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GALOIS MODULE STRUCTURE AND ARTIN L-FUNCTIONS

by

Albrecht FRÖHLICH

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1) Let L and K be number fields, always of finite degree over $\mathbf{Q}$, and L normal over K with Galois group $\mathrm{Gal}(L/K) = \Gamma$. Let χ be a character of Γ. We shall write $\Lambda(s,\chi)$ for the enlarged Artin L-function, to include gamma and exponential factors. It satisfies a functional equation

$$W(\chi)\,\Lambda(s,\chi) = \Lambda(1-s,\overline{\chi}) ,$$

where $\overline{\chi}$ is the complex conjugate of χ and $W(\chi)$, the Artin root number, is of absolute value 1. We write

$$W(L/K,\chi) = W(\chi) = \tau(\chi)\, W_{\infty}(\chi)\, / \, N f(\chi)^{1/2} ,$$

where $W_{\infty}(\chi)$ is a power of $i = \sqrt{-1}$, depending an ramification at infinity, $N f(\chi)^{1/2}$ is the positive square root of the absolute norm of the conductor $f(\chi)$, and $\tau(\chi)$ is the "Galois Gauss-sum" (Hasse).

If χ is real valued then $W(\chi) = \pm 1$. If actually χ comes from a real representation then $W(\chi) = 1$.

Example : Let $\Gamma = H_{4\ell^r}$ be the generalised quaternion group of order $4\ell^r$, where ℓ is an odd prime and $r \geq 1$. For each $j = 1, \dots, r$ there are faithful irreducible representations (of degree 2) of the quotient $H_{4\ell^j}$ of $H_{4\ell^r}$, which have real valued characters ψ_j , but which cannot be realised over the fields of real numbers. If L/K is tame then the root numbers of the distinct charcters $\psi = \psi_j$, with the same j , all coincide (cf. [1] theorem 5). We shall write

$$W(L/K, \psi_j) = W_j(L/K) , \quad (j = 1, \dots, r) .$$

2) Let again L/K be normal and tame, with $\mathrm{Gal}(L/K) = \Gamma$, and let $T : \Gamma \to GL(n, E)$ be a representation of Γ over some number field E with character χ . Let $a \in L$, define the resolvent

$$(a \mid \chi) = \det\left(\sum_{\gamma} a^{\gamma}\, T(\gamma)^{-1} \right) .$$

Let $K(\chi)$ be the field obtained by adjoining the values of χ to K and let $\mathfrak{o}_{K(\chi)}$ be its ring of algebraic integers. Let $\mathfrak{O}$ be the ring of algebraic integers in L . Then the $(a \mid \chi)$, with $a \in \mathfrak{O}$, generate a finitely generated rank one module over $\mathfrak{o}_{K(\chi)}$, contained in $L(\chi)$, which we denote by $(\mathfrak{O} \mid \chi)$. This module is basic for the determination of the Galois module structure of $\mathfrak{O}$.

Let from now on

$$K = \mathbb{Q} .$$

As a special case of a general theorem one has

THEOREM 1. - $(\mathfrak{G} \mid \chi) = \mathfrak{o}_{Q(\chi)}\, \tau(\chi)$ (cf. [3]).

Let $(\mathfrak{G})$ be the class in the class-group $Cl(Z(\Gamma))$ of the integral group-ring $Z(\Gamma)$, given by the $Z(\Gamma)$-module $\mathfrak{G}$. Let $D(Z(\Gamma))$ be the kernel group, i. e. the kernel of $Cl(Z(\Gamma)) \to Cl(\mathfrak{M})$, $\mathfrak{M}$ being a maximal order of $Q(\Gamma)$. Form Theorem 1, one deduces (cf. [3]) :

THEOREM 2. - $(\mathfrak{G}) \in D(Z(\Gamma))$. (Martinet's conjecture).

3) With L/Q as above, let again $\Gamma = H_{4\ell^r}$, whith each $j = 1, \dots, r$ (or rather with the corresponding class of characters $\psi = \psi_j$) there is associated a surjection :

$$\theta_j : D(Z(H_{4\ell^r})) \to \pm 1 ,$$

and these are independent (cf. [2]). Write

$$\theta_j((\mathfrak{G})) = U_j(L) .$$

Then one has (cf. [2]) :

THEOREM 3. - (i) <u>If</u> $\ell \equiv -1 \pmod 4$ <u>then</u>

$$U_j(L) = W_j(L/Q) \quad (j = 1, \dots, r) ,$$

(ii) <u>If</u> $\ell \equiv 1 \pmod 4$ <u>then</u>

$$U_j(L) = 1 \qquad (j = 1, \dots, r) .$$

One can show that given

$$f : [1, \dots, r] \to [\pm 1]$$

there are infinitely many fields L with $W_j(L) = f(j)$.

4) Case (ii) of Theorem 3 shows that the Galois module structure of $\mathfrak{O}$ will not suffice to determine the root numbers. One needs additional structure, and has to consider in the general situation of tame extensions $\mathfrak{O}$ as a "Hermitian Galois module". There is indeed a Hermitian form of L over $Q(\Gamma)$, induced by the trace.

In the new situation one has to construct a Hermitian Class-group $H\,Cl(Z(\Gamma))$, which classifies locally free rank one $Z(\Gamma)$-modules M with a non-singular Hermitian form on the $Q(\Gamma)$-module spanned by M . The element corresponding to $\mathfrak{O}$ will be denoted by $[\mathfrak{O}]$.

Now return to the case $\Gamma = H_{4\ell^r}$. Then $[\mathfrak{O}]$ belong to a certain subgroup G_r of $H\,Cl(Z(H_{4\ell^r}))$ which under the map $H\,Cl(Z(H_{4\ell^r})) \to Cl(Z(H_{4\ell^r}))$ falls into $D(Z(H_{4\ell^r}))$. Let $\mathbb{F}_\ell$ be the field of ℓ element, $\mathbb{F}_\ell^*$ its multiplicative group. There are then canonical surjections $\omega_j : G_r \to \mathbb{F}_\ell$, so that the diagram

$$(D) \qquad \begin{tikzcd} G_r \arrow[r, "\omega_j"] \arrow[d] & \mathbb{F}_\ell^* \arrow[d, "{[x \to x^{(\ell-1)/2} = (\frac{x}{\ell})_2]}"] \\ D(Z(H_{4\ell^r})) \arrow[r, "\theta_j"] & \pm 1 \end{tikzcd}$$

commutes. Write $\omega_j(\mathfrak{O}) = V_j(L)$.

In place of Theorem 3 one now gets the better

THEOREM 4. - $\quad V_j(L) = W_j(L/Q) \quad (j = 1, \dots, r)$.

Note that the distinctions of cases in Theorem 3 now becomes obvious from diagram (D) .

For other surveys of this general topic see Martinet's Bourbaki report (cf. [4]) and the report of my address to the Vancouver congress (cf. [5]).

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LITERATURE

[1] A. FRÖHLICH. - Artin Root numbers, Conductors, and Representations for Generalized Quaternion groups. Proc. London Math. Soc. 28 (1974) 402-438.

[2] A. FRÖHLICH. - Module Invariants and root numbers for quaternion fields of degree $4\ell^r$. Proc. Camb. Phil. Soc. 76 (1974) 393-399.

[3] A. FRÖHLICH. - Arithmetic and Galois module structure for tame extensions, to be published.

[4] J. MARTINET. - Bases normales et constante de l'équation fonctionnelle des fonctions L d'Artin. Sém. Bourbaki 450, 1973/74.

[5] Proceeding of the International Congress of Mathematicians Vancouver, 1974.

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