ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

ADIMURTHI

Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the *n*-laplacian

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 17, n° 3 (1990), p. 393-413

<http://www.numdam.org/item?id=ASNSP_1990_4_17_3_393_0>

© Scuola Normale Superiore, Pisa, 1990, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical Growth for the n-Laplacian

ADIMURTHI

1. - Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary. We are looking for a solution of the following problem:

Let $1 < p \leq n$, find $u \in W_0^{1,p}(\Omega) \backslash \{0\}$ such that

$$
(1.1) \qquad \begin{aligned} \Delta_p u &= f(x,u)|u|^{p-2} \qquad \text{in } \Omega \\ u &\geq 0, \end{aligned}
$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a C^1-function with $f(x,0) = 0$, $f(x,t) \geq 0$ for $t \geq 0$ and of critical growth.

For $p = 2$ and $n \geq 3$, Brézis-Nirenberg [4] have studied the existence and non-existence of solution of (1.1) when f has critical growth of the form $u^{(n+2)/(n-2)} + \lambda u$. A generalization of this result, on the same lines, for the p-Laplacian with $p \leq n$ and $p^2 \leq n$, has been studied by Garcia Azorero-Peral Alonso [7]. When $p = n$, in view of the Trudinger [13] imbedding, a critical growth function $f(x,u)$ behaves like $\exp\left(b|u|^{n/(n-1)}\right)$ for some $b > 0$. In this context, when $p = n = 2$ and Ω is a ball in $\mathbb{R}^2$, existence of a solution of (1.1) has been studied by Adimurthi [1], Atkinson-Peletier [2]. The method used by Atkinson-Peletier is a shooting method and hence cannot be generalized to solve (1.1) in an arbitrary domain. Whereas in Adimurthi [1], (1.1) is solved via variational method which is in the spirit of Brézis-Nirenberg [4] and, based on this method, we prove the following main result in this paper.

Let $f(x,t) = h(x,t) \exp(b|t|^{n/(n-1)})$ be a function of critical growth and $F(x,t)$ be its primitive (see definition (2.1)). For $u \in W_0^{1,n}(\Omega)$, let

$$
(1.2) \qquad J(u) = \frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - \int_{\Omega} F(x,u) \, dx
$$

Pervenuto alla Redazione il 22 Ottobre 1988 e in forma definitiva l'11 Agosto 1989.

$$\lambda_1(u) = \inf\left\{ \int_\Omega |\nabla u|^n \, dx;\ u \in W_0^{1,n}(\Omega),\ \int_\Omega |u|^n \, dx = 1 \right\} \tag{1.3}$$

$$\alpha_n = n\omega_n^{1/(n-1)}, \text{ where } \omega_n = \text{ Volume of } S^{n-1}. \tag{1.4}$$

THEOREM *Let $f(x,t) = h(x,t)\exp(b|t|^{n/(n-1)})$ be a function of critical growth on Ω. Then*

1) $J : W_0^{1,n}(\Omega) \to \mathbb{R}$ *satisfies the Palais-Smale Condition on the interval* $\left(-\infty,\ \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$;

2) *Let $f'(x,t) = \frac{\partial}{\partial t} f(x,t)$ and further assume that*

$$\sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega) \tag{1.5}$$

$$\overline{\lim_{t\to\infty}}\ \inf_{x\in\overline{\Omega}} h(x,t)t^{n-1} = \infty, \tag{1.6}$$

then there exists some $u_0 \in W_0^{1,n}(\Omega)\backslash\{0\}$ such that

$$\begin{aligned} \Delta_n u_0 &= f(x,u_0)u_0^{n-2} && \text{in } \Omega \\ u_0 &\geq 0 \\ u_0 &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.7}$$

The method adopted to solve (1.7) in Brézis-Nirenberg [4] does not work because of the critical growth is of exponential type. Here we adapt the method of artificial constraint due to Nehari [11]. The main idea of the proof is as follows:

Define

$$\frac{a(\Omega,f)^n}{n} = \inf\left\{ J(u);\ \int_\Omega |\nabla u|^n \, dx = \int_\Omega f(x,u)u^{n-1} dx,\ u \not\equiv 0 \right\}, \tag{1.8}$$

then the minimizer of (1.8) is a solution of (1.7).

It has to be noted that α_n is the best constant appearing in Moser's [10] result about the Trudinger's imbedding of $W_0^{1,n}(\Omega)$. In view of this, one expects that J should satisfy the Palais-Smale Condition on $\left(-\infty,\ \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$. Therefore, in order to get a minimizer of (1.8), the question remains to show that

$$a(\Omega,f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1} \tag{1.9}$$

and this has been achieved by showing the following relation

$$\sup_{\int_\Omega |\nabla w|^n \,dx \le 1} \int_\Omega f(x, a(\Omega, f)w)w^{n-1}\,dx \le a(\Omega, f). \tag{1.10}$$

In the forthcoming paper (jointly with Yadava), we discuss the bifurcation and multiplicity results for (1.7) when $n = 2$.

2. - Preliminaries

Let Ω be a bounded domain with smooth boundary. In view of the Trudinger-Moser [13,10] imbedding, we have the following definition of functions of critical growth.

DEFINITION 2.1. Let $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a C^1-function and $b > 0$. Let $f(x,t) = h(x,t)\exp\left(b|t|^{n/(n-1)}\right)$. We say that f is a function of critical growth on Ω if the following holds:

There exist constants $M > 0$, $\sigma \in [0,1)$ such that, for every $\epsilon > 0$, and for every $(x,t) \in \overline{\Omega} \times (0,\infty)$,

(H_1) $f(x,0) = 0$, $f(x,t) > 0$, $f(x,t)t^{n-1} = f(x,-t)(-t)^{n-1}$;

(H_2) $f'(x,t) > \dfrac{f(x,t)}{t}$, where $f'(x,t) = \dfrac{\partial f}{\partial t}(x,t)$;

(H_3) $F(x,t) \le M(1 + f(x,t)t^{n-2+\sigma})$, where

$$F(x,t) = \int_0^t f(x,s)s^{n-2}\,ds$$

is the primitive of f;

(H_4) $\lim\limits_{t\to\infty} \sup\limits_{x\in\overline{\Omega}} h(x,t)\exp\left(-\epsilon t^{n/(n-1)}\right) = 0$,

$$\lim_{t\to\infty} \inf_{x\in\overline{\Omega}} h(x,t)\exp\left(\epsilon t^{n/(n-1)}\right) = \infty.$$

Let $A(\Omega)$ denote the set of all functions of critical growth on Ω.

EXAMPLES. In view of (H_1), it is enough to define f on $\overline{\Omega} \times (0,\infty)$.

1) For $m \ge 1$, $b > 0$, $\beta \ge 0$ and $0 \le \alpha < \dfrac{n}{n-1}$, $f(x,t) = t^m \exp(\beta t^\alpha)\exp\left(bt^{n/(n-1)}\right)$ is in $A(\Omega)$.

2) $f(x,t) = t^2 e^{-t}\exp\left(t^{n/(n-1)}\right)$ is in $A(\Omega)$.

3) Let $f(x,t) = h(x,t)\exp\left(bt^{n/(n-1)}\right)$, satisfying (H_1) and (H_4).

Further assume that $h'(x,t) \geq \dfrac{h(x,t)}{t}$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. Then f is in $A(\Omega)$.

For

$$\frac{f'(x,t)}{f(x,t)} = \frac{h'(x,t)}{h(x,t)} + \frac{nb}{n-1} t^{1/(n-1)} > \frac{1}{t}$$

and hence f satisfy (H_2).

Let $\epsilon > 0$, and $\sigma = \dfrac{1}{n-1}$

$$\begin{aligned} F(x,t) - F(x,\epsilon) &= \frac{n-1}{nb} \int_{\epsilon}^{t} h(x,s) s^{n-1-\sigma} \frac{\mathrm{d}}{\mathrm{d}s} \exp\left(b s^{n/(n-1)}\right) \mathrm{d}s \\ &\leq \frac{n-1}{nb} \left[f(x,t) t^{n-2-\sigma} - f(x,\epsilon)\epsilon^{n-2-\sigma} \right]. \end{aligned}$$

This implies that there exists a constant $M > 0$ such that $F(x,t) \leq M[1 + f(x,t)t^{n-2-\sigma}]$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. This shows that f satisfy (H_3) and hence $f \in A(\Omega)$.

Let $W_0^{1,n}(\Omega)$ be the usual Sobolev space and $f(x,t) = h(x,t) \exp\left(b t^{n/(n-1)}\right)$ be in $A(\Omega)$. For $u \in W_0^{1,n}(\Omega)$, define

$$\|u\|^n = \int_\Omega |\nabla u|^n \,\mathrm{d}x \tag{2.1}$$

$$J(u) = \frac{1}{n}\|u\|^n - \int_\Omega F(x,u)\,\mathrm{d}x \tag{2.2}$$

$$I(u) = \frac{1}{n}\int_\Omega f(x,u)u^{n-1}\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x \tag{2.3}$$

$$\partial B(\Omega, f) = \left\{ u \in W_0^{1,n}(\Omega)\backslash\{0\};\ \|u\|^n = \int_\Omega f(x,u)u^{n-1}\,\mathrm{d}x \right\} \tag{2.4}$$

$$\frac{a(\Omega, f)^n}{n} = \inf\{J(u);\ u \in \partial B(\Omega, f)\} \tag{2.5}$$

$$\lambda_1(\Omega) = \inf\left\{ \|u\|^n;\ \int_\Omega |u|^n\,\mathrm{d}x = 1 \right\} \tag{2.6}$$

$$\alpha_n = n\omega_n^{1/(n-1)}, \text{ where } \omega_n = \text{ Volume of } S^{n-1}.$$

DEFINITION OF MOSER FUNCTIONS. Let $x_0 \in \Omega$ and $R \leq \mathrm{d}(x_0, \partial\Omega)$, where d denotes the distance from x_0 to $\partial\Omega$. For $0 < \ell < R$, define

$$m_{\ell,R}(x, x_0) = \frac{1}{\omega_n^{1/n}} \begin{cases} \left(\log \frac{R}{\ell}\right)^{1-\frac{1}{n}} & \text{if } 0 \leq |x - x_0| \leq \ell \\ \dfrac{\log \frac{R}{r}}{\left(\log \frac{R}{\ell}\right)^{1/n}} & \text{if } \ell \leq r = |x - x_0| \leq R \\ 0 & \text{if } |x - x_0| \geq R. \end{cases}$$

Then it is easy to see that $m_{\ell,R} \in W_0^{1,n}(\Omega)$ and $\|m_{\ell,R}\| = 1$.

For the proof of our theorem, we need the following two results whose proof is found in Moser [10] and P.L. Lions [9] respectively.

THEOREM 2.1 (Moser). 1) *Let* $u \in W_0^{1,n}(\Omega)$, *and* $p < \infty$, *then* $\exp\left(|u|^{n/(n-1)}\right) \in L^p(\Omega)$.

2) $\left(\frac{\alpha_n}{b}\right)^{n-1} = \max\left\{ c^n;\ \sup\limits_{\|w\| \leq 1} \int_\Omega \exp\left(bc^{n/(n-1)}|w|^{n/(n-1)}\right) \mathrm{d}x < \infty \right\}.$

THEOREM 2.2 (P.L. Lions). *Let* $\{u_k;\ \|u_k\| = 1\}$ *be a sequence in* $W_0^{1,n}(\Omega)$ *converging weakly to a non-zero function* u. *Then, for every* $p < (1 - \|u\|^n)^{-1/(n-1)}$,

$$\sup_k \int_\Omega \exp\left(p\alpha_n |u_k|^{n/(n-1)}\right) \mathrm{d}x < \infty.$$

3. - Proof of the Theorem

We need a few lemmas to prove the theorem. The proof of the following lemma is given in the appendix.

LEMMA 3.1. *Let* $f \in A(\Omega)$. *Then we have*

1) *If* $u \in W_0^{1,n}(\Omega)$, *then* $f(x, u) \in L^p(\Omega)$ *for all* $p \geq 0$.

2) $\left(\frac{\alpha_n}{b}\right)^{n-1} = \sup\left\{ c^n;\ \sup\limits_{\|w\| \leq 1} \int_\Omega f(x, cw) w^{n-1}\, \mathrm{d}x < \infty \right\}.$

3) *Let* $\{u_k\}$ *and* $\{v_k\}$ *be bounded sequences in* $W_0^{1,n}(\Omega)$ *converging weakly and for almost every* x *in* Ω *to* u *and* v *respectively. Further assume that*

$$\overline{\lim_{k \to \infty}} \|u_k\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

Then, for every integer $\ell \geq 0$,

$$\lim_{k \to \infty} \int_\Omega \frac{f(x, u_k)}{u_k} v_k^\ell \, \mathrm{d}x = \int_\Omega \frac{f(x, u)}{u} v^\ell \, \mathrm{d}x.$$

4) *Let* $\{u_k\}$ *be a sequence in* $W_0^{1,n}(\Omega)$ *converging weakly and for almost every* x *in* Ω *to* u, *such that*

$$\sup_k \int_\Omega f(x, u_k) u_k^{n-1} \,\mathrm{d}x < \infty.$$

Then, for any $0 \le \tau < 1$,

$$\lim_{k\to\infty} \int_\Omega f(x, |u_k|)\ |u_k|^{n-2+\tau} \,\mathrm{d}x = \int_\Omega f(x, |u|)\ |u|^{n-2+\tau} \,\mathrm{d}x$$

$$\lim_{k\to\infty} \int_\Omega F(x, u_k) \,\mathrm{d}x = \int_\Omega F(x, u) \,\mathrm{d}x.$$

5) $I(u) \ge 0$ *for all* u *and* $I(u) = 0$ *iff* $u \equiv 0$. *Further, there exists a constant* $M_1 > 0$ *such that, for all* $u \in W_0^{1,n}(\Omega)$,

$$\int_\Omega f(x, u) u^{n-1} \,\mathrm{d}x \le M_1(1 + I(u)).$$

LEMMA 3.2. *Let* $f = h \exp\left(b|t|^{n/(n-1)}\right) \in A(\Omega)$ *and define*

$$h_0(t) = \inf_{x\in\overline{\Omega}} h(x,t),\quad M_0 = \sup_{t\ge 0} h_0(t) t^{n-1},\quad R_0 = \sup_{x\in\overline{\Omega}} \mathrm{d}(x, \partial\Omega),$$

and

$$k_0 = \begin{cases} \left(\frac{n}{R_0}\right)^{n/(n-1)} M_0^{-1/(n-1)} & \text{if } M_0 < \infty \\ 0 & \text{if } M_0 = \infty. \end{cases}$$

Let $a \ge 0$ *be such that*

$$\sup_{\|w\|\le 1} \int_\Omega f(x, aw) w^{n-1} \,\mathrm{d}x \le a.$$

If $\frac{k_0}{b} < 1$, *then* $a^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$.

PROOF. From 2) of lemma 3.1, we have $a^n \le \left(\frac{\alpha_n}{b}\right)^{n-1}$. Suppose $a^n = \left(\frac{\alpha_n}{b}\right)^{n-1}$. Let $x_0 \in \Omega$ such that $\mathrm{d}(x_0, \partial\Omega) = R_0$ and $0 < \ell < R_0$. Let

$$m_\ell(x) = m_{\ell,R_0}(x, x_0).$$

be the Moser functions and

$$t = a\omega_n^{-1/n}\left(\log \frac{R_0}{\ell}\right)^{(n-1)/n},$$

then from (3.1) we have

$$\begin{aligned} a &\geq \int_\Omega f(x, am_\ell)m_\ell^{n-1}\,\mathrm{d}x \\ &\geq \int_{B(x_0,\ell)} h_0(am_\ell)m_\ell^{n-1} \ \exp\left(ba^{n/(n-1)}m_\ell^{n/(n-1)}\right)\,\mathrm{d}x \\ &= \frac{h_0(t)t^{n-1}\omega_n R_0^n}{na^{n-1}}. \end{aligned}$$

This implies that

$$\left(\frac{\alpha_n}{b}\right)^{n-1} = a^n \geq \frac{h_0(t)t^{n-1}\omega_n R_0^n}{n}.$$

That is, for all $t \in (0, \infty)$,

$$b \leq \left(\frac{n}{R_0}\right)^{n/(n-1)} \left(h_0(t)t^{n-1}\right)^{-1/(n-1)}$$

and hence

$$b \leq \left(\frac{n}{R_0}\right)^{n/(n-1)} \inf_{t\geq 0}\left(h_0(t)t^{n-1}\right)^{-1/(n-1)} \leq k_0$$

which contradicts the hypothesis $b > k_0$. Hence $a^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$ and this proves the lemma.

LEMMA 3.3. (Compactness Lemma). *Let f be in $A(\Omega)$ and $\{u_k\}$ be a sequence in $W_0^{1,n}(\Omega)$ converging weakly and for almost every x in Ω to a non-zero function u. Further, assume that*

(i) *There exists $C \in \left(0, \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right]$ such that $\lim_{k\to\infty} J(u_k) = C$;*

(ii) $\|u\|^n \geq \int_\Omega f(x,u)u^{n-1}\,\mathrm{d}x$;

(iii) $\sup_k \int_\Omega f(x,u_k)u_k^{n-1}\,\mathrm{d}x < \infty$;

then

$$\lim_{k\to\infty}\int_\Omega f(x,u_k)u_k^{n-1}\,\mathrm{d}x = \int_\Omega f(x,u)u^{n-1}\,\mathrm{d}x.$$

PROOF. From 5) of lemma 3.1, $I(u) > 0$. Therefore, from (ii) we have $J(u) \geq I(u) > 0$ and $J(u) \leq \underline{\lim}_{k\to\infty} J(u_k) = C$. Hence we can choose an $\epsilon > 0$ such that

$$(C - J(u))\ (1+\epsilon)^{n-1} < \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}. \tag{3.2}$$

Let $\beta = \int_\Omega F(x,u)\,dx$. Then, from (iii) and 4) of lemma 3.1, we have

$$\begin{aligned}\lim_{k\to\infty} \|u_k\|^n &= n \lim_{k\to\infty}\left\{J(u_k) + \int_\Omega F(x,u_k)\,dx\right\} \\ &= n(C+\beta).\end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we can choose a $k_0 > 0$ such that, for all $k \geq k_0$,

$$(1+\epsilon)^{n-1}\left(\frac{b}{\alpha_n}\right)^{n-1}\|u_k\|^n < \frac{C+\beta}{C-J(u)} = \left(1 - \frac{\|u\|^n}{n(C+\beta)}\right)^{-1}. \tag{3.4}$$

Now choose p such that

$$(1+\epsilon)^{n-1}\left(\frac{b}{\alpha_n}\right)^{n-1}\|u_k\|^n \leq p^{n-1} < \frac{C+\beta}{C-J(u)}. \tag{3.5}$$

Applying theorem 2.2 to the sequence $\frac{u_k}{\|u_k\|}$ and using (3.3) and (3.5), we have

$$\sup_k \int_\Omega \exp\left[p\alpha_n\left(\frac{u_k}{\|u_k\|}\right)^{n/(n-1)}\right] dx < \infty. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned}&\sup_k \int_\Omega \exp\left((1+\epsilon)^{n-1} b |u_k|^{n/(n-1)}\right) dx \\ &\qquad \leq \sup_k \int_\Omega \exp\left[p\alpha_n\left(\frac{u_k}{\|u_k\|}\right)^{n/(n-1)}\right] dx < \infty.\end{aligned} \tag{3.7}$$

Let

$$M_1 = \sup_{(x,t)\in\overline{\Omega}\times\mathbb{R}} |h(x,t)t^{n-1}| \exp\left(-\epsilon\frac{b}{2}|t|^{n/(n-1)}\right)$$

and $N > 0$. Then from (3.7) we have

$$\int\limits_{|u_k|\geq N} f(x,u_k)u_k^{n-1}\,dx = \int\limits_{|u_k|\geq N} h(x,u_k)u_k^{n-1}\exp\left(b|u_k|^{n/(n-1)}\right)dx \tag{3.8}$$

$$\leq M_1 \int\limits_{|u_k|\geq N} \exp\left(-\epsilon\frac{b}{2}|u_k|^{n/(n-1)}\right)\exp\left[(1+\epsilon)b|u_k|^{n/(n-1)}\right]dx$$

$$= O\left(\exp\left(-\epsilon\frac{b}{2}N^{n/(n-1)}\right)\right).$$

Hence

$$\int\limits_{\Omega} f(x,u_k)u_k^{n-1}\,dx = \int\limits_{|u_k|\leq N} f(x,u_k)u_k^{n-1}\,dx + O\left(\exp\left(-\epsilon\frac{b}{2}N^{n/(n-1)}\right)\right).$$

Now letting $k\to\infty$, and $N\to\infty$ in the above equation, we obtain

$$\lim_{k\to\infty}\int\limits_{\Omega} f(x,u_k)u_k^{n-1}\,dx = \int\limits_{\Omega} f(x,u)u^{n-1}\,dx.$$

This proves the lemma.

LEMMA 3.4. *Let $f\in A(\Omega)$ and assume that*

(i) $\overline{\lim\limits_{t\to\infty}}\, h_0(t)t^{n-1} = \infty$,

where $h_0(t) = \inf\limits_{x\in\overline{\Omega}} h(x,t)$;

(ii) $\sup\limits_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega)$;

then

$$0 < a(\Omega,f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

PROOF. The lemma is proved in several steps.

STEP 1. $a(\Omega,f) > 0$.

Suppose $a(\Omega,f)=0$. Then there exists a sequence $\{u_k\}$ in $\partial B(\Omega,f)$ such that $J(u_k)\to 0$ as $k\to\infty$. Since $J(u_k)=I(u_k)$, hence from 5) of lemma 3.1

$$\sup_k \int\limits_{\Omega} f(x,u_k)u_k^{n-1}\,dx < \infty \tag{3.9}$$

$$\sup_k \|u_k\|^n < \infty. \tag{3.10}$$

Then, by extracting a subsequence, we can assume that $\{u_k\}$ converges weakly and for almost every x in Ω to a function u. Now by Fatou's lemma,

$$0 \leq I(u) \leq \underline{\lim}_{k\to\infty} I(u_k) = \underline{\lim}_{k\to\infty} J(u_k) = 0.$$

Hence $u \equiv 0$. From (3.9) and 4) of lemma 3.1, we have

$$\lim_{k\to\infty} \|u_k\|^n = n \lim_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k)\,\mathrm{d}x \right\} = 0. \tag{3.12}$$

Let $v_k = \frac{u_k}{\|u_k\|}$ and converging weakly to v. Using $u_k \in \partial B(\Omega, f)$, (3.12), 3) of lemma 3.1 and (ii), we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \int_\Omega \frac{f(x,u_k)}{u_k} v_k^n \,\mathrm{d}x \\ &= \int_\Omega f'(x,0) v^n \,\mathrm{d}x < \lambda_1(\Omega) \int_\Omega v^n \,\mathrm{d}x \leq 1, \end{aligned}$$

which is a contradiction. This prove step 1.

STEP 2. For every $u \in W_0^{1,n}(\Omega)\backslash\{0\}$, there exists a constant $\gamma > 0$ such that $\gamma u \in \partial B(\Omega, f)$. Moreover, if

$$\|u\|^n \leq \int_\Omega f(x,u) u^{n-1}\,\mathrm{d}x, \tag{3.13}$$

then $\gamma \leq 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$.

For $\gamma > 0$, define

$$\psi(\gamma) = \frac{1}{\gamma} \int_\Omega f(x, \gamma u) u^{n-1}\,\mathrm{d}x.$$

Then, from 3) of lemma 3.1 and (ii), we have

$$\lim_{\gamma\to 0} \psi(\gamma) = \int_\Omega f'(x,0) u^n \,\mathrm{d}x < \|u\|^n,$$

$$\lim_{\gamma\to\infty} \psi(\gamma) = \infty.$$

Hence there exists $\gamma > 0$ such that $\psi(\gamma) = \|u\|^n$; this implies that $\gamma u \in \partial B(\Omega, f)$. From (H_1) and (H_2), it follows that $\frac{f(x,tu)}{t} u^{n-1}$ is an increasing function for $t > 0$. Hence, if u satisfies (3.13), it follows that $\gamma \leq 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$. This proves step 2.

STEP 3. $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$.

Let $w \in W_0^{1,n}(\Omega)$ such that $\|w\| = 1$. From step 2, we can choose a $\gamma > 0$ such that $\gamma w \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega, f)^n}{n} \leq J(\gamma w) \leq \frac{\gamma^n}{n}\|w\|^n = \frac{\gamma^n}{n};$$

this implies that $a(\Omega, f) \leq \gamma$. Using again the fact that $\frac{f(x, tw)}{t} w^{n-1}$ is an increasing function of t in $(0, \infty)$ and $\gamma w \in \partial B(\Omega, f)$, we have

$$\int_\Omega \frac{f(x, a(\Omega, f)w)}{a(\Omega, f)} w^{n-1}\, \mathrm{d}x \leq \int_\Omega \frac{f(x, \gamma w)}{\gamma} w^{n-1}\, \mathrm{d}x = 1.$$

This implies that

$$\sup_{\|w\| \leq 1} \int_\Omega f(x, a(\Omega, f)w) w^{n-1}\, \mathrm{d}x \leq a(\Omega, f). \tag{3.14}$$

Now from (i), (3.14) and lemma 3.2 we have $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. This proves the lemma.

LEMMA 3.5. *Let $f \in A(\Omega)$ and $u_0 \in \partial B(\Omega, f)$ such that $J'(u_0) \neq 0$ ($J'(u)$ denote the derivative of J at u). Then*

$$J(u_0) > \inf\{J(u);\ u \in \partial B(\Omega, f)\}.$$

PROOF. Choose $h_0 \in W_0^{1,n}(\Omega)$ such that $\langle J'(u_0),\, h_0\rangle = 1$ and, for $\alpha, t \in \mathbb{R}$, define $\sigma_t(\alpha) = \alpha u_0 - t h_0$. Then

$$\lim_{\substack{t \to 0 \\ \alpha \to 1}} \frac{\mathrm{d}}{\mathrm{d}t} J(\sigma_t(\alpha)) = -\langle J'(u_0),\, h_0\rangle = -1$$

and hence we can choose $\epsilon > 0$, $\delta > 0$ such that, for all $\alpha \in [1 - \epsilon,\ 1 + \epsilon]$ and $0 < t \leq \delta$,

$$J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u_0). \tag{3.15}$$

Let

$$\rho_t(\alpha) = \|\sigma_t(\alpha)\|^n - \int_\Omega f(x, \sigma_t(\alpha))\ \sigma_t(\alpha)^{n-1}\, \mathrm{d}x.$$

Since $\frac{f(x,\alpha u_0)}{\alpha}u_0^{n-1}$ is an increasing function of α and using $u_0 \in \partial B(\Omega, f)$, by shrinking ϵ and δ if necessary, we have, for $0 < t \leq \delta$, $\rho_t(1-\epsilon) > 0$ and $\rho_t(1+\epsilon) < 0$. Hence there exists α_t such that $\rho_t(\alpha_t) = 0$. Thefore $\sigma_t(\alpha_t)$ is in $\partial B(\Omega, f)$. Hence from (3.15) we have

$$\begin{aligned}\inf\{J(u);\ u \in \partial B(\Omega, f)\} &\leq J(\sigma_t(\alpha_t)) \\ &< J(\alpha_t u_0) \leq \sup_{t\in\mathbb{R}} J(tu_0) = J(u_0).\end{aligned}$$

This proves the lemma.

PROOF OF THE THEOREM.

1) *Palais-Smale Condition.* Let $C \in \left(-\infty, \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$ and $\{u_k\}$ be a sequence such that

$$\lim_{k\to\infty} J(u_k) = C \qquad (3.16)$$
$$\lim_{k\to\infty} J'(u_k) = 0.$$

Let $h \in W_0^{1,n}(\Omega)$, then we have

$$\langle J'(u_k), h\rangle = \int_\Omega |\nabla u_k|^{n-2}\nabla u_k \cdot \nabla h\,\mathrm{d}x - \int_\Omega f(x,u_k)u_k^{n-2}h\,\mathrm{d}x. \qquad (3.18)$$

Hence we have

$$J(u_k) - \frac{1}{n}\langle J'(u_k), u_k\rangle = I(u_k). \qquad (3.19)$$

CLAIM 1.

$$\sup_k \|u_k\| + \sup_k \int_\Omega f(x,u_k)u_k^{n-1}\,\mathrm{d}x < \infty. \qquad (3.20)$$

Since $\{J(u_k)\}$ and $\{J'(u_k)\}$ are bounded and hence from (3.19), $I(u_k) = O(\|u_k\|)$. Now from 5) of lemma 3.1, we have $\int_\Omega f(x,u_k)u_k^{n-1}\,\mathrm{d}x = O(\|u_k\|)$.

Now from (H_3) it follows that

$$\int_\Omega F(x,u_k)\,\mathrm{d}x = O(\|u_k\|)$$

and, by using the boundedness of $J(u_k)$, we obtain $\|u_k\|^n = O(\|u_k\|)$. This implies (3.20) and hence the claim.

By extracting a subsequence, we can assume that

$$u_k \to u_0 \text{ weakly and for almost all } x \text{ in } \Omega. \tag{3.21}$$

CASE (I). $C \le 0$.

From Fatou's lemma and 5) of lemma 3.1, we have

$$\begin{aligned} 0 \le I(u_0) &\le \underline{\lim}_{k\to\infty} I(u_k) \\ &= \underline{\lim}_{k\to\infty} \left\{ J(u_k) - \frac{1}{n}\langle J'(u_k), u_k\rangle \right\} \\ &= C. \end{aligned}$$

Hence $u_0 \equiv 0$. If $C < 0$, no Palais-Smale sequence exists. If $C = 0$, then from (3.20) and 4) of lemma 3.1 we have

$$\lim_{k\to\infty} \|u_k\|^n = n \lim_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k)\,\mathrm{d}x \right\} = 0.$$

This proves that $u_k \to 0$ strongly.

CASE (II). $C \in \left(0, \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$.

CLAIM 2. $u_0 \not\equiv 0$ and $u_0 \in \partial B(\Omega, f)$.

Suppose $u_0 \equiv 0$. Then, from (3.20) and 4) of lemma 3.1, we have

$$\begin{aligned} \lim_{k\to\infty} \|u_k\|^n &= n \underline{\lim}_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k)\,\mathrm{d}x \right\} \\ &= nC < \left(\frac{\alpha_n}{b}\right)^{n-1}. \end{aligned} \tag{3.22}$$

Hence, from 3) of lemma 3.1 and (3.22), we have

$$\lim_{k\to\infty} \int_\Omega f(x, u_k) u_k^{n-1}\,\mathrm{d}x = \int_\Omega f(x, u_0) u_0^{n-1}\,\mathrm{d}x = 0.$$

This implies that $\lim_{k\to\infty} I(u_k) = 0$ and hence from (3.19)

$$0 < C = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} \left\{ I(u_k) + \frac{1}{n}\langle J'(u_k), u_k\rangle \right\} = 0$$

which is a contradiction. Hence $u_0 \not\equiv 0$. From (3.20) and 4) of lemma 3.1, taking $h \in C_0^\infty(\Omega)$ and letting $k \to \infty$ in (3.19), we obtain

$$\int_\Omega |\nabla u_0|^{n-2} \nabla u_0 \cdot \nabla h \, dx = \int_\Omega f(x, u_0) u_0^{n-2} h \, dx.$$

By density, the above equation holds for all $h \in W_0^{1,n}(\Omega)$. Hence, by taking $h = u_0$, we obtain

$$\|u_0\|^n = \int_\Omega f(x, u_0) u_0^{n-1} \, dx. \tag{3.23}$$

Hence $u_0 \in \partial B(\Omega, f)$ and this proves the claim.

Now from (3.20) and claim 2, $\{u_k, u_0\}$ satisfy all the hypotheses of the compactness lemma 3.3. Hence we have

$$\begin{aligned}
\|u_0\|^n &\leq \varliminf_{k\to\infty} \|u_k\|^n \\
&= n \varliminf_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k) \, dx \right\} \\
&= n \varliminf_{k\to\infty} \left\{ I(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle + \int_\Omega F(x, u_k) \, dx \right\} \\
&= \varliminf_{k\to\infty} \left\{ \int_\Omega f(x, u_k) u_k^{n-1} \, dx + \langle J'(u_k), u_k \rangle \right\} \\
&= \int_\Omega f(x, u_0) u_0^{n-1} \, dx = \|u_0\|^n.
\end{aligned}$$

This implies that u_k converges to u_0 strongly. This proves the Palais-Smale condition.

2) *Existence of Positive Solution.* Since the critical points of J are the solutions of the equation (1.7) and $J(u) = J(|u|)$ for all u in $\partial B(\Omega, f)$ and hence in view of lemma 3.5, it is enough to prove that there exists $u_0 \not\equiv 0$ such that

$$\frac{a(\Omega, f)^n}{n} = J(u_0). \tag{3.24}$$

Let $u_k \in \partial B(\Omega, f)$ such that

$$\lim_{k\to\infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.$$

Since $J(u_k) = I(u_k)$, and hence by 5) of lemma 3.1

$$\sup_k \int_\Omega f(x, u_k) u_k^{n-1} \, dx < \infty, \tag{3.25}$$

$$\sup_k \|u_k\| < \infty. \tag{3.26}$$

Hence we can extract a subsequence such that

$$u_k \to u_0 \text{ weakly and for almost all } x \text{ in } \Omega.$$

CLAIM 3. $u_0 \not\equiv 0$ and

$$\|u_0\|^n \le \int_\Omega f(x, u_0) u_0^{n-1} \, dx. \tag{3.28}$$

Suppose $u_0 \equiv 0$, then from (3.25) and 4) of lemma 3.1

$$\begin{aligned} \lim_{k\to\infty} \|u_k\|^n &= n \lim_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k) \, dx \right\} \\ &= a(\Omega, f)^n. \end{aligned} \tag{3.29}$$

From lemma 3.4, we have $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. Hence, from (3.29) and 3) of lemma 3.1, we have

$$\lim_{k\to\infty} \int_\Omega f(x, u_k) u_k^{n-1} \, dx = 0.$$

This implies that

$$0 < \frac{a(\Omega, f)^n}{n} = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} I(u_k) = 0,$$

which is a contradiction. This proves $u_0 \not\equiv 0$. Suppose (3.28) is false, then

$$\|u_0\|^n > \int_\Omega f(x, u_0) u_0^{n-1} \, dx. \tag{3.30}$$

Now from (3.25), (3.30) and $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, $\{u_k, u_0\}$ satisfy all the hypotheses of lemma 3.3. Hence

$$\lim_{k\to\infty} \int_\Omega f(x, u_k) u_k^{n-1} \, dx = \int_\Omega f(x, u_0) u_0^{n-1} \, dx.$$

This implies that

$$\|u_0\|^n \leq \underline{\lim}_{k\to\infty} \|u_k\|^n = \underline{\lim}_{k\to\infty} \int_\Omega f(x, u_k)u_k^{n-1}\,\mathrm{d}x$$

$$= \int_\Omega f(x, u_0)u_0^{n-1}\,\mathrm{d}x\,.$$

contradicting (3.30). This proves the claim.

Now from (3.28) and step 2 of lemma 3.4, there exists $0 < \gamma \leq 1$ such that $\gamma u_0 \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega, f)^n}{n} \leq J(\gamma u_0) = I(\gamma u_0)$$

$$\leq I(u_0) \leq \underline{\lim}_{k\to\infty} I(u_k)$$

$$= \underline{\lim}_{k\to\infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.$$

This implies that $\gamma = 1$ and $u_0 \in \partial B(\Omega, f)$. Hence $J(u_0) = \frac{a(\Omega, f)^n}{n}$ and this proves the Theorem.

4. Concluding Remarks

REMARK 4.1. (Regularity). From Di-Benedetto [6], Tolksdorf [12] and Gilbarg-Trudinger [8], any solution of (1.7) is in $C^{1,\alpha}(\Omega)$ if $n \geq 3$ and in $C^{2,\alpha}(\overline{\Omega})$ if $n = 2$.

REMARK 4.2. Let $f \in A(\Omega)$ and $f'(x, 0) < \lambda_1(\Omega)$ for all $x \in \overline{\Omega}$. We prove the existence of a solution for (1.7) under the assumption that

$$\overline{\lim_{t\to\infty}} \inf_{x\in\overline{\Omega}} h(x, t)t^{n-1} = \infty. \tag{4.1}$$

The only place where it is used is to show that $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. But, from lemma 3.2, this inequality holds if

$$\frac{k_0}{b} < 1. \tag{4.2}$$

Hence the theorem is true under the less restrictive condition (4.2).

Now the question is what happens if $\frac{k_0}{b} \geq 1$ or the condition (4.1) is not satisfied. In this regard, we have (jointly with Srikanth - Yadava) obtained a partial result, which states that there are functions $f \in A(\Omega)$ such that

$$\lim_{t\to\infty} \inf_{x\in\overline{\Omega}} h(x,t)t^{n-1} = 0$$

for which no solution to problem (1.7) exists if Ω is a ball of sufficiently small radius. In this context, we raise the following question:

Open Problem. Let Ω be a ball and $f \in A(\Omega)$ such that $\sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega)$. Is (4.2) also a necessary condition to obtain a solution to the problem (1.7).

In the case $n = 2$, this question is related to the question of Brézis [3]: "where is the border line between the existence and non-existence of a solution of (1.7)?".

REMARK 4.3. Let $\beta \geq 0$, then by using the norm

$$\left(\int_\Omega |\nabla u|^n \, dx + \beta \int_\Omega |u|^n \, dx \right)^{1/n}$$

in $W_0^{1,n}(\Omega)$, the Theorem still holds if we replace $-\Delta_n u$ by $-\Delta_n u + \beta |u|^{n-2}u$ in the equations (1.7).

Due to this and using a result of Cherrier [5], it is possible to extend the Theorem to compact Riemann surfaces.

ACKNOWLEDGEMENT. I would like to thank Dr. Srikanth and Dr. Borkar for having many helpful discussions during the preparation of this paper.

5. - Appendix

PROOF OF THE LEMMA 3.1.

1) Let $f(x,t) = h(x,t) \exp\left(b|t|^{n/(n-1)}\right) \in A(\Omega)$. From (H_4), for every $\epsilon > 0$, there exists a $C(\epsilon) > 0$ such that

$$|f(x,t)| \leq C(\epsilon) \exp\left((b+\epsilon)|t|^{n/(n-1)}\right)$$

and hence, from theorem 2.1, $f(x,u) \in L^p(\Omega)$ for every $p < \infty$.

2) From (H_4), for every $\epsilon > 0$, there exist positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that

$$|f(x,t)t^{n-1}| \leq C_1(\epsilon) \exp\left(b(1+\epsilon)|t|^{n/(n-1)}\right) \tag{5.1}$$

$$|f(x,t)t^{n-1}| \geq C_2(\epsilon)\exp\left(b(1-\epsilon)|t|^{n/(n-1)}\right) \text{ for } |t| \geq 1. \tag{5.2}$$

Hence, if $c > 0$ such that

$$\sup_{\|w\|\leq 1}\int_\Omega f(x,cw)w^{n-1}\,\mathrm{d}x < \infty,$$

it implies that, for every $\epsilon > 0$,

$$\sup_{\|w\|\leq 1}\int_\Omega \exp\left(b(1-\epsilon)c^{n/(n-1)}|w|^{n/(n-1)}\right)\,\mathrm{d}x < \infty.$$

Therefore, from Theorem 2.1, we have

$$(1-\epsilon)^{n-1}c^n \leq \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

This implies that

$$\sup\left\{c^n;\ \sup_{\|w\|\leq 1}\int_\Omega f(x,cw)w^{n-1}\,\mathrm{d}x < \infty\right\} \leq \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

On the other hand, if $c^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, then by choosing $\epsilon > 0$ such that $(1+\epsilon)^{2n-1}c^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, from Theorem 2.1 and from (5.1), we have

$$\begin{aligned}&\sup_{\|w\|\leq 1}\int_\Omega f(x,(1+\epsilon)cw)w^{n-1}\,\mathrm{d}x\\ &\leq C_1(\epsilon)\sup_{\|w\|\leq 1}\int_\Omega \exp\left[b\left((1+\epsilon)c|w|\right)^{n/(n-1)}\right]\,\mathrm{d}x < \infty\end{aligned}$$

this proves

$$\sup\left\{c^n;\ \sup_{\|w\|\leq 1}\int_\Omega f(x,cw)w^{n-1}\mathrm{d}x < \infty\right\} = \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

3) Since $\overline{\lim_{k\to\infty}}\,\|u_k\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, from 2) we can choose a $p > 1$ such that

$$c_1^p = \sup_k \int_\Omega |f(x,u_k)|^p\,\mathrm{d}x < \infty.$$

Let $\frac{1}{p} + \frac{1}{q} = 1$ and

$$c_2^q = \sup_k \int_\Omega |v_k|^{\ell q}\, dx.$$

Then, for any $N > 0$ and by Holder's inequality,

$$\left| \int_{|u_k|>N} \frac{f(x,u_k)}{u_k} v_k^\ell\, dx \right| \le \frac{1}{N} \int_\Omega |f(x,u_k)|\ |v_k^\ell|\, dx \le \frac{c_1 c_2}{N}.$$

Hence

$$\int_\Omega \frac{f(x,u_k)}{u_k} v_k^\ell\, dx = \int_{|u_k|\le N} \frac{f(x,u_k)}{u_k} v_k^\ell\, dx + O(1/N).$$

By dominated convergence theorem, letting $k \to \infty$ and then $N \to \infty$ in the above equation, it implies that

$$\lim_{k\to\infty} \int_\Omega \frac{f(x,u_k)}{u_k} v_k^\ell\, dx = \int_\Omega \frac{f(x,u)}{u} v^\ell\, dx.$$

4) Let $N > 0$, then

$$\int_{|u_k|>N} f(x,|u_k|)|u_k|^{n-2+\tau}\, dx \le \frac{1}{N^{1-\tau}} \int_\Omega f(x,|u_k|)|u_k|^{n-1}\, dx$$

$$= \frac{1}{N^{1-\tau}} \int_\Omega f(x,u_k) u_k^{n-1}\, dx = O\left(\frac{1}{N^{1-\tau}}\right).$$

Hence

$$\int_\Omega f(x,|u_k|)|u_k|^{n-2+\tau}\, dx = \int_{|u_k|\le N} f(x,|u_k|)|u_k|^{n-2+\tau}\, dx + O\left(\frac{1}{N^{1-\tau}}\right).$$

By dominated convergence theorem, letting $k \to \infty$ and $N \to \infty$ in the above equation, we obtain

$$\lim_{k\to\infty} \int_\Omega f(x,|u_k|)|u_k|^{n-2+\tau}\, dx = \int_\Omega f(x,|u|)|u|^{n-2+\tau}\, dx. \tag{5.3}$$

Now from (H_3),

$$|F(x,t)| \le M(1 + |f(x,t)|\ |t|^{n-2+\sigma})$$

for some $\sigma \in [0, 1)$. Hence, from (5.3) and the dominated convergence theorem,

$$\lim_{k\to\infty} \int_\Omega F(x, u_k)\,\mathrm{d}x = \int_\Omega F(x, u)\,\mathrm{d}x.$$

5) From (H_2) we have, for $t > 0$,

$$\frac{\partial}{\partial t}\left[f(x,t)t^{n-1} - nF(x,t)\right] = \left[f'(x,t) - \frac{f(x,t)}{t}\right] t^{n-1} > 0. \tag{5.4}$$

Therefore from (H_1) and (5.4), $f(x,t)t^{n-1} - nF(x,t)$ is an even positive function and increasing for $t > 0$. This implies that $I(u) \geq 0$ and $I(u) = 0$ iff $u \equiv 0$. From (H_3) we have

$$\begin{aligned} nI(u) &= \int_\Omega \left[f(x,u)u^{n-1} - nF(x,u)\right] \mathrm{d}x \\ &\geq \int_\Omega \left[f(x,u)u^{n-1} - nM(1 + |f(x,u)|\ |u|^{n-2+\sigma})\right] \mathrm{d}x \\ &\geq C_1 + \frac{1}{2} \int_{|u|\geq C_2} f(x,u)u^{n-1}\,\mathrm{d}x \end{aligned}$$

for some constants C_1 and $C_2 > 0$. This implies that there exists a constant $M_1 > 0$ such that

$$\int_\Omega f(x,u)u^{n-1}\,\mathrm{d}x \leq M(1 + I(u)).$$

This proves the lemma 3.1.

REFERENCES

[1] ADIMURTHI, *Positive solutions of the semilinear Dirichlet problem with Critical growth in the unit disc in* $\mathbb{R}^2$, Proc. Indian Acad. Sci., **99**, (1989), pp. 49-73.

[2] F.V. ATKINSON - L.A. PELETIER, *Ground states and Dirichlet problems for* $-\Delta u = f(u)$ *in* $\mathbb{R}^2$, Archive for Rational Mechanics and Analysis, No. 2, **96** (1986), pp. 147-165.

[3] H. BREZIS, *Nonlinear elliptic equations involving the Critical Sobolev exponent - Survey and perspectives*, Directions in partial differential equations, Ed. Crandall etc. (1987), pp. 17-36.

[4] H. BREZIS - L. NIRENBERG, *Positive solutions of non-linear elliptic equations involving critical Sobolev exponents*, Comm Pure Appl. Maths, **36** (1983), pp. 437-477.

[5] P. CHERRIER, *Problems de Neumann non lineaires sur les varietes riemanniennes*, C.R. Acad. Sc. Paris, Serie A, **292** (1984), pp. 225-262.

[6] DI BENEDETTO, *$C^{1,\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis - TMA, No. 8, **7** (1983), pp. 827-850.

[7] J.P. GARCIA AZORERO - I. PERAL ALONSO, *Existence and non-uniqueness for the p-Laplacian*, Comm in P.D.E, **12** (1987), pp. 1389-1430.

[8] D. GILBARG - N. TRUDINGER, *Elliptic Partial differential equations of second order*, 2nd edition, Springer Verlag (1983).

[9] P.L. LIONS, *The Concentration Compactness principle in the calculus of variations, part-I*, Revista mathematica Iberoamericana, No. 1, **1** (1985), pp. 185-201.

[10] J. MOSER, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Maths Jr, No. 11, **20** (1971), pp. 1077-1092.

[11] Z. NEHARI, *On a class of non-linear second order differential equations*, Trans AMS, **95** (1960), pp. 101-123.

[12] P. TOLKSDORF, *Regularity for a more general class of quasilinear elliptic equations*, Jr diff. eqs, **51** (1984), pp. 126-150.

[13] N.S. TRUDINGER, *On imbedding into Orlicz spaces and some applications*, Jr Math Mech, **17** (1967), pp. 473-484.

T.I.F.R. Centre
Post Box No. 1234
Bangalore 560 012
India