Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

Gary M. Lieberman

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 13, nº 3 (1986), p. 347-387

<http://www.numdam.org/item?id=ASNSP_1986_4_13_3_347_0>

The First Initial-Boundary Value Problem for Quasilinear Second Order Parabolic Equations.

GARY M. LIEBERMAN (*)

0. – Introduction.

In a fundamental work [29] Serrin obtained very general conditions under which the Dirichlet problem for a quasilinear elliptic equation with arbitrary smooth boundary values is solvable in a given domain; he also showed that these conditions were sharp in that the problem is not solvable for some (infinitely differentiable) boundary values when they are violated. Prior to [29], existence of solutions for uniformly elliptic equations had been shown by Ladyzhenskaya and Ural'tseva [18] and for uniformly convex domains by Gilbarg [6]. A natural question to ask is, what sort of existence theory can be established for the first initial-boundary value problem for quasilinear parabolic equations? For uniformly parabolic equations this question was answered by Ladyzhenskaya and Ural'tseva [19] at the same time as their work on elliptic problems, and following Serrin's work came several investigations on nonuniformly parabolic problems: [2], [12], [33]. Unfortunately all of these works fall short of the comprehensiveness of [29]. Edmunds and Peletier [2] consider a slight generalization of the uniformly parabolic case while Ivanov [12] considers a parabolic version of Gilbarg's uniformly convex problems; in both cases we shall see that their results are not best possible even within the restricted settings. Trudinger [33] fares better in that he proves solvability under conditions essentially as general as Serrin's, but he does not examine the sharpness of his conditions. (We also mention the work of Iannelli and Vergara Caffarelli [11], Lichnewsky and Temam [20], and Marcellini and Miller [28] on the time-dependent prescribed mean curvature equation. These authors

(*) Supported in part by NSF grant DMS-8315545.
Pervenuto alla Redazione il 29 Aprile 1985.

are concerned with non-classical solutions, in particular the solutions in [20] and [28] do not generally assume their boundary values even in a weak sense, so their results are not strictly comparable to ours. Moreover they only consider time independent boundary values.)

The purpose of this paper is to present sharp parabolic analogs of Serrin's existence and non-existence results. As we shall see, the elliptic methods carry over to the parabolic setting with only minor changes. This fact is apparent in [33] but not in [2] or [12], so we shall repeat many of the details. Moreover we consider somewhat less smooth initial and boundary values (corresponding to the regularity of boundary values for elliptic problems in [21]) than would be used in a strict analog of Serrin's theory.

Of course the key to our existence theory is the establishment of certain *a priori* estimates, and most of our effort is to prove such estimates. After we present some basic results in Section 1, we discuss the crucial estimate, on the spatial gradient of the solution on the lateral boundary, in Section 2. Related to this boundary gradient estimate are some oscillation estimates which imply non-existence of solutions for certain initial-boundary data; these estimates and non-existence results appear in Section 3. Estimates (other than the boundary gradient estimate) needed for the existence program are given in Section 4. These estimates have all appeared before (in [2], [12], [16], [19], [22], [30], [32]) so we shall give all but one in simplified form; the exception is a Hölder estimate of the gradient near the boundary. For our purposes the version of this estimate due to Ladyzhenskaya and Ural'tseva [19] by itself is inadequate so we use also some results of Krylov [16] adapted to the less smooth initial and boundary values we are considering. A somewhat different approach appears in [17]. Finally some existence theorems, extending those in [2], [12] and [33], are given in Section 5.

1. – Statement of problem.

We denote by Ω a cylindrical domain in $\mathbf{R}^{n+1}$ ($n \geqslant 1$), i.e., Ω has the form $\Omega = D \times (0, T)$ for some open, connected $D \subset \mathbf{R}^n$ and some positive T (although most of our discussion is applicable to non-cylindrical domains as well), and we define

$$B\Omega = D \times \{0\}\,, \qquad C\Omega = \partial D \times \{0\}\,, \qquad S\Omega = \partial D \times (0, T)\,,$$

$$\mathcal{P}\Omega = B\Omega \cup C\Omega \cup S\Omega\,.$$

The significance of the letters B, C, S, $\mathcal{T}$ is discussed in [24]. Points in $\mathbf{R}^{n+1}$ will be denoted by $X = (x, t)$, $Y = (y, s)$, etc., and we write

$$D_i = \partial/\partial x^i, \; D_{ij} = \partial^2/\partial x^i \partial x^j \quad \text{for } i, j = 1, \ldots, n.$$

We use a subscript t to denote differentiation with respect to t, and we define $C^{2,1}(\Omega)$ to be the set of all functions u such that u, $D_i u$, $D_{ij} u$, u_t $(i, j = 1, \ldots, n)$ exist and are continuous on Ω. Define the operator P on $C^{2,1}(\Omega)$ by

$$Pu = a^{ij}(X, u, Du) D_{ij} u + a(X, u, Du) - u_t .$$

We observe here and below the convention that repeated indices are to be summed from 1 to n, and $Du = (D_1 u, \ldots, D_n u)$.

Our concern in this paper is with the problem

$$Pu = 0 \text{ in } \Omega , \quad u = \varphi \text{ on } \mathcal{T}\Omega \tag{1.1}$$

for continuous φ under the assumption that P is parabolic, i.e., the matrix $(a^{ij}(X, z, p))$ is positive for all $(X, z, p) \in \Omega \times \mathbf{R} \times R^n$. Additional hypotheses will be imposed on P, φ, and Ω presently. Associated with P are the maximum and minimum eigenvalues, Λ and λ, respectively, of the matrix (a^{ij}) and the Bernstein invariant [1, p. 456]: $\mathcal{E} = a^{ij} p_i p_j$.

Various Hölder norms and spaces will be used; our notation follows [24] and should be compared with that in [7] and [19, Sect. I.1]. We use $| \; |$ to denote the usual Euclidean length

$$|x| = \left(\sum_{i=1}^{n} (x_j)^2 \right)^{\frac{1}{2}}$$

on $\mathbf{R}^n$ as well as the parabolic length

$$|X| = (|x|^2 + |t|)^{\frac{1}{2}}$$

on $\mathbf{R}^{n+1}$, and we denote by $H_0(\Omega)$ the set of all uniformly continuous functions u equipped with the norm

$$|u|_{0;\Omega} = \sup_{\Omega} |u| .$$

For $\alpha \in (0, 1]$ and $\beta \in (0, 2]$ we define

$$
\begin{aligned}
[u]_{\alpha;\Omega} &= \sup\{|u(X) - u(Y)|/|X - Y|^{\alpha} : X \neq Y \quad \text{in } \Omega\}\,, \\
[u]'_{\alpha;D} &= \sup\{|u(x) - u(y)|/|x - y|^{\alpha} : x \in D\,, \quad y \in \partial D\}\,, \\
[u]''_{\alpha;\Omega} &= \sup\{|u(x, t) - u(y, t)|/|x - y|^{\alpha} : x \in D\,, \quad y \in \partial D\,, \quad 0 < t < T\}\,, \\
\langle u \rangle_{\beta;\Omega} &= \sup\{|u(x, t) - u(x, s)|/(t - s)^{\beta/2} : x \in D\,, \quad 0 < s < t < T\}\,,
\end{aligned}
$$

and we suppress the subscripts Ω and D when these are clear from the context. For $a = k + \alpha$ where k is a non-negative integer and $0 < \alpha \leqslant 1$, we denote by $H_a(\Omega)$ the set of all functions u for which all derivatives of the form $D_x^i D_t^j u (j + 2i \leqslant k)$ exist and are continuous with

$$
|u|_a = \sum_{j+2k \leqslant i} |D_x^j D_t^i u|_0 + \sum_{j+2i=k} [D_x^j D_t^i u]_{\alpha} + \sum_{j+2i=k-1} \langle D_x^j D_t^i u \rangle_{1+\alpha}
$$

finite. Setting

$$
d(x) = \operatorname{dist}(x, \partial D)\,, \qquad d^*(X) = \min\{d(x), t^{\frac{1}{2}}\}\,,
$$

we introduce for $\delta > 0$,

$$
\begin{aligned}
&\Omega_\delta = \{X \in \Omega : d^*(X) > \delta\}\,, \qquad \Omega'_\delta = \{X \in \Omega : d(x) < \delta\}\,, \\
&D'_\delta = \{x \in D : d(x) < \delta\}\,.
\end{aligned}
$$

We then denote by $H_a^{(b)}(\Omega)$ the set all of u which restrict to $H_a(\Omega_\delta)$ functions for all $\delta > 0$ and for which

$$
|u|_{a;\Omega}^{(b)} = \sup_{\delta > 0} \delta^{a+b} |u|_{a;\Omega_\delta}
$$

is finite.

For $\gamma \geqslant 0$, we denote by $H_\gamma(S\Omega)$ the space of all functions φ on $S\Omega$ which are restrictions of $H_\gamma(\tilde{\Omega})$ functions for some open $\tilde{\Omega} \supset \bar{\Omega}$, equipped with the norm

$$
|\varphi|_{\gamma;\Omega} = \inf\{|\tilde{\varphi}|_{\gamma;S\tilde{\Omega}} : \tilde{\Omega} \supset \bar{\Omega}\,, \quad \tilde{\varphi} = \varphi \quad \text{on } S\Omega\}\,.
$$

Similarly $H_\gamma(\mathcal{T}\Omega)$ denotes the set of all restrictions to $\mathcal{T}\Omega$ of $H_\gamma(\tilde{\Omega})$ functions, and $H'_\gamma(\Omega)$ denotes the set of all $\varphi \in H_0(\Omega)$ whose restriction to $S\Omega$ lies in $H_\gamma(S\Omega)$ and for which the function φ_0 given by $\varphi_0(x) = \varphi(x, 0)$ has finite norm $[\varphi_0]'_{1,D}$. Other useful collections of functions are

(1) $C^1(\bar{\Omega})$, the set of all functions $u \in H_0(\Omega)$ with $Du \in H_0(\Omega)$,

(2) $W^{2,1}_{n+1}$, the set of all functions u such that $|u|^{n+1}$, $|Du|^{n+1}$, $|D^2u|^{n+1}$, $|u_t|^{n+1}$ are integrable over Ω with respect to the measure $dx\,dt$, and

(3) $C^\infty(\mathcal{T}\Omega)$ the set of all functions on $\mathcal{T}\Omega$ which are restrictions of infinitely differentiable functions (with respect to x and t) in $\mathbb{R}^{n+1}$.

If D is bounded and if there is a function $\psi \in C^1(\mathbb{R}^n)$ such that

$$D = \{x \in \mathbb{R}^n \colon \psi(x) > 0\}$$

and $|D\psi| \neq 0$ on ∂D, we say that $\partial D \in C^1$ or $\partial\Omega \in C^1$. If also $\psi \in H_\gamma$ for some $\gamma > 1$, we say that $\partial D \in H_\gamma$ or $\partial\Omega \in H_\gamma$. When $\partial\Omega \in H_\gamma$ for some $\gamma > 1$, it follows from [24, Sect. 2] that the sets $H_\gamma(S\Omega)$, $H_\gamma(\mathcal{T}\Omega)$, and H'_γ are invariant under H_γ changes of coordinates and that there is an intrinsically defined norm on $H_\gamma(S\Omega)$ which is equivalent to the one we have given.

Next we note that $H_\gamma(S\Omega)$ functions have useful extensions.

LEMMA 1.1. *Let $\varphi \in H_\gamma(S\Omega)$ for some $\gamma \in (1, 2]$. Then for any positive constants M and ε with $M \geqslant \sup|\varphi|$ and any $y \in \partial D$, there is a function*

$$\bar{\varphi} \in C\big(\bar{\Omega}\times[0,1]\big) \cap C^\infty\big(\Omega\times(0,1]\big)$$

and a constant $\Phi = \Phi\big(\varepsilon, M, n, |\varphi|_{\gamma;S\Omega}\big)$ such that

$$\text{(1.2a)}\quad \bar{\varphi}(y, T, 0) = \varphi(y, T)\,,$$

$$\text{(1.2b)}\quad \bar{\varphi}(X, 0) \geqslant \varphi(X) \qquad \textit{for all } X \in S\Omega\,,$$

$$\text{(1.2c)}\quad \bar{\varphi}(X, x^0) = M \textit{ for all } (X, x^0) \in \bar{\Omega}\times[0,1] \qquad \textit{with } |x-y| > \varepsilon\,,$$

$$\text{(1.2d)}\quad |\bar{\varphi}| + \sum_{i=0}^{n} |\bar{\varphi}_i| \leqslant \Phi\,, \qquad \sum_{i,j=0}^{n} |\bar{\varphi}_{ij}| + |\bar{\varphi}_t| \leqslant \Phi |x^0|^{\gamma-2},$$

where subscripts denote differentiation with respect to x^i, x^j, or t, and $\bar{\varphi}$ and its derivatives are evaluated at $(X, x_0) \in \Omega\times(0,1]$ in (1.2d). Moreover, if there is a constant Φ_1 such that

$$\text{(1.3)}\quad |\varphi(x,t) - \varphi(x,T)| \leqslant \Phi_1(T-t) \qquad \textit{for all } x \in \partial D$$

$$\textit{with } |x-y| < \varepsilon\,, \qquad 0 < t < T,$$

then (1.2) *holds, and*

$$|\tilde{\varphi}_t| \leqslant \Phi_1 . \tag{1.4}$$

PROOF. Let $\tilde{\Omega} \supset \bar{\Omega}$ and let $\tilde{\varphi}$ be an $H_\gamma(\tilde{\Omega})$ function with $\tilde{\varphi} = \varphi$ on $S\Omega$ and $|\tilde{\varphi}|_\gamma \leqslant 2|\varphi|_\gamma$. For $\psi \in C^2(\mathbf{R}^n)$ supported in $\{|x-y| < \varepsilon\}$ with $0 \leqslant \psi \leqslant 1$ in $\mathbf{R}^n$ and $\psi(y) = 1$, we set

$$\hat{\varphi}(x, t) = [\tilde{\varphi}(x, T) + |\varphi|_\gamma\, (T-t)^{\gamma/2}]\,\psi(x) + M(1-\psi(x)) . \tag{1.5}$$

An appropriate $\bar{\varphi}$ is given by

$$\bar{\varphi}(X, x^0) = \int\limits_{\mathbf{R}^{n+1}} \hat{\varphi}(x - \eta x^0, t - s(x^0)^2)\, \zeta(\eta, s)\, dY , \qquad (Y = (\eta, s)) ,$$

for a suitable non-negative, compactly supported $\zeta \in C^\infty(\mathbf{R}^{n+1})$ with $\int \zeta\, dY = 1$ and $\zeta(\eta, s) = 0$ for $s \geqslant 0$. Then (1.2*a*, *b*, *c*) are clear. A proof of (1.2*d*) and a suitable ζ are given in [23, Sect. 3]. When (1.3) holds, we replace $|\varphi|_\gamma (T-t)^{\gamma/2}$ by $\Phi_1(T-t)$ in (1.5) to obtain (1.4). ■

Note that Φ actually depends on $|\varphi|_\gamma$ through the quantities

$$|\varphi(\cdot, T)|_\gamma \quad \text{and} \quad \sup_{x \in \partial D} |\varphi(x, t) - \varphi(x, T)|/(T-t)^{\gamma/2} .$$

In addition, by means of a suitable choice of the function ψ and the use of $\varphi(X)$ rather than the term

$$\tilde{\varphi}(x, T) + |\varphi|_\gamma (T-t)^{\gamma/2}$$

in (1.5), we can guarantee that $\bar{\varphi}(X, 0) = \varphi(X)$ for all X in a given closed subset of $\overline{S\Omega} \cup \{|x-y| < \varepsilon\}$.

We also recall from [23, Theorem 1.3 and 4.2(*b*)] properties of a regularized distance.

LEMMA 1.2. *Let* $\partial D \in H_\gamma$ *for some* $\gamma \in (1, 2]$. *Then there is a function* $\varrho \in H_2^{(-\gamma)}$ *such that*

$$1/c(D) \leqslant \varrho/d \leqslant c(D) \quad \text{in } D , \qquad |\varrho|_2^{(-\gamma)} \leqslant c(D) , \qquad |D\varrho| \equiv 1 \quad \text{on } \partial D$$

for some positive constant $c(D)$.

We close this section with a comparison principle based on [8, Theorem 10.1] which seems not to have been stated before in quite this form although it is similar to [3, Theorem 2.16] (cf. [2, Theorem 3]).

LEMMA 1.3. *Let u, v be two functions in $H_0(\Omega) \cap C^{2,1}(\Omega)$ with $u \leqslant v$ on $\mathcal{P}\Omega$, and $Pu > Pv$ in Ω for some parabolic operator P such that a^{ij} is independent of z and there is a non-negative constant μ such that*

$$a(X, z_1, p) - a(X, z_2, p) \leqslant \mu|z_1 - z_2| \quad \text{for all } (X, p) \in \Omega \times \mathbf{R}^n \quad \text{and } z_1 \geqslant z_2 .$$

Then $u < v$ in Ω.

PROOF. Set $w = \exp(-\mu t)(u - v)$, and let X_0 be a point in Ω where w attains its maximum. Since

$$Du = Dv , \qquad a^{ij}(X_0, Du) D_{ij} w \leqslant 0 , \qquad (u_t - v_t) \exp(-\mu t) \geqslant \mu w ,$$

at X_0, it follows that, at X_0,

$$\begin{aligned} 0 &< (Pu - Pv) \exp(-\mu t) \\ &= [a^{ij}(X_0, Du) D_{ij} w + a(X_0, u, Du) - a(X_0, v, Du) - u_t + v_t] \exp(-\mu t) \\ &\leqslant [a(X_0, u, Du) - a(X_0, v, Du) - \mu(u - v)] \exp(-\mu t) \\ &< 0 \qquad \text{if } u(X_0) \geqslant v(X_0) . \end{aligned}$$

Hence $u < v$ at any maximum of w, so $w < 0$ in Ω. ∎

2. – The boundary gradient estimate.

We now give the estimates which are central to our existence program. In order to retain some clarity of expositon, we present details only in some cases with an indication of how to proceed in general.

We begin with some definitions and results essentially in [21], and based on [9] and [29]. Proofs of these results, often simplied from [21], are included for completeness and in order to consider comparison surfaces other than ∂D.

For P as in Section 1 and u an $H_0(\Omega) \cap C^{2,1}(\Omega)$ solution of (1.1), we introduce an auxillary operator $\bar{P}$ by

$$\bar{P}v = \bar{a}^{ij}(X, Dv) D_{ij} v + \bar{a}(X, v, Dv) - v.$$

such that

$$\bar{a}^{ij}(X, Du) = a^{ij}(X, u, Du) , \qquad \bar{a}(X, u, Du) = a(X, u, Du) ,$$

$$\bar{a}(X, z, p) \text{ is non increasing in } z \text{ for fixed } (X, p) \in \Omega \times \mathbf{R}^n .$$

(More general $\bar{P}$ can be used but we see no reason to do so here.) Clearly $\bar{a}^{ij}(X, p) = a^{ij}(X, u(X), p)$; we indicate later a suitable choice for $\bar{a}$. Set $M = \sup_{\Omega} |u|$, let $X_0 \in S\Omega$, and let N contain a set of the form

$$\{X \in \mathbb{R}^{n+1}: |x - x_0| < \varepsilon, t_0 - \eta < t < t_0\}$$

for some positive ε and η; we call N a parabolic neighborhood of X_0. If there are functions $w^{\pm} \in H_0(N \cap \Omega) \cap C^{2,1}(N \cap \Omega)$ such that

$$\pm \bar{P}w^{\pm} < 0 \quad \text{in } N \cap \Omega\,, \tag{2.1a}$$

$$\pm w^{\pm} \geqslant \pm \varphi \quad \text{on } \mathcal{P}\Omega \cap N\,, \tag{2.1b}$$

$$\pm w^{\pm} \geqslant M \quad \text{on } \mathcal{P}N \cap \Omega\,, \tag{2.1c}$$

$$w^{\pm}(X_0) = \varphi(X_0)\,, \tag{2.1d}$$

we call w^+ an upper barrier and w^- a lower barrier at X_0: The comparison principle Lemma 1.3 then implies that

$$w^+ \geqslant u \geqslant w^- \quad \text{in } N\,,$$

so if $w^{\pm}$ are in H_1, we can estimate

$$\sup_{y \in D} \{|u(X_0) - u(y, t_0)|/|x_0 - y|\}\,.$$

In particular if $w^{\pm}$ and u are in $C^1(\bar{\Omega})$, we have

$$|Du(X_0)| \leqslant \max\{|Dw^{\pm}(X_0)|\}\,.$$

Thus a bound on $|Dw^{\pm}(X_0)|$ independent of X_0 gives a bound on $|Du|_{0;S\Omega}$. We shall construct only upper barriers, which we call barriers, for brevity; similar arguments yield lower barriers.

The basic building block of our barriers is a comparison surface $\mathcal{S} = \mathcal{S}^* \times (0, T)$. We assume that $\mathcal{S}^*$ is a surface in $\mathbb{R}^n$ containing x_0 but disjoint from D. We also assume the existence of a positive constant δ_0 such that $\partial \mathcal{S}^*$ is disjoint from $B' = \{x: |x - x_0| < \delta_0\}$ and the function $d(x) = \operatorname{dist}(x, \mathcal{S}^*)$ satisfies $d \in C^2(\overline{B' \cap \Omega})$. (If we wish $\mathcal{S}^*$ to be ∂D, we modify this definition by setting $B' = \Omega'_{\delta_0}$ provided δ_0 is small enough; in this way we can obtain a uniform boundary gradient estimate directly as in [21]). Our barriers will have the form (adapted from [9])

$$w(X) = \bar{\varphi}(X, f\circ d(x)) + Ff\circ d(x) \tag{2.2}$$

where f is a function at our disposal, $\bar{\varphi}$ is as in Lemma 1.1 (with $\varepsilon = \delta_0$ and Ω replaced by $\Omega \cap \{t < t_0\}$), and $F = 1 + 3\Phi + [\varphi_0]'_{1;D} + M$. Concerning f, we assume that there is $\delta \in (0, \delta_0]$ such tht

$$f \in C^1([0, \delta]) \cap C^2(0, \delta)\,, \tag{2.3a}$$

$$f(0) = 0\,, \quad f(\delta) = 1\,, \quad f'(\delta) > 1\,, \tag{2.3b}$$

$$f'' < 0 \quad \text{on } (0, \delta)\,. \tag{2.3c}$$

With $N = \{X \in B' \times (0, T]: d(x) < \delta\}$, (2.3) implies (2.1$b$, c, d) so we need to determine δ and f so that $Q\overline{w} < 0$, in which case

$$|Du(X_0)| \leqslant \Phi + \Phi f'(0) + Ff'(0) \leqslant 2Ff'(0)\,.$$

As in Lemma 1.1 we use subscripts to denote derivatives of $\bar{\varphi}$. Also we omit the arguments of $d, f, \bar{\varphi}$ and their derivatives, and we write $\bar{a}^{ij}$ for $\bar{a}^{ij}(X, Dw)$ and $\bar{a}$ for $\bar{a}(X, w, Dw)$. We also define

$$\mathcal{E}_d = \bar{a}^{ij} D_i d D_j d\,, \qquad \bar{\mathcal{E}} = \mathcal{E}(X, u, Dw)\,,$$

$$W_1 = \bar{a}^{ij}\bar{\varphi}_{ij} + 2f'\bar{a}^{ij}\bar{\varphi}_{i0} D_j d + \mathcal{E}_d[\bar{\varphi}_0(f')^2 + (F + \bar{\varphi}_0) f''],$$

$$W_2 = (F + \bar{\varphi}_0) f' \bar{a}^{ij} D_{ij} d + \bar{a}\,,$$

and observe that $\overline{P}w = W_1 + W_2 - w_t$, $w_t = \bar{\varphi}_t$. Some basic results are collected in the following lemma (cf. [21, Sect. 2.1]).

LEMMA 2.1. *Let γ, M, P, φ, $\bar{\varphi}$, $\overline{P}$, $\mathcal{E}$, and w be as above. Then*

$$\bar{\mathcal{E}} \leqslant 4F^2(f')^2\mathcal{E}_d + 2\bar{\Lambda}\Phi^2\,, \tag{2.4}$$

$$\tfrac{1}{3}Ff' \leqslant |Dw| \leqslant 2Ff'\,, \qquad |Dw| \geqslant f'\,. \tag{2.5}$$

If there are non-negative constants μ_0 and μ_1 such that

$$\Lambda \leqslant \mu_1 \lambda |p|^2, \quad \text{or} \quad 4\Lambda\Phi^2 \leqslant \mathcal{E} \quad \text{and} \quad \Lambda \leqslant \mu_1 E\,, \quad \text{for } |p| \geqslant \mu_0\,, \tag{2.6}$$

then

$$\bar{\mathcal{E}} \leqslant (8 + 8\mu_1 \Phi^2) F^2 (f')^2 \mathcal{E}_d \qquad \text{for } f' \geqslant \mu_0 , \tag{2.7}$$

and for any positive constants c_2 *and* c_3, *we can determine* f *and* δ *so that* (2.3) *is satisfied,* $f' \geqslant c_2$ *on* $(0, \delta)$, *and*

$$\bar{P}w \leqslant W_2 - c_3 (f')^2 f^{\gamma-2} \mathcal{E}_d + \Phi f^{\gamma-2}. \tag{2.8}$$

If also $\langle \varphi \rangle_{2;S\Omega} = \Phi_1$ *is finite, then we can replace* (2.8) *by*

$$\bar{P}w \leqslant W_2 - c_3 (f')^2 f^{\gamma-2} \mathcal{E}_d + \Phi_1 . \tag{2.8$'$}$$

PROOF. We first note that

$$D_i w = \bar{\varphi}_i + (F + \bar{\varphi}_0) f' D_i d . \tag{2.9}$$

Therefore

$$\begin{aligned}\bar{\mathcal{E}} &= \bar{a}^{ij} \bar{\varphi}_i \bar{\varphi}_j + 2f'(F + \bar{\varphi}_0) \bar{a}^{ij} \bar{\varphi}_i D_j d + (F + \bar{\varphi}_0)^2 (f')^2 \mathcal{E}_d \\ &\leqslant 2\bar{a}^{ij} \bar{\varphi}_i \bar{\varphi}_j + 2(F + \bar{\varphi}_0)^2 (f')^2 \mathcal{E}_d \leqslant 2\bar{\Lambda}\Phi^2 + 2(\tfrac{4}{3}F)^2 (f')^2 \mathcal{E}_d .\end{aligned}$$

Since $(\frac{4}{3})^2 < 2$, we have (2.4).

To prove (2.5) we have from (2.9) that

$$\frac{1}{3} F f' \leqslant \left(\frac{2}{3} F\right) f' - \Phi \leqslant |Dw| \leqslant \left(\frac{4}{3} F\right) f' + \Phi \leqslant 2Ff' ,$$

and

$$|Dw| \geqslant (F - \Phi) f' - \Phi \geqslant (F - 2\Phi) f' \geqslant f' .$$

When $\Lambda \leqslant \mu_1 \lambda |p|^2$, (2.7) follows from (2.4), (2.5), and the inequality $\lambda \leqslant \mathcal{E}_d$. When $4\Lambda\Phi^2 \leqslant \mathcal{E}$, (2.7) follows from (2.4).

To obtain (2.8) and (2.8)$'$ we first estimate W_1 by

$$W_1 \leqslant \frac{F}{2} [f'' + (8 + 8\mu_1 \Phi^2) F^2 (f')^2 f^{\gamma-2}] \mathcal{E}_d. \tag{2.10}$$

Since (2.10) is obvious for $\Phi = 0$, we assume $\Phi \neq 0$, so

$$\begin{aligned} 2f' \bar{a}^{ij} \bar{\varphi}_{i0} D_j d &\leqslant \bar{a}^{ij} \bar{\varphi}_{i0} \bar{\varphi}_{j0} / (\Phi f^{\gamma-2}) + \Phi f^{\gamma-2} (f')^2 \mathcal{E}_d \\ &\leqslant \bar{\Lambda} \Phi f^{\gamma-\bar{z}} + \Phi f^{\gamma-2} (f')^2 \mathcal{E}_d . \end{aligned}$$

Combining this inequality with

$$\bar{a}^{ij}\bar{\varphi}_{ij}\leqslant\bar{\Lambda}\Phi f^{\gamma-2}\,, \qquad F+\bar{\varphi}_0\geqslant\frac{F}{2}$$

gives

$$W_1\leqslant 2\bar{\Lambda}\Phi f^{\gamma-2}+\left[\frac{F}{2}f''+2\Phi(f')^2f''^{-2}\right]\mathcal{E}_d\,. \tag{2.11}$$

The proof of (2.10) is completed by using (2.6) to infer that $\bar{\Lambda}\leqslant\mu_1\bar{\mathcal{E}}$, and then using (2.7).

To prove (2.8) and (2.8)′ it therefore suffices to show that for any positive constants μ_2 and μ_3, there is a positive constant δ_1 such that for any $\delta\in(0,\delta_1]$, there is a function f satisfying (2.3a), $f'\geqslant\mu_2$ on $(0,\delta)$, and

$$f''+\mu_3(f')^2f^{\gamma-2}=0\,, \tag{2.12a}$$

$$f(0)=0\,,\quad f(\delta)=1\,. \tag{2.12b}$$

For a given δ, the solution of this boundary value problem is given implicitly by

$$d=\delta H(f)/H(1)\,,\qquad H(f)=\int_0^f\exp\left(\mu_3 g^{\gamma-1}/(\gamma-1)\right)dg\,.$$

Since H is independent of δ and

$$f'=\exp\left(-\mu_3 f^{\gamma-1}/(\gamma-1)\right)H(1)/\delta\geqslant\exp\left(-\mu_3/(\gamma-1)\right)H(1)/\delta\,,$$

we have also $f'\geqslant\mu_2$ if δ is small enough.

We remark that (2.6) is a consequence of the usual conditions:

$$\Lambda=O(\lambda|p|^2)\qquad\text{or}\qquad\Lambda=o(\mathcal{E})\qquad\text{as }|p|\to\infty\,.$$

Also (2.6) can be replaced by $\Lambda=O(\mathcal{E})$ if $|D\varphi|$ is small enough.

From this lemma, it follows that $\bar{P}w<0$ (for suitable f and δ) provided $\mathcal{E}_d$ or $-W_2$ is sufficiently large and positive. In particular if $1=O(\mathcal{E})$ (so that $(f')^2\mathcal{E}_d$ is bounded from below by a constant as $f'\to\infty$) or if φ is independent of time (so that the term $\Phi f^{\gamma-2}$ is not present in (2.8)), then the calculations of [21] (see also [9] and [29]) carry over without essential charge to the parabolic setting.

THEOREM 2.2. *Let $\varphi\in H'_\gamma$ for some $\gamma\in(1,2]$ and suppose that for each*

$x_0 \in \partial D$, there is a ball of fixed radius whose closure meets $\bar{D}$ only at x_0: If also

$$(2.6)' \qquad |p|\Lambda = O(\mathcal{E}),$$

$$(2.13) \qquad 1 = O(\mathcal{E}) \quad \text{or} \quad \varphi_t \equiv 0 \quad \text{on } S\Omega,$$

$$(2.14) \qquad |a| = O(\mathcal{E}),$$

then any solution $u \in C^1(\bar{\Omega}) \cup C^{2,1}(\Omega)$ of (1.1) obeys the estimate

$$(2.15) \qquad |Du|_{0;S\Omega} \leqslant K.$$

PROOF. We take as comparison surface $\mathcal{S}^*$ the surface of the ball in the hypotheses, and we choose $\delta_0 = 1$. With $\bar{a}(X, z, p) = a(X, u(X), p)$ and w given by (2.2), it is readily checked that $W_2 = O(\mathcal{E})$. The proof is completed by using (2.13) to estimate $\Phi f^{\gamma-2}$ and then using Lemma 2.1 to obtain $\bar{P}w < 0$. ■

We note that here and in the rest of this section, condition (2.13) can be localized to

$$(2.13)' \quad 1 = O(\mathcal{E}) \text{ in a neighborhood of any point where } \varphi_t \neq 0.$$

If $\gamma = 2$, we can use a comparison principle based on Krylov's maximum principle [15, Theorem 3.1] for $W^{2,1}_{n+1}$ solutions and the assumption that φ is defined in Ω to relax (2.13) to the pointwise estimate

$$(2.13)'' \qquad |\varphi_t| = O(\mathcal{E}).$$

Moreover the $O(\mathcal{E})$ term in (2.6)$'$, (2.13), and (2.14) can be replaced by $O(|p|h(|p|)\,\mathcal{E})$ where

$$h \text{ is positive and non-increasing on } (0, \infty),$$

$$|p|h(|p|) \geqslant 1 \quad \text{for } |p| \geqslant 1,$$

$$\int_1^\infty \frac{1}{\tau^2 h(\tau)}\, d\tau = \infty,$$

since in this case the differential equation (2.12a) is replaced by

$$f'' + \mu_3 (f')^3 h(\tfrac{1}{2}Ff') f^{\gamma-2} = 0,$$

which is also solvable under our hypothesis (see [26, Section 2] for details).

Thus our Theorem 2.2 includes [2, Theorem 8] and [33, Theorem 6] as special cases. (In fact the version of (2.13) used in [2] is more restrictive than ours.) In comparing our results to those in [2], [12], and [33], we shall not mention these additional considerations again although the results in those references are stated in terms of the more general structure.

If the ball in Theorem 2.2 is replaced by any H_θ surface with $1 < \theta \leqslant 2$ and if we strengthen (2.6)′ to

$$|p|^{3-\theta} \Lambda = O(\mathcal{E}) , \tag{2.6}''$$

then, by introducing the regularized distance ϱ of Lemma 1.2 in place of the ordinary distance d, (2.12a) is replaced by

$$f'' + \mu_3 (f')^{\theta'} \varrho^{\theta' - 2} = 0 , \tag{2.12a}'$$

where $\theta' = \min\{\gamma, \theta\}$ and this equation is solvable under our hypotheses (see [21, Sect. 2.1 and 2.2] for detainls). Similarly (2.14) can be replaced by

$$|a| = O(1 + |p|^{\theta-2} d^{\theta-2})\, \mathcal{E} , \tag{2.14}'$$

since we are again led to (2.12a)′. Moreover the hypotheses (2.6)″, (2.13), (2.14)′ are invariant under a change of variables

$$(x, t) \to (\bar{x}(x, t), t)$$

for $\bar{x} \in H_\theta$, so we infer a boundary gradient estimate in this case for non-cylindrical H_θ domains also.

For convex domains a boundary gradient estimate follows under weaker hypotheses.

THEOREM 2.3. *Let $\varphi \in H'_\gamma$ for some $\gamma \in (1, 2]$ and suppose Ω is convex. If* (2.6), (2.13), *and* (2.14) *hold, then any solution u of* (1.1) *satisfies* (2.15).

PROOF. Now we use a supporting hyperplane for S^* and take $\delta_0 = 1$. With $\bar{a}$ as in Theorem 2.2, we infer from (2.14) and the linearity of d that $W_2 = O(\mathcal{E})$. ■

This theorem generalizes [33, Theorem 5]. An alternative proof is provided by introducing a concave regularized distance in D from [23, Theorem 1.4] and proceeding as in [21, Sect. 2.3].

By strengthening the geometric restriction on Ω, we can relax the growth restriction on a. We say that D satisfies an enclosing sphere condition of radius R at $x_0 \in \partial D$ if there is a ball of radius R containing D such that x_0 is on the surface of this ball. If D satisfies an enclosing sphere condition of radius R at every point of ∂D, we say that D is R-uniformly convex.

THEOREM 2.4. *Let D be R-uniformly convex and let $\varphi \in H'_\gamma$ for some $R > 0$, $\gamma \in (1, 2]$. Suppose* (2.6), (2.13), *and*

$$|a| \leqslant \frac{|p|\mathcal{T}}{R} + O(\mathcal{E}), \tag{2.16}$$

where $\mathcal{T}$ is the trace of (a^{ij}). Then (2.15) *is valid for any solution u of* (1.1).

PROOF. Fix $x_0 \in \partial D$, let $\mathcal{S}^*$ be the (surface of the) enclosing sphere, and set $\delta_0 = R/2$. From (2.9) we have

$$\big||Dw| - (F + \bar{\varphi}_0)f'\big| \leqslant \Phi, \tag{2.17}$$

and by direct computation we have

$$\bar{a}^{ij} D_{ij} d = (\mathcal{E}_d - \overline{\mathcal{T}})/|x - y|$$

where y is the center of the ball. Thus, since $R/2 \leqslant |x - y| \leqslant R$, we have

$$\begin{aligned} W_2 &\leqslant -(F + \bar{\varphi}_0)f'\,\overline{\mathcal{T}}/R + |Dw|\overline{\mathcal{T}}/R + (F + \bar{\varphi}_0)f'\,\mathcal{E}_d/|x - y| + O(\bar{\mathcal{E}}) \\ &\leqslant \Phi\overline{\mathcal{T}}/R + O\big((f')^2\mathcal{E}_d\big) = O\big((f')^2\mathcal{E}_d\big)\,. \end{aligned}$$

By virtue of Lemma 2.1 this estimate gives the boundary gradient estimate. ■

Let us observe that when φ is Lipschitz with respect to t, we can incorporate the Lipschitz quotient $\Phi_1 = \langle\varphi\rangle_{2;S\Omega}$ into (2.16) by using (2.8)′ in place of (2.8).

COROLLARY 2.5. *Let D be R-uniformly convex and let $\varphi \in H'_\gamma$ for some $R > 0$ and $\gamma \in (1, 2]$. If* (2.6) *holds and*

$$\Phi_1 + |a| \leqslant \frac{|p|\mathcal{T}}{R} + O(\mathcal{E}) \tag{2.16$'$}$$

or if $\gamma = 2$ and

$$\Phi_1 + |a| \leqslant \frac{|p|\mathcal{T}}{R'} + O(\mathcal{E}) \tag{2.16$''$}$$

for some $R' > R$, then (2.15) *is valid for any solution u of* (1.1).

Proof. When (2.6) and (2.16)′ hold, we replace (2.8) by (2.8)′ in the proof of Theorem 2.4. When $\gamma = 2$ and (2.16)″ holds, we have

$$W_2 < |Dw|\overline{\mathcal{C}}/R + \Phi\overline{\mathcal{C}}/R + O((f')^2\mathcal{E}_d) + \Phi_1 + \bar{a}$$

and hence (by using (2.11) in place of (2.10))

$$\bar{P}w \leqslant |Dw|\overline{\mathcal{C}}\left(\frac{1}{R'} - \frac{1}{R}\right) + \Phi\overline{\mathcal{C}}/R + 2\overline{\mathcal{C}}\Phi < 0$$

for f' sufficiently large. ■

Corollary 2.5 with (2.16)″ generalizes [12, Theorem 2.1] in which (2,16)″ was strengthened to

$$|a| = o(|p|\Lambda) + O(\mathcal{E}),$$

$$\varphi_t = 0 \quad \text{or} \quad 1 = O(\mathcal{E}) \quad \text{or} \quad |p|\Lambda \to \infty.$$

The geometric restrictions on D we have considered so far are simple conditions on the curvatures of the surface $\mathcal{S}^*$. More general conditions on the curvatures were given by Serrin [29, Sect. 9]. We use the modification [8, (14.43)] of these conditions (see also [31, pp. 850-852]). Suppose for $|p| \neq 0$ the coefficients of P can be decomposed so that

(2.18a) $\quad a^{ij}(X, z, p) = a^{ij}_\infty\left(X, \dfrac{p}{|p|}\right)\Lambda(X, z, p) + a^{ij}_0(X, z, p)$,

(2.18b) $\quad a(X, z, p) \; = a_\infty\left(X, z, \dfrac{p}{|p|}\right)|p|\Lambda(X, z, p) + a_0(X, z, p)$,

(2.18c) $\quad (a^{ij}_\infty)$ is positive semidefinite on $\bar{\Omega} \times S^{n-1}$,

(2.18d) $\quad a_\infty$ is non-increasing in z for fixed $(X, \zeta) \in \tilde{\Omega} \times S^{n-1}$,

(2.18e) $\quad a^{ij}_\infty$ and a_∞ are continuous functions of $(x, \zeta) \in \bar{D} \times S^{n-1}$,

where $S^{n-1} = \{\zeta \in \mathbb{R}^n : |\zeta| = 1\}$, and define

$$\sigma(X, p) = |p|\Lambda_0(X, u(X), p) + |a_0(X, u(X), p)|,$$

Λ_0 being an upper bound on the magnitude of the eigenvalues of (a^{ij}_0).

For $x_0 \in \partial D$, we now write $\mathcal{S}^*(x_0)$ and $\delta_0(x_0)$ to make explicit the dependence on x_0, and we denote by $\partial' D$ the set of all ordered pairs $(y, x_0) \in \partial D \times \partial D$

for which $|y-x_0|<\delta_0(x_0)$. For $(y, x_0)\in\partial' D$, we define $\nu(y)=Dd(y)$,

$$\text{(2.19a)}\quad \varkappa^{\pm}(y,t;x_0)=a^{ij}_{\infty}(y,t,\pm\nu(y))\,D_{ij}d(y)\pm a_{\infty}(y,t,\varphi(y,t),\pm\nu(y)),$$

$$\text{(2.19b)}\quad \varkappa_0=-\sup_{\partial' D\times(0,T)}\varkappa^{\pm}(y,t,x_0)\,.$$

Choosing

$$\bar a(X,z,p)=a_{\infty}\left(X,z,\frac{p}{|p|}\right)|p|\Lambda(X,u,(X),p)+a_0(X,u,p)$$

leads to additional boundary gradient estimates.

THEOREM 2.6. *Let $\varphi\in H'_{\gamma}$ for some $\gamma\in(1,2]$, suppose P can be decomposed according to* (2.18), *and suppose* (2.6) *and* (2.13) *hold. Define $\varkappa_0$ by* (2.19) *and suppose that*

(2.20a) *a^{ij}_{∞} and a_{∞} are Lipschitz functions of (x,ζ) uniformly in (t,z),*

$$\text{(2.20b)}\quad \varkappa_0\geqslant 0\,,\qquad \sigma=O(\mathcal{E})\,,$$

or

$$\text{(2.20)}'\quad \varkappa_0>0\,,\qquad \sigma=o(|p|\Lambda)+O(\mathcal{E})\,.$$

Then any solution u of (1.1) *satisfies* (2.15).

PROOF. Fix $x_0\in\partial D$, and for $x\in D$ with $|x-x_0|<\delta_0$ denote by y the nearest point to $\mathcal{S}^*$ on the intersection of ∂D and by ν the normal to $\mathcal{S}^*$ through x. Note that y is uniquely determined and that $\nu(y)=Dd(y)=Dd(x)$.

From (2.17) we have

$$\text{(2.21)}\quad \big|Dw-|Dw|\,d\big|\leqslant|Dw-(F+\bar\varphi_0)f'Dd|+\big||Dw|-(F+\bar\varphi_0)f'\big|\leqslant 2\Phi\,,$$

so (2.5) implies that

$$\left|\frac{Dw}{|Dw|}-Dd\right|\leqslant\frac{2\Phi}{|Dw|}\leqslant\frac{2\Phi}{f'}\,.$$

To obtain our boundary estimate, we compute:

$$\begin{aligned}W_2&=(F+\bar\varphi_0)f'\Lambda a^{ij}_{\infty}D_{ij}d+|Dw|\bar\Lambda a_{\infty}+(F+\bar\varphi_0)f'a^{ij}_0D_{ij}d+a_0\\&\leqslant|Dw|\bar\Lambda[a^{ij}_{\infty}D_{ij}d+a_{\infty}]+\Phi\bar\Lambda|a^{ij}_{\infty}D_{ij}d|+c_1\sigma\end{aligned}$$

for $c_1=\sup\sum_{i,j=1}^{n}|D_{ij}d|+1$. The key step is an estimate of the term in square brackets:

$$S = a^{ij}_{\infty}\left(X, \frac{Dw}{|Dw|}\right) D_{ij} d(x) + a_{\infty}\left(X, w, \frac{Dw}{|Dw|}\right)$$
$$\leqslant a^{ij}_{\infty}\left(X, \frac{Dw}{|Dw|}\right) D_{ij} d(y) + a_{\infty}\left(x, w, \frac{Dw}{|Dw|}\right) \quad \text{by [6, (14.99)]}.$$

If (2.20a) holds, then there is a constant c_2 such that

$$S \leqslant a^{ij}_{\infty}(y, t, \nu(y)) D_{ij} d(y) + a_{\infty}(y, t, w(X), \nu(y)) + c_2\left(\left|\frac{Dw}{|Dw|} - Dd\right| + |x-y|\right)$$
$$\leqslant -\varkappa_0 + a_{\infty}(y, t, w(X), \nu(y)) - a_{\infty}(y, t, (y, t), \nu(t)) + c_2\left(\frac{2\Phi}{f'} + |x-y|\right).$$

Now $|x-y| = d(x) - d(y) \leqslant d(x)$, so

$$\text{(2.22)} \quad w(X) = \bar{\varphi}(X, f\circ d(x)) + Ff\circ d(x)$$
$$\leqslant \bar{\varphi}(y, t, 0) - \Phi|x-y| - \Phi f\circ d(x) + Ff\circ d(x) \leqslant \bar{\varphi}(y, t, 0) = \varphi(y, t).$$

Also $f'(d)\,d \leqslant f(d) \leqslant 1$ since $f'' < 0$. Using this inequality, (2.22), and the monotonicity of a_{∞} with respect to z yields

$$S \leqslant -\varkappa_0 + c_2(2\Phi + 1)/f'.$$

Thus

$$W_2 \leqslant -|Dw|\bar{\Lambda}\varkappa_0 + \Phi\bar{\Lambda}|a^{ij}_{\infty} D_{ij} d| + c_2(2\Phi + 1)\bar{\Lambda}|Dw|/f' + c_1\sigma = O(\bar{\varepsilon})$$

which leads to a boundary gradient as before.

On the other hand if (2.20)$'$ holds, we can imitate the preceding calculations to infer that

$$S \leqslant -\varkappa_0 + o(1) \qquad \text{as } f' \to \infty,$$

and hence

$$W_2 \leqslant -|Dw|\bar{\Lambda}\varkappa_0 + o(|Dw|\bar{\Lambda}) + O(\bar{\varepsilon}).$$

Since $\varkappa_0 > 0$, we therefore have $W_2 \leqslant O(\bar{\varepsilon})$ for f' large enough. ∎

When $\gamma = 2$ and $\mathcal{S}^* = \partial D$, Theorem 2.6 coincides with [33, Theorem 7]. As with Corollary 2.5, Theorem 2.6 holds without assuming (2.6) if $\gamma = 2$ and (2.20)$'$ is satisfied. Also we can eliminate (2.13) from the hypotheses (for $\gamma \in (1, 2]$) if $\langle\varphi\rangle_{2;\mathcal{S}\Omega}$ is finite and if (2.20) (or (2.20)$'$) is suitably augmented.

COROLLARY 2.7. *Let* $\varphi \in H'_\gamma$ *for some* $\gamma \in (1, 2]$, *suppose* P *can be decomposed according to* (2.18), *and suppose* (2.6). *If* $\langle\varphi\rangle_{2;S\Omega} = \Phi_1$ *is finite and if either* (2.20) *and*

$$\Phi_1 \leqslant \varkappa_0 |p| \Lambda + O(\mathcal{E}) \tag{2.23}$$

or (2.20)′ *and*

$$\Phi_1 \leqslant \varkappa_0 \liminf_{\substack{|p|\to\infty \\ X\in S\Omega, |z|\leqslant M}} |p|\Lambda(X, z, p) \tag{2.23$'$}$$

are satisfied, then (2.15) *is valid for any solution* u *of* (1.1).

Again, when $\gamma = 2$ and (2.20)′, (2.23)′ hold, this corollary is true without (2.6).

We also point out that a simple variant of the decomposition (2.18), described in [8, Problems 14.2-14.4], allows us to recover our previous results as special cases of Theorem 2.6 or Corollary 2.7. In particular, Theorem 2.2 corresponds to Theorem 2.6 with (2.20) holding and a_∞^{ij}, a_∞ all zero.

Elliptic analogs of the theorems in this section appear in [8], [21], [26]. [29], and [33]; the basis of all the results in these works is, of course, [29], Moreover all the boundary gradient estimates of these works have simple analogs in the present work when (2.13) holds as well as more subtle ones like Corollaries 2.5 and 2.7 when (2.13) is not assumed.

The barrier constructions of this section generalize easily to non-Hölder moduli of continuity. As in [26] a boundary gradient estimate is valid for Dini moduli of continuity and a boundary Hölder estimate with arbitrary exponent in $(0, 1)$ is valid for non-Dini moduli. Also, by virtue of the local nature of our arguments, we can obtain boundary gradient estimates for equations with behavior appropriate to diDerent theorems of this section at different points of $S\Omega$; see [29, Sec. 11].

We also point out that if $|a| \leqslant \mu t^{(\gamma-2)/2}$ for some positive constant μ, a boundary gradient estimate can be obtained by adding

$$(2\mu/\gamma)(t^{\gamma/2} - t_0^{\gamma/2})$$

to our (upper) barrier and applying Lemma 1.1 to

$$\varphi + (2\mu/\gamma)(t_0^{\gamma/2} - t^{\gamma/2})$$

when determining $\tilde{\varphi}$. Moreover, by virtue of the local nature of our arguments and the comments after Lemma 1.1, a boundary gradient estimate is valid if we only assume that φ is smooth and u is bounded whenever

$t \geqslant \varepsilon > 0$ with the hypothesized bounds depending on ε; of course the boundary gradient estimate is also dependent on ε.

The estimates of this section apply to fully nonlinear equations using the methods and ideas of Futev [4]. We indicate briefly the necessary modifications when

$$Pu = -u_t + \mathcal{F}(X, u, Du, D^2 u) .$$

The parabolicity of P is expressed in terms of positive functions $\lambda_0(X, z, p, r)$ and $\Lambda_0(X, z, p, r)$ such that

$$\lambda_0 \operatorname{tr}(\eta) < \mathcal{F}(X, z, p, r + \eta) - \mathcal{F}(X, z, p, r) \leqslant \Lambda \operatorname{tr}(\eta)$$

for all positive semidefinite matrices η. We then set

$$\mathcal{E}(X, z, p) = \mathcal{F}(X, z, p, p \otimes p) - \mathcal{F}(X, z, p, 0) ,$$

$$a(X, z, p) = \mathcal{F}(X, z, p, 0) ,$$

and we assume there is a function $\Lambda(X, z, p)$ such that

$$\Lambda_0(X, z, p, \pm mp \otimes p) \leqslant \Lambda(X, z, p)$$

for m and $|p|$ sufficiently large. Fixing $c_1 > 0$ and defining

$$\bar{P}v = -v_t + \mathcal{F}(X, u, Dv, D^2 v) ,$$

we infer from the proof of Lemma 2.1 and (2.21) that

$$\begin{aligned} (2.24) \qquad P\bar{w} \leqslant -\bar{\varphi}_t + O(|p|\Lambda + |a| + \Lambda f^{\gamma-2}) + \\ + \mathcal{F}(X, u, Dw, -c_1 f^{\gamma-2} Dw \otimes Dw) - \mathcal{F}(X, u, Dw, 0) \end{aligned}$$

for a suitable choice of f in our barrier w. Under the hypotheses of Theorem 2.2, we therefore have

$$\bar{P}w \leqslant \mu f^{\gamma-2} \mathcal{E} + \mathcal{F}(X, u, Dw, -c_1 f^{\gamma-2} Dw \otimes Dw) - \mathcal{F}(X, u, Dw, 0)$$

If we make the additional assumption that

$$(2.25) \qquad \mathcal{F}(X, u, p, -mp \otimes p) - \mathcal{F}(X, u, p, 0) \leqslant -Km\mathcal{E}$$

for some position constant K and all large m and p, we obtain $\bar{P}w < 0$ with $c_1 = 2\mu/K$. Moreover we can replace Km by $K(m)$, with K an unbounded increasing function in (2.25) when $\gamma = 2$. Similarly Theorem 2.3 holds in this case because the term $|p|\Lambda$ is not present in (2.24). For the remaining theorems in this section, we refer the reader to [4].

We close this section by observing that in one space dimension, only Theorem 2.2 (or 2.3 which is equivalent because $\lambda = \Lambda$ and D is convex in this case) is applicable. Thus we have boundary gradient estimates for one-dimensional problems provided 1 and a are not too large relative to $\lambda = \Lambda = a^{11}$, or $\varphi_t = 0$ and a is not too large. A supposed counter-example to this last assertion was given in [12, Theorem 5.2] but we have been unable to make sense of the construction in the English version of that work.

3. – Non-existence results.

We now show that (1.1) ceases to be solvable for arbitrary $\varphi \in H_\gamma$ when our structure and geometric conditions are violated. From the proof of these results it follows that (1.1) is not solvable in this case even for arbitrary $\varphi \in C^\infty$.

As with other results of this type, our starting point is a simple variant, Lemma 3.1, of the comparison principle used in Section 2. Our comparison functions are, in fact, taken almost directly from [8, Sect. 14.4], which is based on [29, Chapt. III]; the only difference between the functions there and here is that we add on a suitable multiple of t. Thus our results can hardly be considered new. Nonetheless they seem not to have been noticed before except in a few simple situations ([2, Theorem 17], [12, Theorem 5.1], and [28, Sect. 3]) where the precise relation between the time-dependence of φ and the solvability of (1.1) is obscured.

Throughout this section $\bar{P}$ is as in Section 2 with $\bar{a}(X, z, p) = a(X, u, p)$, and L, M, θ, β denote positive constants with $\beta = \theta/(1 + \theta)$.

We now state our comparison principle, which is essentially [2, Theorem 16] (see also [8, Theorem 14.10] and [29, p. 459] for elliptic versions).

LEMMA 3.1. *Let Γ^* be a relatively open C^1 portion of ∂D with inner normal ν, and set $\Gamma = \Gamma^* \times (0, T)$. If $u \in H_0(\Omega) \cap C^{2,1}(\Omega \cup \Gamma)$ and $v \in H_0(\Omega) \cap C^{2,1}(\Omega)$ satisfy $\bar{P}v < Pu$ in Ω, $v \geqslant u$ on $\mathcal{P}\Omega/\Gamma$, and $\partial v/\partial\nu = -\infty$ on Γ, then $v \geqslant u$ in Ω.*

Let $\delta = \operatorname{diam} D$, $\alpha \in (0, \delta/2)$, and $y \in \partial D$. For $\psi \in C^2(\alpha, \delta)$ and m, $\varkappa$ constants to be chosen, set

$$w(X) = m + \psi(|x - y|) + \varkappa t\,. \tag{3.1}$$

Suppose that $\psi(\delta) = 0$, $\psi'(\alpha) = -\infty$, and $\psi > 0$ on (α, δ). If $\bar{P}w < 0 = Pu$ on the set where $|x - y| > \alpha$ and $|u| > M$, then we choose

$$m = \sup\{u(X) : X \in \mathcal{S}\Omega,\ |x - y| > \alpha\}$$

and $v = w$ in Lemma 3.1 to infer that

$$\sup\{u(X) : X \in \Omega,\ |x - y| = \alpha\} \leqslant \max\{M, m + \psi(\alpha) + \varkappa T\}\,. \tag{3.2}$$

Suppose there is a non-negative constant η such that

$$-\eta + a \leqslant -|p|^\theta \mathcal{E} \quad \text{for } |z| > M,\ |p| > L\,. \tag{3.3}$$

(Note that if also $a \leqslant 0$ and $\eta > 0$, then for any $\eta' \in (0, \eta)$ we have

$$-\eta' + a \leqslant |p|^{\theta/2}(-\eta + a) \leqslant -|p|^{\theta/2}\mathcal{E} \text{ for } |z| > M, \quad |p| > L + (\eta/\eta')^{2/\theta} + 1\,.$$

Thus, in this case, we can assume η to be arbitrarily small.) Then a suitable ψ is given by

$$\psi(r) = K[(\delta - \alpha)^\beta - (r - \alpha)^\beta]$$

for sufficiently large K provided we take $\varkappa = \eta$. Hence (3.2) is valid for this ψ and $\varkappa$.

Now suppose there is a ball with center x_0 and radius R lying entirely in D with y on the boundary of the ball. For $\varepsilon \in (0, R)$ and $\chi_\varepsilon \in C^2(0, R - \varepsilon)$ set

$$w^*(X) = m^* + \chi_\varepsilon(|x - x_0|) + \varkappa^* t\,. \tag{3.1$'$}$$

Suppose also that $\chi_\varepsilon(0) = 0$, $\chi'_\varepsilon(R - \varepsilon) = \infty$, and $\chi''_\varepsilon > 0$ on $(0, R - \varepsilon)$. If $\bar{P}w^* < 0 = Pu$ where $|u| > M$, $|x - y| < \alpha$, $|x - x_0| < R - \varepsilon$ and if

$$m^* = \sup\{u(X) : X \in \Omega \text{ and } |x - y| = \alpha, \quad \text{or } X \in B\Omega \text{ and } |x - y| < \alpha\},$$

then Lemma 3.1 with $v = w^*$ yields

$$\begin{aligned} \sup\{u(X) : X \in \Omega, \quad |x - y| < \alpha, \quad |x - x_0| = R - \varepsilon\} \\ \leqslant \max\{M, m^* + \chi_\varepsilon(R - \varepsilon) + \varkappa^* T\}\,. \end{aligned} \tag{3.2$'$}$$

When also $\chi_\varepsilon(R-\varepsilon)<\chi_0$ for some constant χ_0 independent of ε, we can send ε to zero so

$$(3.2)'' \qquad \sup_{0<t<T} u(y,t) \leqslant \max\{M, m^* + \chi_0 + \varkappa^* T\}\,.$$

In particular if there are constants $\Phi_1 \geqslant 0$ and $R' \in (0, R)$ such that

$$(3.3)' \qquad -\Phi_1 + a + \frac{|p|\mathcal{C}}{R'} \leqslant -|p|^\theta \mathcal{E} \qquad \text{for } |z|>M\,,\ |p|>L$$

then a suitable χ_ε is given by

$$(3.4) \qquad \chi_\varepsilon(r) = K[(R-\varepsilon)^\beta - (R-\varepsilon-r)^\beta]$$

for K sufficiently large provided $\varkappa^* = \Phi_1$, $R - R' > \varepsilon$, and $\alpha > \varepsilon$. (There seems to be some confusion over this point on p. 349 of [6].) Hence (3.2)$''$ holds with $\chi_0 = KR^\beta$. Since (3.3)$'$ implies (3.3) (with η arbitrarily small if $\Phi_1 > 0$ and $\eta = 0$ if $\Phi_1 = 0$), it follows from (3.2) and (3.2)$''$ with $\alpha = (R - R')/2$ that

$$u(y,t) \leqslant \max\{M, m + K[(\delta - R)^\beta + R^\beta] + (\Phi_1 - \eta)T\} \qquad \text{for } t \in (0, T)\,.$$

Thus $\varphi(y,t)$ cannot be prescribed arbitrarily. Moreover by repeating the arguments above with $-u$ in place of u, we can relax (3.3)$'$ to

$$(3.5) \qquad \Phi_1 + |a| \geqslant \frac{|p|\mathcal{C}}{R'} + |p|^\theta \mathcal{E} \qquad \text{for } |z|>M\,,\ |p|>L$$

provided a does not change sign on this set.

These considerations lead to our first non-existence result, which corresponds to the elliptic results [8, Theorem 14.11] and [29, Theorem 16.1] and which extends the parabolic results [2, Theorem 17] and [12, Theorem 5.1].

THEOREM 3.2. *Let D be a bounded domain and let R be the radius of the largest ball contained in D. If there are constants Φ_1 and $R' \in (0, R)$ for which* (3.5) *holds and if a does not change sign when $|z|>M$, $|p|>L$, then there is $\varphi \in C^\infty$ for which* (1.1) *is not solvable. Moreover for any $\varepsilon > 0$ this φ can be chosen so that $|\varphi_t| \leqslant \Phi_1 + \varepsilon$ on S. If $\Phi_1 = 0$, we can choose φ to be time independent.*

Thus Theorem 2.2-2.4 are sharp in that $\mathcal{E}$ cannot be replaced by $|p|^\theta \mathcal{E}$ in (2.6)$'$, (2.13), (2.14), (2.16) and R cannot be replaced by a smaller number in (2.16). Also in Corollary 2.5, $|\varphi_t|_0$ cannot be increased beyond the bounds given by (2.16)$'$ and (2.16)$''$.

Note that we could have used the corresponding comparison functions from [29, Chapt. III] in place of w and w^*, and hence $|p|^\theta$ in (3.5), and in the corresponding conditions to come, can be replaced by $h(|p|)$ for any positive, non-decreasing h with

$$\int_L^\infty \frac{1}{\tau h(\tau)}\, d\tau < \infty .$$

We also note an immediate corollary for the case $n = 1$.

COROLLARY 3.3. *Let $D \subset \mathbf{R}^1$ be an interval. If*

$$\Phi_1 + |a| \geqslant |p|^{\theta+2} a^{11} \quad \text{for } |z| > M,\ |p| > L \tag{3.6}$$

and if a does not change sign, then there is $\varphi \in C^\infty$ for which (1.1) *is not solvable. Moreover for any $\varepsilon > 0$ we can take $|\varphi_t| \leqslant \Phi_1 + \varepsilon$ and, if $\Phi_1 = 0$, we can take φ to be time independent.*

For the time-dependent prescribed mean curvature equation

$$u_t = \frac{1}{n} \mathcal{M}u + h(x)$$

where

$$\mathcal{M}u = \operatorname{div}\left((1 + |Du|^2)^{-\frac12} Du\right),$$

Marcellini and Miller [28, Sect. 3] prove a version of Corollary 3.3 for time-independent boundary values.

Suppose now that the decomposition (2.18) for P is valid with a_∞ inhependent of z and $\sigma = o(|p|\Lambda)$. (The following argument can be modified as in [29, Sect. 18] to allow a_∞ to depend on z.) Suppose also that

$$a \leqslant 0, \quad \mathcal{E} \leqslant \Lambda |p|^{1-\theta} \quad \text{for } |z| > M,\ |p| \geqslant 0 . \tag{3.7}$$

It is a simple calculation to check that $\bar{P}w < 0$ if w is given by (3.1) with

$\varkappa = 0$ and

$$\psi(r) = \beta^{-\beta}\int_r^\delta [\log(\tau/\alpha)]^{-\beta}\, d\tau$$

so (3.2) holds. We now define $\varkappa^-$ by (2.19a) with $\mathcal{S}$ as in Section 2 except that $\mathcal{S}^* \subset \bar{D}$. For R be chosen and χ_ε as for (3.1)* we set

$$w^{**} = m^{**} + \chi_\varepsilon(R - d(x)) + \varkappa^{**} t\,.$$

For a fixed t, we see that

$$Pw^{**} \leqslant -\chi_\varepsilon' \Lambda[\varkappa^-(y,t) + o(1) + \chi_\varepsilon''(\chi_\varepsilon')^{-2-\theta}] - \varkappa^{**}$$

where the $o(1)$ term tends to zero as $R + 1/\chi_\varepsilon' \to 0$. If χ_ε is given by (3.4), then, for any $\eta > 0$ and K sufficiently large depending on η,

$$\bar{P}w^{**} \leqslant -\chi_\varepsilon' \Lambda(\varkappa^-(y,t) - \eta) - \varkappa^{**}\,.$$

Setting

$$\Lambda_\infty = \limsup_{|p|\to\infty} |p|\Lambda\,,$$

we infer that $\bar{P}w^{**} < 0$ if

$$\Lambda_\infty \varkappa^-(y,t) + \Phi_1 > 0 \text{ or } \varkappa^-(y,t) > 0 \qquad \text{for all } t \in (0, T) \tag{3.8}$$

by taking $\varkappa^{**} = \Phi_1$ or 0, respectively. Hence we obtain (3.2)$''$ with $\varkappa^*$ replaced by $\varkappa^{**}$. We thus have an analog of [8, Theorem 14.12] and [29, Theorem 18.1].

THEOREM 3.4. *Suppose P can be decomposed according to* (2.18) *with $\sigma = o(|p|\Lambda)$ and a_∞ independent of z, and suppose* (3.7) *holds. Define $\varkappa^-$ by* (2.19a) (*with $\mathcal{S}^* \subset \bar{D}$*) *and suppose* (3.8) *holds for some $y \in \partial D$. Then there is $\varphi \in C^\infty$ for which* (1.1) *is not solvable. If the first inequality of* (3.8) *holds, then we can take $|\varphi_t| \leqslant \Phi_1 + \varepsilon$, while if the second inequality holds, φ can be made time-independent.*

Theorem 3.4 remains valid upon replacing a by $-a$ in (3.7) and $\varkappa^-$ by $\varkappa^+$ in (3.8). Also if one of the inequalities in (3.8) is valid only for t in some interval (t_0, t_1), then Theorem 3.4 still applies because we can repeat the proof with t_1 in place of T and $\{X \in \mathcal{P}\Omega : t \leqslant t_0\}$ in place of $B\Omega$ (and similarly for Theorem 3.2). Thus we cannot increase $\varkappa_0$ in Theorem 2.6 beyond the bound given there. Moreover conditions (2.23) and (2.23)$'$ of Corollary 2.7 are sharp in that when $|p|\Lambda$ has a limit as $|p| \to \infty$, $|\varphi_t|_0$ cannot be increased beyond the bounds given there, even for positive $\varkappa_0$.

4. – The other estimates.

For our existence program, we need several estimates in addition to the boundary gradient estimate of Section 2. The simplest ones are bounds on the size of the solution and its gradient. Since these bounds, especially the gradient bound, have been proved under farily general conditions elsewhere (e.g., [2, Sects. 3 and 5], [12, Sect. 3], [22], [30, Sect. 6]), we give them here only in simplified form.

In our first lemma, $C^1(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$ denotes the set of all functions g on $\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$ such that g_x, g_z, g_p (but not necessarily g_t) exist and are continuous on $\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$, and δ is the operator given by $\delta g = g_z + |p|^{-2}p\cdot g_x$.

LEMMA 4.1. *Let $u \in H_0(\Omega) \cap C^{2,1}(\Omega)$ be a solution of* (1.1), *and let μ_0, μ_1, μ_2, μ_3 be non-negative constants.*

(*a*) *If*

$$a(X, z, p)z/|z| \leqslant (\mu_1|p| + \mu_2)\,\mathcal{E}(X, z, p)/|p|^2 + \mu_3(|z| + 1) \qquad \textit{for } |p|, |z| \neq 0\,, \tag{4.1}$$

then

$$|u|_0 \leqslant \exp(\mu_3 T)\Big(|\varphi|_0 + \mu_2 \exp\big((\mu_1 + 1)\operatorname{diam} D\big)\Big)\,. \tag{4.2}$$

(*b*) *If $u \in C^1(\bar{\Omega})$ and if there are functions a^{ij}_* and c^i such that*

$$(a^{ij}_*) \textit{ is positive definite with minimum eigenvalue } \lambda^*\,, \tag{4.3a}$$

$$a^{ij} = a^{ij}_* + \tfrac{1}{2}(p_i c^j + p_j c^i)\,, \tag{4.3b}$$

$$a^{ij}_*,\ c^i,\ \textit{and } a \textit{ are in } C^1(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n)\,, \tag{4.3c}$$

$$\big(|p|^2/4\lambda^*\big)\sum_{i,j}(\delta a^{ij}_*)^2 + \delta a \leqslant \mu_0\,, \tag{4.3d}$$

then

$$|Du|_{0;\Omega} \leqslant \exp(\mu_0 T/2)\,|Du|_{0;\mathcal{P}\Omega}\,. \tag{4.4}$$

PROOF. For (a), we assume first that $\mu_2 > 0$ and that D lies in the slab $0 < x^1 < d_0 = \operatorname{diam} D$. Defining

$$\bar{P}v = a^{ij}(X, u, Dv)D_{ij}v + a(X, u, Dv) + \mu_3(v - u) - v_t,$$

$$v(X) = \exp(\mu_3 t)\Big(|\varphi|_0 + 1 + \mu_2[\exp((\mu_1 + 1)d_0) - \exp((\mu_1 + 1)x^1)]\Big) - 1,$$

it is readily checked (cf. [8, Theorem 10.3]) that

$$\bar{P}v < \bar{P}u \text{ in } \Omega^+ = \{X \in \Omega : u(X) > 0\}, \quad v \geqslant u \text{ on } \mathcal{P}\Omega^+,$$

$$\bar{P}(-v) > \bar{P}u \text{ in } \Omega^- = \{X \in \Omega : u(X) < 0\}, \quad -v \leqslant u \text{ on } \mathcal{P}\Omega^-.$$

Hence by Lemma 1.3, we have $|u| \leqslant v$ in Ω, which implies the desired result for $\mu_2 > 0$. For the case $\mu_2 = 0$, we let μ_2 tend to zero.

For (b), we apply the operator $D_k u D_k$ to the equation $Pu = 0$ and set $v = |Du|^2$. Following the proof of [8, Theorem 15.2], we see that v is a weak solution of the inequality

$$D_i(a^{ij}D_j v) + b^i D_i v + \mu_0 v - v_t \geqslant 0\,;$$

hence $\bar{v} = \exp(-\mu_0 t)v$ is a weak solution of

$$D_i(a^{ij}D_j\bar{v}) + b^i D_i\bar{v} - \bar{v}_t \geqslant 0\,,$$

and thus $|\bar{v}|_0 \leqslant |\bar{v}|_{0;\mathcal{P}\Omega}$ by [19, Theorem III.7.2]. ■

We mention that a global gradient bound can be obtained without assuming $u \in C^1$ by combining the $[u]''_1$ estimate from Section 2 and a suitable interior gradient bound as in [26, Sect. 4]. As the interior gradient bounds have not yet appeared explicitly in a suitable form, we shall not pursue this matter further.

The final estimate needed is a Hölder estimate for Du. For $\varphi \in C^{2,1}(\mathcal{P}\Omega)$ such an estimate was proved by Ladyzhenskaya and Ural'tseva [19, Theorems V.5.2 and VI.2.3] under certain assumptions on the coefficients of P, of which we single out a Lipschitz condition with respect to t. More recently Krylov [16, Theorem 4.2] proved a Hölder estimate for the normal derivative of the solution on $S\Omega$ (again for $C^{2,1}$ data) under slightly different hypotheses, in particular without the Lipschitz condition. Using Krylov's result in Ladyzhenskaya and Ural'tseva's method gives the full Hölder gradient estimate without the Lipschitz condition for $C^1(\bar{\Omega})$ solutions with $C^{2,1}$ data. We use a variant of this combination (cf. the remarks following

[27, Theorem 2.4]) to relax the hypotheses to H_0 solutions with bounded gradient, H_γ data, and even some unboundedness of the coefficients of P as mentioned in Section 2.

In order to apply Krylov's ideas, we make some observations. First by virtue of the global gradient bound, the matrix $(a^{ij}(X)) = (a^{ij}(X, u, Du))$ satisfies

$$\lambda|\xi|^2 \leqslant a^{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \tag{4.5}$$

for all $X \in \Omega$, with known positive constants Λ and λ. Setting

$$L = a^{ij}D_{ij} - D_t\,, \tag{4.6}$$

we see that $|Lu|$ is bounded by a known constant. For our purposes, it is useful (and in fact crucial) to generalize this bound to $|Lu|\,(d^*)^{2-\gamma}$ being boinded. (Recall that $d^*(X) = \min\{\operatorname{dist}(x, \partial D), t^{\frac12}\}$.) We now use the regularized distance ϱ of Lemma 1.2 and the function $\bar\varphi$ of Lemma 1.1 (obtained by taking $\psi \equiv 1$ in the proof) to obtain an extension $\tilde\varphi$ of φ to all of Ω with $\tilde\varphi \in H_2^{(-\gamma)}$ and $|\tilde\varphi|_2^{(-\gamma)} \leqslant C(n, |\varphi|_\gamma)$ by setting $\tilde\varphi(X) = \bar\varphi(X, \varrho(x))$. Replacing u by $u - \tilde\varphi$ and locally flattening ∂D (with ϱ as the x^n-coordinate), we may assume that

$$|Lu| \leqslant \lambda F_1(\min\{x^n, t^{\frac12}\})^{\gamma-2}, \qquad |u| \leqslant Hx^n \quad \text{for } |x| < 1,\ x^h > 0,\ 0 < t < T$$

with known constants F_1 and H.

Also for positive constants τ, ϱ, R we introduce the sets

$$B^+ = \{|x| < 1,\ x^n > 0,\ 0 < t < \tau\}\,,$$
$$G(\varrho, R) = \{|x'| < R,\ 0 < x^n < \varrho R,\ \tau - R^2 < t < \tau\}\,,$$
$$G'(\varrho, R) = \{|x'| < R,\ \varrho R < x^n < 2\varrho R,\ \tau - R^2 < t < \tau\}\,,$$

where $x' = (x^1, \dots, x^{n-1})$. Under these hypotheses, we begin our proof ot the Hölder gradient estimate. Our first step is a comparison result based on [16, Lemma 2.1] and the idea of Caffarelli for elminating Krylov's added independent variables.

LEMMA 4.2. *Let γ, F_1, λ, Λ, R, τ be positive constants with $\Lambda \geqslant \lambda$, $1 < \gamma < 2$, $10R^2 \leqslant \min\{1, \tau\}$ and set $\varrho = \lambda/[2 + (2n+4)\Lambda]$. (So $G(\varrho, 2R) \subset B^+$ and $x^n < t^{\frac12}$ in $G(\varrho, 2R)$.) Define L by* (4.6) *and suppose* (4.5) *holds. If $u \in C^{2,1}(B^+)$*

satisfies

(4.7) $$Lu \leq \lambda F_1 (x^n)^{\gamma-2}, \ u \geq 0 \quad in\ G(\varrho, 2R),$$

then

(4.8) $$\inf_{G'(\varrho,2R)} u/x^n \leq 4 \inf_{G(\varrho,R} u/x^n + 4(2\varrho R)^{\gamma-1} F_1/(\gamma-1).$$

PROOF. We set

$$w_1 = \big(1 - x^n/(2\varrho R)\big)\, x^n + |x'|^2 R^{-2} x^n + (\tau - t) R^{-2} x^n,$$

$$w_2 = [(2\varrho R)^{\gamma-1} - (x^n)^{\gamma-1}] x^n/(\gamma-1),$$

$$A = \inf_{G'(\varrho,2R)} u/x^n,$$

$$w = u - Ax^n + Aw_1/4 + F_1 w_2.$$

A simple calculation shows that

$$Lw \leq 0 \text{ in } G(\varrho, 2R), \quad w \geq 0 \text{ on } \mathcal{P}G(\varrho, 2R),$$

so the maximum principle imples that $w \geq 0$ in $G(\varrho, 2R)$. Since $G(\varrho, R) \subset G(\varrho, 2R)$ and $w_1 \leq 3x^n$ in $G(\varrho, R)$ we have

$$u \geq [A/4 - (2\varrho R)^{\gamma-1} F_1/(\gamma-1)]\, x^n \quad \text{in } G(\varrho, R)$$

which leads easily to the desired result. ■

From Lemma 4.2 we infer a preliminary oscillation estimate for u/x^n (cf. [16, Theorem 4.2]).

LEMMA 4.3. *Let γ, λ, Λ, ϱ, τ, F_1, L be as in Lemma 4.2. If $u \in C^{2,1}(B^+)$ satisfies*

(4.9) $$|Lu| \leq \lambda F_1 (x^n)^{\gamma-1} \quad |u|/x^n \text{ bounded}$$

on the set of all $X \in B^+$ with $x^n < t^{\frac{1}{2}}$, then there are positive constants β and C depending only on γ, λ, Λ, n such that

(4.10) $$\operatorname*{osc}_{G(\varrho,R)} u/x^n \leq C\Big(\tau^{-\beta/2} \operatorname*{osc}_{G(\varrho,(\tau/2)^{\frac{1}{2}})} u/x^n + F_1\Big) R^\beta$$

for any R such that $0 < R^2 \leq \frac{1}{2} \min\{1, \tau\}$.

PROOF. It suffices to prove the inequality when $R^2 \leqslant 1/10 \min\{1, \tau\}$. Fix such an R and for $i = 1, 2$ set

$$m_i = \inf_{G(\varrho, iR)} u/x^n\,, \qquad M_i = \sup_{G(\varrho, iR)} u/x^n\,,$$

$$\Sigma = \{|x'| < R, \quad \varrho R < x^n < \tfrac{3}{2}\varrho R, \quad \tau - 4R^2 < t < \tau - 2R^2\},$$

$$\Sigma' = G'(\varrho, 2R)\,, \quad \Sigma'' = \{X \in G(\varrho, 2R): x^n > \tfrac{1}{2}\varrho R\}\,.$$

Since $|Lu| \leqslant \lambda F_1(\varrho R/2)^{\gamma-2}$ in Σ'', we can apply the weak Harnack inequality [10, Theorem 3.1] to $u - m_2 x^n$ in Σ''. Thus there are positive constants C and $\varkappa$ depending only on λ, Λ and n such that

$$\begin{aligned}\left(\frac{1}{|\Sigma|}\int_\Sigma (u - m_2 x^n)^\varkappa\, dx\right)^{1/\varkappa} &\leqslant C\Big(\inf_{\Sigma'} (u - m_2 x^n) + \varrho^{\gamma-2} R^\gamma F_1\Big)\\ &\leqslant C\Big(\psi_\varrho(m_1 - m_2) + \big(\psi(2\varrho)^{\gamma-1}/(\gamma-1) + \varrho^{\gamma-2}\big)F_1 R^{\gamma-1}\Big)R\\ &\leqslant C(n, \lambda, \Lambda, \gamma)(m_1 - m_2 + F_1 R^{\gamma-1})R\end{aligned}$$

by Lemma 4.2. Adding to this inequality the corresponding one for $M_2 x^n - u$ (which is proved in the same way) and observing that

$$\int_\Sigma (x^n)^\varkappa\, dX = c(n, \varkappa) R^\varkappa |\Sigma|\,,$$

we conclude that

$$M_2 - m_2 \leqslant C(n, \lambda, \Lambda, \gamma)\left(M_2 - m_2 - (M_1 - m_1) + F_1 R^{\gamma-1}\right).$$

Applying [8, Lemma 8.23] completes the proof. ■

Next we estimate $\tau^{-\beta/2} \operatorname*{osc}_{G(\varrho, (\tau/2)^{\frac{1}{2}})} u/x^n$. When $\tau \geqslant \frac{1}{4}$, this quantity can be bounded in terms of H. When $\tau < \frac{1}{4}$, we proceed as in [16, Lemma 2.2] using a barrier argument similar to the one in Lemma 4.2.

LEMMA 4.4. *Let γ, F_1, λ, Λ, τ, L be an in Lemma 4.2 with $\tau < \frac{1}{4}$. If $u \in C^{2,1}(B^+)$ and if there is a non-negative constant H such that*

$$Lu \leqslant \lambda F_1\big(\min\{x^n, t^{\frac{1}{2}}\}\big)^{\gamma-2}, \quad u \geqslant -Hx^n \text{ in } B^+\,, \tag{4.11a}$$

$$u(x, 0) \geqslant 0 \quad \textit{for } |x| < 1\,, \tag{4.11b}$$

then

$$u(X) \geqslant -C(n, \lambda, \Lambda, \gamma)(F_1 + H)\,\tau^{(\gamma-1)/2} x^n \tag{4.12}$$

whenever $(x^n)^2 \leqslant t < \tau$, $|x| < \frac{1}{2}$, $x^n > 0$.

PROOF. We begin by showing that for any $R \in (0, \tau^{\frac{1}{2}}]$ and x_0 with $x_0^n = 0$ and $|x_0| < \frac{1}{2}$, we have

$$u \geqslant -C(n, \lambda, \Lambda, \gamma)(F_1 + H)R^\gamma \tag{4.13}$$

on the set $\Sigma_R = \{|x - x_0| < R,\ 0 < t < R^2,\ x^n > 0\}$. To prove this estimate, let $\varepsilon \in (R, \frac{1}{2}]$ and introduce the functions

$$w_1 = \begin{cases} [(\gamma+1)R^{\gamma-1} - (x^n)^{\gamma-1}]x^n/(\gamma-1) - 2\lambda t^{\gamma/2} & \text{if } x^n < R\,, \\ R^\gamma + R^{\gamma-1}x^n/(\gamma-1) + 2\gamma t^{\gamma/2} & \text{if } x^n \geqslant R\,, \end{cases}$$

$$w_2 = |x - x_0|^2/\varepsilon + (2n+2)\Lambda t/\varepsilon.$$

Defining

$$\Sigma = \{|x - x_0| < \varepsilon, \quad x^n > 0, \quad 0 < t < R^2\}\,,$$

we easily check that $w = u + F_1 w_1 + H w_2$ satisfies

$$Lw \leqslant 0 \text{ a.e. in } \Sigma\,, \quad w \geqslant 0 \text{ on } \mathcal{P}\Sigma\,.$$

Since $w \in W^{2,1}_{n+1}(\Sigma)$, the maximum principle [15, Theorem 3.1] implies that $w \geqslant 0$ in Σ. (Alternatively we can approximate w_1 by suitable $C^{2,1}$ functions and apply the classical maximum principle as in [16]). Therefore $w \geqslant 0$ in Σ_R, and so

$$u \geqslant -C(n, \lambda, \Lambda, \gamma)(F_1 + H)(R^\gamma + R^2/\varepsilon) \text{ in } \Sigma_R\,.$$

We now choose $\varepsilon = 2^{1-\gamma}R^{2-\gamma}$ and observe that $R < \varepsilon < \frac{1}{2}$, $R^2/\varepsilon = 2^{\gamma-1}R^\gamma$ to infer (4.13).

To complete the proof, we denote by H_1 the coefficient of R^γ in (4.13). By taking $R = t^{\frac{1}{2}} = x^n$ in that inequality, we infer that

$$u(X) \geqslant -H_1(x^n)^\gamma \quad \text{if } x^n = t^{\frac{1}{2}} \leqslant \tau^{\frac{1}{2}}\,.$$

For $R \in (0, \tau^{\frac{1}{2}}]$ and $\varepsilon \in (R, \frac{1}{2})$ we define

$$w_3 = [R^{\gamma-1} - (x^n)^{\gamma-1}]x^n/(\gamma-1)\,, \qquad \Sigma^+ = \{X \in \Sigma : x^n < t^{\frac{1}{2}}\}.$$

It is readily checked that

$$\overline{w} = u + (F_1 + H_1)w_3 + Hw_2 + H_1(x^n)^\gamma$$

satisfies $L\overline{w}\leq 0$ in Σ^+, $\overline{w}\geq 0$ on $\mathcal{P}\Sigma^+$, so preceeding as before and recalling the definition of H_1 yields

$$u\geq -C(n, \lambda, \Lambda, \gamma)(F_1+H)(R^\gamma+R^2/\varepsilon) \quad \text{in } \Sigma^+\cap\Sigma_R\,.$$

The proof is now completed by fixing X and then taking $R=(\tau^{(\gamma-1)/2}x^n)^{1/\gamma}$ and $\varepsilon=2^{1-\gamma}R^{2-\gamma}$. ■

Since $\beta\leq\gamma-1$, applying this lemma to u and $-u$ gives the required estimate on $\tau^{-\beta/2}\operatorname*{osc}_{G(\varrho,(\tau/2)^{\frac12})} u/x^n$ when $0<\tau<\min\{T, \frac14\}$. We therefore obtain a Hölder estimate (with exponent β) on u/x^n by virtue of Lemma 4.3 and 4.4. Back in the original domain (and with the original boundary values) we have a Hölder estimate on $(u-\tilde{\varphi})/\varrho$. This estimate implies that the normal derivative $D_\nu u$ exists on $S\Omega$ (in fact the existence follows from Lemma 4.3) and that $D_\nu u$ is Hölder continuous on $S\Omega$; we shall use the sharper estimate.

COROLLARY 4.5. *Let γ, F, K, λ, Λ, T be positive constants with $\Lambda\geq\lambda$ and $1<\gamma\leq 2$. Let $\partial\Omega\in H_\gamma$, define L by* (4.6), *and suppose* (4.5) *holds for all $X\in\Omega$. If $u\in H_0(\Omega)\cap C^{2,1}(\Omega)$ satisfies*

$$|Lu|\leq F(d^*)^{\gamma-2}\,,\qquad |u|+|Du|\leq K \text{ in } \Omega\,, \tag{4.14a}$$

$$|u|_{\gamma;\mathcal{P}\Omega}\leq K, \tag{4.14b}$$

then the normal derivative $D_\nu u$ exists on $S\Omega$. Moreover there are positive constants $\beta=\beta(n, \lambda, \Lambda, \gamma)$ and $C=C(F, K, T, n, \lambda, \Lambda, \gamma)$ and a function $v\in H_2^{(-\beta-1)}(\Omega)$ such that

$$|u-v|\leq C(d^*)^{\beta+1} \text{ in } \Omega\,,\qquad |v|_2^{(-\beta-1)}\leq C\,. \tag{4.15}$$

PROOF. The existence of $D_\nu u$ follows from the preceding remarks. For (4.15) we take β from Lemma 4.3. A simple modification of the proof of [23, Theorem 4.2(b)] (see also [24, Lemma 2.3]) implies that there is a function $v\in H_2^{(-\beta-1)}$ with $|v|_2^{(-\beta-1)}\leq C$ and

$$v=u \text{ on } \mathcal{P}\Omega\,,\qquad D_\nu v=D_\nu u \text{ on } S\Omega$$

because $|D_\nu u|_{\beta;S\Omega}\leq C$. From the remarks preceding this corollary, we infer the first inequality of (4.15) provided $d(x)<t^{\frac12}$, so it remains to establish the inequality

$$|u-v|\leq Ct^{(\beta+1)/2} \text{ on } \Sigma=\{X\in\Omega\colon d(x)>t^{\frac12}\}\,.$$

However for the function v, we have

$$|L(u-v)| \leqslant Ct^{(\beta-1)/2} \text{ in } \Sigma, \qquad |u-v| \leqslant Ct^{(\beta+1)/2} \text{ on } \mathcal{P}\Sigma.$$

The maximum principle applied to $\pm(u-v) + 2Ct^{(\beta+1)/2}$ gives the desired result. ■

The Hölder gradient estimate now follows by combining Corollary 4.5 with the interior Hölder gradient estimate of Ladyzhenskaya and Ural'tseva [19] in the sharp form given in [32]. We carry out the details for operators in the divergence form

$$Pu = \operatorname{div}\left(A(X, u, Du)\right) + B(X, u, Du) - u_t, \tag{4.16}$$

in which case $a^{ij} = \partial A^i/\partial p_j$.

THEOREM 4.6. *Let $\partial\Omega \in H_\gamma$, $\varphi \in H_\gamma(\mathcal{P}\Omega)$ for some $\gamma \in (1, 2]$, and let P have the form* (4.16) *in Ω. If $u \in H_0(\Omega) \cap C^{2,1}(\Omega)$ is a solution of* (1.1) *and if there are positive constants I, λ_K, μ_K such that*

$$|u|_0 + |Du|_0 \leqslant K, \tag{4.17a}$$

$$a^{ij}(X, u, Du)\xi_i\xi_j \geqslant \lambda_K|\xi|^2 \quad \textit{for all } X \in \Omega,\ \xi \in \mathbb{R}^n, \tag{4.17b}$$

$$|A_p| + (d^*)^{\gamma-2}\left(|A_z| + |A_x| + |B|\right) \leqslant \mu_K \textit{ in } \Omega, \tag{4.18}$$

then there are positive constants

$$C = C(\gamma, K, \lambda_K, \mu_K, n, |\varphi|_\gamma, \Omega) \quad \textit{and} \quad \alpha = \alpha(\gamma, \lambda_K, \mu_K, n)$$

such that

$$[Du]_{\alpha;\Omega} \leqslant C. \tag{4.19}$$

PROOF. (In this proof we denote by $c_1, c_2, \ldots$ constants depending only on the same quantities as C in (4.19).) Setting $L = a^{ij}(X, u, Du)D_{ij} - D_t$, we have

$$|Lu| \leqslant c_1(d^*)^{\gamma-2}.$$

Thus for β and v from Corollary 4.5 and $w = u - v$, it follows that

$$|w| \leqslant c_2(d^*)^{\beta+1}, \qquad |Dw| \leqslant c_2.$$

Moreover $u_k = D_k u$, $k = 1, \dots, n$, is a weak solution (see [19, p. 436] or [32]) of the equation

$$D_i(a^{ij} D_j u_k + f_k^i) - D_t u_k = 0 \text{ in } \Omega$$

with f_k^i satisfying $|f_k^i| \leqslant c_3 (d^*)^{\gamma-2}$.

For $r > 0$ and $X_0 \in \mathbb{R}^{n+1}$, denote by $Q(r, X_0)$ the cylinder

$$\{X : |x - x_0| < r,\ t_0 - r^2 < t < t_0\},$$

and for r, R, R' constants satisfying $0 < r < R \leqslant R'$, let $Q(r) = Q(r, X_0)$ and $Q(R) = Q(R, X_0)$ be (concentric) cylinders in $\Omega_{R'}$. Since $|f_k^i| \leqslant c_3 (R')^{\gamma-2}$ in $Q(R)$, [28, Theorem 2.2] implies that there is a constant $\sigma = \sigma(\lambda_K, \mu_K, n) \in (0, 1)$ such that

$$\operatorname*{osc}_{Q(r)} u_k \leqslant (c_4/n)\Big[\operatorname*{osc}_{Q(R)} u_k + R(R')^{\gamma-2}\Big](r/R)^\sigma$$

and hence

$$\operatorname*{osc}_{Q(r)} Du \leqslant c_4\Big[\operatorname*{osc}_{Q(R)} Du + R(R')^{\beta-1}\Big](r/R)^\sigma. \tag{4.20}$$

Furthermore we have $\operatorname*{osc}_{Q(\varrho)} Dv \leqslant c_5 \varrho^\beta$ for $\varrho = r$ or R and therefore

$$\operatorname*{osc}_{Q(r)} Dw \leqslant c_6\Big[\operatorname*{osc}_{Q(R)} Dw + R^\alpha\Big](r/R)^\alpha \tag{4.21}$$

for $\alpha = \min\{\beta, \sigma\}$ and $c_6 = (c_4 + 1)(2c_5 + 1)$.

Now let $\Sigma = Q(R'', X_1)$ be an arbitrary cylinder in Ω_R, and let X, Y be in Σ_ε for some $\varepsilon \in (0, R')$. If $|X - Y| < \varepsilon$, then (4.21) with $r = |X - Y|$ $R = \varepsilon$ gives

$$\varepsilon^\alpha |Dw(X) - Dw(Y)|\, |X - Y|^{-\alpha} \leqslant (2c_6)\left(|Dw|_{0;\Sigma_\varepsilon} + \varepsilon^\alpha\right),$$

an inequality which is obvious if $|X - Y| \geqslant \varepsilon$. So if we multiply this inequality by ε, take the supremum over $\varepsilon > 0$, and set

$$[w]^*_{\alpha+1;\Sigma} = \sup_{\varepsilon > 0} \varepsilon^{\alpha+1}[Dw]_{\alpha;\Sigma_\varepsilon},$$

we obtain

$$[w]^*_{\alpha+1;\Sigma} \leqslant 2c_6\left(|Dw|^{(1)}_{0;\Sigma} + (R'')^{1+\alpha}\right). \tag{4.22}$$

To proceed we note the following interpolation inequality, which is proved just like [8, (6.86)]:

$$|Dw|^{(1)}_{0;\Sigma} \leqslant (2/\mu)\, |w|_{0;\Sigma} + 2^{1+\alpha}\mu^\alpha [w]^*_{\alpha+1;\Sigma} \tag{4.23}$$

for all $\mu \in (0, \frac{1}{2}]$. Combining (4.22) and (4.23), with suitable μ, gives

$$|Dw|^{(1)}_{0;\Sigma} \leqslant c_7\big(|w|_{0;\Sigma} + (R'')^{1+\alpha}\big)\,. \tag{4.24}$$

Now for X_1 and X_2 in Ω, we set $R' = \frac{1}{2}\min\{d^*(X_1), d^*(X_2)\}$. If $|X_1 - X_2| < R'/2$, we use (4.24) with $R'' = R'$ and $X = X_1$ to infer

$$|Dw|^{(1)}_{0;\Sigma} \leqslant c_8 (R')^{1+\alpha}.$$

Next we set $R = R'/2$ and note that $\Sigma_R = Q(R, X_1)$, so

$$|Dw|^{(1)}_{0;\Sigma} \geqslant \tfrac{1}{2} R \operatorname*{osc}_{Q(R,X_1)} Dw\,.$$

Hence

$$\operatorname*{osc}_{Q(R,X_1)} Dw \leqslant 8c_8 R^\alpha,$$

and therefore (4.21) with $r = |X_1 - X_2|$ and $X_0 = X_1$ yields

$$|Dw(X_1) - Dw(X_2)| \leqslant c_6(8c_8 + 1)\, |X_1 - X_2|^\alpha\,.$$

On the other hand if $|X_1 - X_2| > R'/2$, (4.24) implies that

$$|Dw(X_1) - Dw(X_2)| \leqslant |Dw(X_1)| + |Dw(X_2)| \leqslant 4c_7(c_2 + 1)\, |X_1 - X_2|^\alpha\,.$$

The combination of these last two inequalities gives (4.19). ■

The proof of Theorem 4.6 is based on [21, Chapter III] which, in turn, is based on [8, Lemma 6.20] in its use of the rate at which a solution of a differential equation goes to zero near the boundary. A slightly different but equivalent approach is given in [34, Theorem 6.1]. Note that the techniques of [5] cannot be applied directly in the parabolic case without assuming some smoothness of a^{ij} with respect to t; however, these techniques can be combined with the methods described here just as in [26, Sect. 5].

As we have already remarked, (4.19) can be obtained more easily if $|A_z|$, $|A_x|$ and $|B|$ are bounded and if $u \in C^1(\bar{\Omega})$. In this case the estimate on $[D_\nu u]_{\beta;S\Omega}$ implies that $Du \in H_\beta(\mathcal{S}\Omega)$ and [32, Theorem 4.2] gives the estimates. Our approach proves the continuity of Du.

When P is in general form the Hölder gradient estimate is proved in the same way with [32, Theorem 2.3] replacing [32, Theorem 2.2].

THEOREM 4.7. *Let $\partial\Omega \in H_\gamma$, $\varphi \in H_\gamma(\mathcal{S}\Omega)$ for some $\gamma \in (1, 2]$. Let $u \in C(\bar{\Omega}) \cap C^{2,1}(\Omega)$ be a solution of* (1.1). *If there are positive constants K, γ_K, μ_K such that* (4.17) *holds and*

$$|a^{ij}| + |a^{ij}_p| + (d^*)^{\gamma-2}\left(|a^{ij}_z| + |a^{ij}_x| + |a|\right) \leqslant \mu_K \text{ in } \Omega, \tag{4.25}$$

Then (4.19) *holds with α also depending on K.*

Note that the estimates on the derivatives of a^{ij} are only used to obtain (4.20) from [32, Theorem 2.3].

If Ω is a noncylindrical domain with $\partial\Omega \in H_\gamma$ in the sense of [24, Sect. 2], the proofs of Theorems 4.6 and 4.7 are applicable without change. Moreover a global Hölder gradient bound for solutions of fully nonlinear equations follows by using an interior Hölder gradient bound based on the parabolic version of [32, Theorem 5.1] (and the difference quotient arguments of [27] to relax the smoothness of u to $u \in C^{2,1}$) in place of [32, Theorems 2.2 and 2.3]. This bound includes the corresponding result in [17, Theorem 3] with the rest of that result contained in the remarks after Theorem 2.2 and a parabolic version of [34, Theorem 7.2]; the parabolic version of this theorem proceeds via the observations in [30, Sect. 6].

5. – Existence theorems.

We now infer solvability of (1.1) under appropriate conditions from the estimates of Sections 2 and 4. Only a few selected existence results will be given. Suitable parabolic versions of the elliptic results of e.g., [8, Sect. 15.5] and [29, Sect. 14 and Chapt. IV] are also easily obtained.

The basic tools for these results is a consequence of the Leray-Schauder fixed point theorem.

LEMMA 5.1. *Let $\partial\Omega \in H_\gamma$ and $\varphi \in H_\gamma(\mathcal{S}\Omega)$ for some $\gamma \in (1, 2)$. Suppose*

(5.1a) *a^{ij}, a lie in $H_\alpha(\Omega \times \mathbf{R} \times \mathbf{R}^n)$ for some $\alpha \in (0, 1)$,*

(5.1b) *a^{ij} depends Lipschitz continuously on (x, z, p) uniformly in Ω,*

(5.1c) $(a^{ij}) > 0$.

Suppose also that there are functions b^{ij}, b defined on $\Omega \times \mathbf{R} \times \mathbf{R}^n \times [0, 1]$

such that

(5.2*a*) $b^{ij}(\cdot\,;\tau), b(\cdot\,;\tau)$ *lie in* $H_:(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ *for each* $\tau\in[0,1]$,

(5.2*b*) b^{ij} *depends Lipschitz continuously on* (x,z,p) *uniformly in* $\Omega\times[0,1]$,

(5.2*c*) $(b^{ij})>0$,

(5.2*d*) $b^{ij}(\cdot\,;1)=a^{ij}, \quad b(\cdot\,;1)=a$,

(5.2*e*) *as functions of* τ, $b^{ij}(\cdot\,;\tau)$ *and* $b(\cdot\,;\tau)$ *map* $[0,1]$ *continuously into* $H_\alpha(\Omega\times\mathbb{R}\times\mathbb{R}^n)$.

Set

$$P_\tau v = b^{ij}(X,v,Dv;\tau)D_{ij}v + b(X,v,Dv;\tau) - v_t$$

and suppose also that the problem $P_0 v=0$ *in* Ω, $v=0$ *on* $\mathcal{P}\Omega$ *has only the zero solution. If there is a constant* M *such that for every* $\tau\in[0,1]$, *and solution* $v\in H^{(-\gamma)}_{2+\alpha}$ *of* $P_\tau v=0$ *in* Ω, $v=\tau\varphi$ *on* $P\Omega$ *obeys the estimate* $|v|_1\leqslant M$, *then* (1.1) *has a solution* $u\in H^{(-\gamma)}_{2+\alpha}$.

PROOF. Since this result is fairly standard (except for the $H^{(-\gamma)}_{2+\alpha}$ setting), we skecth the proof, based on [8, Theorems 11.4 and 11.8] and [21, Lemma 1.2]. For $\delta\in(1,\gamma)$ to be chosen, we define a map $T\colon H_\delta\times[0,1]\to H_\delta$ by saying $u=T(v,\tau)$ if u is the unique solution in $H^{(-\delta)}_{2+\alpha}$ of

$$b^{ij}(X,v,Dv;\tau)D_{ij}u + b(X,v,Dv;\tau)=0 \text{ in } \Omega, \qquad u=\tau\varphi \text{ on } \mathcal{P}\Omega.$$

The existence of u follows from [25, Theorem 11.3]. It is readily checked that T is continuous, and [24, Lemma 5.2] guarantees that T is compact. Also by hypothesis $T(v,0)=0$ for all $v\in H_\delta$. If $(v,\tau)\in H_\delta\times[0,1]$ satisfies $v=T(v,\tau)$, it is readily checked from [25, Theorem 11.3] that $v\in H^{(-\gamma)}_{2+\alpha}$ and also that $|v|_1\leqslant M$. Hence by Theorem 4.7 $|v|_\delta<M_\delta$ for some $\delta\in(1,\gamma)$ and some constant M_δ independent of v and τ. Hence by [8, Theorem 11.6], there is $u\in H_\delta$ for which $u=T(u,1)$ and therefore u is the desired solution. ∎

Using now the classical Schauder theory of Friedman [3, Chapt. 3], we can improve the regularity of the solution for smoother data.

COROLLARY 5.2. *Under the hypotheses of Lemma* 5.1 *if also* $\partial\Omega\in H_{2+\alpha}$, $\varphi\in H_{2+\alpha}(\mathcal{P}\Omega)$, *and* $P\varphi=0$ *on* $C\Omega$, *then* (1.1) *has a solution* $u\in H_{2+\alpha}$.

Note that the condition $P\varphi = 0$ on $C\Omega$ is necessary for u to be in $H_{2+\alpha}$. An earlier version [2, Theorems 1 and 2] of this corollary was proved using only Friedman's Schauder theory; this version required that the map T in the proof of Lemma 5.1 had as range space $H_{2+\alpha\delta}$ and hence a much stronger condition than just $P\varphi = 0$ on $C\Omega$ was needed. Ivanov [12, Theorem 1.1] (see also [13, Theorem 2.1.3]) essentially proved Corollary 5.2 using Friedman's Schauder theory and also the L^p Schauder theory of Solonnikov [19, Chapt. IV]. (Also Ivanov did not have available the Krylov boundary estimates we used in Section 4, so his hypotheses are somewhat stronger than ours.) As well as allowing H_γ initial and boundary data, our approach has the advantage of only using Schauder theory in Hölder spaces.

For our purposes, it suffices to choose $b^{ij}(X, z, p; \tau) = (1-\tau)\delta^{ij} + \tau a^{ij}(X, z, p)$, where δ^{ij} is the Kronecker δ, and $b(X, z; p; \tau) = \tau a(X, z, p)$; then $P_0 v = 0$ in Ω, $v = 0$ on $\mathcal{P}\Omega$ has only the zero solution by [3, Theorem 3.7]. Other choices for P_τ, suitable to other structure conditions, can be found in [2, Sect. 1], and in [8, Sect. 15.5] and [29, Sect. 14] for elliptic problems.

We now present our existence theorems.

THEOREM 5.3. *Let $\partial\Omega \in H_\gamma$ and $\varphi \in H_\gamma(\mathcal{P}\Omega)$ for some $\gamma \in (1, 2)$. Suppose P satisfies the structure conditions* (4.1), (4.3), (5.1), *and*

$$1 = O(\mathcal{E}) \text{ or } \varphi \text{ is time independent} \tag{5.3}$$

$$|p|\Lambda + |a| = O(\mathcal{E})\,. \tag{5.4}$$

If D satisfies a uniform exterior sphere condition, then (1.1) *has a solution $u \in H_{2+\alpha}^{(-\gamma)}$.*

PROOF. In Lemma 5.1, we observe that the estimates provided by Theorems 2.2 and 4.1 will be valid with constants independent of τ. ■

In light of the remarks following Theorem 2.2, we see that this theorem (augmented by Corollary 5.2 and a suitable extension of Lemma 4.1) includes [2, Theorem 14 and 15] and hence [12, Theorem 4.2] and [19, Theorem VI.4.1] as special cases. Elliptic analogs of our result include [8, Theorem 15.13], [21, Theorem 4.2], and [29, Theorem 14.1].

THEOREM 5.4. *Let $\partial\Omega \in H_\gamma$ and $\varphi \in H_\gamma(\mathcal{P}\Omega)$ for some $\gamma \in (1, 2)$ with Ω R-uniformly convex. Suppose P satisfies the structure conditions* (4.1), (4.3), *and* (5.1). *If either* (*a*) *P satisfies the structure conditions* (5.3),

$$\Lambda = O(\lambda|p|^2) \quad \text{or } \Lambda = o(\mathcal{E}), \tag{5.5a}$$

$$|a| \leqslant \frac{|p|\mathcal{C}}{R} + O(\mathcal{E}), \tag{5.5b}$$

or

(*b*) *P satisfies* (5.3), $\Phi_1 = \langle\varphi\rangle_{2;S\Omega}$ *is finite and*

$$|a| + \Phi_1 \leqslant \frac{|p|\mathcal{C}}{R} + O(\mathcal{E}), \tag{5.5$'$}$$

or

(*c*) $\varphi \in H_2$ *and there is* $R' > R$ *such that*

$$|a| + \Phi_1 \leqslant \frac{|p|\mathcal{C}}{R'} + O(\mathcal{E}), \tag{5.5$''$}$$

then (1.1) *has a solution* $u \in H_{2+\alpha}^{(-\gamma)}$.

Theorem 5.4(*c*) is a stronger version of [12, Theorem 4.1]. An elliptic analog is [8, Theorem 15.14].

Our final result is a parabolic version of Jenkins and Serrin's sharp criterion [14] for the solvability of the Dirichlet problem for the minimal surface equation. To state this result succintly we define

$$H_\gamma^M = \{\varphi \in H_\gamma(\mathcal{P}\Omega) : \langle\varphi\rangle_{2;S\Omega} \leqslant M\}.$$

THEOREM 5.5. *Let* $\partial\Omega \in C^2$, *let* $\gamma \in (1, 2)$ *and denote by* H *the mean curvature of* $\partial\Omega$. *Then the problem*

$$u_t = \frac{1}{n}\mathcal{M}u = \frac{1}{n}\operatorname{div}\left((1 + |Du|^2)^{-\frac{1}{2}} Du\right) \qquad \text{in } \Omega, \ u = \varphi \text{ on } \mathcal{P}\Omega \tag{5.6}$$

is solvable for arbitrary $\varphi \in H_\gamma^M$ *if and only if*

$$M \leqslant \frac{n-1}{n} \min_{\partial D} H. \tag{5.7}$$

Note that we have used (2.22). When $\varphi \in H_2$, (5.7) can be replaced by a pointwise inequality between $|\varphi|$ and H; the sufficiency of this form of (5.7) for the solvability of (5.6) is due to Trudinger [33, Corollary 4].

We also consider the family of related problems

$$(5.6)_\tau \qquad u_t = \frac{1}{n}(1 + |Du|^2)^\tau \mathcal{M}u \quad \text{in } \Omega\,, \; u = \varphi \text{ on } \mathcal{P}\Omega$$

for τ a real parameter; always assuming that $H \geqslant 0$ on ∂D. For $\tau < 0$, $(5.6)_\tau$ is solvable for arbitrary $\varphi \in H_\gamma^M$ if and only if $M = 0$; for $\tau = 0$, $(5.6)_\tau$ becomes (5.6) and hence is solvable for arbitrary $\varphi \in H_\gamma^M$ if and only if (5.7) holds; for $0 < \tau < \frac{1}{2}$, $(5.6)_\tau$ is solvable for arbitrary $\varphi \in H_\gamma^M$ provided M is finite; and finally for $\tau \geqslant \frac{1}{2}$, $(5.6)_\tau$ is solvable for any $\varphi \in H_\gamma$. Note that the elliptic analog of $(5.5)_\tau$, for any τ, is

$$(5.6)' \qquad \mathcal{M}u = 0 \text{ in } D\,, \quad u = \varphi \text{ on } \partial D\,,$$

so the parabolic problem has the same solvability characteristic as the elliptic one if $\tau \geqslant \frac{1}{2}$ (which corresponds to $1 = O(\mathcal{E})$) or if φ is time independent.

We close by mentioning that the techniques of [8, Sect. 14.5] can be used to modify our boundary gradient estimates so that modulus of continuity estimates at the boundary for solutions of (1.1) result when φ is merely continuous. In conjunction with suitable interior gradient and interior Hölder gradient estimates, these boundary estimates can be used to show that (1.1) is solvable with continuous boundary values. Unfortunately various technical complications (e.g., the restrictions on φ when $1 \neq O(\mathcal{E})$ in Section 2) arise which sometimes prevent the consideration of arbitrary continuous boundary values. For this reason we shall not pursue this matter further.

Acknowledgments. The author would like to thank the Centre for Mathematical Analysis at the Australian National University for its hospitality during his Faculty Improvement Leave, at which time most of this work was written. Thanks are also due Iowa State University for providing this leave.

REFERENCES

[1] S. Bernstein, *Sur les équations du calcul des variations*, Ann. Scuola Norm Sup. Pisa, Cl. Sci., **29** (1912), pp. 431-485.

[2] D. Edmunds - L. Peletier, *Quasilinear parabolic equations*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci., (3) **25** (1971), pp. 399-421.

[3] A. FRIEDMAN, *Partial differential equations of parabolic type*, Krieger Press Malabar, Florida, 1983.
[4] FUTEV, to appear.
[5] M. GIAQUINTA - E. GIUSTI, *Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form*, J. Reine Angew. Math., **351** (1984), pp. 55-65.
[6] D. GILBARG, *Boundary value problems for nonlinear elliptic equations in n variables*, in: Nonlinear Problems, pp. 151-159, University of Wisconsin Press, Madison, Wisconsin, 1963.
[7] D. GILBARG - L. HÖRMANDER, *Intermediate Schauder estimates*, Arch. Rational Mech. Anal., **74** (1980), pp. 297-318.
[8] D. GILBARG - N. S. TRUDINGER, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.
[9] E. GIUSTI, *Boundary behavior of nonparametric minimal surface*, Indiana Univ. Math. J., **22** (1975), pp. 435-444.
[10] M. GRUBER, *Harnack inequalities for solution of general second order parabolic equations and estimates of their Hölder constants*, Math. Z., **185** (1984), pp. 23-43.
[11] M. IANNELLI - G. VERGARA CAFFARELLI, *On the boundary value problem for surfaces of prescribed mean curvature*, Sympos. Math., **14** (1974), pp. 473-480.
[12] A. V. IVANOV, *The first boundary value problem for second order quasi-linear parabolic equations*, Zap. Nauchn. Sem. Leningrad, Otdel Mat. Inst. Steklov. (LOMI), **38** (1973), pp. 10-32 [Russian]. English translation in J. Soviet Math., **8** (1977), pp. 354-372.
[13] A. V. IVANOV, *Quasilinear degenerate and nonuniformly elliptic and parabolic equations of second order*, Trudy Mat. Inst. Steklov., **160** (1982), pp. 1-285 [Russian]. English translation, Proc. Math. Inst. Steklov, **160** (1984), pp. 1-287.
[14] H. JENKINS - J. SERRIN, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. Reine Angew. Math., **229** (1968), pp. 170-187.
[15] N. V. KRYLOV, *On the maximum principle for nonlinear parabolic and elliptic equations*, Izv. Akad. Nauk SSSR, **42** (1978), pp. 1050-1062 [Russian]. English translation in Math. USSR Izv., **13** (1979), pp. 335-347.
[16] N. V. KRYLOV, *Boundedly inhomogeneous elliptic and parabolic equations in a domain*, Izv. Akad. Nauk SSSR, **47** (1983), pp. 75-108 [Russian]. English translation in Math. USSR Izv., **22** (1984), pp. 67-98.
[17] N. V. KRYLOV, *On estimates for the derivatives of solutions of nonlinear parabolic equations*, Dokl. Akad. Nauk SSSR, **274** (1984), pp. 23-26 [Russian]. English translation in Soviet Math. Dokl., **29** (1984), pp. 14-17.
[18] O. A. LADYZHENSKAYA - N. N. URAL'TSEVA, *Linear and quasilinear elliptic solutions*, Izd. Nauka, Moscow, 1964 [Russian]. English translation, Academic Press, New York, 1968; 2nd Russian ed., 1973.
[19] O. A. LADYZHENSKAJA - V. A. SOLONNIKOV - N. N. URAL'TSEVA, *Linear and quasilinear equations of parabolic type*, Izd. Nauka, Moscow, 1967 [Russian]. English translation, Amer. Math. Soc., Providence, R.I., 1968.
[20] A. LICHNEWSKY - R. TEMAM, *Pseudosolutions of the time-dependent minima surface problem*, J. Differential Equations, **30** (1978), pp. 340-364.
[21] G. M. LIEBERMAN, *The quasilinear Dirichlet problem with decreased regularity at the boundary*, Comm. Partial Differential Equations, **6** (1981), pp. 437-497.
[22] G. M. LIEBERMAN, *Interior gradient estimates for nonuniformly parabolic equations*, Indiana Univ. Math. J., **32** (1983), pp. 579-601.

[23] G. M. LIEBERMAN, *Regularized distance and its applications*, Pacific J. Math., **117** (1985), pp. 329-352.

[23] G. M. LIEBERMAN, *Intermediate Schauder theory for second order parabolic equations.* I: *Estimates*, J. Differential Equations, to appear.

[25] G. M. LIEBERMAN, *Intermediate Schauder theory for second order parabolic equations.* II: *Existence, uniqueness and regularity*, J. Differential Equations, to appear.

[26] G. M. LIEBERMAN, *The Dirichlet problem for quasilinear elliptic equations with continuously differentiable boundary data*, to appear.

[27] G. M. LIEBERMAN - N. S. TRUDINGER, *Nonlinear oblique boundary value problems for nonlinear elliptic equations*, to appear.

[28] P. MARCELLINI - K. MILLER, *Asymptotic growth for the parabolic equations of prescribed mean curvature*, J. Differential Equations, **51** (1984), pp. 326-358.

[29] J. SERRIN, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A, **264** (1969), pp. 413-495.

[30] J. SERRIN, *Gradient estimates for solutions of nonlinear elliptic and parabolic equations*, in: Contributions to Nonlinear Functional Analysis, pp. 565-501, Academic Press, New York, 1971.

[31] L. SIMON, *Interior gradient bounds for nonuniformly elliptic equations*, Indiana Univ. Math. J., **25** (1976), pp. 821-855.

[32] N. S. TRUDINGER, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math., **21** (1968), pp. 205-226.

[33] N. S. TRUDINGER, *The boundary gradient estimate for quasilinear elliptic and parabolic differential equations*, Indiana Univ. Math. J., **21** (1972), pp. 657-670.

[34] N. S. TRUDINGER, *Fully nonlinear uniformly elliptic equations under natural structure conditions*, Trans. Amer. Math. Soc., **278** (1983), pp. 751-769.

Department of Mathematics
400 Carver Hall
Iowa State University
Ames, Iowa 50011