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THE TRANSCENDENCE OF DEFINITE INTEGRALS OF ALGEBRAIC FUNCTIONS

by

David William MASSER

Many numbers interesting to transcendence theorists appear as definite integrals of certain algebraic functions. For example, we have for algebraic $\alpha \neq 0$

$$\frac{1}{4}\pi = \int_0^\infty (1+x^2)^{-1}dx \ , \ \log \alpha = \int_1^\alpha x^{-1}dx \ ,$$

the integral mentioned by Siegel in [12]

$$\frac{1}{3}(\log 2 + \frac{\pi}{\sqrt{3}}) = \int_0^1 (1+x^3)^{-1}dx \ ,$$

and the following integrals involving values of the classical gamma function

$$2^{-8/3}\pi^{-1}(\Gamma(\tfrac{1}{3}))^3 = \int_1^\infty (x^3-1)^{-1/2}dx \ , \ 2^{-3/2}\pi^{-1/2}(\Gamma(\tfrac{1}{4}))^2 = \int_1^\infty (x^3-x)^{-1/2}dx \ .$$

To describe the most general such integral it is convenient to use geometrical language. Let F be a polynomial in two variables with algebraic coefficients, and let $\mathcal{C}$ be the curve in complex space $\mathbb{C}^2$ defined by the equation $F(x,y) = 0$. Take a rational function $D(x,y)$ on $\mathcal{C}$ with algebraic coefficients, and form the differential $\delta = D(x,y)dx$ on $\mathcal{C}$. If $\mathfrak{p}$ is any path on the Riemann surface $\mathfrak{R}$ of $\mathcal{C}$ whose endpoints have algebraic coordinates, we can define the definite integral $\delta(\mathfrak{p})$ of δ along $\mathfrak{p}$ provided $\mathfrak{p}$ does not pass through any poles of δ. Clearly all the above integrals are examples of this general construction. Thus from the point of view of transcendence theory it would be useful to know the arithmetic nature of the number $\delta(\mathcal{P})$.

If the curve $\mathcal{C}$ is rational this knowledge comes from the basic theory of linear forms in logarithms of algebraic numbers, and in [10] van der Poorten gave necessary and sufficient conditions for $\delta(\mathcal{P})$ to be a transcendental number. But for general $\mathcal{C}$ it seems difficult even to formulate a conjecture, and we shall say no more about this problem.

Instead we shall simplify the question by restricting attention to closed paths or loops $\mathcal{L}$ on $\mathcal{R}$. By way of example, let us calculate the integrals $\delta(\mathcal{L})$ when $\mathcal{C}$ is either rational or elliptic.

In the first case, we may identify the rational curve $\mathcal{C}$ with the complete complex plane. Then

$$\delta(\mathcal{L}) = \int f(z)dz$$

where the integral is taken over a closed contour and $f(z)$ is a rational function with algebraic coefficients. Clearly the residue of $f(z)$ at any pole is an algebraic number, and it follows that $\delta(\mathcal{L})$ is an algebraic multiple of $2\pi i$. Hence $\delta(\mathcal{L})$ is either zero or transcendental, and these two possibilities are easily distinguished.

A similar argument shows that for general $\mathcal{C}$ the number $\delta(\mathcal{L})$ is an algebraic multiple of $2\pi i$ if $\mathcal{L}$ is trivial ; i.e. homologous to a single point of $\mathcal{R}$. Henceforth we shall consider only non-trivial loops. We may also exclude the case of exact differentials δ , since $\delta(\mathcal{L}) = 0$ whenever $\delta = dG$ for some rational function $G = G(x,y)$ on $\mathcal{C}$.

Next suppose $\mathcal{C}$ is an elliptic curve, given without loss of generality in Weierstrass normal form

$$y^2 = 4x^3 - g_2x - g_3$$

for algebraic numbers g_2, g_3 with $g_2^3 \neq 27g_3^2$. This curve is parametrized by the formulae

$$x = \wp(z) \ , \ y = \wp'(z) \ ,$$

where $\wp(z)$ is the Weierstrass elliptic function with invariants g_2, g_3. Accordingly we obtain

$$\delta(\mathcal{L}) = \int f(z)dz$$

where the integral is a path from some point z_0 to $z_0 + \omega$ and $f(z)$ is an elliptic function with respect to the period lattice Λ of $\wp(z)$. Here ω is a period of Λ that vanishes if and only if the loop $\mathcal{L}$ is trivial. Now $f(z)$ has a canonical expression as a linear combination of functions $\zeta^{(r)}(z-u)$ $(r \geq 0)$, where $\zeta(z)$ denotes the Weierstrass quasi-periodic function and the finitely many numbers u are related to the poles of $f(z)$. See for example [13] p. 450. The algebraic definition of δ can be translated into statements about this expansion, and by explicitly integrating by means of the sigma function, we find that

$$\delta(\mathcal{L}) = \alpha\omega + \beta\eta + \gamma(2\pi i) \ ,$$

where

$$\eta = \zeta(z_0+\omega) - \zeta(z_0)$$

is independent of z_0, and α, β, γ are algebraic numbers. We deduce immediately from a result of Coates [4] that $\delta(\mathcal{L})$ is either zero or transcendental. In addition, the generalization of this result proved in the author's thesis [6] shows that $\delta(\mathcal{L})$ is actually transcendental unless $\alpha = \beta = \gamma = 0$. But it can easily be verified by referring back to the above canonical expression for $f(z)$ that this condition holds only if δ is exact.

Thus we have shown that when $\mathcal{C}$ is an elliptic curve the number $\delta(\mathcal{L})$ is transcendental if $\mathcal{L}$ is non-trivial and δ is non-exact. This does not quite remain true for curves of arbitrary genus $g > 1$, as we shall see below. To describe the results so far obtained in this case we first recall the well-known classification of differentials. An arbitrary differential is called of the third kind ; it becomes of the second kind if its residue at every pole vanishes ; while if it has no poles at all it is of the first kind.

In 1941 Schneider [11] established the following result for curves of genus $g \geq 1$. If δ is a non-exact differential of the second kind there is a loop $\mathcal{L}$ such that $\delta(\mathcal{L})$ is transcendental. Since $\delta(\mathcal{L})$ now depends only on the homology class of $\mathcal{L}$, this is equivalent to the assertion that at least one of $\delta(\mathcal{L}_1),\ldots,\delta(\mathcal{L}_{2g})$ is transcendental, where $\mathcal{L}_1,\ldots,\mathcal{L}_{2g}$ constitute a basis for the homology group. In fact, if this basis is suitably chosen, Schneider proved the stronger statement that at least one of the numbers $\delta(\mathcal{L}_1),\ldots,\delta(\mathcal{L}_g)$ is transcendental. This implies the above result for elliptic curves when the differentials are of the second kind.

By taking curves of the form

$$y^r = x^s(1-x)^t$$

for rational integers r,s,t Schneider deduced the transcendence of the value $B(u,v)$ of the classical beta function at all positive rational non-integral u,v.

My own recent results concern the case $g = 2$. Let us call the curve $\mathcal{C}$ of genus 2 simple if there is no non-constant rational map from $\mathcal{C}$ to an elliptic curve. This amounts to saying that the Jacobian $\mathcal{J}$ of $\mathcal{C}$ is not isogenous to a product of two elliptic curves. If $\mathcal{J}$ does split in this way then the value of $\delta(\mathcal{L})$ is a linear combination, with algebraic coefficients, of $2\pi i$ and certain periods of the corresponding two elliptic functions. Since much work has already been done on such expressions (Baker [1], [2] ; Coates [3], [4], [5] ; Masser [6],

[8]), we shall assume that $\mathcal{C}$ is simple. Our main result is the following.

THEOREM. If $\mathcal{C}$ is simple, $\mathcal{L}$ is non-trivial, and δ is non-exact of the second kind, then $\delta(\mathcal{L})$ is transcendental.

The proof will appear in [9]. Thus we see that, in the notation introduced above to describe Schneider's theorem, both of the numbers $\delta(\mathcal{L}_1)$, $\delta(\mathcal{L}_2)$ are transcendental when $g = 2$. Incidentally the simplicity of $\mathcal{C}$ is necessary for the validity of the Theorem. One may see this by general arguments, or by writing down an exemple as follows. If $\mathcal{C}$ is the curve $y^2 = 1-x^6$, with the obvious map to the elliptic curve $v^2 = 1-u^3$, we have

$$\int (1 - x\sqrt{3})y^{-1}dx = 0$$

where the integral is over any real loop starting at $(x,y) = (1,0)$, proceeding to $(-1,0)$ via non-negative values of y, and returning via non-positive values of y.

In [7] I had proved this Theorem for differentials of the first kind, when the condition that δ is non-exact reduces simply to $\delta \neq 0$. If δ is of the third kind, I can prove that $\delta(\mathcal{L})$ is either zero or transcendental ; here of course the conditions of non-triviality or non-exactness are irrelevant. However, the proof turns out to be much simpler than the proof of the above Theorem. It remains an interesting problem to decide whether $\delta(\mathcal{L})$ is zero or transcendental in this case by extending the Theorem to differentials of the third kind.

The solution of this problem would have the following consequence for the values of the beta function. Consider the numbers $B(m/5, n/5)$ as m, n run through all positive integers. It is quickly verified that these span a vector space V of dimension at most 6 over the field of algebraic numbers. Let $\beta_0, \ldots, \beta_5$ be the elements of any spanning set for V with

$$\beta_0 = B(1,1) = 1 \ , \ \beta_5 = 2i \sin(\pi/5)B(1/5,4/5) = 2\pi i \ .$$

The extended Theorem would imply that the dimension of V is exactly 6 ; in other words, $\beta_0,\ldots,\beta_5$ are linearly independent over the field of algebraic numbers.

The Theorem itself gives the linear independence of $\beta_0,\ldots,\beta_4$, while the easier result on differentials of the third kind shows that any linear combination of $\beta_1,\ldots,\beta_5$ with algebraic coefficients is either zero or transcendental.

We conclude by mentioning some of the features of the proof of the Theorem. We use the Abelian functions associated with the curve $\mathcal{C}$. These are functions $A(\mathbf{z}) = A(z_1,z_2)$, meromorphic on the complex space $\mathbb{C}^2$, satisfying

$$A(\mathbf{z}+\boldsymbol{\omega}) = A(\mathbf{z})$$

for all $\boldsymbol{\omega}$ lying in some lattice Λ of $\mathbb{C}^2$. A differential δ of the first kind on $\mathcal{C}$ corresponds to a linear function $\boldsymbol{\alpha}.\mathbf{z}$ on $\mathbb{C}^2$. A loop $\mathcal{L}$ corresponds to a period $\boldsymbol{\omega}$ of Λ , and then $\delta(\mathcal{L}) = \boldsymbol{\alpha}.\boldsymbol{\omega}$. This is the situation of [7]. More generally, a differential δ of the second kind corresponds to a meromorphic function $H(\mathbf{z})$ quasi-periodic with respect to Λ . That is, the difference

$$\eta(\boldsymbol{\omega}) = H(\mathbf{z}+\boldsymbol{\omega}) - H(\mathbf{z})$$

is independent of $\mathbf{z}$ for each $\boldsymbol{\omega}$ in Λ . Then if $\mathcal{L}$ corresponds to the period $\boldsymbol{\omega}$, we have $\delta(\mathcal{L}) = \eta(\boldsymbol{\omega})$.

One of the difficulties of the proof concerns certain division values associated with $H(\mathbf{z})$. A typical expression is

$$\varphi(r,q) = H(r\boldsymbol{\omega}/q) - r\eta(\boldsymbol{\omega})/q \qquad (1 \leq r < q)$$

where q,r are coprime integers. Without much trouble it can be shown that as q becomes large $\varphi(r,q)$ is an algebraic number of degree at most cq^4 . It is relatively easy to deduce that its height does not exceed $q^{cq^{\varkappa}}$ for some $\varkappa \geq 6$, and this crude estimate suffices for the proof of the result on differentials of the third kind. For the Theorem, however, we need an estimate with $\varkappa < 16/3$, and in fact we obtain one with $\varkappa = 5$. Probably the true order of magnitude is about c^{q^4}.

Note that if δ is of the first kind, then $H(\mathbf{z})$ is linear and $\varphi(r,q) = 0$, so that these difficulties do not enter into [7].

In addition the proof makes use of an elimination procedure to be found in [8], and it also employs Gelfond's well-known criterion for transcendence. Under the assumption that $\eta(\boldsymbol{\omega})$ is algebraic for some $\boldsymbol{\omega} = (\omega_1, \omega_2)$ with $\omega_2 \neq 0$, say, we construct an auxiliary function and deduce from Gelfond's criterion that ω_1/ω_2 is an algebraic number α . This contradicts the main theorem of [7] for the differential of the first kind corresponding to the linear function $z_1 - \alpha z_2$.

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