

Stochastic stability of the Ekman spiral

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Abstract. Consider the stochastic Navier-Stokes-Coriolis equations in $\mathbb{T}^2 \times (0, b)$ subject to Dirichlet boundary conditions as well as the Ekman spiral which is a stationary solution to the deterministic equations. It is proved that the stochastic Navier-Stokes-Coriolis equation admits a weak martingale solution. Moreover, as an stochastic analogue of the existing deterministic stability results for the Ekman spiral, stochastic stability of the Ekman spiral is proved by considering stationary martingale solutions.

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1. Introduction

Consider the Navier-Stokes equations in the rotational setting on an infinite layer $\mathbb{R}^2 \times (0, b) =: \Omega$ for some $b > 0$,

$$\left\{ \begin{array}{ll} \partial_t u - \nu \Delta u + \omega(e_3 \times u) + (u \cdot \nabla)u + \nabla p = 0, & t > 0, x \in \Omega, \\ \operatorname{div} u = 0, & t > 0, x \in \Omega, \\ u(t, x_1, x_2, 0) = 0, & t > 0, x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \\ u(t, x_1, x_2, b) = e_1 \cdot u_b, & t > 0, x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \\ u(0, \cdot) = u_0, & x \in \Omega. \end{array} \right. \quad (1.1)$$

Here $\omega \in \mathbb{R}$ denotes the speed of rotation, $\nu > 0$ the viscosity of the fluid, $u_b \in \mathbb{R}$ is a constant, and $e_i, i = 1, 2, 3$, denotes the i -th unit vector in \mathbb{R}^3 . Equations (1.1) are sometimes also called Navier-Stokes-Coriolis equations because of the additional Coriolis forcing term. It is a remarkable fact that (1.1) admits an explicit stationary solution (u_b^E, p_b^E) given by

$$u_b^E(x_1, x_2, x_3) = \tilde{u}_b \begin{pmatrix} 1 - e^{-\frac{x_3}{\delta}} \cos\left(\frac{x_3}{\delta}\right) \\ e^{-\frac{x_3}{\delta}} \sin\left(\frac{x_3}{\delta}\right) \\ 0 \end{pmatrix}, \quad p_b^E(x_1, x_2, x_3) = -\omega \tilde{u}_b x_2$$

with $\delta = \frac{b}{k\pi}$, $k \in \mathbb{Z}$, and

$$\tilde{u}_b := u_b(1 - e^{-\frac{b}{\delta}})^{-1} \text{ if } k \text{ is even and } \tilde{u}_b := u_b(1 + e^{-\frac{b}{\delta}})^{-1} \text{ if } k \text{ is odd.}$$

This stationary solution of equation (1.1) is called in honour of the swedish oceanographer V. W. Ekman, the *Ekman spiral*; see [9]. It describes rotating boundary layers between a geostrophic flow and a solid boundary subject to Dirichlet boundary conditions. Starting from this particular stationary solution, it is a natural problem to investigate its stability properties.

Deterministic perturbations of the Ekman spiral by functions u solving the above equation (1.1) have been considered by many authors. In fact, set

$$w := u - u_b^E, \quad \text{and} \quad q := p - p_b^E.$$

Then the pair (w, q) formally satisfies the equations

$$\left\{ \begin{array}{l} \partial_t w - \nu \Delta w + \omega e_3 \times w \\ \quad + (u_E \cdot \nabla)w + w_3 \partial_3 u_b^E + (w \cdot \nabla)w + \nabla q = 0, \quad t > 0, x \in \Omega, \\ \operatorname{div} w = 0, \quad t > 0, x \in \Omega, \\ w(t, x_1, x_2, 0) = 0, \quad t > 0, x_1, x_2 \in \mathbb{R}, \\ w(t, x_1, x_2, b) = 0, \quad t > 0, x_1, x_2 \in \mathbb{R}, \\ w(0, x) = w_0, \quad x \in \Omega, \end{array} \right. \quad (1.2)$$

where $w_0 = u_0 - u_b^E$. It was shown by Desjardin, Dormy and Grenier in [8] that the Ekman spiral is stable in the sense that if for all initial data $w_0 \in L^2_\sigma(D)$ there exists a global weak solution w to (1.2), then

$$\|w(t)\|_2 \leq \|w_0\|_2 \quad \text{for all } t > 0, \quad (1.3)$$

holds true provided the Reynolds number Re is small enough. Furthermore, it was shown by Hess in [16] that for every initial data $w_0 \in L^2_\sigma(D)$ there exists a global weak solution to (1.2) and that this solution satisfies

$$\lim_{t \rightarrow \infty} \|w(t)\|_2 = 0.$$

with an exponential rate. Note that this notion of stability corresponds to the definition of asymptotical stability in dynamical systems. It was proved moreover in [17] that the Ekman spiral in the half space \mathbb{R}^3_+ is nonlinearly stable with respect to L^2 -perturbations provided the corresponding Reynolds number is small enough. Moreover, the decay rate could be computed in terms of the decay of the corresponding linear problem.

For further stability results concerning the Navier-Stokes-Coriolis equations defined on all of \mathbb{R}^3 we refer to [14] and [18].

Observe that the above stability results concern the situation of a fixed speed of rotation ω , whereas e.g. Rousset's result [23] concerns the situation, where ω tends

to ∞ . Very recently, Gallay and Roussier-Michon considered in [12] the Navier-Stokes-Coriolis equations in an infinite layer with periodic boundary conditions in the vertical directions. They proved the existence and uniqueness of a global mild solution to the above system for arbitrarily large data belonging to a certain function space, provided ω is large enough, and investigated its long-time behavior. For related results see also [20] and [3].

The aim of this paper is twofold: first we introduce a *stochastic analogue* of (1.1) in domains D of the form $D := \mathbb{T}^2 \times (0, b)$ by adding an exterior additive stochastic forcing term to (1.1) using a function-space valued Wiener process. Secondly, as the stochastic analogue of the above deterministic stability results we study the *stochastic stability* of the associated stochastic partial differential equation by considering stationary martingale solutions.

More precisely, we consider in the following the stochastic Navier-Stokes-Coriolis equations in the space $L^{2,\text{per}}(D)^3$, by restricting the problem to (x_1, x_2) -periodic solutions of (1.1). Indeed, note that since the Ekman spiral is independent of the variables (x_1, x_2) , it can be viewed as a (x_1, x_2) -periodic function belonging to $L^{2,\text{per}}(D)$.

The stochastic Navier-Stokes equations in the rotational setting can be formulated as

$$\left\{ \begin{array}{l} du_t = [v\Delta u_t - \omega(e_3 \times u_t) - (u_t \cdot \nabla)u_t + \nabla p_t] dt + dW_t \\ \operatorname{div} u_t = 0 \\ u_t(x_1, x_2, 0) = 0 \\ u_t(x_1, x_2, b) = e_1 \cdot u_b, \end{array} \right. \quad (1.4)$$

where (W_t) is a Wiener process on the subspace $H \subset L^{2,\text{per}}(D)^3$ of the closure of smooth (x_1, x_2) -periodic vector-fields having divergence zero. We will reformulate (1.4) as a stochastic evolution equation and consider the resulting stochastic partial differential equation for the velocity

$$du_t = [vA_S u_t - \omega\Pi(e_3 \times u_t) - \Pi(u_t \cdot \nabla u_t)] dt + dW_t \quad (1.5)$$

on the space H . Here $\Pi: L^{2,\text{per}}(D)^3 \rightarrow H$ denotes the Helmholtz projection and $A_S = \Pi\Delta$ the Stokes operator in H . For a precise definition of Π and the Stokes operator in this context, we refer to Section 2. Equation (1.5) can be regarded as a stochastic Navier-Stokes-Coriolis equation.

Stochastic Navier-Stokes equations are studied already for quite some time; we refer to the pioneering works of Bensoussan and Temam [4] and Vishik and Fursikov [26] for existence of weak solutions. Of particular interest to us is the seminal paper [10] by Flandoli and Gatarek concerning the existence of (stationary) martingale solutions of stochastic Navier-Stokes equations.

Similarly to the deterministic case, for a solution u_t of (1.5), we consider

$$u_t = v_t + u_b^E,$$

where u_b^E denotes the Ekman spiral. Then v_t is a solution of the semilinear stochastic evolution equation

$$dv_t = [(vA_S + B)v_t - \Pi(v_t \cdot \nabla v_t)] dt + dW_t \tag{1.6}$$

where B is defined by

$$Bv := -\omega \Pi(e_3 \times v) - \Pi(u_b^E \cdot \nabla v) - \Pi(v_3 \partial_3 u_b^E), \quad v \in D(A_S).$$

In Section 3 we present the main result of this paper concerning existence of weak martingale solutions of (1.6). Our proof is very much inspired by the proof of Theorem 3.2 in [10], the main differences to [10] are due to the additional Ekman and Coriolis terms and the different boundary conditions for the Stokes operator. For simplicity we restrict ourselves to the case of Wiener-processes (W_t) having finite trace covariance Q .

As a stochastic analogue of the deterministic stability of the Ekman spiral, we study in Section 4 the stochastic stability of (1.5) by considering stationary martingale solutions. In particular, the time-invariant distribution μ of a stationary martingale solution can be interpreted as the long-time statistics of random fluctuations of (1.5) around the Ekman spiral. In Corollary 4.3 we prove the existence of stationary martingale solutions with invariant distribution μ satisfying the exponential moment estimate

$$\int e^{\varepsilon \|u\|_H^2} d\mu(u) < \infty$$

for small ε . This means that the fluctuations of (1.5) around the Ekman spiral have exponential tails.

Similar moment estimates for stochastic Navier-Stokes equations in the classical setting, *i.e.* without rotation, can be found *e.g.* in [5, 24] and [22].

2. Deterministic Navier-Stokes-Coriolis equations

Let $D := \mathbb{T}^2 \times (0, b)$ for some $b > 0$. We first introduce Sobolev spaces consisting of functions being periodic in (x_1, x_2) . Given a fixed number $l > 0$, let \mathcal{D} be the space of all functions $f : \overline{D} \rightarrow \mathbb{C}$ which can be expressed as

$$f(x', x_3) = \sum_{k \in J} \hat{f}_k(x_3) e^{iz \langle k, x' \rangle}$$

for some finite subset J of \mathbb{Z}^2 and some $\hat{f}_k \in C^\infty([0, b])$, where $z = \frac{\pi}{l}$. Then $\{e^{iz \langle k, \cdot \rangle} : k \in \mathbb{Z}\}$ is an orthogonal set in $L^2(T)$ with $T = [-l, l]^2$. Note that the coefficients \hat{f}_k of f are uniquely determined by the partial Fourier series of f by

$$\hat{f}_k(x_3) = \frac{1}{(2\pi)^2} \int_T f(x', x_3) e^{-iz \langle k, x' \rangle} dx', \quad k \in \mathbb{Z}^2.$$

We define $\mathcal{D}_{0,\sigma}$, the space of divergence free test functions, as

$$\mathcal{D}_{0,\sigma} := \{f \in [\mathcal{D}]^3 : \operatorname{div} f = 0 \text{ in } D \text{ and } f = 0 \text{ on } \partial D\}.$$

Given $m \in \mathbb{N}_0$ and $f \in D$, we set

$$\|f\|_2^2 = \int_0^b \int_T |f(x', x_3)|^2 dx' dx_3 \quad \text{and} \quad \|f\|_{m,2}^2 = \sum_{|\alpha| \leq m} \|D^\alpha f\|_2^2.$$

Then our basic function spaces are defined as the closures of \mathcal{D} or $\mathcal{D}_{0,\sigma}$ with respect to $\|\cdot\|_{m,2}$ as

$$H^{m,2,\text{per}} = \overline{[\mathcal{D}]^3}^{\|\cdot\|_{m,2}}, \quad L_\sigma^{2,\text{per}} = \overline{\mathcal{D}_{0,\sigma}}^{\|\cdot\|_2}, \quad H_{0,\sigma}^{1,2,\text{per}} = \overline{\mathcal{D}_{0,\sigma}}^{\|\cdot\|_{1,2}}.$$

We next introduce the Helmholtz projection from $L^{2,\text{per}}(D)$ onto $L_\sigma^{2,\text{per}}(D)$. Note first that the existence of the Helmholtz projection for spaces of the form $L^2(D)$ is well known; see *e.g.* [25] or [11]. Adapting their arguments to partial Fourier series one obtains the following result. For each $u \in L^{2,\text{per}}(D)$, there exists a unique $v \in L_\sigma^{2,\text{per}}(D)$ satisfying $u = v + \nabla p$ for some scalar $p \in H^{1,2,\text{per}}(D)$ and a constant $C > 0$ such that

$$\|v\|_2 + \|\nabla p\|_2 \leq C\|u\|_2$$

Hence, the mapping

$$u \in L^{2,\text{per}}(D) \mapsto v = \Pi u \in L_\sigma^{2,\text{per}}(D)$$

defines a bounded linear operator Π from $L^{2,\text{per}}(D)$ onto $L_\sigma^{2,\text{per}}(D)$, which is called the Helmholtz projection. The Stokes operator A_S in $L_\sigma^{2,\text{per}}(D)$ is then defined by

$$A_S u := \Pi \Delta u \quad \text{with} \quad D(A_S) := H_{0,\sigma}^{1,2,\text{per}}(D) \cap H^{2,2,\text{per}}(D).$$

Note that the Stokes operator A_S could be defined alternatively also by variational methods, see *e.g.* [25]. Observe, however, that then the above characterization of the domain $D(A_S)$ for mixed periodic-Dirichlet boundary data is not obvious.

Adapting the arguments given in [1, 2] or [15] to the mixed periodic-Dirichlet setting, we see that the Stokes operator has the following spectral properties.

Lemma 2.1. *Let $0 < \varepsilon < \frac{\pi}{2}$ and $\lambda \in \Sigma_{\pi-\varepsilon} := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi - \varepsilon\}$. Then for $f \in L_\sigma^{2,\text{per}}(D)$ there exists a unique $u \in D(A_S)$ satisfying $(\lambda - \nu A_S)u = f$ and a constant $C = C(\varepsilon) > 0$ such that*

$$|\lambda| \|u\|_2 + \nu \|u\|_{2,2} \leq C \|f\|_2. \tag{2.1}$$

The above lemma implies in particular that the Stokes operator A_S generates an analytic semigroup on $L_\sigma^{2,\text{per}}(D)$ and that the set $\mathbb{C} \setminus (-\infty, 0]$ is contained in the resolvent set of A_S . Moreover, since the estimate (2.1) is uniform in $\lambda \in (0, \infty)$, it follows that

$$0 \in \varrho(A_S) \quad \text{and} \quad \nu \|(A_S)^{-1} f\|_{2,2} \leq C \|f\|_2$$

for some constant $C > 0$. A more detailed analysis even shows that $\mathbb{C} \setminus (-\infty, -\frac{\pi^2}{b^2}] \subset \varrho(A_S)$.

The precise description of $D(A_S)$ will be important in step 3 of the proof of our main result, where the tightness of the sequence P^n of projection on the span of the first n eigenvectors of the Stokes operator is proved by the compactness of certain embeddings.

Finally, for $V := H_{0,\sigma}^{1,2,\text{per}}(D)$, equipped with the equivalent norm $\|u\|_V^2 := \langle -A_S u, u \rangle_H$, consider operators $B : V \rightarrow H$ for which there exist constants $\omega_0 > 0, \omega_1 \geq 0$ satisfying

$$\langle (\nu A_S + B)u, u \rangle_H \leq -\omega_0 \|u\|_V^2 + \omega_1 \|u\|_H^2, \quad u \in D(A_S).$$

It then follows by standard perturbation theory that $\nu A_S + B$ with domain $D(A_S)$ generates an analytic semigroup on H .

3. Stochastic Navier-Stokes-Coriolis equations

We start this section by adding noise as an independent exterior forcing term to (1.1) and obtain the equation

$$\left\{ \begin{array}{l} du_t = [\nu \Delta u_t - \omega(e_3 \times u_t) - (u_t \cdot \nabla)u_t + \nabla p_t]dt + dW_t \\ \operatorname{div} u_t = 0 \\ u_t(x_1, x_2, 0) = 0 \\ u_t(x_1, x_2, b) = e_1 \cdot u_b. \end{array} \right. \quad (3.1)$$

Here, $(W_t)_{t \geq 0}$ is an H -valued Q -Wiener process, where $H := L_\sigma^{2,\text{per}}(D)$, defined on an underlying stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ (see [6, Chapter 4]). Note that the covariance operator Q of $(W_t)_{t \geq 0}$ necessarily has finite trace.

Applying the Helmholtz projection Π introduced in Section 2 to (3.1) leads to the following semilinear stochastic evolution equation

$$du_t = [\nu A_S u_t - \omega \Pi(e_3 \times u_t) - \Pi(u_t \cdot \nabla u_t)] dt + dW_t \quad (3.2)$$

in H . Let $V' = H_{0,\sigma}^{-1,2,\text{per}}(D)$ be the (topological) dual space of V . Identifying H with its dual H' we have that $V \hookrightarrow H \equiv H' \hookrightarrow V'$ densely and continuously.

We denote by ${}_{V'}\langle u, v \rangle_V$, $u \in V'$, $v \in V$, the dualization of V' with V . Note that ${}_{V'}\langle u, v \rangle_V = \langle u, v \rangle_H$ if $u \in H$.

As described above, for a solution u_t of (1.5), we consider

$$u_t = v_t + u_b^E,$$

where u_b^E denotes the Ekman spiral. Then v_t is a solution of the semilinear stochastic evolution equation

$$dv_t = [(vA_S + B)v_t - \Pi(v_t \cdot \nabla v_t)] dt + dW_t \tag{3.3}$$

where B is defined by

$$Bv := -\omega \Pi(e_3 \times v) - \Pi(u_b^E \cdot \nabla v) - \Pi(v_3 \partial_3 u_b^E), \quad v \in D(A_S).$$

We will study in the following the equation (1.6) not only for B defined as above, but for linear operators B satisfying the following two assumptions:

(A1) $B : V \rightarrow H$ is a bounded linear operator such that there exist $\omega_0 > 0$, $\omega_1 \geq 0$ with

$$\langle (vA_S + B)u, u \rangle_H \leq -\omega_0 \|u\|_V^2 + \omega_1 \|u\|_H^2, \quad u \in D(A_S).$$

(A2) The covariance operator Q of $(W(t))_{t \geq 0}$ has finite trace.

Remarks 3.1. a) We note that (A2) implies that there exists an orthonormal basis $(f_k)_{k \geq 1}$ of H consisting of eigenvectors of Q with corresponding eigenvalues $(\mu_k^2)_{k \geq 1}$ such that $(W_t)_{t \geq 0}$ can be represented as

$$W_t = \sum_{k=1}^{\infty} \mu_k f_k \cdot \beta_k(t),$$

where $(\beta_k(t))_{t \geq 0}$, $k \geq 1$, are independent, 1-dimensional Brownian motions on (Ω, \mathcal{F}, P) .

b) Note also that (A1) implies as described in Section 2 that the operator $vA_S + B$ with domain $D(A_S)$ generates an analytic C_0 -semigroup $(T_t)_{t \geq 0}$ on H satisfying $\|T(t)\| \leq e^{(\omega_1 - \frac{\omega_0^2}{b^2} \omega_0)t}$ for all $t \geq 0$.

We now introduce the notion of a martingale solution to equation (1.6).

Definition 3.2. A martingale solution of (1.6) is a pair $((v_t)_{t \geq 0}, (\tilde{W}_t)_{t \geq 0})$, defined on a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{P})$ satisfying

- (i) $(\tilde{W}_t)_{t \geq 0}$ is an $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ Wiener-process with covariance Q ,
- (ii) $v. : [0, T] \times \tilde{\Omega} \rightarrow H$ is progressively measurable,
- (iii) $v. \in L^\infty([0, T]; H) \cap L^2([0, T]; V) \quad \tilde{P} - a.s.$,

(iv) For all $u \in V$ and all $t \in [0, T]$ $\tilde{P} - a.s.$

$$\begin{aligned} \langle v_t, u \rangle_H &= \langle v_0, u \rangle_H + \int_0^t v_s \langle (v A_S + B)v_s, u \rangle_V ds \\ &\quad - \int_0^t \langle \Pi(v_s \cdot \nabla v_s), u \rangle_H ds + \langle \tilde{W}_t, u \rangle_H. \end{aligned}$$

The following theorem is the main result of this section.

Theorem 3.3. *Assume (A1) and (A2). Let $\xi : \Omega \rightarrow H$ be \mathcal{F}_0 -measurable, square-integrable and independent of $(W_t)_{t \geq 0}$. Then for all $T > 0$ there exists a martingale solution $((v_t)_{t \geq 0}, (\tilde{W}_t)_{t \geq 0})$ of (1.6) satisfying $\tilde{P} \circ v_0^{-1} = P \circ \xi^{-1}$. Moreover,*

$$\tilde{E} \left(\sup_{0 \leq t \leq T} \|v_t\|_H^2 + \int_0^T \|v_t\|_V^2 dt \right) < \infty. \tag{3.4}$$

The following proof of Theorem 3.3 follows closely the proof of [10, Theorem 3.2]. To this end, denote by $(e_k)_{k \geq 1}$ an orthonormal basis of H consisting of eigenvectors of A_S .

Step 1: Galerkin approximations

Denote by

$$P_n : H \rightarrow \text{span}\{e_k : k \leq n\} \subset H, x \mapsto \sum_{k=1}^n \langle x, e_k \rangle e_k$$

the orthogonal projection onto the linear subspace of H given as the linear span of the first n eigenvectors and consider for fixed n the following stochastic evolution equation:

$$\begin{aligned} dv_t^n &= [P_n(v A_S + B)v_t^n + P_n F_n(v_t^n, v_t^n)] dt + dP_n W_t, \\ v_0^n &= P_n \xi_0. \end{aligned} \tag{3.5}$$

Here,

$$F_n(u, v) := \Theta_n(\|u\|_H)(u \cdot \nabla)v, \quad u, v \in P_n H,$$

for some continuously differentiable cutoff-function $\Theta_n : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$1_{[-n, n]} \leq \Theta_n \leq 1_{[-n-1, n+1]}.$$

Note that for $u, v \in P_n H$ we have

$$\langle P_n F_n(u, v), v \rangle_H = \langle F_n(u, v), v \rangle_H = \langle \Theta_n(\|u\|_H)(u \cdot \nabla)v, v \rangle_H = 0. \tag{3.6}$$

The above equation (3.5) can be viewed as a finite dimensional stochastic differential equation having locally Lipschitz continuous coefficients of at most linear

growth. Now [21, Theorem 1 in Section V.1] implies that (3.5) has a unique strong solution satisfying in addition the following moment estimate

$$E\left(\sup_{0 \leq t \leq T} \|v_t^n\|_H^2\right) < \infty.$$

Step 2: Energy estimate

In this step we will derive the uniform energy estimate

$$E\left(\sup_{0 \leq t \leq T} \|v_t^n\|_H^2 + \int_0^T \|v_t^n\|_V^2 dt\right) \leq M < \infty \quad (3.7)$$

for some finite constant M independent of n . In order to establish (3.7), note that by Ito's formula

$$\begin{aligned} \frac{1}{2}e^{-\omega_1 t} \|v_t^n\|_H^2 &= \frac{1}{2}\|P_n \xi_0\|_H^2 + \int_0^t e^{-\omega_1 s} \langle v_s^n, dP_n W_s \rangle_H \\ &\quad + \int_0^t e^{-\omega_1 s} \left(\langle v_s^n, P_n(vA_s + B)v_s^n + P_n F_n(v_s^n, v_s^n) \rangle_H - \omega_1 \|v_s^n\|_H^2 \right) ds \\ &\quad + \frac{1}{2\omega_1} (1 - e^{-\omega_1 t}) \operatorname{tr}_H(P_n \circ Q). \end{aligned} \quad (3.8)$$

Taking into account (3.6) and assumption (A1) we obtain

$$\begin{aligned} \langle v_s^n, P_n(vA_s + B)v_s^n + P_n F_n(v_s^n, v_s^n) \rangle_H &= \langle v_s^n, (vA_s + B)v_s^n \rangle_H \\ &\leq -\omega_0 \|v_s^n\|_V^2 + \omega_1 \|v_s^n\|_H^2. \end{aligned}$$

Inserting the last inequality into (3.8) and rearranging terms we obtain

$$\begin{aligned} &\frac{1}{2}e^{-\omega_1 t} \|v_t^n\|_H^2 + \omega_0 \int_0^t e^{-\omega_1 s} \|v_s^n\|_V^2 ds \\ &\leq \frac{1}{2}\|P_n \xi_0\|_H^2 + \frac{1}{2\omega_1} (1 - e^{-\omega_1 t}) \operatorname{tr}_H(P_n \circ Q) + \int_0^t e^{-\omega_1 s} \langle v_s^n, dP_n W_s \rangle_H \end{aligned}$$

and thus

$$\begin{aligned} &E\left(\sup_{0 \leq t \leq T} \|v_t^n\|_H^2 + 2\omega_0 \int_0^T \|v_s^n\|_V^2 ds\right) \\ &\leq e^{\omega_1 T} E\left(\|P_n \xi_0\|_H^2\right) + \frac{e^{\omega_1 T}}{\omega_1} \operatorname{tr}_H(P_n \circ Q) \\ &\quad + e^{\omega_1 T} E\left(\sup_{0 \leq t \leq T} \left| \int_0^t e^{-\omega_1 s} \langle v_s^n, dP_n W_s \rangle \right|\right). \end{aligned} \quad (3.9)$$

Since the stochastic integral $\int_0^t e^{-\omega_1 s} \langle v_s^n, dP_n W_s \rangle, t \geq 0$, is a continuous $(\mathcal{F}_t)_{t \geq 0}$ -martingale, Doob's maximal inequality implies that

$$\begin{aligned} E \left(\sup_{0 \leq t \leq T} \left| \int_0^t e^{-\omega_1 s} \langle v_s^n, dP_n W_s \rangle \right| \right) &\leq 2E \left(\left(\int_0^T e^{-\omega_1 s} \langle v_s^n, dP_n W_s \rangle \right)^2 \right)^{\frac{1}{2}} \\ &= 2E \left(\int_0^T e^{-2\omega_1 s} \|\sqrt{Q} v_s^n\|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq 2\|\sqrt{Q}\|_{\text{op}} \sqrt{T} \cdot E \left(\sup_{0 \leq t \leq T} \|v_t^n\|_H^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} e^{-\omega_1 T} E \left(\sup_{0 \leq t \leq T} \|v_t^n\|_H^2 \right) + 2e^{\omega_1 T} \|\sqrt{Q}\|_{\text{op}}^2 T. \end{aligned}$$

Here, $\|\sqrt{Q}\|_{\text{op}}$ denotes the usual operator norm of \sqrt{Q} on H . Inserting the last inequality into (3.9) we obtain the energy estimate

$$\begin{aligned} &E \left(\sup_{0 \leq t \leq T} \|v_t^n\|_H^2 + 4\omega_0 \int_0^T \|v_s^n\|_V^2 ds \right) \\ &\leq 2e^{\omega_1 T} E \left(\|P_n \xi_0\|_H^2 \right) + 2 \frac{e^{\omega_1 T}}{\omega_1} \text{tr}_H(P_n \circ Q) + 4e^{2\omega_1 T} T \|\sqrt{Q}\|_{\text{op}}^2 \\ &\leq 2e^{\omega_1 T} E \left(\|\xi_0\|_H^2 \right) + 2 \frac{e^{\omega_1 T}}{\omega_1} \text{tr}_H(Q) + 4e^{2\omega_1 T} T \|\sqrt{Q}\|_{\text{op}}^2 =: M \end{aligned}$$

uniformly in n .

Step 3: Tightness of $P^n := P \circ (v \cdot)^{-1}$

We next show that $P^n, n \geq 1$, is tight on the space $L^2([0, T]; H)$. By Prohorov's theorem, it is sufficient to show that

$$\sup_n E \left(\|v \cdot^n\|_{W^{\beta,2}([0,T], D((-A_S)^{-\frac{\alpha}{2}}))} \right) < \infty \tag{3.10}$$

for some $\alpha > 3$ and $\beta \in (0, \frac{1}{2})$ since the embedding

$$L^2([0, T]; V) \cap W^{\beta,2}([0, T]; D(A_S^{-\frac{\alpha}{2}})) \hookrightarrow L^2([0, T]; H)$$

is compact (see [10, Theorem 2.1]). Here, for a given Banach space X , the space $W^{\beta,p}([0, T]; X)$ denotes the Slobodeckii space defined to be the space consisting of all $u \in L^p([0, T]; X)$ satisfying

$$\|u\|_{W^{\beta,p}([0,T];X)}^p := \int_0^T \int_0^T \frac{|u(s) - u(t)|_X^p}{|s - t|^{1+\beta p}} ds dt < \infty,$$

and the space $L^2([0, T]; V) \cap W^{\beta, 2}([0, T]; D(A_S^{-\frac{\alpha}{2}}))$ is endowed with the natural norm

$$\|u\|_{L^2([0, T]; V)} + \|u\|_{W^{\beta, p}([0, T]; D((-A_S)^{-\frac{\alpha}{2}})}.$$

Note that [10, Theorem 2.1] can be applied, since the embedding $V \hookrightarrow H$ is compact. We postpone the proof of (3.10) to Lemma 3.4.

Since P^n is tight, actually tight on $L^2([0, T]; H) \cap C([0, T]; D((-A_S)^{-\frac{\alpha}{2}}))$ for all $\alpha > 3$, we may choose a subsequence, again denoted by (P^n) , converging weakly to some probability measure P on $L^2([0, T]; H) \cap C([0, T]; D((-A_S)^{-\frac{\alpha}{2}}))$.

Step 4: *The limit $n \rightarrow \infty$*

This part of the proof is completely analogous to Step 3 of the proof of [10, Theorem 3.1]. We hence only give a brief sketch of it.

Fix $\alpha > 3$. Then Skorohod’s embedding theorem (see [19]) implies that there exists a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{P})$ and previsible processes $\tilde{v}^n, v \in L^2([0, T]; H) \cap C([0, T], D(A_S^{-\frac{\alpha}{2}}))$ with

$$P \circ (v^n)^{-1} = \tilde{P} \circ (\tilde{v}^n)^{-1}, \quad n \in \mathbb{N}$$

and $\tilde{v}^n \rightarrow \tilde{v}$, $\tilde{P} - a.s$ in $L^2([0, T]; H) \cap C([0, T], D((-A_S)^{-\frac{\alpha}{2}}))$. The uniform energy estimate (3.7) implies by Fatou’s lemma, the estimate

$$\tilde{E} \left(\sup_{0 \leq t \leq T} \|\tilde{v}_t\|_H^2 + \int_0^T \|\tilde{v}_t\|_V^2 dt \right) < \infty.$$

Hence, $\tilde{v} \in L^2([0, T]; V)$. In order to identify the limiting process \tilde{v} as a martingale solution of (1.5) note that for all n

$$\tilde{M}_t^n := \tilde{v}_t^n - P_n \xi_0 - \int_0^t P_n (v A_S + B) \tilde{v}_s^n ds - \int_0^t P_n F_n(\tilde{v}_s^n, \tilde{v}_s^n) ds,$$

$t \in [0, T]$, is a continuous square-integrable martingale w.r.t $\tilde{\mathcal{G}}_t^n := \sigma(\tilde{v}_t^n : s \in [0, t])$ converging \tilde{P} -a.s. to

$$\tilde{M}_t := \tilde{v}_t - \xi_0 - \int_0^t (v A_S + B) \tilde{v}_s ds - \int_0^t (\tilde{v}_s \cdot \nabla \tilde{v}_s) ds$$

for all $t \in [0, T]$ in $D(A_S^{-\frac{\alpha}{2}})$ for all $\alpha > 3$.

This implies that \tilde{M}_t , $t \in [0, T]$, is a martingale w.r.t. the filtration $\tilde{\mathcal{G}}_t := \sigma(\tilde{v}_t : s \in [0, t])$. The martingale representation theorem implies that there exists a Q -Wiener process $(\tilde{W}_t)_{t \geq 0}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that

$$\langle \tilde{M}_t, u \rangle = \langle \tilde{W}_t, u \rangle \quad \tilde{P} - a.s., \quad u \in V.$$

The proof is complete. □

We finally prove the tightness of P^n .

Lemma 3.4. *Let $\alpha > 3$, $\beta \in (0, \frac{1}{2})$. Then*

$$\sup_n E \left(\|v_t^n\|_{W^{\beta,2}([0,T]; D((-A_S)^{-\frac{\alpha}{2}}))} \right) < \infty.$$

Proof. We decompose the unique strong solution v_t^n of (3.5) as

$$\begin{aligned} v_t^n &= P_n \xi_0 + \int_0^t P_n (\nu A_S + B) v_s^n ds + \int_0^t P_n F_n(v_s^n, v_s^n) ds + P_n W_t \\ &=: I_1 + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

It is then sufficient to show the desired estimate for each term separately. Clearly, for I_1 , we obtain

$$\|I_1\|_{W^{\beta,2}([0,T]; D((-A_S)^{-\frac{\alpha}{2}}))} = \sqrt{T} \|P_n \xi_0\|_{D((-A_S)^{-\frac{\alpha}{2}})} \leq C \cdot \sqrt{T} \cdot \|\xi_0\|_H$$

for some uniform constant $C > 0$, so that

$$\sup_n E \left(\|I_1\|_{W^{\beta,2}([0,T]; D((-A_S)^{-\frac{\alpha}{2}}))} \right) < \infty.$$

By Lemma 2.1 in [10],

$$\|P_n W\|_{W^{\beta,2}([0,T]; D(A_S^{-\frac{\alpha}{2}}))}^2 \leq C(\beta, 2) \cdot T \cdot \text{tr}_H(\sqrt{Q} \circ P_n \circ \sqrt{Q})$$

for some uniform constant $C(\beta, 2)$. Thus

$$\sup_n E \left(\|I_4(\cdot)\|_{W^{\beta,2}([0,T]; D(A_S^{-\frac{\alpha}{2}}))} \right) < \infty.$$

Concerning I_2 , note that $\dot{I}_2(t) = P_n(\nu A_S + B)v_t^n$, and thus

$$\|\dot{I}_2(t)\|_{V'} = \|P_n(\nu A_S + B)v_t^n\|_{V'} \leq C \|v_t^n\|_V$$

for some universal constant C , because

$$\begin{aligned} \langle P_n(\nu A_S + B)v_t^n, u \rangle_H &= \nu \langle A_S v_t^n, P_n u \rangle_H + \langle B v_t^n, P_n u \rangle_H \\ &= \nu \langle A_S v_t^n, u \rangle_H + \langle B v_t^n, P_n u \rangle_H \\ &\leq \nu \|v_t^n\|_V \cdot \|u\|_V + \|B v_t^n\|_H \|P_n u\|_H \\ &\leq C \|v_t^n\|_V \cdot \|u\|_V. \end{aligned}$$

Here we used assumption (A.1) and the fact that $\|P_n u\|_H \leq \|u\|_H \leq C \|u\|_V$ holds for some uniform constant $C > 0$. Consequently,

$$\sup_n E \left(\int_0^T \|\dot{I}_2(t)\|_{V'}^2 dt \right) \leq C \sup_n E \left(\int_0^T \|v_t^n\|_V^2 dt \right) < \infty,$$

by using the energy estimate (3.7). Since $\|I_2(\cdot)\|_{W^{\beta,2}([0,T];D((-A_S)^{-\frac{\alpha}{2}}))}^2$ can obviously be estimated from above by $\|I_2(\cdot)\|_{W^{1,2}([0,T];V')}^2$ up to some uniform constant, we obtain the desired uniform estimate

$$\sup_n E\left(\|I_2(\cdot)\|_{W^{\beta,2}([0,T];D(A_S^{-\frac{\alpha}{2}}))}\right) < \infty.$$

Finally, note that

$$\begin{aligned} \langle P_n F_n(v_t^n, v_t^n), u \rangle_H &= \langle \Theta_n(\|v_t^n\|_H) v_t^n \cdot \nabla v_t^n, P_n u \rangle_H \\ &\leq \int_{\Omega_b} |v_t^n \cdot \nabla v_t^n|_{\mathbb{R}^3} dx \cdot \|P_n u\|_{\mathbb{R}^3} \|\infty \\ &\leq C \|v_t^n\|_H \cdot \|v_t^n\|_V \|u\|_{D((-A_S)^{\frac{\alpha}{2}})} \end{aligned}$$

for some uniform constant, since

$$\|P_n u\|_{\mathbb{R}^3} \|\infty \leq C \|P_n u\|_{D((-A_S)^{\frac{\alpha}{2}})}$$

for $\alpha > 3$. Hence,

$$\|P_n F_n(v_t^n, v_t^n)\|_{D(A_S^{-\frac{\alpha}{2}})} \leq C \|v_t^n\|_H \|v_t^n\|_V$$

for some uniform constant. This implies that I_3 is differentiable with respect to t in $D(A_S^{-\frac{\alpha}{2}})$ and that

$$\|\dot{I}_3(t)\|_{D((-A_S)^{-\frac{\alpha}{2}})} \leq C \|v_t^n\|_H \|v_t^n\|_V.$$

Thus

$$\|I_3\|_{W^{1,2}([0,T];D(A_S^{-\frac{\alpha}{2}}))}^2 \leq C \sup_{0 \leq t \leq T} \|v_t^n\|_H^2 \cdot \int_0^T \|v_t^n\|_V^2 dt,$$

and

$$E\left(\|I_3(\cdot)\|_{W^{1,2}([0,T];D((-A_S)^{-\frac{\alpha}{2}}))}\right) \leq \frac{C}{2} E\left(\sup_{0 \leq t \leq T} \|v_t^n\|_H^2 + \int_0^T \|v_t^n\|_V^2 dt\right)$$

is uniformly bounded in n which finally implies the desired estimate for I_3 . Summarizing, the lemma is proven. \square

We now may apply Theorem 3.3 to the stochastic evolution equation (1.6), with B given by

$$Bv = \omega \Pi(e_3 \times v) - \Pi(u_b^E \cdot \nabla v) - \Pi(v_3 \partial_3 u_b^E).$$

Since B satisfies assumption (A.1), we obtain from Theorem 3.3 that for any $\mu \in \mathcal{M}_1(H)$ satisfying $\int \|x\|_H^2 d\mu(x) < \infty$, there exists a martingale solution $((v_t)_{t \geq 0}, (W_t)_{t \geq 0})$ of (1.6) satisfying $\tilde{P}_0 \circ v_0^{-1} = \mu$ and the energy estimate

$$E\left(\sup_{0 \leq t \leq T} \|v_t\|_H^2 + \int_0^T \|v_s\|_V^2 ds\right) < \infty.$$

Defining the stochastic process

$$u_t := v_t + u_b^E, \quad t \geq 0,$$

it follows that $((u_t)_{t \geq 0}, (W_t)_{t \geq 0})$ is a martingale solution of (1.6). Thus, the following corollary holds

Corollary 3.5. *Let $\mu \in \mathcal{M}_1(H)$ with $\int \|x\|_H^2 d\mu(x) < \infty$. Then there exists a weak martingale solution of the stochastic Navier-Stokes-Coriolis-Ekman equation (1.6) with initial distribution μ .*

4. Stationary martingale solutions and invariant measures

We recall that a stochastic process $(X_t)_{t \geq 0}$ is called *stationary* if the distribution of the time-shifted process $(X_{s+\cdot})_{t \geq 0}$ is independent of s , i.e.,

$$P \circ (X_{s+\cdot})^{-1} = P \circ (X_\cdot)^{-1}, \quad s \geq 0.$$

Stationarity implies in particular that the distribution of X_t is independent of t . Hence, $\mu = P \circ X_0^{-1}$ is called an *invariant probability measure*.

A particular example for a stationary process is a Markov process $((X_t)_{t \geq 0}, (P_x)_{x \in E})$ on a state space (E, \mathcal{E}) having an invariant measure μ in the sense that for any \mathcal{E} -measurable bounded function F

$$\int E_x(F(X_t))\mu(dx) = \int F(x)\mu(dx), \quad t \geq 0. \quad (4.1)$$

In this case, the distribution of the shifted process $(X_{s+t})_{t \geq 0}$ with respect to the probability measure

$$P_\mu(A) := \int P_x(A)\mu(dx)$$

will be independent of s .

In the case of the stochastic Navier-Stokes-Coriolis equation we are yet far away from the construction of a full Markov process, however using the same techniques as in Section 3, we are able to construct a stationary martingale solution under additional assumptions on the coefficients. To this end, consider again first the perturbed stochastic Navier-Stokes equation (1.6) for general B satisfying assumption (A.1) with $\omega_1 < \frac{\pi^2}{b^2}\omega_0$. Then

$$\langle (vA_S + B)u, u \rangle_H \leq -\tilde{\omega}_0 \|u\|_V^2, \quad u \in D(A_S).$$

with $\tilde{\omega}_0 := \omega_0 - \frac{b^2}{\pi^2}\omega_1 > 0$.

Theorem 4.1. *There exists a stationary martingale solution $((v_t)_{t \geq 0}, (W_t)_{t \geq 0})$ of (1.6) satisfying*

$$E \left(\sup_{0 \leq t \leq T} \|v_t\|_H^2 + \int_0^T \|v_t\|_V^2 dt \right) < \infty, \quad T > 0. \tag{4.2}$$

Moreover, the invariant distribution $\mu = P \circ v_t^{-1}$ satisfies the moment estimates

$$\int e^{\varepsilon \|x\|_H^2} \mu(dx) < \infty \quad \text{for } \varepsilon < \varepsilon_0 := \frac{\tilde{\omega}_0}{\|Q\|_{\text{op}}} \frac{\pi^2}{b^2} \tag{4.3}$$

and

$$\int \|x\|_V^2 \mu(dx) < \infty. \tag{4.4}$$

Proof. Similar to the proof of Theorem 3.3, we consider the finite dimensional Galerkin approximation

$$dv_t^n = [P_n(vA_S + B)v_t^n + P_n F_n(v_t^n, v_t^n)] dt + dP_n W_t. \tag{4.5}$$

Equation (4.5) has a unique strong solution $v^n(t, v_0)$ for any initial condition $v_0 \in P_n H$. Moreover, the associated transition semigroup

$$P_t^n F(v_0) := E(F(v^n(t, v_0))) \quad t \geq 0,$$

has the Feller property, i.e. $P_t^n(C_b(P_n H)) \subset C_b(P_n H)$, where $C_b(P_n H)$ denotes the space of bounded continuous functions on $P_n H$.

Itô's formula, applied to $\frac{1}{2} \|v^n(t, v_0)\|_H^2$, yields the estimate

$$\begin{aligned} \frac{1}{2} \|v^n(t, v_0)\|_H^2 &\leq \frac{1}{2} \|v_0\|_H^2 + \int_0^t \langle v_s^n, dP_n W_s \rangle_H \\ &\quad - \tilde{\omega}_0 \int_0^t \|v^n(s, v_0)\|_V^2 ds + \frac{t}{2} \cdot \text{tr}_H(P_n \circ Q). \end{aligned}$$

In particular, taking expectations we conclude that

$$E \left(\frac{1}{t} \int_0^t \|v^n(s, v_0)\|_V^2 ds \right) \leq \frac{1}{2t\tilde{\omega}_0} \|v_0\|_H^2 + \frac{1}{2\tilde{\omega}_0} \text{tr}_H(P_n \circ Q), \tag{4.6}$$

which implies that the family $\mu_n(t, A) := \frac{1}{t} \int_0^t P(v^n(s, v_0) \in A) ds$, $A \in \mathcal{B}(P_n H)$, $t \geq 0$, of the mean occupation time measures is tight on $P_n H$. The Krylov-Bogoliubov Theorem (see [7]) implies the existence of an invariant probability measure μ_n on $P_n H$ for (P_t^n) , i.e.,

$$\int_{P_n H} P_t^n F d\mu_n = \int_{P_n H} F d\mu_n$$

satisfying the moment estimate

$$\int_{P_n H} \|v\|_V^2 \mu_n(dv) \leq \frac{1}{2\tilde{\omega}_0} \operatorname{tr}_H(P_n \circ Q) \leq \frac{1}{2\tilde{\omega}_0} \operatorname{tr}_H(Q) \tag{4.7}$$

uniformly in n . Moreover, we construct in the usual way a stationary martingale solution $((v_t^n)_{t \geq 0}, (W_t)_{t \geq 0})$ with distribution μ_n . In Lemma 4.2 below we will prove that for $\varepsilon < \varepsilon_0 = \frac{\tilde{\omega}_0}{\|Q\|_{\text{op}}} \frac{\pi^2}{b^2}$, μ_n also satisfies the exponential moment estimate

$$\int_{P_n H} e^{\varepsilon \|v\|_H^2} \mu_n(dv) \leq C_1(\varepsilon) \tag{4.8}$$

for some finite constant $C_1(\varepsilon)$ independent of n .

We next show that the family P_n of distributions of $(v_t^n)_{t \leq 0}$ is tight on $L^2_{\text{loc}}([0, T]; H) \cap C([0, T]; D(A_S^{-\frac{\alpha}{2}}))$ for any $\alpha > 3$ and any finite T . To this end, it suffices to recall from the proof of Theorem 3.3 that for all $T > 0$

$$\sup_n E\left(\|v^n(\cdot, v_0)\|_{W^{\beta,2}([0,T]; D((-A_S)^{-\frac{\alpha}{2}}))}\right) < \infty$$

for any $\alpha > 3$ and $\beta \in (0, \frac{1}{2})$, see (3.10). Now, similarly to Step 4 in the proof of Theorem 3.3, we find a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{F}_t)_{t \geq 0}, \tilde{P})$ and previsible processes \tilde{v}^n, \tilde{v} such that $\tilde{P} \circ (\tilde{v}^n)^{-1}$ coincides with the distribution of v^n . In particular, \tilde{v}^n is stationary for all n , hence $\tilde{v}_t = \lim_{n \rightarrow \infty} \tilde{v}_t^n$ is stationary too. Similar to the proof of Theorem 3.3 we can show that there exists a Q -Wiener process $(\tilde{W}_t)_{t \geq 0}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that

$$\langle \tilde{W}_t, u \rangle =_{\tilde{V}} \langle \tilde{v}_t - \tilde{v}_0 - \int_0^t (v A_S + B) \tilde{v}_s ds - \int_0^t \Pi(\tilde{v}_s \cdot \nabla \tilde{v}_s) ds, u \rangle_{\tilde{V}} \quad \tilde{P}\text{-a.s.}, u \in V.$$

which implies the assertion of the theorem. □

It remains to prove the exponential moment estimate (4.8).

Lemma 4.2. *Let $t_k \uparrow \infty$ and μ_n be an accumulation point of the mean occupation time measures $\frac{1}{t_k} \int_0^{t_k} P(v^n(s, v_0) \in A) ds$, $A \in \mathcal{B}(P_n H)$, $k \geq 1$, for some $v_0 \in P_n H$. Then, for $\varepsilon < \varepsilon_0 = \frac{\tilde{\omega}_0}{\|Q\|_{\text{op}}} \frac{\pi^2}{b^2}$, there exists a finite constant $C_1(\varepsilon)$, independent of n , such that*

$$\int_{P_n H} e^{\varepsilon \|v\|_H^2} d\mu_n(v) \leq C_1(\varepsilon).$$

Proof. Since for any deterministic initial condition v_0 , $\|v_0\|_H \in \mathcal{L}^m(P)$ for all m , it follows that

$$E\left(\sup_{0 \leq t \leq T} \|v^n(t, v_0)\|_H^{2m}\right) < \infty \quad \text{for all } T.$$

Ito's formula implies for all $m \geq 1$

$$\begin{aligned} \|v^n(t, v_0)\|_H^{2m} &\leq \|v_0\|_H^{2m} \\ &\quad + m \int_0^t \|v^n(s, v_0)\|_H^{2(m-1)} (\text{tr}_H(P_n \circ Q) - 2\tilde{\omega}_0 \|v^n(s, v_0)\|_V^2) ds \\ &\quad + 2m(m-1) \int_0^t \|v^n(s, v_0)\|_H^{2(m-2)} \langle Qv^n(s, v_0), v^n(s, v_0) \rangle_H ds \\ &\quad + 2m \int_0^t \|v^n(s, v_0)\|_H^{2(m-1)} \langle v^n(s, v_0), dP_n W_s \rangle_H. \end{aligned}$$

Since $\langle Qv^n(s, v_0), v^n(s, v_0) \rangle_H \leq \|Q\|_{\text{op}} \|v^n(s, v_0)\|_H^2$ we obtain that

$$\begin{aligned} &\frac{1}{t} \int_0^t \|v^n(s, v_0)\|_V^2 \|v^n(s, v_0)\|_H^{2(m-1)} ds \\ &\leq \frac{1}{t} \frac{1}{2\tilde{\omega}_0 m} \|v_0\|_H^{2m} \\ &\quad + \frac{1}{t} \int_0^t \frac{\text{tr}(P_n \circ Q) + 2(m-1)\|Q\|_{\text{op}}}{2\tilde{\omega}_0} \cdot \|v^n(s, v_0)\|_H^{2(m-1)} ds \\ &\quad + \frac{1}{t} \int_0^t \frac{1}{\tilde{\omega}_0} \|v^n(s, v_0)\|_H^{2(m-1)} \langle v^n(s, v_0), dP_n W_s \rangle_H. \end{aligned}$$

Taking expectations we conclude that for $m \in \mathbb{N}$

$$\begin{aligned} &E\left(\frac{1}{t} \int_0^t \|v^n(s, v_0)\|_H^{2(m-1)} \|v^n(s, v_0)\|_V^2 ds\right) \\ &\leq \frac{1}{t} \frac{1}{2\tilde{\omega}_0 m} \|v_0\|_H^{2m} + \frac{\text{tr}(P_n \circ Q) + 2(m-1)\|Q\|_{\text{op}}}{2\tilde{\omega}_0} \cdot E\left(\frac{1}{t} \int_0^t \|v^n(s, v_0)\|_H^{2(m-1)} ds\right). \end{aligned} \tag{4.9}$$

It is now standard to conclude that (4.9) implies for the limiting measure μ_n first that

$$\int \|v\|_H^{2m} d\mu_n(v) \leq C(m), \quad m \in \mathbb{N}$$

for some finite constant $C(m)$ independent of n and then, by taking the limit $t \rightarrow \infty$ in (4.9),

$$\begin{aligned} &\int \|v\|_V^2 \|v\|_H^{2(m-1)} d\mu_n(v) \\ &\leq \frac{\text{tr}(P_n \circ Q) + 2(m-1)\|Q\|_{\text{op}}}{2\tilde{\omega}_0} \int \|v\|_H^{2(m-1)} d\mu_n(v), \quad m \in \mathbb{N}. \end{aligned} \tag{4.10}$$

For $\varepsilon < \varepsilon' < \varepsilon_0$ let $M_\varepsilon \in \mathbb{N}$ be such that

$$\varepsilon \text{tr}_H(Q) \leq (\varepsilon' - \varepsilon)2(m-1)\|Q\|_{\text{op}} \quad \text{for all } m \geq M_\varepsilon.$$

Since $\|v\|_H^2 \leq \frac{b^2}{\pi^2} \|v\|_V^2$, we obtain for $M \in \mathbb{N}$, $M \geq M_\varepsilon \in \mathbb{N}$, that

$$\sum_{k=M_\varepsilon}^M \frac{\varepsilon^k}{k!} \int \|v\|_H^{2k} d\mu_n(v) \leq \varepsilon' \frac{b^2}{\pi^2} \frac{\|Q\|_{\text{op}}}{\tilde{\omega}_0} \sum_{k=M_\varepsilon-1}^{M-1} \frac{\varepsilon^k}{k!} \int \|v\|_H^{2k} d\mu_n(v),$$

and thus

$$\sum_{k=0}^M \frac{\varepsilon^k}{k!} \int \|v\|_H^{2k} d\mu_n(v) \leq \frac{1}{1 - \varepsilon'/\varepsilon_0} \sum_{k=0}^{M_\varepsilon-1} \frac{\varepsilon^k}{k!} \int \|v\|_H^{2k} d\mu_n(v).$$

The right hand side above does not depend on M , so that we may take the limit $M \rightarrow \infty$ to obtain the exponential moment estimate

$$\int e^{\varepsilon \|v\|_H^2} d\mu_n(v) \leq \frac{\varepsilon_0}{\varepsilon_0 - \varepsilon'} \sum_{k=0}^{M_\varepsilon-1} \frac{\varepsilon^k}{k!} C(k) =: C_1(\varepsilon).$$

for some finite constant $C_1(\varepsilon)$ independent of n . □

Finally, consider again the stochastic evolution equation (1.6) with

$$Bv := -\omega \Pi(e_3 \times v) - \Pi(u_b^E \cdot \nabla v) - \Pi(v_3 \cdot \partial_3 u_b^E).$$

If

$$\frac{\tilde{u}_b}{v} \left(\delta - be^{-\frac{b}{\delta}} - \delta e^{-\frac{b}{\delta}} \right) < \frac{1}{\sqrt{2}}, \tag{4.11}$$

then B satisfies assumption (A.1) with $\omega_1 < \frac{\pi^2}{b^2} \omega_0$, so that

$$\langle (vA_S + B)u, u \rangle_H \leq -\tilde{\omega}_0 \|u\|_V^2,$$

with $\tilde{\omega}_0 = \omega_0 - \frac{b^2}{\pi^2} \omega_1$. Hence, there exists a stationary martingale solution $((v_t)_{t \geq 0}, (W_t)_{t \geq 0})$ of (1.6) satisfying (4.2) and its distribution $P \circ v_t^{-1}$ on H satisfies the moment estimates (4.3) and (4.4).

Considering the stochastic process

$$u_t = v_t + u_b^E, \quad t \geq 0,$$

we verify that $((u_t)_{t \geq 0}, (W_t)_{t \geq 0})$ is a stationary martingale solution of (1.5), which proves the following corollary.

Corollary 4.3. *Assume that equation (4.11) holds. Then there exists a stationary martingale solution $((u_t)_{t \geq 0}, (W_t)_{t \geq 0})$ of the stochastic Navier-Stokes-Coriolis-Ekman equation (1.6). Its invariant distribution $\mu = P \circ u_t^{-1}$ satisfies the moment estimates (4.3) and (4.4).*

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