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On the order of $\zeta(\frac{1}{2} + it)$

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1. – Introduction.

Important for number theory, the problem of bounding the Riemann zeta-function $\zeta(s)$ in the critical strip $0 < \operatorname{Re} s < 1$ stimulated a lot of work on exponential sums of the type

$$\sum_{M}^{M_1} e(f(m)) \tag{1.1}$$

where $f(x)$ is a real smooth function on $[M, M_1]$. Since $\zeta(s)$ can be well approximated by finite sums

$$\sum_{M}^{M_1} m^{-s}, \qquad s = \sigma + it, \tag{1.2}$$

the sums (1.1) with $f(x) = (2\pi)^{-1} t \log x$ are of special interest. Three basic techniques for bounding sums (1.1) are known (cf. [9]):

1) Weyl-Hardy-Littlewood method;

2) Van der Corput method;

3) Vinogradov method.

In this paper we develop a new method which uses a bit of each of these three techniques. Our main result is

THEOREM. *For any* $t \geq 1$ *and* $\varepsilon > 0$, *we have*

$$\zeta(\tfrac{1}{2} + it) = O(t^{\Theta+\varepsilon}) \tag{1.3}$$

with $\Theta = 9/56$, *the constant implied in* O *depending on* ε *alone.*

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The sharpest result hitherto proved was due to G. Kolesnik [6];

$$\Theta = 139/858 = 0.162004\ldots.$$

This was obtained by an extension of the Van der Corput method (exponent pairs theory in several variables). Using computers, on the basis of certain conjectures, S. W. Graham and G. Kolesnik [4] predicted that the best constant one can ever obtain by that method is $\Theta = 0.1618\ldots$ while we have $\Theta = 9/56 = 0.16071\ldots$.

The method works for general sums (1.1), so we carry out the arguments in a relatively general setting until the end of Section 5 where we specify $f(x) = t\log x$ in order to use Theorem 4.1 whose proof is elementary. This restriction would not be necessary if we had extended Theorem 4.1 to the relevant form. But a proof of such a result would require the highly advanced technique of the spectral theory of automorphic functions [2]. It was our wish to avoid this at least in the most spectacular case of the sums (1.2).

Our method seems to work for the divisor problem as well. Since this would require substantial modifications we do not claim any results.

The authors express their thanks to J.D. Vaaler for his helpful suggestions concerning the proof of Lemma 2.3.

Notation and conventions.

$e(z) = \exp[2\pi i z]$,

f', f'', f''' denotes the derivatives of order 1, 2 and 3,

$f^{(j)}$ denotes the derivative of order j,

$f = O(g)$ means $|f| \leqslant cg$ with some unspecified constant c, not necessarily the same in each formula,

$f \ll g$ means $f = O(g)$,

$f \sim g$ means $c_1 \leqslant f/g \leqslant c_2$ with some positive unspecified constants c_1, c_2,

$[x] = \max\{k \in \mathbb{Z};\ k \leqslant x\}$,

$\|x\| = \inf\{|x - k|;\ k \in \mathbb{Z}\}$,

$\bar{a}/c$ means d/c where d is a solution of the congruence $ad \equiv 1 (\mathrm{mod}\ c)$,

■ indicates the end of a proof or it means the result is easy.

2. – Basic lemmas.

In this section we present in a general setting some principles applied throughout the paper.

LEMMA 2.1. *Let $N < M$ and $\omega(x)$ be a real function such that $|\omega'(x)| \leqslant \Omega$ throughout the interval $[N, M]$. For any complex numbers a_n we have*

$$\Big|\sum_{N<n\leqslant M} a_n e\big(\omega(n)\big)\Big| \leqslant \Big|\sum_{N<n\leqslant M} a_n\Big| + 2\pi\Omega\int_N^M \Big|\sum_{N<n\leqslant x} a_n\Big|\,dx\,.$$

PROOF. Follows by partial integration. ■

LEMMA 2.2. *Let $M \leqslant N < N_1 \leqslant M_1$ and a_n be any complex numbers. We then have*

$$\Big|\sum_{N<n\leqslant N_1} a_n\Big| \leqslant \int_{-\infty}^{\infty} K(\theta)\Big|\sum_{M<m\leqslant M_1} a_m e(\theta m)\Big|\,d\theta$$

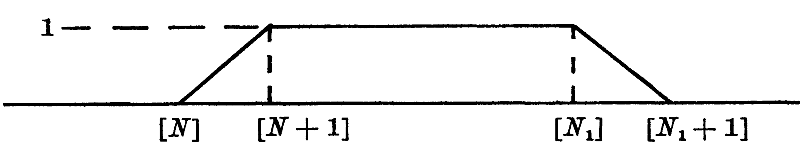
with $K(\theta) = \min\{M_1 - M + 1, (\pi|\theta|)^{-1}, (\pi\theta)^{-2}\}$, so the L_1-norm of $K(\theta)$ is

$$\int_{-\infty}^{\infty} K(\theta)\,d\theta \leqslant 3\log(2 + M_1 - M)\,.$$

PROOF. The sum on the left-hand side is equal to

$$\sum_{M<m\leqslant M_1} a_m \chi(m)$$

where $\chi(m)$ is the function whose graph is



The Fourier transform of $\chi(m)$ satisfies $|\hat{\chi}(\theta)| \leqslant \min\{N_1 - N + 1, (\pi|\theta|)^{-1}, (\pi\theta)^{-2}\}$. This completes the proof. ■

LEMMA 2.3. *Let $\mathscr{P}$ be a set of points $\mathfrak{p} \in \mathbf{R}^K$ and let $b(\mathfrak{p})$, for $\mathfrak{p} \in \mathscr{P}$, be arbitrary complex numbers. Let $\delta_1, \ldots, \delta_K$, $T_1, \ldots, T_K$ be positive numbers.*

Then we have

$$\int_{-T_1}^{T_1}\dots\int_{-T_K}^{T_K}\Big|\sum_{\mathfrak{p}\in\mathscr{P}} b(\mathfrak{p})\, e(\mathfrak{p}\cdot\mathfrak{t})\Big|^2 dt_1\dots dt_K \leqslant \prod_{k=1}^{K}(2T_k+\delta_k^{-1})\sum_{\substack{\mathfrak{p}\in\mathscr{P}\\ |p_j-p'_j|\leqslant\delta_j}}\sum_{\mathfrak{p}'\in\mathscr{P}}|b(\mathfrak{p})\,b(\mathfrak{p}')|\,.$$

PROOF. Let δ, T be positive numbers. There exists a function $f(t)$, $t\in\mathbb{R}$, such that

$$f(t)\geqslant\begin{cases}1 & \text{if } |t|\leqslant T\,,\\ 0 & \text{if } |t|>T\,,\end{cases}$$

and such that its Fourier transform $\hat f(u)$ has compact support in $|u|\leqslant\delta$ and

$$\hat f(0)=2T+\delta^{-1}\,.$$

Up to a change of variables, this is the well-known Beurling-Selberg function (for a full account of this and related functions we refer to the expository paper by J.D. Vaaler [10]). Since

$$\int_{-T_1}^{T_1}\dots\int_{-T_K}^{T_K}\Big|\sum_{\mathfrak{p}\in\mathscr{P}} b(\mathfrak{p})\, e(\mathfrak{p}\cdot\mathfrak{t})\Big|^2 dt_1\dots dt_K\leqslant\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} f_{\delta_1}(t_1)\dots f_{\delta_K}(t_K)\Big|\sum_{\mathfrak{p}\in\mathscr{P}} b(\mathfrak{p})\, e(\mathfrak{p}\cdot\mathfrak{t})\Big|^2 dt_1\dots dt_K$$

the result follows. ■

LEMMA 2.4. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two sets of points $\mathfrak{x}\in\mathbb{R}^K$ and $\mathfrak{y}\in\mathbb{R}^K$ respectively and let $a(\mathfrak{x})$ for $\mathfrak{x}\in\mathscr{X}$ and $b(\mathfrak{y})$ for $\mathfrak{y}\in\mathscr{Y}$ be arbitrary complex numbers. Let $X_1,\dots,X_K$, $Y_1,\dots,Y_K$ be positive numbers. Define the bilinear forms*

$$\mathscr{B}(\mathfrak{b};\mathfrak{X})=\sum_{\substack{\mathfrak{y}\in\mathscr{Y}\\ |y_k-y'_k|\leqslant(2X_k)^{-1},\ k=1,\dots,K}}\sum_{\mathfrak{y}'\in\mathscr{Y}}|b(\mathfrak{y})\,b(\mathfrak{y}')|\,,$$

$$\mathscr{B}(\mathfrak{a};\mathfrak{Y})=\sum_{\substack{\mathfrak{x}\in\mathscr{X}\\ |x_k-x'_k|\leqslant(2Y_k)^{-1}}}\sum_{\mathfrak{x}'\in\mathscr{X}}|a(\mathfrak{x})\,a(\mathfrak{x}')|$$

and

$$\mathscr{B}(\mathfrak{a},\mathfrak{b};\mathfrak{X},\mathfrak{Y})=\sum_{\substack{\mathfrak{x}\in\mathscr{X}\\ |x_k|\leqslant X_k}}\sum_{\substack{\mathfrak{y}\in\mathscr{Y}\\ |y_k|\leqslant Y_k}}a(\mathfrak{x})\,b(\mathfrak{y})\,e(\mathfrak{x}\cdot\mathfrak{y})\,.$$

We have

$$|\mathscr{B}(\mathfrak{a},\mathfrak{b};\mathfrak{X},\mathfrak{Y})|^2\leqslant(2\pi^2)^K\prod_{k=1}^{K}(1+X_kY_k)\,\mathscr{B}(\mathfrak{b};\mathfrak{X})\,\mathscr{B}(\mathfrak{a};\mathfrak{Y})\,.$$

PROOF. We begin with the integral formula

$$(2.3)\qquad e(xy)=\frac{\pi y}{\sin(2\pi\varepsilon y)}\int_{x-\varepsilon}^{x+\varepsilon}e(ty)\,dt\,.$$

Put $\varepsilon_k=(4Y_k)^{-1}$ and for $\mathfrak{y}=(y_1,\dots,y_k)\in\mathfrak{Y}$ put

$$w(\mathfrak{y})=\prod_{k=1}^{K}\frac{\pi y_k}{\sin(2\pi\varepsilon_k y_k)}\,,$$

so, for $\mathfrak{y}$ with $|y_k|\leqslant Y_k$, we have

$$(2.4)\qquad |w(\mathfrak{y})|\leqslant\pi^K Y_1\dots Y_K\,.$$

By (2.3), letting $b^*(\mathfrak{y})=b(\mathfrak{y})\,w(\mathfrak{y})$ and $T_k=\varepsilon_k+X_k$ we proceed as follows

$$|\mathscr{B}(\mathfrak{a},\mathfrak{b};\mathfrak{X},\mathfrak{Y})|=\Big|\sum_{\mathfrak{x}\in\mathscr{X}}a(\mathfrak{x})\int_{x_1-\varepsilon_1}^{x_1+\varepsilon_1}\dots\int_{x_K-\varepsilon_K}^{x_K+\varepsilon_K}\Big(\sum_{\mathfrak{y}\in\mathscr{Y}}b^*(\mathfrak{y})\,e(\mathfrak{t}\cdot\mathfrak{y})\Big)\,dt_1\dots dt_K\Big|$$
$$\leqslant\int_{-T_1}^{T_1}\dots\int_{-T_K}^{T_K}\sum_{\substack{\mathfrak{x}\in\mathscr{X}\\|x_k-t_k|\leqslant\varepsilon_k}}|a(\mathfrak{x})|\Big|\sum_{\mathfrak{y}\in\mathscr{Y}}b^*(\mathfrak{y})\,e(\mathfrak{t}\cdot\mathfrak{y})\Big|dt_1\dots dt_K\leqslant\Big(\int\dots\int|\sum_{\mathfrak{x}}|^2\Big)^{1/2}\Big(\int\dots\int|\sum_{\mathfrak{y}}|^2\Big)^{1/2}$$

by Cauchy's inequality. Here we have

$$\int\dots\int\Big|\sum_{\mathfrak{x}}\Big|^2\leqslant 2^K\varepsilon_1\dots\varepsilon_K\sum_{\substack{\mathfrak{x}\in\mathscr{X}\\|x_k-x'_k|\leqslant 2\varepsilon_k}}\sum_{\mathfrak{x}'\in\mathscr{X}}|a(\mathfrak{x})\,a(\mathfrak{x}')|$$

and

$$\int\dots\int\Big|\sum_{\mathfrak{y}}\Big|^2\leqslant\prod_{k=1}^{K}(2T_k+\delta_k^{-1})\sum_{\substack{\mathfrak{y}\in\mathscr{Y}\\|v_j-v'_j|\leqslant\delta_j}}\sum_{\mathfrak{y}'\in\mathscr{Y}}|b^*(\mathfrak{y})\,b^*(\mathfrak{y}')|$$

for any $\delta_k>0$, by Lemma 2.3. Finally taking $\delta_k=(2X_k)^{-1}$, by (2.4) one completes the proof. ∎

The last result admits an obvious modification. Suppose that for some k's all x_k's take integral values. Then the norms $|y_k|$ and $|y_k-y'_k|$ can be replaced by $\|y_k\|$ and $\|y_k - y_k'\|$ respectively and we put $Y_k = 1$ for such k's. Such a modified form of Lemma 2.4 can in fact be deduced from the lemma itself by an appeal to the inequality

$$\|a-b\| \leqslant |a-b| .$$

In the next five lemmas we prepare ourselves to appy Poisson's summation to sums which are rather short. The arguments are delicate, though standard. Being unable to quote sources precise enough, we provide proofs in full detail. A reader experienced with the stationary phase method may find his own proofs easier.

LEMMA 2.5. *For $y > 0$ we have*

$$\int_0^\infty e(x^3 - 3yx)\, dx = (6\sqrt{y})^{-1/2} \exp[\pi i/4]\, e(-2y^{3/2}) + O(y^{-1}) .$$

PROOF. The integral is known as the Airy-Hardy integral, see for example [5]. One can show that its asymptotic expansion is convergent for $y \geqslant 1$ giving the result. For $0 < y < 1$ the assertion is trivial.

Let us give a direct proof for $y \geqslant 1$. We have

$$x^3 - 3yx = -2y^{3/2} + 3\sqrt{y}(x-\sqrt{y})^2 + (x-\sqrt{y})^3 .$$

Hence

$$\int_0^\infty e(x^3 - 3yx)\, dx = e(-2y^{3/2})\, y^{1/2} \int_{-1}^\infty e((y^{3/2}(3t^2 + t^3))\, dt$$
$$= e(-2y^{3/2})\, y^{1/2}(I_1 + I_2) + O(y^{-1})$$

where

$$I_1 = \int_0^1 e(y^{3/2}(3t^2 + t^3))\, dt$$

and

$$I_2 = \int_0^1 e(y^{3/2}(3t^2 - t^3))\, dt .$$

By a change of variable $3t^2+t^3=u$, we get

$$I_1=\int_0^4 e(y^{3/2}u)(6t+3t^2)^{-1}\,du\,.$$

We have

$$(6t+3t^2)^{-1}=\frac{1}{2\sqrt{3u}}+f(u)\,,$$

with $f(u)=c_0+c_1\sqrt{u}+c_2u+\ldots$, $f(z^2)$ analytic in $|z|\leqslant 4$, so

$$I_1=\frac{1}{2\sqrt{3}}\int_0^4 e(y^{3/2}u)\,\frac{du}{\sqrt{u}}+O(y^{-3/2})$$

$$=\frac{1}{2\sqrt{3}}\int_0^\infty e(y^{3/2}u)\,\frac{du}{\sqrt{u}}+O(y^{-3/2})\,.$$

Similarly, we deduce that

$$I_2=\frac{1}{2\sqrt{3}}\int_0^\infty e(y^{3/2}u)\,\frac{du}{\sqrt{u}}+O(y^{-3/2})\,.$$

Hence

$$\int_0^\infty e(x^3-3xy)\,dx=3^{-1/2}y^{-1/4}e(-2y^{3/2})\int_0^\infty e(u)\,\frac{du}{\sqrt{u}}+O(y^{-1})\,.$$

But

$$\int_0^\infty e(u)\,u^{-1/2}\,du=2^{-1/2}\exp\,[\pi i/4]$$

completing the proof. ■

COROLLARY. *For $\mu>0$, $c>0$ and $h>0$ we have*

$$\int_0^\infty e\left(\mu x^3-\frac{h}{c}x\right)dx=\left(\frac{c}{12\mu h}\right)^{1/4}\exp\,[\pi i/4]\,e\left(-2\mu^{-1/2}\left(\frac{h}{3c}\right)^{3/2}\right)+O\left(\frac{c}{h}\right),$$

the constant implied in O being absolute. ■

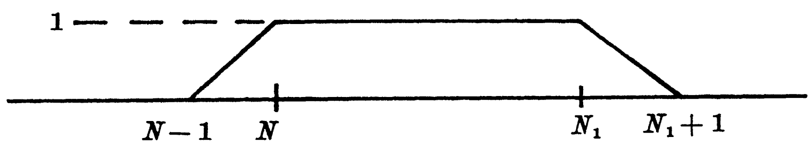
LEMMA 2.6. *Let $\mu > 0$, $c \geqslant 1$, $1 \leqslant N < N_1 \leqslant 2N$ and let $\delta(h)$ be the characteristic function of the interval $[3\mu c N^2, 3\mu c N_1^2]$. For any real $h \neq 0$ we have*

$$\int_N^{N_1} e\left(\mu x^3 - \frac{h}{c}x\right)dx = \delta(h)\left(\frac{c}{12\mu h}\right)^{1/4}\exp[\pi i/4]\, e\left(-2\mu^{-1/2}\left(\frac{h}{3c}\right)^{3/2}\right)$$
$$+ O\left(\min\left\{(\mu N)^{-1/2}, \left|3\mu N^2 - \frac{h}{c}\right|^{-1}\right\}\right) + O\left(\min\left\{(\mu N)^{-1/2}, \left|3\mu N_1^2 - \frac{h}{c}\right|^{-1}\right\}\right)$$
$$+ O(c|h|^{-1})$$

the constant implied in O being absolute.

PROOF. This follows from Lemmas 4.2 and 4.4 of [9] and from the Corollary to Lemma 2.5. ■

LEMMA 2.7. *Let $g(x)$ be the function whose graph is*



Suppose that either $h < 2\mu c N^2$ or $h > 4\mu c N_1^2$. We then have

$$\int g(x)\, e\left(\mu x^3 - \frac{h}{c}x\right)dx \ll c(\mu c N^2 + |h|)^{-1},$$

where the constant implied in $\ll$ is absolute.

PROOF. By partial integration our integral is equal to

$$-\frac{1}{2\pi i}\int\left(\frac{cg(x)}{3\mu c x^2 - h}\right)' e\left(\mu x^3 - \frac{h}{c}x\right)dx \ll \int\left|\left(\frac{cg(x)}{3\mu c x^2 - h}\right)'\right| dx \ll \frac{c}{\mu c N^2 + |h|}$$

because the derivative is monotonic in three subintervals. ■

LEMMA 2.8. *For $|h| > 4\mu c N_1^2$ we have*

$$\int g(x)\, e\left(\mu x^3 - \frac{h}{c}x\right)dx \ll (1 + \mu N^2)\, c^2 h^{-2},$$

where the constant implied in $\ll$ is absolute.

PROOF. Our integral splits into 3 parts (use the formula from the proof of Lemma 2.7) $I_1 - I_2 + I_3$, say, where

$$I_1 = \frac{6\mu c^2}{2\pi i}\int\left(\frac{xg(x)}{(3\mu cx^2-h)^2}\right)e\left(\mu x^3-\frac{h}{c}x\right)dx \ll \mu c^2 N^2 h^{-2},$$

$$I_2 = \frac{1}{2\pi i}\int\limits_{N-1}^{N}\left(3\mu x^2-\frac{h}{c}\right)^{-1}e\left(\mu x^3-\frac{h}{c}x\right)dx \ll c^2 h^{-2},$$

by Lemma 4.3 of [4], and similarly

$$I_3 = \frac{1}{2\pi i}\int\limits_{N_1}^{N_1+1}\left(3\mu x^2-\frac{h}{c}\right)^{-1}e\left(\mu x^3-\frac{h}{c}x\right)dx \ll c^2 h^{-2}.$$

Gathering these together three estimates one completes the proof. ■

LEMMA 2.9 (Poisson's summation). *Let $f(x)$ be a continuous function compactly supported in $(-\infty, \infty)$ and let c, d be integers, $c \geqslant 1$. We then have*

$$\sum_{n\equiv d\,(\mathrm{mod}\,c)} f(n) = \frac{1}{c}\sum_h e\left(-\frac{dh}{c}\right)\hat{f}\left(\frac{h}{c}\right)$$

where $\hat{f}(y)$ is the Fourier transform of $f(x)$. ■

Denote the Gauss sums

$$G(a, l; c) = \sum_{d\,(\mathrm{mod}\,c)} e\left(\frac{ad^2+ld}{c}\right).$$

LEMMA 2.10. *If $c \geqslant 1$ and $(a, c) = 1$, then*

$$G(a, l; c) = \begin{cases} e\left(-\dfrac{\bar{a}}{c}\dfrac{l^2}{4}\right)G(a, 0; c) & \text{if } l \equiv 0 \pmod 2 \\[2ex] e\left(-\dfrac{\bar{a}}{c}\dfrac{l^2-1}{4}\right)G(\bar{a}, \bar{a}, ; c) & \text{if } l \equiv 1 \pmod 2 \end{cases}$$

and

$$|G(a, l; c)| \leqslant (2c)^{1/2}.\quad ■$$

3. – Incomplete Gauss sums.

Now we are ready to estimate the exponential sum

$$S(\mu;\, a,\, b;\, c) = \sum_{N<n\leqslant N_1} e\left(\mu n^3 + \frac{a}{c}n^2 + \frac{b}{c}n\right)$$

which can be regarded as an incomplete (perturbed) Gauss sum (mod c).

LEMMA 3.1. *Let* $c\geqslant 1$, $(a, c) = 1$, $c\leqslant N < N_1 \leqslant 2N$ *and* $0<\mu\leqslant N^{-2}$. *We then have*

$$|S(\mu;\, a,\, b;\, c)| \leqslant \left|\sum_{3\mu cN^2<h<3\mu cN_1^2} 2(\pm 1)^h(\mu ch)^{-1/4} e\left[2\mu^{-1/2}\left(\frac{h}{3c}\right)^{3/2} + \frac{\bar{a}}{c}\left(\frac{b+h}{2}\right)^2\right]\right| + O(N^{1/2}\log N + \mu^{-1}N^{-2})\,,$$

the constant implied in O being absolute.

PROOF. We have

$$\begin{aligned} S(\mu;\, a,\, b;\, c) &= \sum_n g(n)\, e\left(\mu n^3 + \frac{a}{c}n^2 + \frac{b}{c}n\right) + O(1) \\ &= \sum_{d\,(\mathrm{mod}\, c)} e\left(\frac{ad^2+bd}{c}\right) \sum_{n\equiv d\,(\mathrm{mod} c)} g(n)\, e(\mu n^3) + O(1) \\ &= \frac{1}{c}\sum_h G(a,\, b+h;\, c)\int g(x)\, e\left(\mu x^3 - \frac{h}{c}x\right) dx + O(1) \end{aligned}$$

by Poisson's formula. The terms with $|h|\geqslant 16c$ contribute

$$\ll c^{1/2} \ll N^{1/2}$$

by Lemma 2.8. The terms with $|h|<16c$ and $h\notin[2\mu cN^2,\, 4\mu cN_1^2]$ contribute

$$\ll c^{1/2}\log N + \mu^{-1}c^{-1/2}N^{-2} \ll N^{1/2}\log N + \mu^{-1}N^{-2}$$

by Lemma 2.7. For the remaining terms with $2\mu cN^2\leqslant h\leqslant 4\mu cN_1^2$ we write

$$\int g(x)\, e\left(\mu x^3 - \frac{h}{c}x\right) dx = \int_N^{N_1} e\left(\mu x^3 - \frac{h}{c}x\right) dx + O(1)$$

and we appeal to Lemma 2.6. The leading term i.e. the term attached to

$\delta(h)$ gives rise to the main term while the total error is

$$\ll c^{-1/2}\sum_{2\mu cN^2<h<4\mu cN_1^2}\left(1+ch^{-1}+\min\left\{(\mu N)^{-1/2},\left|3\mu N^2-\frac{h}{c}\right|^{-1}+\left|(3\mu N_1^2-\frac{h}{c}\right|^{-1}\right\}\right)$$
$$\ll \mu c^{1/2}N^2+c^{1/2}+(\mu cN)^{-1/2}+c^{1/2}\log N\ll N^{1/2}\log N$$

unless $16\mu cN^2<1$ in which case the summation over h is void and the final bound remains valid. ■

REMARKS. A similar idea of using Gauss' sums $G(a, l; c)$ to evaluate incomplete Gauss' sums $S(0; a, 0; c)$ is applied in [3] in a different context.

4. – The distribution of certain fractions.

The problem considered in this section seems to be of independent interest; thus the results obtained are given in the status of theorems.

Let $A\geqslant 1$, $C\geqslant 1$, $\Delta_1>0$, $\Delta_2>0$ and let $\mathcal{N}(\Delta_1, \Delta_2; A, C)$ be the number of pairs $\{a/c, a_1/c_1\}$ such that

$$a, a_1\sim A\,,\qquad c, c_1\sim C\,,\qquad (a, c)=(a_1, c_1)=1\,, \tag{4.1}$$

$$\left\|\frac{\bar a}{c}-\frac{\bar a_1}{c_1}\right\|<\Delta_1 \tag{4.2}$$

and

$$|ac-a_1c_1|<\Delta_2AC\,. \tag{4.3}$$

THEOREM 4.1. *We have*

$$\mathcal{N}(\Delta_1, \Delta_2; A, C)\ll(AC+\Delta_1\Delta_2A^2C^2+\Delta_1^2A^2C^2+\Delta_2A^2+\Delta_2C^2)(AC)^\varepsilon$$

the constant implied in $\ll$ *depending on* ε *and on the constants implied in* (4.1).

Our proof (by induction with respect to certain parameters) forces us to consider a more general problem. For a given pair of relatively prime positive integers r, s put

$$\mathcal{P}(r, s)=\{a/c;\ a\sim A,\ c\sim C,\ (a, c)=1,\ a\equiv O(\operatorname{mod} r),\ c\equiv O(\operatorname{mod} s)\}\,.$$

Let $\mathcal{M}(\Delta_1, \Delta_2; r, s)$ be the number of pairs $\{a/c, a_1/c_1\}$ from $\mathcal{P}(r, s)$ satisfying (4.2) and (4.3). We are going to prove, by induction on rs, the following

THEOREM 4.2. *For any* $r \geqslant 1$, $s \geqslant 1$, $(r, s) = 1$, $A \geqslant 1$, $C \geqslant 1$, $\Delta_1 > 0$ *and* $\Delta_2 > 0$ *we have*

$$\mathcal{M}(\Delta_1, \Delta_2; r, s) \ll \lambda(\varepsilon)(rs)^{-1-\varepsilon}(AC)^{2\varepsilon}(AC + \Delta_1\Delta_2A^2C^2 + \Delta_1^2A^2C^2 + \Delta_2A^2 + \Delta_2C^2)\,,$$

the constant $\lambda(\varepsilon)$ *depending on* ε *and on the constants implied in* (4.1).

On taking $r = s = 1$ one infers Theorem 4.1.

In order to prove Theorem 4.2 we need 3 lemmas. Denote $d = (a_1, c)$ and $d_1 = (a, c_1)$, so $(d, d_1) = 1$ and $(dd_1, rs) = 1$. Put

$$\mathscr{a}_1 = a_1/d\,, \qquad \mathscr{a} = a/d_1\,,$$
$$\mathscr{c}_1 = c_1/d_1\,, \qquad \mathscr{c} = c/d\,,$$

thus

$$(\mathscr{a}\mathscr{a}_1, \mathscr{c}\mathscr{c}_1) = 1\,, \tag{4.4}$$

$$\mathscr{a}_1 \equiv \mathscr{a} \equiv 0 \pmod r \quad \text{and} \quad \mathscr{c}_1 \equiv \mathscr{c} \equiv 0 \pmod s\,.$$

Let $\mathcal{M}_{dd_1}(\Delta_1, \Delta_2; r, s)$ be the number of those pairs $\{a/c, a_1/c_1\}$ pertaining to d, d_1. We have

$$\mathcal{M}(\Delta_1, \Delta_2; r, s) = \sum_{\substack{(d,d_1)=1\\(dd_1,rs)=1}}\sum \mathcal{M}_{dd_1}(\Delta_1, \Delta_2; r, s)\,. \tag{4.5}$$

By (4.4) it follows that $\begin{bmatrix}\mathscr{a}_1\\ \mathscr{c}_1\end{bmatrix}$ and $\begin{bmatrix}\mathscr{a}\\ \mathscr{c}\end{bmatrix}$ are $SL(2, \mathbb{Z})$ equivalent, i.e.

$$\begin{bmatrix}\mathscr{a}_1\\ \mathscr{c}_1\end{bmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{bmatrix}\mathscr{a}\\ \mathscr{c}\end{bmatrix} \tag{4.6}$$

for some $\tau = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL(2, \mathbb{Z})$. In fact τ is in the congruence subgroup $\Gamma(r, s) = \{\tau;\, \beta \equiv O(\mathrm{mod}\, r),\, \gamma \equiv O(\mathrm{mod}\, s)\}$. All such τ's are given by

$$\begin{pmatrix}\alpha + \mathscr{a}_1\mathscr{c}k & \beta - \mathscr{a}_1\mathscr{a}k\\ \gamma + \mathscr{c}_1\mathscr{c}k & \delta - \mathscr{a}\mathscr{c}_1k\end{pmatrix}$$

with $k \in \mathbb{Z}$, therefore one can find τ (unique) with

$$-\tfrac{1}{2}\mathscr{c}\mathscr{c}_1 < \gamma \leqslant \tfrac{1}{2}\mathscr{c}\mathscr{c}_1\,. \tag{4.7}$$

First we deal with the pairs which differ by a « trivial » transformation $\tau=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, i.e. a transformation with $\alpha\beta\gamma\delta=0$. Let $\mathscr{M}^0_{dd_1}(\Delta_1,\Delta_2;r,s)$ denote the number of these « trivial » pairs.

LEMMA 4.1. *We have*

$$\mathscr{M}^0_{dd_1}(\Delta_1,\Delta_2;r,s)\ll(rs)^{-1}(AC+\Delta_2A^2+\Delta_2C^2)\,, \tag{4.8}$$

the constant implied in $\ll$ *depending on those implied in* (4.1) *only.*

PROOF. Consider the four cases:

Case 1: $\alpha=0$. This implies $\beta\gamma=-1$, $\beta=1$, $\gamma=-1$, $a_1=c=1$, $r=s=1$ giving at most $O(AC/rs)$ pairs. ■

Case 2: $\delta=0$. This is similar to case 1 giving at most $O(AC/rs)$ pairs. ■

Case 3: $\beta=0$. This simplies $\alpha\delta=1$, $\alpha=1$, $\delta=1$ and

$$a_1=a\,,\qquad c_1\equiv c(\operatorname{mod} a)\,.$$

The pairs on the diagonal $a_1=a$, $c_1=c$ contribute

$$\#\{a_1=a,\,c_1=c\}\ll AC/rs\,.$$

For the remaining pairs, we have by (4.3)

$$0<|c-c_1|<\Delta_2AC(add_1)^{-1}\,.$$

Hence the number of such points is estimated by

$$\#\{a,c,c_1;\ a\equiv O(\operatorname{mod} r),\ c\equiv c_1\equiv O(\operatorname{mod} s),\ c\equiv c_1(\operatorname{mod} a)\,,$$

$$0<|c-c_1|<\Delta_2ACa^{-1}\}\ll\sum_{a\equiv O(\operatorname{mod} r)}\Delta_2AC^2a^{-2}s^{-2}\ll\Delta_2C^2r^{-1}s^{-1}\,.$$

The total number of pairs from case 3 is $\ll(AC+\Delta_2C^2)(rs)^{-1}$. ■

Case 4: $\gamma=0$. This is similar to Case 3 giving a similar bound. ■

Now, we cound pairs which differ by a transformation $\tau=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ with $\alpha\beta\gamma\delta\neq 0$. Let $\mathscr{M}^*_{dd_1}(\Delta_1,\Delta_2;r,s)$ denote the number of these pairs.

LEMMA 4.2. *We have*

$$\mathcal{M}^*_{dd_1}(\Delta_1, \Delta_2; r, s) \ll \Delta_1(\Delta_1 + \Delta_2)(rs)^{-1}(AC)^{2+\varepsilon} \tag{4.9}$$

the constant implied in $\ll$ *depending on* ε *and on those constants implied in* (4.1).

PROOF. A simple computation shows that

$$a_1 c_1 - ac = \alpha\gamma a^2 + 2\beta\gamma ac + \beta\delta c^2 = \gamma a a_1 + \beta c c_1 . \tag{4.10}$$

In particular one deduces from (4.10) that

$$a_1 c_1 - ac \equiv \gamma a a_1 (\mathrm{mod}\; cc_1) ,$$

$$(a\bar{a}) a_1 c_1 - (a_1 \bar{a}_1) ac \equiv \gamma a a_1 (\mathrm{mod}\; cc_1) ,$$

$$\bar{a} c_1 - \bar{a}_1 c \equiv \gamma (\mathrm{mod}\; cc_1) ,$$

and

$$\frac{\gamma}{cc_1} \equiv \frac{\bar{a}}{c} - \frac{\bar{a}_1}{c_1} \equiv \left(\frac{\bar{a}}{c} - \frac{\bar{a}_1}{c_1}\right) dd_1 (\mathrm{mod}\, 1) .$$

Next, by (4.2) and (4.7) one gets

$$\frac{|\gamma|}{cc_1} = \left\|\frac{\gamma}{cc_1}\right\| = \left\|\left(\frac{\bar{a}}{c} - \frac{\bar{a}_1}{c_1}\right) dd_1\right\| \leqslant \left\|\frac{\bar{a}}{c} - \frac{\bar{a}_1}{c_1}\right\| dd_1 \leqslant \Delta_1\, dd_1 ,$$

thus

$$|\gamma| \leqslant \Delta_1 cc_1 \ll \Delta_1 C^2 . \tag{4.11}$$

This together with (4.10) yields

$$|\beta| \leqslant \Delta_1 a a_1 + O(AC^{-1}) \ll \Delta_1 A^2 . \tag{4.12}$$

Here was discarded the term $O(AC^{-1})$ because if it dominated then $A \gg |\beta| C \gg C$, so $\Delta_1 A^2 \gg |\gamma| C^{-2} A^2 \gg C^{-2} A^2 \gg AC^{-1}$ by (4.11). Now, by (4.10) and (4.3) we infer

$$\left(\frac{a}{c}\right)^2 + 2\frac{\beta}{\alpha}\frac{a}{c} + \frac{\beta\delta}{\alpha\gamma} \ll \frac{\Delta_2}{|\alpha\gamma|}\frac{dA}{d_1 C} ,$$

whence

$$\left(\frac{a}{c} - z_1\right)\left(\frac{a}{c} - z_2\right) \ll \frac{\Delta_2}{|\alpha\gamma|}\frac{dA}{d_1 C}$$

where z_1, z_2 are the roots of the relevant quadratic equation (real or complex):

$$z_j = -\frac{1}{\alpha}\left(\beta \pm \sqrt{\frac{-\beta}{\gamma}}\right).$$

Here one factor is $\geqslant \frac{1}{2}|z_1 - z_2| = |\alpha|^{-1}|\beta/\gamma|^{1/2}$, thus the other one satisfies

$$\left|\frac{a}{c} - z_j\right| \ll \frac{\Delta_2}{\sqrt{|\beta\gamma|}}\frac{dA}{d_1 C}. \tag{4.13}$$

The points a/c are (rsd^2C^{-2})-spaced, so (4.13) implies

$$\#\{a/c\} \ll 1 + \frac{\Delta_2}{\sqrt{|\beta\gamma|}}\frac{AC}{dd_1 rs}. \tag{4.14}$$

Finally, by (4.11), (4.12) and (4.14) we conclude that

$$\mathcal{M}^*_{dd_1}(\Delta_1, \Delta_2; r, s) \ll \sum_{\tau \in \Gamma(r,s)} \#\{a/c\} \ll \sum_{\substack{1 \leqslant \beta \ll \Delta_1 C^2 \\ \beta \equiv O(\bmod r)}} \sum_{\substack{1 \leqslant \gamma \ll \Delta_1 A^2 \\ \gamma \equiv O(\bmod s)}} \left(1 + (\beta\gamma)^{-1/2}\Delta_2 AC\right)(AC)^\varepsilon$$

which leads to (4.9). ■

Lemmas 4.1 and 4.2 are useful only for small dd_1. For large dd_1 we intend to apply an induction hypothesis. With this in mind we establish

LEMMA 4.3. *We have*

$$\mathcal{M}^2_{dd_1}(\Delta_1, \Delta_2; r, s) \leqslant 2^4 \mathcal{M}(2\Delta_1, 2\Delta_2; rd_1, sd)\mathcal{M}(2\Delta_1, 2\Delta_2; rd, sd_1).$$

PROOF. This follows by applying Cauchy's inequality to

$$\mathcal{M}_{dd_1}(\Delta_1, \Delta_2; r, s) \leqslant \sum_x \sum_y \#\{a/c \in \mathcal{P}(rd_1, sd);\ \|\bar{a}/c - x\| < \Delta_1,\ |ac - y| < \Delta_2 AC\}$$
$$\#\{a_1/c_1 \in \mathcal{P}(rd, sd_1);\ \|\bar{a}_1/c_1 - x\| < \Delta_1,\ |a_1c_1 - y| < \Delta_2 AC\}$$

where x ranges over Δ_1-spaced points and y ranges over $\Delta_2 AC$-spaced points. ■

Now, we are ready to prove Theorem 4.2 by induction on rs. If $rs \geqslant AC$

we have trivially

$$\mathcal{M}(\Delta_1, \Delta_2; r, s) \ll \left(\frac{AC}{rs}\right)^2,$$

so the assertion is obvious. Let us assume that $rs < AC$. If $dd_1 \geqslant 2$ by the induction hypothesis and by Lemma 4.3 one has

$$\mathcal{M}_{dd_1}(\Delta_1, \Delta_2; r, s) \leqslant 2^4 \lambda(\varepsilon)(rsdd_1)^{-1-\varepsilon}(AC)^{2\varepsilon}\,\mathcal{Q}(\Delta_1, \Delta_2; A, C) \tag{4.15}$$

where, for notational simplicity, we put

$$\mathcal{Q}(\Delta_1, \Delta_2; A, C) = AC + \Delta_1\Delta_2A^2C^2 + \Delta_1^2A^2C^2 + \Delta_2A^2 + \Delta_2C^2\,.$$

Since

$$\sum_{dd_1 > D}(dd_1)^{-1-\varepsilon} \leqslant \zeta^2\left(1 + \frac{\varepsilon}{2}\right)D^{-\varepsilon/2} \leqslant \left(1 + \frac{2}{\varepsilon}\right)^2 D^{-\varepsilon/2},$$

we obtain by (4.15)

$$\begin{aligned} \sum_{dd_1 > D}\sum \mathcal{M}_{dd_1}(\Delta_1, \Delta_2; r, s) &\leqslant 2^4\lambda(\varepsilon)\left(1 + \frac{2}{\varepsilon}\right)^2 D^{-\varepsilon/2}\,(rs)^{-1-\varepsilon}(AC)^{2\varepsilon}\,\mathcal{Q}(\Delta_1, \Delta_2; A, C) \\ &\leqslant \frac{1}{2}\,\lambda(\varepsilon)(rs)^{-1-\varepsilon}(AC)^{2\varepsilon}\,\mathcal{Q}(\Delta_1, \Delta_2; A, C) \end{aligned} \tag{4.16}$$

on taking $D = D(\varepsilon)$ such that $2^4(1 + 2/\varepsilon)^2 D^{-\varepsilon/2} = \frac{1}{2}$.

For $dd_1 \leqslant D$ we apply Lemmas 4.1 and 4.2 getting first

$$\mathcal{M}_{dd_1}(\Delta_1, \Delta_2; r, s) \ll (rs)^{-1}(AC)^{\varepsilon_1}\,\mathcal{Q}(\Delta_1, \Delta_2; A, C)$$

for any $\varepsilon_1 > 0$, and then

$$\begin{aligned} \sum_{dd_1 < D}\mathcal{M}_{dd_1}(\Delta_1, \Delta_2; r, s) &\leqslant \varrho(\varepsilon)(rs)^{-1}(AC)^{\varepsilon}\,\mathcal{Q}(\Delta_1, \Delta_2; A, C) \\ &\leqslant \tfrac{1}{2}\lambda(\varepsilon)(rs)^{-1-\varepsilon}(AC)^{2\varepsilon}\,\mathcal{Q}(\Delta_1, \Delta_2; A, C) \end{aligned} \tag{4.17}$$

by taking $\lambda(\varepsilon)$ sufficiently large; $\lambda(\varepsilon) \geqslant 2\varrho(\varepsilon)$.

Combining (4.5), (4.16) and (4.17) one completes the proof of the induction step and of Theorem 4.2. ∎

Remarks. Theorem 4.1 is important for estimating exponential sums that are related to the Riemann zeta-function, cf. Section 5. When more general sums are treated, one is faced with a similar problem, the condition (4.3) being replaced by

$$\left|f(c)\,g\left(\frac{a}{c}\right) - f(c_1)\,g\left(\frac{a_1}{c_1}\right)\right| \leqslant \Delta_2 FG$$

where $f(x)$ and $g(x)$ are some smooth functions; $f \ll F$, $g \ll G$. In order to study the most interesting exponential sums of type

$$\sum_m e(m^r X)$$

it suffices to consider $f(x) = x^\varkappa$ and $g(x) = x^\lambda$. But even then, unless $\varkappa = 2\lambda \neq 0$, the elementary arguments which we have used to prove Theorem 4.1 fail. In such circumstances one should try to apply the rather advanced technique of spectral theory of automorphic functions. Using the Kuznetsov trace formula for sums of Kloosterman sums together with the large sieve inequality for the Fourier coefficients of Maass cusp forms (cf. [2]) one is able to extend Theorem 4.1 to the case $\varkappa^2 + \lambda^2 \neq 0$.

5. – Proof of theorem.

We first consider the sums of type

$$S_f(M) = \sum_{m \sim M} e(f(m)) \tag{5.1}$$

where $f(x)$ is a smooth function such that

$$|f^{(j)}(m)| \sim FM^{-j} \tag{5.2}$$

for $m \sim M$ and $j \geqslant 1$ with some $F \geqslant 1$ such that

$$F^{3/7} < M < F^{1/2} \tag{5.3}$$

with the aim of showing that

$$S_f(M) \ll F^{\Theta+\varepsilon} M^{1/2}, \quad \Theta = \frac{9}{56}\,. \tag{5.4}$$

We begin by applying Weyl's method to reduce the problem to the estimation of cubic exponential sums. For $n \geqslant 1$ we have

$$S_f(M) = \sum_{m \sim M} e(f(m+n)) + O(n)$$

and by Taylor expansion

$$f(m+n) = f(m) + f'(m)n + \tfrac{1}{2}f''(m)n^2 + \tfrac{1}{6}f'''(m)n^3 + O(n^4 M^{-4} F) .$$

Now average the result over n in $(N, 2N]$ for some N with

$$F^{-1/3} M < N < F^{-1/2} M^{3/2} . \tag{5.5}$$

Since the error term $O(n^4 M^{-4} F)$ is bounded, we conclude by Lemma 2.1 that

$$S_f(M) \ll N^{-1} \sum_{m \sim M} \Big| \sum_{N < n \leqslant N_1} e\big(f'(m)n + \tfrac{1}{2}f''(m)n^2 + \tfrac{1}{6}f'''(m)n^3\big)\Big| + N ,$$

with some $N_1 \leqslant 2N$ independent of m and n. The innermost sum is considered as a Gauss' sum perturbed by the factor $e\big(\tfrac{1}{6}f'''(m)n^3\big)$. Traditionally at this point one applies Cauchy's inequality a number of times until a linear polynomial is reached. This procedure sets the limit $\Theta = \frac{1}{6}$ in (5.4). We depart from Weyl's method so that Cauchy's inequality is not used. The key idea is to evaluate the perturbed Gauss' sum rather than to estimate it. A direct use of Poisson's summation is not recommended because of the great variation of terms, the worst one being the quadratic term.

Each middle coefficient $\frac{1}{2}f''(m)$ has a rational approximation

$$\left|\frac{1}{2}f''(m) + \frac{a}{c}\right| \leqslant \frac{1}{cN} \tag{5.6}$$

with $1 \leqslant c \leqslant N$ and $(a, c) = 1$. Let $m(a/c)$ be the solution of

$$-\frac{1}{2}f''(m) = \frac{a}{c} .$$

Then each m satisfying (5.6) can be written as

$$m = [m(a/c)] + l$$

with $|l| \leqslant L(c)$ where $L(c) \sim (cFN)^{-1} M^3$. Notice that $L(c) \gg 1$ by (5.5). The terms pertaining to the fraction a/c with small denominators,

$$1 \leqslant c \leqslant C_0$$

say, when treated trivially, contribute to $S_f(M)$

$$\ll N^{-1} \sum_{1\leqslant c\leqslant C_0} L(c) \sum_{a\sim cFM^{-2}} N \ll C_0 MN^{-1}.$$

Now assume that $C_0 < c \leqslant N$. Denote

$$b = b(a, c) = [cf'([m(a/c)])]$$

and

$$\mu = \mu(a/c) = \tfrac{1}{6} f'''(m(a/c)) .$$

We find the following approximations to the extreme coefficients

$$\tfrac{1}{6} f'''(m) = \mu + O((cMN)^{-1})$$

and

$$\begin{aligned} f'(m) &= f'([m(a/c)]) - 2ac^{-1} l + O(l^2 FM^{-3}) \\ &= \frac{b-2al}{c} + O\left(\frac{1}{c} + \frac{M^3}{c^2 FN^2}\right). \end{aligned}$$

The second error term is $\ll c^{-1}$ provided

$$C_0 = F^{-1} M^3 N^{-2}$$

which we henceforth assume. By Lemma 2.1 we conclude

$$S_f(M) \ll \sum_{\substack{C_0 < c \leqslant N \\ (a,c)=1}} c^{-1} \sum_{a\sim cFM^{-2}} \sum_{|l|\leqslant L} \left| \sum_{N<n\leqslant N_1} e\left(\mu n^3 - \frac{a}{c} n^2 + \frac{b-2al}{c} n\right)\right| + N + C_0 MN^{-1}$$

with some $N_1 \leqslant 2N$ independent of the variables of summation. Hence, for some A, C and L with

$$C_0 < C < N, \quad A = CFM^{-2}, \quad L \sim (CFN)^{-1} M^3$$

we obtain

$$S_f(M) \ll (\log N)\, C^{-1} \sum_{c\sim C} \sum_{\substack{a\sim A \\ (a,c)=1}} \sum_{|l|\leqslant L} |S(\mu;\, a, b-2al;\, c)| + N + C_0 MN^{-1}.$$

The innermost sum $S(\mu;\, a, b + 2al;\, c)$ is an incomplete Gauss' sum $(\mathrm{mod}\, c)$ whose order of magnitude in case $\mu = 0$ (no perturbation) can be exactly determined in a number of ways. Yet, when μ is not too large the Fourier technique of completing the sum works well. This technique has been applied to prove Lemma 3.1. Now we use this result. The error terms from Lemma 3.1 contribute to $S_f(M)$

$$\begin{aligned} &\ll C^{-1}\sum_a\sum_c\sum_l (N^{1/2} + \mu^{-1}N^{-2})(\log N)^2 \\ &\ll C^{-1}ACL(N^{1/2} + F^{-1}M^3N^{-2})(\log N)^2 \\ &\ll (MN^{-1/2} + F^{-1}M^4N^{-3})(\log N)^2\,. \end{aligned}$$

The main term $\sum_h$ from Lemma 4.1 can be simplified a bit in two steps. First remove the factor $2(\mu ch)^{-1/4}$ by using Lemma 2.1 (partial summation) and then replace the constraint $3\mu cN^2 < h < 3\mu cN_1^2$ by a weaker one $h \sim H$ with

$$H = CFM^{-3}N^2 \qquad (1 < H < C)$$

by using Lemma 2.2. The resulting inequality is

$$S_f(M) \ll C^{-3/2}(FN)^{-1/2}M^{3/2}(\log N)^2 \sum_{c\sim C}\;\sum_{\substack{a\sim A\\(a,c)=1}}\;\sum_{|l|\leqslant L}$$

$$\cdot\left|\sum_{h\sim H}{}^{*}\, e\left(\frac{\bar a}{c}\frac{h^2}{4} + \frac{\bar a b + 2l + c\eta}{c}\frac{h}{2} - \frac{\nu\, h^{3/2}}{c}\right)\right| + (N + MN^{-1/2} + F^{-1}M^4N^{-3})(\log N)^2$$

where $\sum^*$ is restricted to h with a fixed parity,

$$\nu = \nu(a, c) = \tfrac{2}{3}(3\mu c)^{-1/2} = \tfrac{4}{3}\big(2cf'''(m(a/c))\big)^{-1/2} \sim (CF)^{-1/2}M^{3/2}\,,$$

and η is a real number which does not depend on the variables of summation. By Hölder's inequality we get

$$(5.7)\qquad S_f(M) \ll \left(\frac{M^9}{C^3F^2N^5}\right)^{1/4}(\log N)^2\Big(\sum_c\sum_a\sum_l\Big|\sum_h{}^{*}\Big|^4\Big)^{1/4} + (N + MN^{-1/2} + F^{-1}M^4N^{-3})(\log N)^2\,.$$

Here the sum $\sum\sum\sum|\sum^*|^4$ can be regarded as a bilinear form of type $\mathscr{B}(\mathfrak{a}, \mathfrak{b};\, \mathfrak{X}, \mathfrak{Y})$ considered in Lemma 2.4, where $a(\mathfrak{x})$ is the multiplicity of representations of $\mathfrak{x}$ in the form

$$\mathfrak{x} = (x_1, x_2, x_3) = \left(\tfrac{1}{4}\sum_1^4(-1)^j h_j^2,\ \tfrac{1}{2}\sum_1^4(-1)^j h_j,\ \sum_1^4(-1)^j h_j^{3/2}\right)$$

with relevant h_j and $b(\mathfrak{y})$ is the multiplicity of representations of $\mathfrak{y}$ in the form

$$\mathfrak{y}=(y_1, y_2, y_3)=\left(\frac{\bar a}{c}, \frac{\bar a b+2l+c\eta}{c}, \frac{\nu}{c}\right).$$

Therefore

$$\mathfrak{X}=(X_1, X_2, X_3)\sim(H^2, H, H^{3/2})$$

and

$$\mathfrak{Y}=(Y_1, Y_2, Y_3)=(1, 1, Y_3)$$

with

$$Y_3\sim F^{-1/2}C^{3/2}M^{-3/2}\geqslant H^{-1/2}.$$

By Lemma 2.4 (see the remarks after the proof) we get

$$\left(\sum_c\sum_a\sum_l\left|\sum_h{}^*\right|^4\right)^2\ll H^{9/2}Y_3\mathscr{B}(\mathfrak{b};\mathfrak{X})\mathscr{B}(\mathfrak{a};\mathfrak{Y})\tag{5.8}$$

where, in our particular context, $\mathscr{B}(\mathfrak{b};\mathfrak{X})$ is the number of pairs $\{(a, c, l), (a_1, c_1, l_1)\}$ such that simultaneously

$$\left\|\frac{\bar a}{c}-\frac{\bar a_1}{c_1}\right\|\ll H^{-2},\tag{5.9}$$

$$\left\|\frac{\bar a b+2l}{c}-\frac{\bar a_1 b_1+2l_1}{c_1}\right\|\ll H^{-1},\tag{5.10}$$

$$\left\|\frac{\nu(a, c)}{c}-\frac{\nu(a_1, c_1)}{c_1}\right\|\ll H^{-3/2}\tag{5.11}$$

and $\mathscr{B}(\mathfrak{a};\mathfrak{Y})$ is the number of h_j's, $1\leqslant j\leqslant 8$ such that simultaneously

$$\sum_1^4(h_j^2-h_{j+4}^2)\ll 1,\tag{5.12}$$

$$\sum_1^4(h_j-h_{j+4})\ll 1,\tag{5.13}$$

and

$$\sum_1^4(h_j^{3/2}-h_{j+4}^{3/2})\ll H^{1/2}.\tag{5.14}$$

The second system is investigated, among other things, in a separate paper [1] with the result

$$\mathscr{B}(\mathfrak{a};\,\mathfrak{Y}) \ll H^{4+\varepsilon}\,. \tag{5.15}$$

It remains to estimate $\mathscr{B}(\mathfrak{b};\,\mathfrak{X})$. By (5.10) we deduce that

$$\left\|\frac{\bar{a}b}{c} - \frac{\bar{a}_1 b_1}{c_1}\right\| \ll \frac{L}{C} \tag{5.16}$$

(notice that $LH \sim N$, so $LC^{-1} \sim NC^{-1}H^{-1} \gg H^{-1}$) and that

$$\#\{l, l_1\} \ll (C + L)\,LH^{-1}\,.$$

Therefore

$$\mathscr{B}(\mathfrak{b};\,\mathfrak{X}) \ll (C + L)\,LH^{-1}\mathscr{N}(A, C, H) \tag{5.17}$$

where $\mathscr{N}(A, C, H)$ stands for the number of pairs $\{(a, c), (a_1, c_1)\}$ satisfying (5.9), (5.16) and (5.11). Since the numbers $b = b(a, c) = [cf'([m(a/c)])]$ depend on a, c in a very complex way, we are forced to discard (5.16). This is a place at which our method is wasteful.

So far our arguments have been applicable to quite general sums $S_f(M)$. Now specifying to

$$f(x) = t \log x\,, \tag{5.18}$$

so $F = t$, we find that

$$m(a/c) = \left(\frac{ct}{2a}\right)^{1/2}$$

and

$$c^{-1}\nu(a, c) = \tfrac{2}{3}(2ac)^{-3/4}t^{1/4}\,,$$

thus, $c^{-1}\nu(a, c)$ is a function of ac. The condition (5.11) is equivalent to

$$|ac - a_1 c_1| \ll \Delta_2 AC \tag{5.19}$$

with $\Delta_2 = (AC)^{3/4}H^{-3/2}t^{-1/4} = t^{-1}M^3N^{-3}$. By Theorem 4.1 we infer

$$\mathscr{N}(A, C, H) \ll (AC + \Delta_1\Delta_2 A^2C^2 + \Delta_1 A^2C^2 + \Delta_2 A^2 + \Delta_2 C^2)(AC)^{\varepsilon}$$

with $\Delta_1 = H^{-2}$. Here we have

$$AC \ll tM^{-2}C^2$$

and

$$\Delta_1\Delta_2 A^2C^2 \ll t^{-1}M^5N^{-7}C^2 .$$

The other three terms are smaller, therefore we conclude

$$\mathcal{N}(A, C, H) \ll (FM^{-2} + F^{-1}M^5N^{-7})C^2F^{\varepsilon} . \tag{5.20}$$

Finally we argue as follows. By (5.8), (5.15), (5.17) and (5.20) one gets

$$\sum_c\sum_a\sum_l\Big|\sum_h\Big|^4 \ll C^3M^{-9}N^7(FM^{-2}N^2 + F^{-1}M^5N^{-5})^{1/2}F^{3+\varepsilon} .$$

This together with (5.7) yields

$$\begin{aligned} S_f(M) &\ll N^{1/2}(FM^{-2}N^2 + F^{-1}M^5N^{-5})^{1/8}F^{1/4+\varepsilon} \\ &\qquad + (N + MN^{-1/2} + F^{-1}M^4N^{-3})(\log N)^2 \\ &\ll (M^{1/2}F^{9/56} + M^{1/2}F^{1/7} + MF^{-1/7})F^{\varepsilon} \\ &\ll M^{1/2}F^{9/56+\varepsilon} \end{aligned}$$

on taking

$$N = MF^{-2/7} .$$

This completes the proof of (5.4) for M in (5.3) and $f(x)$ defined by (5.18).

For $1 \leqslant M \leqslant F^{3/7}$ we appeal to the exponent pair theory, cf. [7] and [8], giving

$$S_f(M) \ll M^{\lambda}(FM^{-1})^{\varkappa} .$$

The pair $(\varkappa, \lambda) = (1/9, 13/18)$ yields

$$S_f(M) \ll M^{1/2}(FM)^{1/9} \ll M^{1/2}F^{10/63} \tag{5.21}$$

which is stronger than (5.4). Finally (5.4) and (5.21) imply our theorem, cf. [9]. ■

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