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Translation Invariant Operators on Lorentz Spaces

LEONARDO COLZANI

Let Ω be a locally compact non compact group equipped with left Haar measure, and let $0 < p < +\infty$, $0 < q \leq +\infty$. We prove that every left or right translation invariant linear operator bounded on the Lorentz space $L^{p,q}(\Omega)$ is also bounded on $L^p(\Omega)$. (A similar result for compact groups, $0 < p < 2$, and $q = +\infty$, has been recently proved by A.M. Shteinberg.) We also characterize all left or right translation invariant linear operators bounded on $L^{p,q}(\Omega)$, when $0 < p < 1$ and $0 < q \leq +\infty$, and we obtain sharp estimates for Bochner-Riesz means on the Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^N)$, when $0 < p \leq 1$ and $0 < q \leq +\infty$.

Let Ω be a locally compact group, equipped with left Haar measure, and let $L^{r,s}_{p,q}(\Omega), 0 < p, r < +\infty$ and $0 < q, s \leq +\infty$, denote the space of left translation invariant linear operators that map the Lorentz space $L^{p,q}(\Omega)$ into $L^{r,s}(\Omega)$ continuously. In particular, operators in the spaces $L^{r,r}_{p,p}(\Omega), L^{r,\infty}_{p,p}(\Omega)$, and $L^{r,\infty}_{p,1}(\Omega)$, are known as operators of strong type, weak type, and restricted weak type (p, r) respectively.

Inclusion relations between the spaces $L^{r,s}_{p,q}(\Omega)$ have been the object of intensive study. For example, the inclusion $L^{p,u}(\Omega) \subseteq L^{p,v}(\Omega)$ if $u \leq v$, immediately gives us the inclusion $L^{p,r}_{p,q}(\Omega) \subseteq L^{p,t}_{p,s}(\Omega)$ when $s \leq q$ and $r \leq t$. It is often a deeper problem to show that such inclusion relations are strict, and the following result of M. Cowling and J. Fournier ([C-F]) is a good example of this.

THEOREM A. *For every locally compact infinite group Ω and every p, with $1 < p < 2$ or $2 < p < +\infty$, one has the strict inclusions*

$$L^{p,r}_{p,q}(\Omega) \not\subseteq L^{p,s}_{p,q}(\Omega) \text{ if } 1 < q < 2 \text{ and } q \leq r < s,$$

and

$$L^{p,r}_{p,q}(\Omega) \not\subseteq L^{p,r}_{p,s}(\Omega) \text{ if } 2 < r \leq +\infty \text{ and } r/(r-1) \leq s < q \leq r.$$

A particularly interesting case of this theorem, and indeed the prototype of it, is the result of M. Zafran ([Z]): If $1 < p < 2$ there exists a convolution

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operator on $\mathbb{R}^N$ of weak type (p,p) but not of strong type (p,p). That is, if $1 < p < 2$, there exists a convolution operator Φ satisfying

$$a) \qquad \|\Phi f\|_{p,\infty} \leq \|f\|_p$$

for all functions f in $L^p(\Omega)$; yet the stronger inequality

$$b) \qquad \|\Phi f\|_p \leq c\|f\|_p$$

is false whatever the value of c.

The main aim of this paper, however, is to prove that if we strengthen a) to the inequality

$$a') \qquad \|\Phi f\|_{p,\infty} \leq \|f\|_{p,\infty},$$

then we also obtain inequality b) (with $c = 1$). More generally we want to prove the following.

THEOREM B. *Let Ω be a locally compact non compact group (equipped with left Haar measure), and let $0 < p < +\infty$ and $0 < q \leq +\infty$. Then every left (right) translation invariant linear operator that maps the Lorentz space $L^{p,q}(\Omega)$ into $L^{p,q}(\Omega)$ continuously, also maps $L^p(\Omega)$ into $L^p(\Omega)$.*

Actually we prove a similar theorem not only for convolution operators on non compact groups, but also for translation invariant sublinear operators on some measure spaces a bit more general than the locally compact non compact groups. Unfortunately our techniques do not always apply to the compact groups. However, after this paper was completed, we learned that in 1982 G. Pisier conjectured that every convolution operator bounded on $L^{p,\infty}(\Omega)$ is automatically bounded also on $L^p(\Omega)$ if the group Ω is compact and $0 < p < 2$ ([P]). This conjecture has been recently proved by A.M. Shteinberg ([Sh.]). More precisely, we have the following.

THEOREM C. *Let Ω be a compact group and let $0 < p < 2$. Then every translation invariant sublinear operator that maps the Lorentz space $L^{p,\infty}(\Omega)$ into $L^0(\Omega)$ continuously, also maps $L^p(\Omega)$ into $L^p(\Omega)$. ($L^0(\Omega)$ is the space of measurable functions on Ω with the topology of the convergence in measure.)*

This paper is divided into four sections. Section 1 contains some definitions and preliminary material, and in the main section, section 2, we prove theorem B along with some extensions and corollaries.

In section 3 we give a complete characterization of the bounded left or right translation invariant operators on the spaces $L^{p,q}(\Omega)$, when $0 < p < 1$, $0 < q \leq +\infty$, and Ω is any locally compact, not necessarily non compact, group. Extending previous results of S. Sawyer, J. Peetre, and D. Oberlin for L^p spaces (see [Sa], [B-L] pg. 170, and [O] respectively), we prove that every right translation invariant linear operator bounded on $L^{p,q}(\Omega)$ is given by the

left convolution with a discrete measure $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}$ where $\{x_k\}$ is a sequence of points in Ω, and the sequence $\{\alpha_k\}$ is in the sequence space $\ell^{p,q}$ if $0 < q \leq p$, or in ℓ^p if $p \leq q \leq +\infty$. The characterization of the left translation invariant linear operators is similar.

It is widely acknowledged that when $p \leq 1$ a "good sobstitute" of L^p spaces is given by the corresponding Hardy spaces. Hence in section 4 we consider briefly translation invariant linear operators in the Hardy-Lorentz spaces $H^{p,q}(\mathbb{R}^N)$, and we study in some more details the Bochner-Riesz means of distributions in these spaces. In [S-T-W] E. Stein, M. Taibleson, and G. Weiss, proved that when $0 < p < 1$ the Bochner-Riesz means at the "critical index" $N/p - (N+1)/2$ do not map the Hardy space $H^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, but they do map $H^p(\mathbb{R}^N)$ into $L^{p,\infty}(\mathbb{R}^N)$. (In particular this can be considered as an extension of Zafran's theorem to the case $0 < p < 1$). We shall complement this result by showing that when $0 < p \leq 1$ and $p < q \leq +\infty$, the Bochner-Riesz means at the critical index do not map the Hardy space $H^{p,q}(\mathbb{R}^N)$ into $L^{p,\infty}(\mathbb{R}^N)$. Perhaps this result is not directly related to the main theme of this paper, but, originally, the study of these end point results for convolution operators motivated the rest of this paper.

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1. - Preliminaries

Let $(\Omega, \mathcal{B}, \mu)$ denote a measure space. The Lorentz space $L^{p,q}(\Omega, \mathcal{B}, \mu) = L^{p,q}(\Omega), 0 < p < +\infty, 0 < q \leq +\infty$, is the set of all measurable functions f on Ω with quasi-norm

$$\|f\|_{p,q} = \left\{ q/p \int_0^{+\infty} (s^{1/p} f^*(s))^q \frac{ds}{s} \right\}^{1/q} < +\infty,$$

where f^* is the nonincreasing rearrangement of f. It will be convenient for us to use an equivalent definition of $\|f\|_{p,q}$, namely

$$\|f\|_{p,q} = \left\{ q \int_0^{+\infty} (t\mu(\{x : |f(x)| > t\})^{1/p})^q \frac{dt}{t} \right\}^{1/q}.$$

To check that these two expressions are the same, simply make the substitution $s = \mu(\{x : |f(x)| > t\})$ and then integrate by parts. As usual, instead of $L^{p,p}(\Omega)$

and $\|f\|_{p,p}$ we simply write $L^p(\Omega)$ and $\|f\|_p$ respectively, and we denote by $L^\infty(\Omega)$ the set of all measurable essentially bounded functions f on Ω, with $\|f\|_\infty = \operatorname{Ess.Sup}_{x\in\Omega} |f(x)|$. Also, following the standard notation, we let $\ell^{p,q}$ and ℓ^p denote the spaces $L^{p,q}(N)$ and $L^p(N)$, where N is the set of positive integers equipped with the counting measure, and we write $\|\{a_k\}\|_{p,q}$ and $\|\{a_k\}\|_p$ to denote the corresponding quasi-norms of a sequence $\{a_k\}$.

It is well known that if $1 < p < +\infty$ and $1 \le q \le +\infty$, or $p = q = 1$, then $L^{p,q}(\Omega)$ is a Banach space, and $\|\cdot\|_{p,q}$ is equivalent to a norm. However, for other values of p and q, $L^{p,q}(\Omega)$ is only a quasi-Banach space. In particular, if $0 < q < 1 \le p$ then $\|\cdot\|_{p,q}$ is equivalent to a q-norm, and if $0 < p < 1$ and $0 < q \le +\infty$ then $\|\cdot\|_{p,q}$ is equivalent to a r-norm, with $r = \operatorname{Min}\{p, q\}$.

Lorentz spaces have a close relation with the real method of interpolation, and in the sequel we shall use the following well known results. If $0 < q, r, s \le +\infty, 0 < p_0 < p < p_1 < +\infty$, and if $(1-\theta)/p_0 + \theta/p_1 = 1/p$, then $(L^{p_0,r}(\Omega), L^{p_1,s}(\Omega))_{\theta,q} = L^{p,q}(\Omega)$. If $0 < p < +\infty, 0 < r < q < s \le +\infty$, and if $(1-\theta)/r + \theta/s = 1/q$, then $(L^{p,r}(\Omega), L^{p,s}(\Omega))_{\theta,q} = L^{p,q}(\Omega)$.

Also, it is well known that the dual space of $L^{p,q}(\Omega), 1 < p < +\infty, 0 < q < +\infty$, can be naturally identified with the space $L^{p',q'}(\Omega)$, where $p' = p/(p-1)$ and $q' = q/(q-1)$ if $1 < q < +\infty, q' = +\infty$ if $0 < q \le 1$.

For all these properties, and more on Lorentz spaces, see for example [B-L], [H], and [S-W,2]. A proof that $\|\cdot\|_{p,\infty}$ is a p-norm when $0 < p < 1$ is contained in [K] and [S-T-W]. A slight modification of this proof shows that $\|\cdot\|_{p,q}$ is a p-norm also if $0 < p < 1$ and $p \le q \le +\infty$.

It turns out that Lorentz spaces, as many other quasi-Banach spaces, admit some sort of "atomic decomposition". This case is very simple.

Let f be a function in $L^{p,q}(\Omega), 0 < p < +\infty, 0 < q \le +\infty$. Then there exists a sequence of measurable functions $\{g_j\}$ such that:

i) $\|g_j\|_\infty \le 1$,

ii) $\mu(\operatorname{Supp} g_j) \le 2^j$,

iii) $\mu(\operatorname{Supp} g_j \cap \operatorname{Supp} g_k) = 0$ if $j \ne k$,

iv) $f = \sum_{j=-\infty}^{+\infty} f^*(2^j) g_j$ and $\|f\|_{p,q} \approx \{\sum_{j=-\infty}^{+\infty} (2^{j/p} f^*(2^j))^q\}^{1/q}$.

The series $\sum_{j=-\infty}^{+\infty} f^*(2^j) g_j$ converges in the topology of $L^{p,q}(\Omega)$ if $0 < q < +\infty$, or, more generally, in the topology of $L^{p_0}(\Omega) + L^{p_1}(\Omega)$ if $0 < p_0 < p < p_1 < +\infty$ and $0 < q \le +\infty$.

This atomic decomposition is obtained simply by defining

$$\begin{aligned} g_j(x) &= (f^*(2^j))^{-1} f(x) && \text{if } f^*(2^{j+1}) \le |f(x)| < f^*(2^j), \\ &= 0 && \text{otherwise.} \end{aligned}$$

We shall use this decomposition in the sequel. Here we want only to

remark that from this decomposition we can easily obtain the well known imbedding of the Lorentz space $L^{p,q}(\Omega)$ into $L^{p,r}(\Omega)$ if $q < r$.

Throughout this paper we assume that the measure space $(\Omega, \mathcal{B}, \mu)$ has a family $\{T_\alpha\}$ of measure preserving transformations; i.e. each T_α is a measurable transformation of Ω into Ω, and for every A in $\mathcal{B}, \mu(T_\alpha^{-1}A) = \mu(A)$. We also assume that the family $\{T_\alpha\}$ has the following "mixing property": If A and B are measurable sets of finite measure, n is a positive integer, and if $\varepsilon > 0$, there exist $T_1, T_2, ..., T_n$, such that for every $k = 1, 2, ..., n$, we have

$$\mu(T_k^{-1}A \cap \bigcup_{j \neq k} T_j^{-1}A) < \varepsilon, \text{ and } \mu(T_k^{-1}B \cap \bigcup_{j \neq k} T_j^{-1}B) < \varepsilon.$$

Note that this forces $\mu(\Omega)$ to be $+\infty$.

If f is a measurable function on Ω, we write $T_\alpha f(x)$ for $f(T_\alpha x)$. An operator Φ on measurable functions commutes with the family of translations $\{T_\alpha\}$ if for every T_α and every f, in the domain of definition of Φ, we have $\Phi T_\alpha f = T_\alpha \Phi f$. Φ is sublinear if for every complex number λ and every function f and g we have $|\Phi \lambda f| = |\lambda||\Phi f|$, and $|\Phi(f+g)| \leq |\Phi f| + |\Phi g|$. An operator Φ that satisfies the above properties is called a translation invariant sublinear operator. Translation invariant linear operators are defined similarly.

The example we have in mind is a locally compact non compact group Ω with the family of Borel sets $\mathcal{B}$ and left Haar measure μ. $\{T_\alpha\}$ is the family of left translations. However, in this case the space $(\Omega, \mathcal{B}, \mu)$ is equipped with another natural family of translations: the right translations. We shall use both type of translations in what follows, but we want to emphasize that only the left translations are measure preserving transformations if the group Ω is non unimodular. In any case, there is a close relation between left and right translation invariant sublinear operators. Let us write the group operation additively, and let us denote by Λ the modular function of $(\Omega, \mathcal{B}, \mu)$. If $0 < p < +\infty$, we define the operator I_p on measurable functions on Ω by $I_p f(x) = \Lambda^{1/p}(-x) f(-x)$. Then it is straightforward to verify that I_p is an isometry of $L^{p,q}(\Omega), 0 < p < +\infty, 0 < q \leq +\infty$. Also, if Φ is a left (right) translation invariant sublinear operator on $L^{p,q}(\Omega), I_p \Phi I_p$ is a right (left) translation invariant sublinear operator on $L^{p,q}(\Omega)$, and the operator quasi-norms of Φ and $I_p \Phi I_p$ are equal. Using this result we shall be able to deduce general properties of the right translation invariant sublinear operators from the corresponding properties of the left translation invariant ones.

An important class of left (right) translation invariant linear operators on a locally compact group is given by right (left) convolutions. If ν is a complex measure and f is an integrable function,

$$f * \nu(x) = \int_\Omega \Lambda(-y) f(x-y) \mathrm{d}\nu(y),$$

and

$$\nu * f(x) = \int_\Omega f(-y+x) \mathrm{d}\nu(y).$$

2. - Operators on Lorentz Spaces

We are now ready to state the main results of this paper.

THEOREM 2.1. *Let $0 < q < p < +\infty, 0 < q < 1$, and let Φ be a bounded (not necessarily translation invariant) sublinear operator on the Lorentz space $L^{p,q}(\Omega)$. Then Φ can be extended to a bounded operator of $L^{p,r}(\Omega)$ into $L^{p,r}(\Omega)$, for every r with $q \leq r \leq \mathrm{Min}\{1, p\}$.*

THEOREM 2.2. *Let $1 \leq q < p < +\infty$, and let Φ be a bounded linear operator on the Lorentz space $L^{p,q}(\Omega)$. Then if the adjoint of Φ commutes with translations, Φ can be extended to a bounded operator of $L^{p,r}(\Omega)$ into $L^{p,r}(\Omega)$, for every r with $q \leq r \leq p$.*

THEOREM 2.3. *Let $0 < p < q \leq +\infty$, and let Φ be a bounded translation invariant sublinear operator on the Lorentz space $L^{p,q}(\Omega)$. Then Φ also maps $L^{p,r}(\Omega)$ into $L^{p,r}(\Omega)$, for every r with $p \leq r \leq q$.*

Note that when $(\Omega, \mathcal{B}, \mu)$ is a locally compact group equipped with left Haar measure, then a linear operator Φ is left translation invariant if and only if its adjoint Φ^* is left translation invariant. Hence, putting together theorems 2.1, 2.2, and 2.3, we obtain theorem B in the introduction. Let us now pass to the proof of these results.

PROOF OF THEOREM 2.1. Let f be in $L^{p,q}(\Omega)$. Then f is also in $L^{p,r}(\Omega)$, and we can decompose f as $\sum_{j=-\infty}^{+\infty} f^*(2^j) g_j$, with $\|g_j\|_\infty \leq 1, \mu(\mathrm{Supp}.g_j) \leq 2^j$, and $\{\sum_{j=-\infty}^{+\infty} (2^{j/p} f^*(2^j))^r\}^{1/r} \approx \|f\|_{p,r}$. Note that for every j and every q, $\|g_j\|_{p,q} \leq 2^{j/p}$. Then, since the operator Φ is sublinear and the space $L^{p,r}(\Omega)$ is r-normed, we have

$$\begin{aligned}
\|\Phi f\|_{p,r}^r &\leq c \sum_{j=-\infty}^{+\infty} (f^*(2^j) \|\Phi g_j\|_{p,r})^r \\
&\leq c \sum_{j=-\infty}^{+\infty} (f^*(2^j) \|\Phi g_j\|_{p,q})^r \\
&\leq c \sum_{j=-\infty}^{+\infty} (f^*(2^j) \|g_j\|_{p,q})^r \\
&\leq c \sum_{j=-\infty}^{+\infty} (2^{j/p} f^*(2^j))^r \\
&\leq c \|f\|_{p,r}.
\end{aligned}$$

Since the space $L^{p,q}(\Omega)$ is dense in $L^{p,r}(\Omega)$, the theorem follows.

PROOF OF THEOREM 2.2. This theorem is an immediate consequence of theorem 2.3 and the duality relations between Lorentz spaces.

We now come to the proof of theorem 2.3. In order to present the idea behind this proof, we begin by proving a particular case of it. As we shall see the basic idea involves expressing the L^p quasi-norm of a function g in terms of the $L^{p,\infty}$ quasi-norm of a related function f.

Suppose that $(\Omega, \mathcal{B}, \mu)$ is a locally compact non compact group, and Φ is a right convolution operator with a kernel with compact support. Then Φ is a left translation invariant operator that maps functions with compact support into functions with compact support. We shall prove that if Φ maps $L^{p,\infty}(\Omega)$ into $L^{p,\infty}(\Omega)$, then it also maps $L^p(\Omega)$ into $L^p(\Omega)$. Let g be a function with compact support, and let $\{x_k\}$ be a sequence of points in Ω such that the supports of the functions $\{g(x_k+\cdot)\}$ are mutually disjoint, and the same is true for the functions $\{\Phi g(x_k+\cdot)\}$. Finally let $f = \sum_{k=1}^{+\infty} k^{-1/p} g(x_k+\cdot)$. We claim that $\|f\|_{p,\infty} = \|g\|_p$. Indeed, using Riemann sums for approximating the integral of the distribution function of $|g|^p$, we have

$$t^p\mu(\{x : |f(x)| > t\}) = \sum_{k=1}^{+\infty} t^p\mu(\{x : |g(x)|^p > kt^p\}) \leq \int_\Omega |g(x)|^p \mathrm{d}\mu(x),$$

but also

$$\operatorname*{Lim}_{t\to 0} t^p\mu(\{x : |f(x)| > t\}) = \int_\Omega |g(x)|^p \mathrm{d}\mu(x).$$

Analogously $\|\Phi g\|_p = \|\Phi f\|_{p,\infty}$. Then if the operator Φ is bounded on $L^{p,\infty}(\Omega)$ we have

$$\|\Phi g\|_p = \|\Phi f\|_{p,\infty} \leq c\|f\|_{p,\infty} = c\|g\|_p.$$

Note also that the quasi-norm of Φ as an operator on $L^{p,\infty}(\Omega)$ majorizes the quasi-norm of Φ as an operator on $L^p(\Omega)$.

Let us now pass to the proof of the general result. We start by establishing a couple of lemmas.

LEMMA 2.4. *If f is a function in $L^p(\Omega)$ and $\varepsilon > 0$, then there exists n_0 such that for every $n > n_0$ and every t with $n^{-2} < t < n^{-1}$ we have*

$$0 \leq \|f\|_p - \{\sum_{k=1}^{n^3} t\mu(\{x : |f(x)|^p > kt\})\}^{1/p} < \varepsilon.$$

If f is not in $L^p(\Omega)$ and $M > 0$, then there exists n_0 such that for $n > n_0$ and

$n^{-2} < t < n^{-1}$ *we have*

$$\{\sum_{k=1}^{n^3} t\mu(\{x : |f(x)|^p > kt\})\}^{1/p} > M.$$

PROOF. The first part of the lemma easily follows from the inequalities

$$\begin{aligned} 0 &\leq \int_0^{+\infty} \mu(\{x : |f(x)|^p > s\})\mathrm{d}s - \sum_{k=1}^{n^3} t\mu(\{x : |f(x)|^p > kt\}) \\ &\leq \int_0^t \mu(\{x : |f(x)|^p > s\})\mathrm{d}s + \int_{(n^3+1)t}^{+\infty} \mu(\{x : |f(x)|^p > s\})\mathrm{d}s. \end{aligned}$$

The proof of the second part of the lemma is similar, and we omit the details.

LEMMA 2.4. *Let g be a simple function, $0 < p < +\infty, p \leq q \leq +\infty$, and $n > 1$. Then there exist $T_1, T_2, ..., T_n$, such that*

$$\|\sum_{k=1}^{+\infty} k^{-1/p} T_k g\|_{p,q} \leq c(\mathrm{Log}\ n)^{1/q} \|g\|_p.$$

PROOF. Let A be the support of g, and let $\varepsilon > 0$. There exist $T_1, T_2, ..., T_n$, such that for every $k, \mu(T_k^{-1}A \cap \bigcup_{j \neq k} T_j^{-1}A) < \varepsilon$. Denote by A_k the set $T_k^{-1}A - \bigcup_{j \neq k} T_j^{-1}A$. Then if $q = p$ and $1 \leq p < +\infty$, since the A_k's are mutually disjoint,

$$\begin{aligned} &\|\sum_{k=1}^{n} k^{-1/p} T_k g\|_p \\ &\leq \|\sum_{k=1}^{n} k^{-1/p} \chi_{A_k} T_k g\|_p + \sum_{k=1}^{n} k^{-1/p} \|(1 - \chi_{A_k}) T_k g\|_p \\ &\leq (\sum_{k=1}^{n} k^{-1})^{1/p} \|g\|_p + \varepsilon \|g\|_\infty \sum_{k=1}^{n} k^{-1/p} \\ &\leq c(\mathrm{Log}\ n)^{1/p} \|g\|_p \end{aligned}$$

if ε is small enough. If $0 < p \leq 1$ and again $q = p$, then we simply have

$$\|\sum_{k=1}^{n} k^{-1/p} T_k g\|_p^p \leq (\sum_{k=1}^{n} k^{-1}) \|g\|_p^p.$$

This takes care of the case $q = p$. We consider now the case $q = +\infty$. If $t \geq \sum_{k=1}^{n} k^{-1/p}\|g\|_\infty$ we obviously have

$$\mu(\{x : |\sum_{k=1}^{n} k^{-1/p}T_k g(x)| > t\}) = 0.$$

If $t < \sum_{k=1}^{n} k^{-1/p}\|g\|_\infty$ we have

$$\begin{aligned}
&\mu(\{x : |\sum_{k=1}^{n} k^{-1/p}T_k g(x)| > t\}) \\
&\leq \mu(\{x : |\sum_{k=1}^{n} k^{-1/p}\chi_{A_k}(x)T_k g(x)| > t\}) + \\
&\qquad \mu(\{x : |\sum_{k=1}^{n} k^{-1/p}(1-\chi_{A_k}(x))T_k g(x)| > 0\}) \\
&\leq \sum_{k=1}^{n} \mu(\{x : |k^{-1/p}\chi_{A_k}(x)T_k g(x)| > t\}) + n\varepsilon \\
&\leq t^{-p}\sum_{k=1}^{n} t^p \mu(\{x : |g(x)| > kt^p\}) + n\varepsilon \\
&\leq t^{-p}\|g\|_p^p + n\varepsilon \\
&\leq 2t^{-p}\|g\|_p^p
\end{aligned}$$

if ε is small enough. The lemma is then proved also in the case $q = +\infty$. The remaining case, $p < q < +\infty$, now follows by convexity. Indeed for every f in $L^p(\Omega)$, and in particular for $f = \sum_{k=1}^{n} k^{-1/p}T_k g$, we have

$$\begin{aligned}
\|f\|_{p,q} &\leq (q/p)^{1/q}\|f\|_p^{p/q}\|f\|_{p,\infty}^{1-p/q} \\
&\leq c((\text{Log } n)^{1/p}\|g\|_p)^{p/q}(\|g\|_p)^{1-p/q} \\
&\leq c(\text{Log } n)^{1/q}\|g\|_p.
\end{aligned}$$

PROOF OF THEOREM 2.3. Since $L^{p,r}(\Omega)$ is an interpolation space between $L^p(\Omega)$ and $L^{p,q}(\Omega)$, it is enough to prove the theorem for the particular case $r = p$.

Suppose the operator Φ is not bounded on $L^p(\Omega)$. Then for every $M > 0$ there exists a simple function g, with $\|g\|_p = 1$, but $\|\Phi g\|_p > M$. Denote by A the support of g, and by $B(s)$ the set $\{x : |\Phi g(x)| > s\}$. By lemma 2.4 it is possible to choose n so big that $\{\sum_{k=1}^{n^3} t\mu(B(k^{1/p}t^{1/p}))\}^{1/p} > M$, for every t with $n^{-2} < t < n^{-1}$. Also it is possible to choose $T_1, T_2, ..., T_{n^3}$, such that for every

$k = 1, 2..., n^3, \mu(T_k^{-1}A \cap \bigcup_{j\neq k} T_j^{-1}A) < \varepsilon$, and $\mu(T_k^{-1}B(n^{-\delta}) \cap \bigcup_{j\neq k} T_j^{-1}B(n^{-\delta})) < \varepsilon$, where ε and δ are two positive constants we shall specify later. Using g and the T_k's, we construct the function $f = \sum_{k=1}^{n^3} k^{-1/p}T_k g$. Then, if ε is small enough, by lemma 2.5 we have that $\|f\|_{p,q} \leq c(\text{Log } n)^{1/q}$. On the other hand we shall prove that $\|\Phi f\|_{p,q} > cM(\text{Log } n)^{1/q}$, and this will contraddict the boundedness of Φ on $L^{p,q}(\Omega)$.

Let $D(k,t) = T^{-1}B(k^{1/p}t^{1/p}) - \bigcup_{j\neq k} T^{-1}B(n^{-\delta})$. Then, if x is in $D(k,t)$ and $n^{-2} < t < n^{-1}$,

$$\begin{aligned}|\Phi f(x)| &\geq k^{-1/p}|\Phi T_k g(x)| - \sum_{j\neq k} j^{-1/p}|\Phi T_j g(x)| \\ &\geq t^{1/p} - n^{-\delta}\sum_{j\neq k} j^{-1/p} \\ &\geq 2^{-1}t^{1/p}\end{aligned}$$

if δ is big enough. Also, for the same range of $t, D(j,t) \cap D(k,t)$ is empty if $j \neq k$, and, by the assumption on the T_k's, $\mu(D(k,t)) > \mu(B(k^{1/p}t^{1/p})) - \varepsilon$. Then

$$\begin{aligned}\mu(\{x : |\Phi f(x)| > 2^{-1}t^{1/p}\}) &\geq \sum_{k=1}^{n^3} \mu(D(k,t)) \\ &> \sum_{k=1}^{n^3} \mu(B(k^{1/p}t^{1/p})) - n^3\varepsilon \\ &> t^{-1}M^p - n^3\varepsilon \\ &> 2^{-1}t^{-1}M^p\end{aligned}$$

if ε is small enough. (Observe that the choice of ε and δ depends only on n and M.) To conclude

$$\begin{aligned}\|\Phi f\|_{p,q} &= \{q\int_0^{+\infty} (s\mu(\{x : |\Phi f(x)| > s\})^{1/p})^q \frac{ds}{s}\}^{1/q} \\ &= 2^{-1}\{q/p\int_0^{+\infty} (t\mu(\{x : |\Phi f(x)| > 2^{-1}t^{1/p}\}))^{q/p}\frac{dt}{t}\}^{1/q} \\ &> 2^{-1}\{q/p\int_{n^{-2}}^{n^{-1}} (2^{-1}M^p)^{q/p}\frac{dt}{t}\}^{1/q} \\ &> cM(\text{Log } n)^{1/q}.\end{aligned}$$

When $q < +\infty$ the above proof can be simplified a bit. The idea is to split Φg into $a + b$, with a simple, $\|a\|_p > M$, and $\|b\|_{p,q} < \varepsilon$. The technical details of the proof became easier. In the next remarks we shall show with examples that essentially no hypothesis in our theorems can be weakened.

REMARK 2.6. It is clear that theorem 2.1 holds for any measure space $(\Omega, \mathcal{B}, \mu)$ and sublinear operator Φ. However in theorems 2.2 and 2.3 the assumption that the operator Φ commutes with some kind of translation semms essential, as the following example shows. Let $1 < p < +\infty, 1 \leq q \leq +\infty$, and let $1/p + 1/p' = 1/q + 1/q' = 1$. If h is a function in $L^{p,q}(\Omega) - \bigcup_{r<q} L^{p,r}(\Omega)$, and if g is a function in $L^{p',q'}(\Omega) - \bigcup_{r'<q'} L^{p',r'}(\Omega)$, define

$$\Phi f(x) = (\int_\Omega f(y)g(y)d\mu(y))h(x).$$

Then Φ is a linear operator bounded on $L^{p,q}(\Omega)$, but not bounded on $L^{p,r}(\Omega)$ if $r \neq q$. (This example is essentially taken from [S-W,1].)

REMARK 2.7. We proved in theorem 2.3 that every translation invariant sublinear operator bounded on $L^{p,q}(\Omega), p < q$, is automatically bounded also on $L^p(\Omega)$. The converse is not true if $1 \leq p < +\infty$. Consider indeed the following example. Let $(\Omega, \mathcal{B}, \mu)$ be a locally compact group with non atomic left Haar measure, let U be a non empty open set in Ω of positive finite measure, and let $1 \leq p < +\infty$. Then the operator

$$\Phi f(x) = \{\int_U |f(x+y)|^p d\mu(y)\}^{1/p}$$

is sublinear, commutes with left translations, it is also bounded on $L^p(\Omega)$, but it is not bounded on $L^{p,q}(\Omega)$ if $q > p$.

Examples of left translation invariant linear operators bounded on $L^p(\Omega)$, but not on $L^{p,q}(\Omega)$ if $q < p < 1$, are also given in the next section.

REMARK 2.8. For translation invariant linear operators on the N-dimensional torus $T^N (= \mathbb{R}^N/Z^N)$ one can quite easily prove the following result.

Let φ be a continuous function on $\mathbb{R}^N$. For every $\varepsilon > 0$ and f in $L^{p,q}(T^N), 1 < p < +\infty, 1 \leq q \leq +\infty$, define

$$\Phi_\varepsilon f(x) = \sum_{k \in Z^N} \varphi(\varepsilon k)\hat{f}(k)\exp(2\pi i k \cdot x).$$

Then if $\|\Phi_\varepsilon f\|_{p,q} \leq c_1\|f\|_{p,q}$, with c_1 independent on ε and f, we also have $\|\Phi_\varepsilon f\|_p \leq c_2\|f\|_p$, with c_2 independent on ε and f.

The proof of this result is by transference between $\mathbb{R}^N$ and T^N. First one has to transfer in a natural way the operators $\{\Phi_\varepsilon\}$ on $L^{p,q}(T^N)$ to continuous operators on $L^{p,q}(\mathbb{R}^N)$. By the theorems proved these operators are also bounded on $L^p(\mathbb{R}^N)$. Then one has to transfer back there operators from $L^p(\mathbb{R}^N)$ to $L^p(T^N)$. Details are as in [S-W,2], VII.3. Then at least for this class of convolution operators the theorem of A.M. Shteinberg can be extended to the case $2 < p < +\infty$.

3. - The case $0 < p < 1$

In all this section we let $(\Omega, \mathcal{B}, \mu)$ denote a locally compact group Ω, not necessarily non compact, with Borel sets $\mathcal{B}$, and left Haar measure μ. The modular function of $(\Omega, \mathcal{B}, \mu)$ is denoted by Λ, and we write the group operation additively.

In [O] D. Oberlin proved that if $0 < p < 1$, then the bounded right translation invariant linear operators on the space $L^p(\Omega)$ are precisely the operators Φ of the form

$$\Phi f(x) = \sum_{k=1}^{+\infty} \alpha_k f(-x_k + x),$$

where $\{x_k\}$ is a sequence of points in Ω, and $\{\alpha_k\}$ is a sequence of complex numbers with $\sum_{k=1}^{+\infty} |\alpha_k|^p < +\infty$. (See also [Sa] and [B-L], pg. 170, for the case of operators on the torus and the real line respectively.) Later N. Kalton proved that these operators are the only ones bounded on $L^{p,\infty}(\Omega)$, at least when Ω is a metrizable compact group. (See [K].)

The purpose of this section is to extend these results and characterize the right and the left translation invariant linear operators that are bounded on the Lorentz spaces $L^{p,q}(\Omega)$, with $0 < p < 1$ and $0 < q \leq +\infty$. The results are the following.

THEOREM 3.1. *The right translation invariant linear operators bounded on the Lorentz space $L^{p,q}(\Omega), 0 < p < 1, 0 < q \leq +\infty$, are precisely the operators Φ of the form*

$$\Phi f(x) = \sum_{k=1}^{+\infty} \alpha_k f(-x_k + x),$$

where $\{x_k\}$ is a sequence of points in Ω, and $\{\alpha_k\}$ is a sequence of complex numbers in the Lorentz space $\ell^{p,q}$ if $0 < q \leq p$, or in ℓ^p if $p \leq q \leq +\infty$.

THEOREM 3.2. *The left translation invariant linear operators bounded on the Lorentz space $L^{p,q}(\Omega), 0 < p < 1, 0 < q \leq +\infty$, are precisely the operators Φ of the form*

$$\Phi f(x) = \sum_{k=1}^{+\infty} \alpha_k \Lambda^{1/p}(x_k) f(x + x_k),$$

where $\{x_k\}$ is a sequence of points in Ω, and $\{\alpha_k\}$ is a sequence of complex numbers in the Lorentz space $\ell^{p,q}$ if $0 < q \leq p$, or in ℓ^p if $p \leq q \leq +\infty$.

If x is a point in Ω, we denote by δ_x the unit mass measure at x. A discrete measure is a measure of the form $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}$, where $\{x_k\}$ is a sequence of points in Ω, and $\{\alpha_k\}$ is a sequence of complex numbers with

$\sum_{k=1}^{+\infty} |\alpha_k| < +\infty$. If f is in $L^1(\Omega)$, then

$$(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f(x) = \sum_{k=1}^{+\infty} \alpha_k f(-x_k + x).$$

Thus the right translation invariant linear operators bounded on $L^{p,q}(\Omega), 0 < p < 1, 0 < q \leq +\infty$, are left convolution operators with discrete measures. In the sequel we shall abuse notation slightly. To perform convolutions freely, by $L^{p,q}(\Omega)$ we shall often mean $L^{p,q}(\Omega) \cap L^1(\Omega)$. The proof of theorems 3.1 and 3.2 is a bit involved, also because we do not have a complete analog of theorem B for compact groups. Therefore, to make it more digestible, we break it up into several easy lemmas.

LEMMA 3.3. *Let Φ be a bounded right translation invariant linear operator on $L^{p,q}(\Omega), 0 < p < 1, 0 < q \leq +\infty$. Then there exists a discrete measure $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}$, such that for every f in $L^{p,q}(\Omega)$ we have $\Phi f = (\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f$.*

PROOF. Suppose first that the group Ω is compact, and choose an r, with $p < r < 1$. Then if Φ maps $L^{p,q}(\Omega)$ into itself, it also maps $L^{r,\infty}(\Omega)$ into $L^0(\Omega)$. By the Shteinberg theorem, or even by the Nikishin-Stein theorem and interpolation (see [GC-RF], VI), Φ is bounded on $L^r(\Omega)$, and by Oberlin's result Φ is the left convolution with a discrete measure. If the group Ω is not compact and Φ is bounded on $L^{p,q}(\Omega)$, then, by theorem B, Φ is also bounded on $L^p(\Omega)$, and again from the Oberlin result Φ is the left convolution with a discrete measure.

LEMMA 3.4. *Let $0 < p < 1, 0 < q \leq +\infty$, and let g be a function on Ω with $\|g\|_\infty \leq 1$, and $\mu(\text{Supp}.g) \leq s$. Then*

$$\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * g\|_{p,q} \leq c s^{1/p} \|\{\alpha_k\}\|_{p,q}.$$

PROOF. Choose $0 < p_0 < p < p_1 < 1$. Then, if $i = 0, 1$,

$$\begin{aligned}\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * g\|_{p_i} &= \|\sum_{k=1}^{+\infty} \alpha_k g(-x_k + \cdot)\|_{p_i} \\ &\leq \{\sum_{k=1}^{+\infty} |\alpha_k|^{p_i} \|g(-x_k + \cdot)\|_{p_i}^{p_i}\}^{1/p_i} \\ &\leq s^{1/p_i} \|\{\alpha_k\}\|_{p_i}.\end{aligned}$$

Hence by interpolation we also have

$$\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * g\|_{p,q} \leq c s^{1/p} \|\{\alpha_k\}\|_{p,q}.$$

LEMMA 3.5. *Let $0 < p < 1$ and $p \le q \le +\infty$. If for every f in $L^p(\Omega)$ we have $\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q} \le c\|f\|_p$, then the sequence $\{\alpha_k\}$ is in $\ell^{p,q}$. Conversely, for every sequence $\{\alpha_k\}$ in $\ell^{p,q}$,*

$$\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q} \le c\|\{\alpha_k\}\|_{p,q}\|f\|_p.$$

Note that this lemma characterizes those discrete measures that convolve $L^p(\Omega)$ into $L^{p,q}(\Omega)$, a result of some independent interest. A proof of this lemma for metrizable compact groups appears in [K]. Our proof is essentially a bilinear interpolation argument, applies to every group, and it is perhaps more elementary.

PROOF. The first part of the lemma follows by testing the measure $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}$ against characteristic functions of very small open sets. The details, easy but not completely trivial, are as in [O] (but see also the proof of the next lemma).

To prove the second part of the lemma decompose a function f in $L^p(\Omega)$ as $\sum_{j=-\infty}^{+\infty} f^*(2^j) g_j$, with $\|g_j\|_\infty \le 1, \mu(\mathrm{Supp}.g_j) \le 2^j$, and $\|f\|_p \approx \{\sum_{j=-\infty}^{+\infty} (2^{j/p} f^*(2^j))^p\}^{1/p}$. Then, since the space $L^{p,q}(\Omega)$ is p-normed, using the previous lemma we obtain

$$\begin{aligned}
\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q}^p &\le c \sum_{j=-\infty}^{+\infty} (f^*(2^j))^p \|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * g_j\|_{p,q}^p \\
&\le c\|\{\alpha_k\}\|_{p,q}^p \sum_{j=-\infty}^{+\infty} (2^{j/p} f^*(2^j))^p \\
&\le c\|\{\alpha_k\}\|_{p,q}^p \|f\|_p^p.
\end{aligned}$$

LEMMA 3.6. *Let $0 < p < 1$ and $0 < q \le +\infty$. If for every f in $L^{p,q}(\Omega)$ we have $\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q} \le c\|f\|_{p,q}$, then the sequence $\{\alpha_k\}$ is in $\ell^{p,q}$ if $0 < q \le p$, or in ℓ^p if $p \le q \le +\infty$.*

PROOF. By testing the measure $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}$ against characteristic functions of very small open sets we easily see that the sequence $\{\alpha_k\}$ is at least in $\ell^{p,q}$. Then the lemma follows in the case $0 < q \le p$. If $q = +\infty$, or if $p < q \le +\infty$ and the group Ω is non compact, and if the measure $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}$ convolves $L^{p,q}(\Omega)$ into itself, then by theorems B or C this measure also convolves $L^p(\Omega)$ into itself. We thus conclude that the sequence $\{\alpha_k\}$ is in ℓ^p, and the lemma

is proved also in this case. To complete the proof of the lemma we need to consider the case of a compact group, and $p < q < +\infty$. We can suppose the group Ω infinite and its Haar measure non atomic. This is what we shall need in the rest of the proof.

We start by assuming, without loss of generality, that $\|\{\alpha_k\}\|_{p,q} \leq 1/2$, and that the sequence $\{|\alpha_k|\}$ is non increasing. We also assume that all the terms of this sequence are different from zero, otherwise the proof is easier. Let n be an arbitrary large integer. Our goal is to estimate $\sum_{k=1}^{n} |\alpha_k|^p$. Let ε be a positive constant much smaller than $\{\sum_{k=1}^{n} |\alpha_k|^p\}^{1/p}(\text{Log } |\alpha_n|^{-1})^{1/q-1/p}$. Then it is possible to find $m > n$ such that $\|\{\alpha_{m-1+k}\}\|_{p,q} < \varepsilon$, and it is also possible to find an open set U such that the sets $x_1 + U, x_2 + U, ..., x_{m-1} + U$, are mutually disjoint. Let $\mu(U) = \delta$, and let $\eta = \delta|\alpha_n|^{2p}$. We can construct a function f supported in U such that

$$\begin{aligned} f^*(t) &= \eta^{-1/p} && \text{if } 0 < t < \eta, \\ &= t^{-1/p} && \text{if } \eta \leq t < \delta, \\ &= 0 && \text{if } \delta \leq t. \end{aligned}$$

Hence if $0 < r < +\infty$, $\|f\|_{p,r} = (1 + r/p\text{Log } \delta/\eta)^{1/r} \approx (\text{Log } |\alpha_n|^{-1})^{1/r}$, and if $\delta^{-1/p}|\alpha_k| < t < \eta^{-1/p}|\alpha_k|$ for every $k = 1, 2, ..., n$,

$$\begin{aligned} \mu(\{x : |\sum_{k=1}^{m-1} \alpha_k f(-x_k + x)| > t\}) &\geq \sum_{k=1}^{n} \mu(\{x : |f(x)| > t/|\alpha_k|\}) \\ &= t^{-p} \sum_{k=1}^{n} |\alpha_k|^p. \end{aligned}$$

Since $(\delta^{-1/p}, \eta^{-1/p}|\alpha_n|) \subseteq \bigcap_{k=1}^{n} (\delta^{-1/p}|\alpha_k|, \eta^{-1/p}|\alpha_k|)$, we thus have

$$\begin{aligned} &\|(\sum_{k=1}^{m-1} \alpha_k \delta_{x_k}) * f\|_{p,q} \\ &\geq \{q \int_{\delta^{-1/p}}^{\eta^{-1/p}|\alpha_n|} (t\mu(\{x : |\sum_{k=1}^{m-1} \alpha_k f(-x_k + x)| > t\})^{1/p})^q \frac{dt}{t}\}^{1/q} \\ &\geq \{\sum_{k=1}^{n} |\alpha_k|^p\}^{1/p} \{q \int_{\delta^{-1/p}}^{\eta^{-1/p}|\alpha_n|} \frac{dt}{t}\}^{1/q} \\ &= \{\sum_{k=1}^{n} |\alpha_k|^p\}^{1/p} (q\text{Log } |\alpha_n|^{-1})^{1/q}. \end{aligned}$$

Also, by the previous lemma,

$$\|(\sum_{k=m}^{+\infty} \alpha_k \delta_{x_k} * f\|_{p,q} \leq c\|\{\alpha_{m-1+k}\}\|_{p,q}\|f\|_p$$
$$\leq c\varepsilon(\text{Log } |\alpha_n|^{-1})^{1/p}.$$

Collecting these two estimates, and because of our definition of ε we thus obtain

$$\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q}$$
$$\geq c\|(\sum_{k=1}^{m-1} \alpha_k \delta_{x_k}) * f\|_{p,q} - c\|(\sum_{k=m}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q}$$
$$\geq c\{\sum_{k=1}^{n} |\alpha_k|^p\}^{1/p}(\text{Log } |\alpha_n|^{-1})^{1/p} - c\varepsilon(\text{Log } |\alpha_n|^{-1})^{1/p}$$
$$\geq c\{\sum_{k=1}^{n} |\alpha_k|^p\}^{1/p}(\text{Log } |\alpha_n|^{-1})^{1/q}$$
$$\geq c\{\sum_{k=1}^{n} |\alpha_k|^p\}^{1/p}\|f\|_{p,q}.$$

The lemma then follows.

LEMMA 3.7. *Let* $0 < p < 1$ *and* $p \leq q \leq +\infty$. *Then*

$$\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q} \leq c\|\{\alpha_k\}\|_p\|f\|_{p,q}.$$

PROOF. This is an easy consequence of the p-triangle inequality for the space $L^{p,q}(\Omega)$.

LEMMA 3.8. *Let* $0 < p < 1, 0 < q \leq +\infty$, *and let* $|\beta_k| \leq 1, k = 1, 2, ..., n$. *Then*

$$\|(\sum_{k=1}^{n} \beta_k \delta_{x_k}) * f\|_{p,q} \leq c\ n^{1/p}\|f\|_{p,q}.$$

PROOF. Choose $0 < p_0 < p < p_1 < 1$. By the above lemma we have

$$\|(\sum_{k=1}^{n} \beta_k \delta_{x_k}) * f\|_{p_i} \leq n^{1/p_i}\|f\|_{p_i}, \qquad i = 0, 1.$$

The lemma then follows by interpolation.

LEMMA 3.9. *Let* $0 < q \leq p < 1$. *Then*

$$\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q} \leq c\|\{\alpha_k\}\|_{p,q}\|f\|_{p,q}.$$

PROOF. We can suppose that the sequence $\{|\alpha_k|\}$ is non increasing. Then we can write $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k} = \sum_{k=0}^{+\infty} |\alpha_{2^k}| \sum_{j=2^k}^{2^{k+1}-1} \beta_j \delta_{x_j}$, with $|\beta_j| \leq 1$ for every j, and $\{\sum_{k=0}^{+\infty} (2^{k/p}|\alpha_{2^k}|)^q\}^{1/q} \approx \|\{\alpha_k\}\|_{p,q}$. Using the previous lemma and the fact that $L^{p,q}(\Omega)$ is q-normed we thus have

$$\begin{aligned}
\|(\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}) * f\|_{p,q}^q &\leq c \sum_{k=0}^{+\infty} |\alpha_{2^k}|^q \|(\sum_{j=2^k}^{2^{k+1}-1} \beta_j \delta_{x_j}) * f\|_{p,q}^q \\
&\leq c \sum_{k=0}^{+\infty} (2^{k/p}|\alpha_{2^k}|)^q \|f\|_{p,q}^q \\
&\leq c \|\{\alpha_k\}\|_{p,q}^q \|f\|_{p,q}^q.
\end{aligned}$$

PROOF OF THEOREM 3.1. By lemmas 3.3 and 3.6 every right translation invariant linear operator Φ bounded on $L^{p,q}(\Omega)$ is the left convolution with a discrete measure $\sum_{k=1}^{+\infty} \alpha_k \delta_{x_k}$, with $\{\alpha_k\}$ in $\ell^{p,q}$ if $0 \leq q \leq p$, or with $\{\alpha_k\}$ in ℓ^p if $p \leq q \leq +\infty$. Conversely, by lemmas 3.7 and 3.9 every such measure convolves $L^{p,q}(\Omega)$ into itself.

PROOF OF THEOREM 3.2. Let Φ be a left translation invariant linear operator bounded on $L^{p,q}(\Omega)$, and let I_p be the operator defined by $I_p f(x) = \Lambda^{1/p}(-x) f(-x)$. Then $I_p \Phi I_p$ is a right translation invariant linear operator bounded on $L^{p,q}(\Omega)$, and, by the previous theorem, $I_p \Phi I_p$ has the form $I_p \Phi I_p f(x) = \sum_{k=1}^{+\infty} \alpha_k f(-x_k + x)$, for appropriate $\{x_k\}$ and $\{\alpha_k\}$. Hence, by an easy computation,

$$\Phi f(x) = \sum_{k=1}^{+\infty} \alpha_k \Lambda^{1/p}(x_k) f(x + x_k).$$

4. - Bochner-Riesz means

In this section $(\Omega, \mathcal{B}, \mu)$ is the Euclidean space $\mathbb{R}^N$ equipped with Lebesgue measure. Here we want to consider briefly translation invariant linear operators on certain Hardy spaces with Lorentz quasi-norms, and study in some more details the Bochner-Riesz means of distributions in these spaces. Let us then start with some definitions.

If ψ is an indefinitely differentiable function on $\mathbb{R}^N$ with compact support and $\int_{\mathbb{R}^N} \psi(x) dx = 1$, and if f is a tempered distribution, we define the maximal function $\mathcal{M} f(x) = \operatorname{Sup}_{t>0}\{|\psi_t * f(x)|\}$, where as usual $\psi_t(x) = t^{-N}\psi(t^{-1}x)$. We say

that a tempered distribution f is the Hardy space $H^{p,q}(\mathbb{R}^N), 0 < p < +\infty, 0 < q \leq +\infty$, if the maximal function $\mathcal{M}f$ is in the Lorentz space $L^{p,q}(\mathbb{R}^N)$, and we set $\|f\|_{H^{p,q}} = \|\mathcal{M}f\|_{p,q}$.

It is possible to show that this definition of $H^{p,q}(\mathbb{R}^N)$ is essentially independent of the test function ψ, and also that these spaces admit many other important characterizations. A general reference on the spaces $H^p(\mathbb{R}^N) = H^{p,p}(\mathbb{R}^N)$ is of course [F-S]. The first explicit definition of the spaces $H^{p,q}(\mathbb{R}^N)$ is, as far as we know, in [F-R-S].

Theorem B in the introduction has the following analog for Hardy spaces.

THEOREM 4.1. *Every translation invariant linear operator that maps the Hardy space $H^{p,q}(\mathbb{R}^N), 0 < p < +\infty, 0 < q \leq +\infty$, into $H^{p,q}(\mathbb{R}^N)$ continuously, also maps $H^p(\mathbb{R}^N)$ into $H^p(\mathbb{R}^N)$.*

It is easy to reduce the proof of this theorem to theorem B in the introduction (and also $H^{p,q}(\mathbb{R}^N) = L^{p,q}(\mathbb{R}^N)$ if $1 < p < +\infty$!). Therefore we consider this theorem proved, and we turn our attention to the study of the Bochner-Riesz means in the spaces $H^{p,q}(\mathbb{R}^N)$.

The Bochner-Riesz means of index $\delta \geq 0$ of a distribution in $H^{p,q}(\mathbb{R}^N)$ are defined by $(R^\delta f)\hat{}\,(\eta) = (1 - |\eta|^2)_+^\delta \hat{f}(\eta)$, where of course $\hat{}$ denotes the Fourier transform. In [S-T-W] E. Stein, M. Taibleson, and G. Weiss, proved that when $0 < p < 1$ and $\delta = N/p - (N+1)/2$ (the "critical index"), these means are not bounded on $H^p(\mathbb{R}^N)$; however they do map $H^p(\mathbb{R}^N)$ into $H^{p,\infty}(\mathbb{R}^N)$. (Actually they proved much more than this.) We want here to complete this result with the following.

THEOREM 4.2. *Let $0 < p \leq 1$ and $p < q \leq +\infty$. Then the Bochner-Riesz means of index $\delta = N/p - (N+1)/2$ do not map $H^{p,q}(\mathbb{R}^N)$ into $H^{p,\infty}(\mathbb{R}^N)$.*

Before we start with the proof notice that since the Bochner- Riesz means at the critical index are not bounded on $H^p(\mathbb{R}^N)$, the case $q = +\infty$ of this theorem is an immediate consequence of theorem 4.1. Not so for the case $p < q < +\infty$.

PROOF. For notational convenience we restrict our attention only to the case $N = 1$ and $1/2 < p \leq 1$. Let $a = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ (a is just the simplest atom), and let $f = \sum_{k=2}^{+\infty} k^{-1/p}(\text{Log } k)^{-1/q-\varepsilon} a(\cdot - x_k)$, where $0 < \varepsilon < 1/p - 1/q$ and $\{x_k\}$ is a very sparse sequence of points (e.g. $x_k = \exp(\exp(\exp k))$). To prove the theorem we only have to show that f is in $H^{p,q}(\mathbb{R})$, but $R^{1/p-1}f$ is not in $L^{p,\infty}(\mathbb{R})$.

To prove that f is in $H^{p,q}(\mathbb{R})$ we split f into $f_0 + f_1$, where $f_0 = \sum_{k=2}^{n} k^{-1/p}(\text{Log } k)^{-1/q-\varepsilon} a(\cdot - x_k)$ and $f_1 = f - f_0$. Let $1/2 < p_0 < p$. Then, since $\|\mathcal{M}a\|_{p_0} \leq c$, we easily obtain $\|\mathcal{M}f_0\|_{p_0} \leq cn^{1/p_0 - 1/p}(\text{Log } n)^{-1/q-\varepsilon}$, and also $\|\mathcal{M}f_1\|_\infty \leq cn^{-1/p}(\text{Log } n)^{-1/q-\varepsilon}$. Hence, by a standard interpolation argument, $\|f\|_{H^{p,q}} = \|\mathcal{M}f\|_{p,q} \leq c$.

To estimate $R^{1/p-1}f$ one has to use the fact, proved in [S-T-W], that $|R^{1/p-1}a(x)| \approx (1+|x|)^{-1/p}$. Hence if the sequence $\{x_k\}$ is very sparse, to avoid "big" interferences between the summands of the series $\sum_{k=2}^{+\infty} k^{-1/p}(\text{Log } k)^{-1/q-\varepsilon} R^{1/p-1}a(\cdot - x_k)$, we have

$$|R^{1/p-1}f(x)| \approx \sum_{k=2}^{+\infty} k^{-1/p}(\text{Log } k)^{-1/q-\varepsilon}(1+|x-x_k|)^{-1/p}.$$

Note that $\{(1+|\cdot - x_k|)^{-1/p}\}$ is a family of functions equidistributed, and with supports "essentially" disjoint, and that the sequence $\{k^{-1/p}(\text{Log } k)^{-1/q-\varepsilon}\}$ is not in ℓ^p. Using techniques similar to those developed in the proof of theorem 2.3 it is now possible to show that the above series is not in $L^{p,\infty}(\mathbb{R})$. We omit the details.

Added in Proof

After this paper was submitted the journal "Functional Analysis and its Applications" published a paper of A.M. Shteinberg with title "*Translation-invariant operators in Lorentz spaces*" (Funct. Anal. Appl. **20** (1986), 166-168). This short note contains the statements of the main results of this paper with no proofs.

Also P. Siögren has recently obtained some nice results on the translation invariant operators on weak L^1. (P. Sjögren, *Translation invariant operators on weak* L^1. Preprint.)

REFERENCES

[B-L] J. BERGH, J. LÖFSTRÖM. *Interpolation spaces*. Springer Verlag, Berlin, Heidelberg, New York, 1976.

[C-F] M. COWLING, J. FOURNIER, *Inclusion and noninclusion of spaces of convolution operators*. Trans. Am. Math. Soc. **221** (1976), 59-95.

[F-R-S] C. FEFFERMAN, N. RIVIERE, Y. SAGHER. *Interpolation of H^p spaces, the real method*. Trans. Amer. Math. Soc. **191** (1974), 75-82.

[F-S] C. FEFFERMAN, E. STEIN. *H^p spaces of several variables*. Acta Math. **129** (1972), 137-193.

[GC-RF] J. GARCIA-CUERVA, J.L. RUBIO DE FRANCIA. *Weighted norm inequalities and related topics*. North Holland, Amsterdam, New York, Oxford, 1985.

[H] R. HUNT. *On $L(p,q)$ spaces*. L'Ens. Math. **12** (1966), 249-275.

[K] N. KALTON. *Operators on $L^p, 0 < p < 1$*. Trans. Amer. Math. Soc. **259** (1980), 319-355.

[O] D. OBERLIN. *Translation invariant operators on* $L^p(G), 0 < p < 1$. (I) Michigan Math. J. 23 (1976), 119-122. (II) Can. J. Math. **29** (1977), 226-230.

[P] *Open problems presented at the first seminar Poland-GDR on operator ideals and geometry of Banach spaces*. Forshungsgebnisse, 1982, N/82/27.

[Sa] S. SAWYER. *Maximal inequalities of weak type*. Ann. Math. **84** (1966), 157-174.

[Sh] A.M. SHTEINBERG. *On shift-invariant operators*. Russian Math. Surveys **39** (1984), 215-216.

[S-T-W] E. STEIN, M. TAIBLESON, G. WEISS. *Weak type estimates for maximal operators on certain* H^p *classes*. Suppl. Rendiconti Circ. Mat. Palermo **1** (1981), 81-97.

[S-W,1] E. STEIN, G. WEISS. *An extension of a theorem of Marcinkiewicz and some of its applications*. J. Math. Mec. **8** (1959), 263-284.

[S-W,2] E. STEIN, G. WEISS. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, 1971.

[Z] M. ZAFRAN. *Multiplier transformations of weak type*. Ann. Math. **101** (1975), 34-44.

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