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# A REMARK ON VECTOR FIELDS ON OPEN MANIFOLDS

by

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Summary . A partial answer is given to the question in what circumstances for a vector field X on an open manifold M there exists a neighborhood U in $C^0$ – Whitney topology , such that for every $Y \in U$ and compact $K \subset M$ the closure of positive semitrajectory $\overline{\bigcup_{t \geq 0} Y_t(K)}$ is compact .

Let M be an open differentiable manifold / i.e. non compact without boundary , with a countable basis / and let $X^1(M)$ be the class of all $C^1$ vector fields on M endowed with $C^0$-Whitney topology which is given by the neighborhoods of zero

$$\left\{ X \in X^1(M) \quad : \quad \|X(p)\| \leq \varepsilon(p) \right\}$$

where $\varepsilon$ is a real positive continuous function on M and $\| \ \|$ is a Riemannian complete metric on M .

Throughout the paper X will denote a complete $C^1$ vector field on M and $\{X_t\}_{t \in R}$ will be the corresponding flow generated by X . The positive semitrajectory of a point $p \in M$ will be denoted by $O_X(p) = \{X_t(p) : t \geq 0\}$, its $\omega$ – limit set by $\omega^X(p)$ and $\alpha$ – limit set by $\alpha^X(p)$.

Most of the notations and definitions used here are as in [1], [3]

DEFINITION 1. We say that $X \in \mathcal{X}^1(M)$ is C-stable iff there exists a neighborhood U of X such that for every $Y \in U$ and every compact $K \subset M$ the closure of semitrajectory of K $\overline{\bigcup_{t \geq 0} Y_t(K)}$ is compact.

EXAMPLES. Let $M = R^2$ and let flow $\{X_t\}_{t \in R}$ be given by

$$\frac{d}{dt}x^1 = x^2 \qquad \frac{d}{dt}x^2 = -x^1$$

The vector field $(x^2, -x^1)$ is not C-stable. It is easy to see that the vector field on $R^2$ given by

$$\frac{d}{dt}x^1 = -x^1 \qquad \frac{d}{dt}x^2 = -x^2$$

is C-stable.

We ask : Which conditions imply C-stability of X ?

In this paper we give a partial answer to this question. Proofs and ideas we use in this paper are similar to those used in studying the phenomenon of $\Omega$-explosion / see [2] /.

DEFINITION 2. A compact invariant subset A of M is called an attractor iff there exists a neighborhood V of A such that for every $p \in V$ its $\omega$-limit set $\omega^X(p)$ is contained in A. The domain of attraction of A is a maximal subset D of M such that for every $p \in D$, $\omega^X(p) \subset A$.

DEFINITION 3. Let A be an attractor. We say that A is uniformly asymptotically stable if for every neighborhood U of A there exists neighborhood V of A such that $V \subset U$ and $X_t(V) \subset V$ for every $t \geq 0$.

Wilson [4] proved that if A is uniformly asymptotically stable and D is a domain of attraction of A then there exists a smooth function $L : D \longrightarrow R^+$ such that :

I. $L|_A = 0$

II. For any $c \in R^+$ $\mathrm{dist}(L^{-1}(c), A) < +\infty$ and

$\lim_{c \to 0} \mathrm{dist}(L^{-1}(c), A) = 0$ here dist denotes the Hausdorff distance .

III. $\lim_{p_n \to \infty} L(p_n) = +\infty$ and $\lim_{p_n \to \partial D} L(p_n) = +\infty$

and $p_n \to \infty$ means that $\mathrm{dist}(p_n, A)$ tends to $+\infty$

IV. If $p \in D \setminus A$ then $\frac{d}{dt} L(X_t(p)) < 0$

The function L is called Lyapunov function .

DEFINITION 4 . A filtration for X is a collection

$$\{M_i \ : i = 1,2,\ldots\}$$

of submanifolds of M / with boundaries / such that for every i

1. $M_i$ is compact and $M_i \subset \mathrm{Int}\, M_{i+1}$
2. $X_t(M_i) \subset M_i$ for every $t \geqslant 0$
3. X is transverse to the boundary $\partial M_i$ of $M_i$
4. $\bigcup_{i \in N} M_i = M$

It is clear that if a filtration for X exists then for every compact $K \subset M$ $\overline{\bigcup_{t \geqslant 0} X_t(K)}$ is compact.

The following property which follows immediately from Definition 4 shows that the existence of filtration is an open property .

LEMMA 1 . If $\{M_i : i = 1,2,\ldots\}$ is a filtration for X then there is a $C^0$ – neighborhood U of X such that $\{M_i : i = 1,2,\ldots\}$ is also a filtration for every $Y \in U$ .

In virtue of this Lemma the existence of a filtration implies C-stability of X .

LEMMA 2. Let $X \in \mathfrak{X}^1(M)$, $\overline{\bigcup_{t \geq 0} X_t(K)}$ be compact for every compact $K \subset M$ and let A be an attractor with domain of attraction M. Then there is an uniformly assymptotically stable set B such that $A \subset B$ .

PROOF. We present in detail an argument from [2] .

Let $B = \overline{\{p \in M \ : \ \alpha^X(p) \cap A \neq \emptyset\}}$. By definition, B is invariant and closed. Let W be the compact neighborhood of A. If $p \in M$ and $\alpha^X(p) \cap A \neq \emptyset$ then there exists $t < 0$ such that $X_t(p) \in W$ hence $p \in \bigcup_{t \geq 0} X_t(W)$ so the set B is compact / being a closed subset of compact $\overline{\bigcup_{t \geq 0} X_t(W)}$ / . It is clear that B is an attractor. We show that B is uniformly asymptotically stable . Let U be the compact neighborhood of A. By definition of the set B, all points of $U \setminus B$ have their $\alpha$-limit sets empty. For every positive real number r denote $A_r = \bigcap_{0 \leq t \leq r} X_t(U)$. We note that the compact sets $A_r$ are nested . We show that for sufficiently large r , $X_t(A_r) \subset \operatorname{Int} A_1$ for $0 \leq t \leq 1$ . Consider the sets

$$V_r = \bigcup_{0 \leq t \leq 1} X_t(A_r) \setminus \operatorname{Int} A_1$$

which are a nested family of compact sets with empty intersection hence there exists r such that $V_r = \emptyset$. For such r and $0 \leq T \leq 1$ $X_T(A_r) \subset A_r$ . This implies that $X_T(A_r) \subset A_r$ for every $T \geq 0$ . We put $\operatorname{Int} A_r = V$ . Then $B \subset V \subset U$ and $X_t(V) \subset V$ for every $t \geq 0$ .

THEOREM 1 . Let $X \in \mathfrak{X}^1(M)$ and let for every compact $K \subset M$ the closure of positive semitrajectory $\overline{\bigcup_{t \geqslant 0} X_t(K)}$ and the set $F = \bigcup_{p \in M} \omega^X(p)$ be compact. Then the vector field X is C-stable .

PROOF. F is an attractor with domain of attraction M . By Lemma 2 there is an uniformly asymptotically stable set B with domain of attraction M. Therefore [4] there exists a smooth Lyapunov function L for B. We define $M_1 = L^{-1}([0 , 1])$. $M_1$ is a compact / being a closed , bounded subset of Riemannian manifold with complete metric / submanifold and $X_t(M_1) \subset M_1$ for every $t \geqslant 0$ . Now define a sequence of submanifolds $\{M_i : i = 1,2,...\}$ by putting $M_i = X_{-i}(M_1)$. It is then clear that $\{ M_i : i = 1,2,...\}$ is a filtration for X, and by Lemma 1 , X is C stable .

Suppose that for a vector field X the set $F = \bigcup_{p \in M} \omega^X(p)$ is a union of compact , invariant , isolated subset $\omega_i$ i.e.

$$(*) \qquad F = \omega_1 \cup \omega_2 \cup \omega_3 \cup \dots$$

Let $W^s \omega_i = \{p \in M : \omega^X(p) \subset \omega_i\}$ and $W^u \omega_i = \{p \in M : \alpha^X(p) \subset \omega_i\}$ and define in $\{ \omega_i : i = 1,2,...\}$ the relation

$$\omega_i \leqslant \omega_j \quad \text{iff} \quad \overline{W^s \omega_i} \cap \overline{W^u \omega_j} \neq \emptyset$$

LEMMA 3. Let $\omega_1$ , $\omega_2$ , $\omega_3$ be any sets appearing in (*) and suppose that there exists a point $p_0 \in \omega_2$ such that $\overline{O_X(p_0)} = \omega_2$ If $\omega_1 \geqslant \omega_2 \geqslant \omega_3$ then in every neighborhoods U and V of X and $\omega_1$ respectively there are $Y \in U$ and $p \in V$ such that $\omega^Y(p) \subset \omega_3$

This Lemma is consequence lemma 9 from [2] .

THEOREM 2 . Let X be a vector field , $F = \bigcup_{p\in M} \omega^X(p) = \omega_1 \cup \omega_2 \cup \dots$ be a union of infinitely many compact , invariant , isolated sets and let for every $\omega_i$ there exists $p_i \in \omega_i$ such that $\overline{O_X(p)} = \omega_i$ . Moreover let $\overline{\bigcup_{t\geqslant 0} X_t(K)}$ be compact for every compact K . Then X is stable iff no infinite sequence

$$\omega_{i_1} \geqslant \omega_{i_2} \geqslant \omega_{i_3} \geqslant \dots \quad \text{exists} .$$

PROOF. If suffices to show the existence of a filtration for X We define a set

$$A_1 = \Big\{ \omega_k : \text{there is a sequence } \omega_{i_1} , \dots , \omega_{i_j} \text{ such that } \omega_{i_1} = \omega_1 , \ \omega_{i_j} = \omega_k \text{ and } \omega_{i_1} \geqslant \omega_{i_2} \geqslant \dots \geqslant \omega_{i_j} \Big\}$$

Due to our asumption there is no infinite sequence

$$\omega_{i_1} \geqslant \omega_{i_2} \geqslant \omega_{i_3} \geqslant \dots$$

and since $\overline{\bigcup_{t\geqslant 0} X_t(K)}$ is compact for every compact K , $A_1$ is finite and $C_1 = \bigcup_{\omega_i\in A_1} \omega_i$ is compact, therefore there is a neighborhood V of $C_1$ such that for every $p \in V$ $\omega^X(p)$ is contained in $C_1$ . Hence $C_1$ is an attractor with a domain of attraction $D_1 = \bigcup_{\omega_i\in A_1} W^s \omega_i$ . Let $B_1 = \{ p\in D_1 : \alpha^X(p) \cap C_1 \neq \emptyset \}$ The set $B_1$ is compact invariant / see Lemma 2 / and $B_1 \subset D_1$ by definition of the relation $\leqslant$. In the some way as in Lemma 2 we may show that $B_1$ is uniformly asymptotically stable . Let $L_1$ be a Lyapunov function for $B_1$ and $M_1 = L_1^{-1}([0 , 1])$.

Starting with $M_1$ we construct a filtration $M_1 , M_2 , M_3 , \dots$ by induction . Suppose that the submanifolds $M_1, M_2, M_3 , \dots M_k$ are already done . To define $M_{k+1}$ , put $N_k = X_{-1}(M_k)$.

We choose a set $\omega_{i_0}$ not contained in $N_k$ and define

$$A_{k+1} = \left\{ \omega_k : \text{there is a sequence } \omega_{i_1}, \ldots, \omega_{i_j} \text{ such that } \omega_{i_1} = \omega_{i_0} \text{ and } \omega_k = \omega_{i_j} \text{ and } \omega_{i_1} \geqslant \omega_{i_2} \geqslant \ldots \geqslant \omega_{i_j} \right\}$$

Again

$$C_{k+1} = \bigcup_{\omega_i \in A_{k+1}} \omega_i \cup B_k \quad \text{is an attractor.}$$

Let $B_{k+1}$ be the uniformly asymptotically stable set such that $C_{k+1} \subset B_{k+1}$ and let $L_{k+1}$ be a Lyapunov function for $B_{k+1}$. Put $M_{k+1} = L_{k+1}^{-1}\left([0 , c_k]\right)$ where $c_k$ is choosen such that $N_k \subset M_{k+1}$

The sequence $M_1$ , $M_2$ , $M_3$ , ... of compact submanifolds is a filtration for $X$ , for it is clear that $X_t(M_i) \subset M_i$ for $t \geqslant 0$ and $X$ is transverse to $\partial M_i$ . To verify that $\bigcup_{i=1} M_i = M$ , take $p \in M$, by assumption $\omega^X(p) \neq \emptyset$ , hence there exists k such that p is contained in domain of attraction of $B_k$ . By our construction $X_{-i}(M_k) \subset M_{k+i}$ so there exists l such that $p \in M_l$ , hence $\bigcup_{i=1} M_i = M$ .

Suppose now that there exists an infinite sequence $\omega_{i_1} \geqslant \omega_{i_2} \geqslant \omega_{i_3} \geqslant \ldots$ . We will show then , that in every neighborhood of the vector X there is a vector field Y and a point $p \in M$ such that $\omega^Y(p) = \emptyset$ . Since the closure of the semitrajectory of every compact set is compact ,we may choose from the sequence $\omega_{i_1}, \omega_{i_2}$ , ... a sequence $\widetilde{\omega}_{i_1} \geqslant \widetilde{\omega}_{i_2} \geqslant \widetilde{\omega}_{i_3} \geqslant \ldots$ such that if only $k > l+1$ then

$$\overline{W^s \widetilde{\omega}_{i_k}} \cap \overline{W^u \widetilde{\omega}_{i_l}} = \emptyset$$

Then choose for every $\tilde{\omega}_{i_k}$ a neighborhood $V_{i_k}$ of $\tilde{\omega}_{i_k}$ such that $V_{i_k} \cap V_{i_l} = \emptyset$ if $k \neq l$. By Lemma 3 there are a vector field $Y_1$ and $p_{i_1} \in V_{i_1}$ such that $\omega^{Y_1}(p_{i_1}) \subset \tilde{\omega}_{i_3}$ $Y_1 = X$ off $V_{i_1}$ and $\sup_{p \in M} \| Y_1(p) - X(p) \|$ is arbitrarily small. Similarly we change the vector field $Y_1$ on $V_{i_3}$ so that for this new vector field $Y_2$ $\omega^{Y_2}(p_{i_1}) \subset \tilde{\omega}_{i_4}$ and then repeat this procedure for $V_{i_4}$, $V_{i_5}$, ... . In this way we get a vector field $Y$ which is arbitrarily near to $X$ and such that $\omega^{Y}(p_{i_1}) = \emptyset$ This proves the necessity part of our theorem.

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