, $w = \sum_{i=1}^{n} \alpha_i dx_i$ one says that w has a *non-degenerate zero* if the matrix $\left(\frac{\partial \alpha_i}{\partial x_j}(0)\right)$ has rank n. In this case w has an algebraically isolated zero and by Theorem 5.1 one sees that $\left(\frac{\partial \alpha_i}{\partial x_j}(0)\right)$ is a constant times the hessian of a function. Let $i(w)$ be the index of this matrix.

Theorem 5.4 [18]: *If w is a C^∞ integrable 1-form with non-degenerate zero, then w has a C^∞ first integral in each of the two cases:*

(a) $i(w) \neq 2, n-2$

(b) $i(w)$ *is equal to* 2 *or* $n-2$ *and the germ of the foliation does not have any holonomy.*

Note: If $i(w) \neq 2, n-2$ the foliation cannot have holonomy and if $i(w)$ equals 2 or $n-2$ there are examples which do have holonomy.

If $w(0) = 0$ and w has a first integral, then $dw(0) = 0$. When $dw(0) \neq 0$ there is also a fairly good way to classify integrable 1-forms.

Theorem 5.5 [7]: *If w is an integrable 1-form on $\mathbf{R}^n$ with $dw(0) \neq 0$, then there is a fibration $\pi : \mathbf{R}^n \longrightarrow \mathbf{R}^2$ such that w is π-basic. That is, there is a 1-form $\bar{w}$ on $\mathbf{R}^2$ such that $w = \pi^* \bar{w}$.*

There is a unique vector field X on $\mathbf{R}^2$ satisfying $\bar{w} = X \lrcorner\, d\bar{w}$. Therefore the study of integrable 1-forms w satisfying $dw(0) \neq 0$ reduces to the study of this type of vector field (see the discussion related to 2.5, part c).

The proof of Theorem 5.5 is based on Theorem 3.8 (b) and (c). Let $\Sigma = \{w = 0\}$. Off of Σ dw is decomposable since the integrability condition implies dw is a multiple of w. For a non-zero form dw, decomposability off Σ implies decomposability on Σ because $\dim \operatorname{Ker} dw \leq n-2$ and decomposability is equivalent to equality. Therefore, by Theorem 3.8, dw is π-basic for some fibration $\pi : \mathbf{R}^n \longrightarrow \mathbf{R}^2$. One checks that the fibers of π are contained in Ker w and that w is π-basic as well.

The situation is more complex when $dw(0) = 0$ and w does not have a first integral. This case has been studied by Camacho and Lins Neto in [1], [2], [11]. Their point of view is that of structural stability. Since an integrable 1-form defines a codimension one foliation off the set $\Sigma = \{w = 0\}$, one can make the following definitions.

Definition 5.6: Integrable 1-forms w, w' are said to be topologically equivalent if there is a homeomorphism which sends $\Sigma(w)$ onto $\Sigma(w')$ and which sends the leaves of the w foliation onto the leaves of the w' foliation.

Definition 5.7: A C^∞ integrable 1-form w is C^r structurally stable if there is a neighborhood of w in the C^r topology on integrable 1-forms which consists of forms topologically equivalent to w.

There are similar definitions for germs of integrable forms.

If f is a Morse function of index unequal to 2 or n-2, then df is structurally stable. In fact, if w satisfies the hypothesis of Theorme 5.4 (a) then w is C^1 structurally stable [15].

The differentials of Morse functions of index 2, n-2 are not structurally stable as integrable 1-forms since arbitrarily small perturbations of them exist which have holonomy. Among the set of integrable 1-forms with first integrals those with a non-degenerate zero are dense; one sees that if w has a first integral and is structurally stable then $w = gdf$ where f is a Morse function.

There are, however, structurally stable integrable 1-forms which do not have first integrals. In [11], a class of C^2 structurally stable 1-forms were found which had vanishing 1-jets. These examples of C^2 structurally stable integral 1-forms which are not C^1 structurally stable contrast with the situation for vector fields where structural stability implies a hyperbolic 1-jet. In [2], one finds classes of C^{2r} structurally stable integrable 1-forms which are not C^{r-1} structurally stable for $r > 1$. These examples have foliations which arise as the orbit foliation of Lie group actions.

For p-forms with $p > 1$ it is not clear what the best definition of integrable should be. One possibility is that off of $\Sigma = \{w = 0\}$, w should be a decomposable p-form. Using this definition, Medeiros [15] showed that Theorem 5.5 could be generalized to p-forms for $p > 1$.

Theorem 5.8: If w is the germ of an integrable p-form with $dw(0) \neq 0$ then there exists a fibration $\pi : \mathbf{R}^n \longrightarrow \mathbf{R}^{p+1}$ such that w is π-basic.

This reduces the study of these p-forms on $\mathbf{R}^n$ to that of the type of vector field on $\mathbf{R}^{p+1}$ discussed after Theorem 2.5 (c).

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