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GENERATORS AND ALMOST TOPOLOGICAL ISOMORPHISMS

by

Manfred Denker

Summary:

For many (classical) measure theoretic dynamical systems there are natural m.t. isomorphisms which are at the same time homeomorphisms between sets of measure one or residual sets. Among many other examples the $2x \bmod 1$ transformation, irrational rotations, Meshalkin's isomorphism of Bernoulli shifts and axiom-A-diffeomorphisms are naturally isomorphic to a suitable subshift and these correspondences explain the situation of this paper. Isomorphisms of this type are called almost topological and will be studied in forthcoming papers by M.Keane and myself. Here some of the results about invariants and about generators defining such isomorphisms are presented.

In topological dynamics the notion of (measure theoretic) isomorphism (m.t. isomorphism) does not make any sense since no topological property is invariant, while on the other hand in measure theoretic ergodic theory (topological) conjugacy is irrelevant for the isomorphism problem since all topological invariants are independent of the probabilistic structure. This not very surprising observation gets of greater value and interest when looking at examples of isomorphisms. Many of them arise in a natural context in ergodic theory and analysing their properties they should be arranged in between conjugacy and isomorphism. It is not quite apparent what the right definition for an analogue isomorphism notion should be when trying to get "at least some" topological invariants and - at the same time - to keep the isomorphism classes "large enough". (From the viewpoint of conjugacy this means enlarging the conjugacy classes by reducing the number of topological invariants.) From the measure theoretic point of view it seems to be reasonable to demand the validity of Rohlin's and Krieger's generator theorems, since they are the most general isomorphism theorems in some sense. It is important to remark here that a generator in this context must get another meaning, because the existence of such a generator should give an isomorphism of the desired type. A theorem like this clearly tells something about the largeness of the isomorphism classes. On the other hand, in topological dynamics some of the most important topological invariants for conjugacy are transitivity, mixing, almost periodicity and strict ergodicity. Clearly one should try to leave them invariant for our notion.

Having in mind these two principles, it should not be forgotten that a reasonable definition should be applicable to many of the natural examples in ergodic theory. For this reason one is led to study some examples first.

Consider the $2x \bmod 1$ transformation on $\Omega = [0,1[$ which is invariant with respect to Lebesgue measure. The unique dyadic representation for a.e. $x \in \Omega$ defines a.e. a homeomorphism $\varphi : \Omega \to \{0,1\}^{\mathbb{N}}$ which commutes with the shift on $\{0,1\}^{\mathbb{N}}$. More precisely, φ is defined on the residual set of all points having a unique dyadic representation.

Next let T be an irrational rotation on the unit circle X in the plane. Every partition $\alpha = (A_0, A_1)$ of X, where each A_i $(i = 0,1)$ is an interval of positive length is a generator for T. The restriction map of $\varphi : X \to \{0,1\}^{Z}$, defined by $\varphi(x) = (\omega_i)_{i\in\mathbb{Z}}$ iff $T^i x \in A_{\omega_i}$, to the set X_0 of all points whose orbit never meets the boundary of the A_i's, actually is a homeomorphism on the dense G_δ-set X_0, which also is a set of measure 1.

Our next example is concerned with the continued fraction expansion. Let $X = {}^{\mathbb{R}}/_{\mathbb{Z}}$, $Tx = x + \beta \bmod 1$ with β irrational. For $0 < \alpha \leq 1$ define $N(\alpha) := [1/\alpha]+1$ where $[x]$ denotes the largest integer $\leq x$, and $S(\alpha) \in]0,1]$

by the equation

$$\alpha = [N(\alpha) - S(\alpha)]^{-1}.$$

Hence the continued fraction expansion of α can be written as

$$\alpha = \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots}}}$$

where $n_i = N(S^{i-1}(\alpha))$ $(i \geq 1)$, and for $\alpha_i = S^{i-1}(\alpha)$ we have

$$\alpha_i = \cfrac{1}{n_i - \cfrac{1}{n_{i+1} - \cfrac{1}{\ddots}}}$$

Now consider the irrational β (i.e. there exist infinitely many $n_i \neq 2$). Let $\Omega = \prod_{i \geq 1} \{0, 1, \ldots, n_i - 1\}$ and

$$\Omega_\beta := \Omega \setminus \bigcup_{i,k \geq 1} [n_i - 1, n_{i+1} - 2, \ldots, n_{i+k-1} - 2, n_{i+k} - 1]_i^{i+k}$$

Define a transformation $S_\beta : \Omega_\beta \to \Omega_\beta$ by

$$S_\beta(\omega) = (\omega_1 + 1, \omega_2, \omega_3, \ldots) \text{ if } \omega \in \Omega_\beta \setminus \bigcup_{k=1}^{\infty} [n_1 - 2, \ldots, n_{k-1} - 2, n_k - 1]_1^k$$

and $S_\beta(\omega) = (0, \ldots, 0, \underset{(k+1)\text{st place}}{\omega_{k+1} + 1}, \omega_{k+2}, \ldots)$ if $\omega \in [n_1 - 2, \ldots, n_{k-1} - 2, n_k - 1]_1^k$.

(Ω_β, S_β) is called an irrational adding machine and it is shown by M.Keane [4] that there exists a map

$$\varphi : \Omega_\beta \to X$$

which is a homeomorphism on a dense G_δ.

Again let $X = [0,1[$. Suppose $\beta_1,\ldots,\beta_n$ are given such that $\beta_i > 0$ and $\sum\beta_i=1$, and let τ be a permutation of $\{1,\ldots,n\}$. Define a map $T : X \to X$ by cutting X into intervals $I_1,\ldots,I_n$ of lengths $\beta_1,\ldots,\beta_n$ and re-arranging the pieces according to τ. This is called an interval exchange transformation (Keane [6]). T is called irrational if there is no nontrivial τ-invariant subset of $\{1,\ldots,n\}$ and $\beta_1,\ldots,\beta_n$ are irrational in the usual sense (i.e. $k_i \in \mathbb{Z}$, $k_o = \sum\beta_i k_i \Rightarrow k_o = k_i$ for every i). It follows from [6] that the partition $\alpha = (I_1,\ldots,I_n)$ is a generator if T is irrational and that the natural map $\varphi : X \to \{0,1\}^{\mathbb{Z}}$ is a homeomorphism on a residual set of Lebesgue measure one.

So far every isomorphism was defined on a residual set. The argument of our fifth example gives only an a.e. defined isomorphism. M.Keane pointed out to me that this isomorphism can be extended to a residual set. In fact, this is a particular case of a general theorem contained in our forthcoming paper. Consider Meshalkin's example of an isomorphism of the Bernoulli shifts $p = (1/4,1/4,1/4,1/4)$ and $q = (1/2,1/8,1/8,1/8,1/8)$ over the alphabets $0,\ldots,3$ and $0,\ldots,4$. The isomorphism φ is defined as follows: On $\Omega_1 = \{0,\ldots,3\}^{\mathbb{Z}}$ consider the homeomorphism defined by the correspondence $0 \longleftrightarrow {0 \atop 0}$, $1 \longleftrightarrow {0 \atop 1}$, $2 \longleftrightarrow {1 \atop 0}$. and $3 \longleftrightarrow {1 \atop 1}$ and on $\Omega_2 = \{0,\ldots,4\}^{\mathbb{Z}}$ consider $0 \longleftrightarrow 0$, $1 \longleftrightarrow 100$; $2 \longleftrightarrow 101$; $3 \longleftrightarrow 110$ and $4 \longleftrightarrow 111$. Now every $\omega \in \Omega_1$ has a representation of the form

$$\omega = \left(\ldots, \begin{matrix}\omega^1_{-1}\\ \omega^0_{-1}\end{matrix}, \begin{matrix}\omega^1_{0}\\ \omega^0_{0}\end{matrix}, \begin{matrix}\omega^1_{1}\\ \omega^0_{1}\end{matrix}, \ldots\right)$$

and $\varphi(\omega) = (\eta_i)_{i\in\mathbb{Z}} \quad \left(\eta_i = \begin{cases} \eta^0_i\ \eta^1_i\ \eta^2_i & (\eta^0_i=1) \\ \eta^0_i & (\eta^0_i=0) \end{cases}\right)$

is defined as follows:

If $\omega^0_i=0$ then $\eta_i = \eta^0_i = 0$. If $\omega^0_i=1$ then $\eta_i = \eta^0_i\ \eta^1_i\ \eta^2_i$ where $\eta^0_i=1$, $\eta^1_i = \omega^1_i$ and $\eta^2_i = \omega^1_k$ such that $k > i$, $\omega^0_k = 0$ and $\sum_{j=i+1}^{k-1} (\omega^0_j - 1/2) = 0$ and k is minimal.

φ is well defined a.e. since by the recurrence properties of a symmetric random walk on $\mathbb{Z}$ there are infinitely many k's such that in $\omega^0_i,\ldots,\omega^0_{i+k}$ more ones than zeros appear and conversely (note that the random variables $\omega \to \omega^0_i$ are independent and symmetric). It is not hard to see that φ-restricted to suitable chosen sets of measure 1 - is a homeomorphism. For details see Meshalkin [8] or Keane [3], [5].

Our next example comes from differentiable dynamics and has a quite general nature. Let $T : X \to X$ be a homeomorphism on a compact metric space admitting a Markov partition $\alpha = (A_1,\ldots,A_s)$, for example axiom-A-diffeomorphisms and homeomorphisms like hyperbolic torus automorphisms and topological Markov chains([1],[2],[9]). If X_o denotes the set of all points whose orbit does not intersect the boundary of any A_i, the map $\varphi : X_o \to \{1,\ldots,s\}^{\mathbb{Z}}$ is a homeomorphism where φ is defined by $\varphi(x)_j = k$ iff $T^j x \in A_k$.

If T is transitive, Bowen [1] has shown that there exists a unique measure with maximal entropy, which is positive on open sets and for which the partition α consists of continuity sets, and therefore φ carries this measure to some measure on a subshift. The physical interpretation as well as the torus automorphisms allow to consider the Bowen measure as the natural given one and since the boundaries of a Markov partition lie in the contracting and expanding directions the map φ can be derived in a natural way knowing these directions and the expansive constant.

All the preceding examples of m.t. isomorphisms carry some additional structure which will be given in the following definitions. We always consider compact metric spaces, their Borel fields and the various probabilities on the completions. A transformation T on Ω in Ω' is called almost-continuous (with respect to some probability m) if it is defined and continuous a.e. Then (Ω, F, m, T) is an almost-continuous m.t.dynamical system (a.c. m.t. system).
Note that this definition is somewhat artificial. Since it is possible to neglect a set of measure zero, the definition could have been formulated just be requiring that Ω and Ω' are metric spaces. However, as can be seen in the examples there is always a natural underlying compact space.
Two a.c.m.t. systems are called almost-topologically isomorphic if there exists an m.t.isomorphism φ such that φ and φ^{-1} are almost-continuous. This is the weakest property which all the considered isomorphisms have in common. It

should be remarked here that other authors have considered this kind of isomorphy; in [7] K.Krickeberg gives exactly the same definition, though he uses it for other purposes than it is done here. All examples even lead to a stronger notion: Let φ be an almost-topological isomorphism. φ is said to be strongly almost-topological if φ and φ^{-1} are defined and continuous on residual sets of measure one. Both definitions sharpen the notion of m.t.isomorphism and are derived from it. On the other hand, from a purely topological viewpoint, two systems (Ω,T) and (Ω',T'), where T and T' are defined and continuous on residual sets, are called almost-topological conjugate if there exist invariant residual sets $\Omega_o \subset \Omega$ and $\Omega'_o \subset \Omega'$, such that $(\Omega_o|T|\Omega_o)$ and $(\Omega'_o|T|\Omega'_o)$ are conjugate in the usual sense. They are called strongly almost-topological conjugate if in addition the conjugacy transports at least one measure.

As explained in the beginning, there are two problems to be handled, that is finding the topological invariants and proving a generator theorem for the "correct" definition of a generator.

Some topological properties are known in ergodic theory and topological dynamics which obviously are not invariants for almost-topological isomorphy (conjugacy), like topological entropy, weak ergodicity and weak minimality, since these properties do not depend on residual sets only. But the following is immediate. Let φ be an almost topological isomorphism for $(\Omega,\mathcal{F},m,T)$ and $(\Omega',\mathcal{F}',m',T')$ where m' is positive on open sets or let φ be an almost topological conjugacy for (Ω,T) and (Ω',T'). Then if $x \in \operatorname{def}\varphi$ has a dense orbit (is almost periodic) then φx has a dense orbit (is almost periodic) also. Let $\mathcal{E}$ (resp. $\mathcal{C}$) denote the class of all m.t.dynamical systems (topological dynamical systems) where the transformation is continuous and where in the first case the measures are ergodic and positive on open sets. The above observation implies the following fact:

Proposition 1:

In both classes $\mathcal{E}$ and $\mathcal{C}$ topological transitivity, topological mixing and minimality are almost-topological invariants.

While this proposition holds for the weak form of a.t. conjugacy, the next theorem holds for the strong from. (I believe that it is true only in this case.)

Theorem 1:

Strict ergodicity is a strongly almost topological conjugacy invariant for the class $\mathcal{C}$.

Proof:

It is not hard to show that for a sequence f_n of continuous functions on a compact metric space converging pointwise to a limit function f on a residual set there must be a point of continuity for f. Now let (Ω,T) and (Ω',T') be strongly almost topologically conjugate dynamical systems in $\mathcal{C}$ and suppose (Ω,T) is strictly ergodic with invariant probability m. Denote by φ the isomorphism. By definition m is transported by φ into $\mu = \varphi m$. Choose a μ-continuity open set V. Then $\lim \frac{1}{n} \sum_{t=o}^{n-1} 1_V T'^t = \mu(V)$ exists on a residual set. Hence any invariant ergodic measure $\nu \neq \mu$ on Ω' satisfies $\nu(V) = \mu(V)$ which shows strict ergodicity of (Ω',T').

Other invariance properties will be discussed elsewhere (for example other recurrence properties, topological entropy for noncompact sets and the natural map between the sets of invariant measures) and I shall now present the results concerning the second problem for invertible T.

The problem of finding an isomorphism to a subshift for a system (Ω,F,m,T) (or (Ω,T)) is in most cases that of constructing a point separating partition. In order to turn this isomorphism into an almost-topological one, the definition of a topological generator for (Ω,F,m,T) (and for a residual set $\Omega_o \subset \Omega$ in case of a top.dyn. system (Ω,T)) is given first. In the first case a topological generator is just a measurable partition $\alpha = (A_1,A_2,..)$ such that $m(\partial A_i) = 0$ $(i \in \mathbb{N})$ and such that for almost all $x \in \Omega$

$$x \in \bigcap_{k\in\mathbb{Z}} T^k \overline{A_{i_k}} \Rightarrow \{x\} = \bigcap_{k\in\mathbb{Z}} T^k \overline{A_{i_k}}.$$

In the second case one requires that the partition $\alpha = (A_1,A_2,...)$ satisfies $\Omega = \overline{\bigcup \operatorname{int} A_k}$, $\operatorname{int} A_k \neq \emptyset$ and that for $x \in \Omega_o$

$$x \in \bigcap_{n\in\mathbb{Z}} \bigcap_{k=-n}^{n} \overline{T^k \operatorname{int} A_{i_k}} \Rightarrow \{x\} = \bigcap_{n\in\mathbb{Z}} \bigcap_{k=-n}^{n} \overline{T^k \operatorname{int} A_{i_k}}$$

Let Ω_α denote the set of all points whose orbit never meets the boundary of any element of the generator α. It is easy to see that a topological generator $\alpha = (A_1, A_2, \ldots)$ always defines a natural homeomorphism $\varphi : \Omega_o \cap \Omega_\alpha \xrightarrow{\text{into}} \mathbb{N}^{\mathbb{Z}}$ where Ω_o is a suitable chosen set of measure one or the set of points which are separated by α. In case α is a finite generator for Ω, φ^{-1} has a continuous extension onto $\overline{\varphi(\Omega_o \cap \Omega_\alpha)}$ so that (Ω, T) becomes a factor of a suitable subshift.

One has to distinguish between the different cases when studying the existence of topological generators. First let (Ω, T) be a topological dynamical system.

Theorem 2:

If the set of aperiodic points of T is dense in Ω, then there exists a topological generator $\alpha = (A_1, A_2, \ldots)$ for Ω.

The proof of this fact uses a topological version of Rohlin's lemma and some standard techniques. It is also possible to choose the generator α in such a way that measure theoretic entropies can be compared, and that it becomes an m-continuity partition for a given measure. In this last case the generator gives rise to a strongly almost-topological isomorphism, and the natural question in this context is that of finding finite topological generators. Here our two main results are stated. (They have some obvious corollaries!) Their proofs use the well-known techniques for generator theorems and some more easy facts.

Theorem 3.:

Let $(\Omega, \mathcal{F}, m, T)$ be ergodic with a.c. transformation T and let $h_m(T) < \infty$. Then there exists a finite topological generator α where $[\exp h_m(T)] + 1$ can be prescribed for the number of atoms of α. Thus $(\Omega, \mathcal{F}, m, T)$ is almost-topological isomorphic to some finite subshift.

Theorem 4:

Let $(\Omega, \mathcal{F}, m, T)$ have a transformation T which is continuous on a residual set of measure one. If m is ergodic with finite entropy then there exists a finite topological generator α for $(\Omega, \mathcal{F}, m, T)$ and for a residual set in Ω. The number of atoms in α can be bounded by $[\exp h_m(T)] + 1$. Thus α defines a strongly almost topological isomorphism onto some finite subshift.

Addendum:

After this paper had been written, the results of M.Keane and myself presented here were put into a final form. Because of the discussion of a valuable isomorphism notion in this article we used the notion of strongly almost-topological isomorphy but called it almost-topological isomorphy (or finitary isomorphy in accordance with the literature).
I should like to add to the results announced here that we also started to study a new kind of theorems in our paper. For an almost-topological isomorphism between symbolic systems we defined the expected coding time and, if it is finite, we show that convergences in distribution for the coordinate processes correspond, provided some mixing condition is satisfied.

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