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Existence Results for Embedded Minimal Surfaces of Controlled Topological Type, I.

JÜRGEN JOST (*)

Introduction.

A minimal surface M in a three dimensional manifold X can either be considered as a parametric representation $f\colon S \to X$ with $f(S) = M$, where S is a two dimensional domain and f is conformal and harmonic, or as a submanifold of X with vanishing mean curvature.

Whereas the parametric approach furnished the first existence results, it was later on criticised that solutions produced from this point of view usually are not embedded submanifolds and even immersed only by additional considerations and under an additional hypothesis, namely that they are minimizing (cf. [A1], [A2], [Gu], [O]). Taking up this criticism, recently methods from geometric measure theory were able to prove existence results for embedded minimal surfaces of striking generality (Hardt-Simon [HS], Taylor [TJ], Pitts [P]), at the expense, however, of having no control at all over the topological type of their solutions. Whereas physical considerations make it reasonable to look for the absolute minimum of area over all topological types (as in [A], [HS], [TJ]), from a mathematical point of view it might also be desirable and useful for applications to find solutions of a problem with more specified properties.

In the present context this means to search for embedded minimal surfaces of a prescribed topological type. Several interesting results of this type were already obtained.

If Γ is a Jordan curve on the boundary of a strictly convex set in $\mathbb{R}^3$, it was shown that Γ bounds an embedded minimal disk by Gulliver-Spruck [GS] (under the additional hypothesis that the total curvature of Γ does not exceed 4π), Tomi-Tromba [TT1], Almgren-Simon [AS], and Meeks-

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Yau [MY1]. The methods of [AS] and [MY1] are both of a rather general nature and worked for general Riemannian manifolds and admitted several extensions, the first one to minimal surfaces of higher genus in Riemannian manifolds ([MSY]), the other one to closed solutions of genus O and to a free boundary value problem for minimal disks ([MY2]) and recently also to surfaces of higher genus in Riemannian manifolds ([FHS]).

We also mention [SS] and [GJ2] where embedded minimal spheres in S^3 resp. disks with a free boundary on a strictly convex surface were obtained by saddle point arguments.

It was found out that when trying to prove the existence of an embedded minimal surface of given topological type for some prescribed boundary value problem it is usually necessary to assume the existence of a suitable barrier that is convex or at least of nonnegative mean curvature.

On the other hand, recently some interesting parametric existence results were obtained that arose the question whether they could be improved by showing the existence of an embedded solution. Tomi and Tromba [TT2] provided geometric conditions on a Jordan curve Γ in $\mathbb{R}^3$ that ensure the existence of a minimal surface of given genus $g > 0$ bounded by Γ. Tolksdorf [Td] considered a boundary configuration consisting of a Jordan curve $\Gamma \subset \mathbb{R}^3$ and a free boundary ∂K with $K \cap \Gamma = \emptyset$ and showed that this configuration always bounds a minimal disk in $\mathbb{R}^3 \setminus K$ with holes, i.e. having Γ as a fixed boundary and free boundary curves on ∂K the number of which is not a priori prescribed.

In the present article, we shall use the approach of Almgren-Simon, namely to minimize area only among embedded surfaces, and prove various boundary regularity results (for fixed and free boundaries) (§ 3-5) and finally, with the help of additional approximation arguments (partially making use of results of [MY1] and [MY3]), deduce in § 6 existence results for embedded minimal surfaces of controlled topological type. These results in particular imply that in the situations considered in [TT2] one can produce embedded minimal surfaces and on the other hand that for the configuration of [Td] one can also construct embedded solutions, provided we assume the existence of a suitable barrier containing Γ as mentioned before.

We want to display some typical examples that illustrate our results. Let Γ be a Jordan curve on some catenoid C in $\mathbb{R}^3$, and suppose Γ is not contractible to a point on C; let Z be a cylinder with the same axis as C, and $\Gamma \cap Z = \emptyset$. If Γ is contained in the interior of this cylinder, then there is an embedded minimal surface of the topological type of the annulus, having Γ as a fixed boundary and a free boundary curve on Z. If on the other hand Z is in the interior of the catenoid, then there is again an embedded minimal surface with a fixed boundary Γ and one or more free boundary curves on Z

and this surface is topologically a disk with holes. In this case, it is not possible to fix the number of free boundary curves a priori. This is more obvious when looking at the example where Γ is a Jordan curve contained in the unit sphere in $\mathbb{R}^3$ and $K = \{x \in \mathbb{R}^3\colon |x| \leqslant \frac{1}{2}\}$. Although in this case Γ can bound a disk in the complement of K, it might not bound a minimal disk in $\mathbb{R}^3/K$, for example if Γ is a great circle. Even if $\Gamma = \{x = (x^1, x^2, x^3) \in \mathbb{R}^3\colon |x| = 1, x^3 = \frac{1}{2}\}$ and consequently bounds a minimal disk in $\mathbb{R}^3 \setminus K$, there exists a minimal annulus with fixed boundary Γ and a free boundary curve on ∂K that has less area, and this annulus will be the minimal surface produced by our method.

An example for the situations considered in [TT2] is a solid torus T with boundary of nonnegative mean curvature (with respect to the interior normal) in $\mathbb{R}^3$, and Γ a Jordan curve on ∂T that represents twice the generator of $\pi_1(T)$. We shall show that Γ bounds an embedded minimal Möbous strip contained in the interior of T.

Of course, these are only very special cases, and for more general results (in particular also for minimal surfaces in Riemannian manifolds) we refer to § 6.

All solutions are obtained by a minimizing procedure. In a consecutive paper, we shall combine the present methods with saddle point arguments.

1. – Minimizing among embedded surfaces in bodies with positive mean curvature.

Let A be a closed subset of some threedimensional complete Riemannian manifold X. A need not be compact. We assume that ∂A is a surface of class C^2 and has positive mean curvature with respect to its interior normal. Let Γ be a Jordan curve of class C^2 on ∂A. Let K be another closed subset (possibly empty) of X with boundary surface ∂K of class C^2. We assume that the angle between ∂A and ∂K is always less than $\pi/2$, when measured in $A \cap (X/K)$ (i.e. if ν_A and ν_K denote the resp. outward unit normal vectors, $\nu_A \cdot \nu_K > 0$ in $\partial A \cap \partial K$). Moreover, $\Gamma \cap K = \emptyset$.

Later on, by an approximation argument, we shall also treat the case where ∂A has only nonnegative mean curvature in the sense of [MY2] and the angle between ∂A and ∂K is only assumed to be less or equal to $\pi/2$.

We denote by $\mathcal{M}(g, k)$ the collection of all embedded surfaces M with boundary of class C^2 in $X \setminus \mathring{K}$ of genus g and connectivity k with $\partial M \subset \Gamma \cup \partial K$ and meeting ∂K transversally. The variable g, will also be used to indicate whether M is orientable or not. Note that k just denotes the number of boundary curves of M, and one of those coincides with Γ while the other ones (if any) lie on K. Likewise, $\mathcal{M}(g)$ denotes the collection of such surfaces of genus (and orientability) g with an arbitrary (finite) number of boundary curves that satisfy the conditions required above.

We want to minimize area among surfaces in $\mathcal{M}(g, k)$ or $\mathcal{M}(g)$ that are contained in A. This constraint, namely that we consider only surfaces in A, might lead to complications which however can be avoided with the help of the following trick.

We denote the distance function in X by $d(\cdot,\cdot)$ and define

$$A_\delta := \{x \in X : d(x, A) \leqslant \delta\}$$

If $\delta_0 > 0$ is chosen small enough, then for all δ, $0 \leqslant \delta \leqslant \delta_0$, ∂A_δ is still of class C^2 and has positive mean curvature. In particular, we choose δ_0 so small that geodesic rays emanating from ∂A into the direction of the exterior normal never intersect in $A_{\delta_0} \setminus A$. Furthermore, we can also achieve that the angle between ∂K and ∂A is less than $\pi/2$ (in the same sense as above) if $0 < \delta \leqslant \delta_0$.

We now choose a minimizing sequence of surfaces in $\mathcal{M}(g, k)$ (or $\mathcal{M}(g)$) that are contained in A_{δ_0} and claim that w.l.o.g. we can assume that they are in fact all contained in A itself. Then a limit of such a sequence will not be affected by the constraint since we can perform all sufficiently small variations. In particular, as in [AS, § 1], we shall get a stationary varifold in this way. In case $K \neq \emptyset$, « stationary » here of course means with respect to variations that move ∂K into itself.

In order to prove the claim we shall construct out of each such surface in A_{δ_0} another surface contained in A whose area is not bigger than the area of the original one.

Let M be such a surface. We first want to modify the situation in such a way that M intersects ∂A transversally.

We first want to achieve that the surface meets ∂A transversally at Γ. For this, we have to impose some topological restriction on our minimizing sequence. Let n_M be the normal of Γ in M, and n_A be one normal vector of Γ in ∂A (this vector can be chosen consistently since ∂A as a boundary is orientable). We then require that the curves in S^2 that are obtained as Gauss images of n_M and n_A, resp., have interesection number zero mod 2. Under this assumption, after an arbitrary slight perturbation of M we can assume that it meets ∂A transversally along Γ. It is important to note that this topological condition will remain unaffected under all replacement argument that will eventually be performed on our minimizing sequence (cf. §§ 3 and 6).

We choose η_0 so small that the nearest point project on onto Γ is of class C^2 in T_{η_0} and choose a function $\mathbb{R}^+ \to [0, 1]$ of class C^2 with

$$g(0) = 0$$
$$g(t) = 1 \qquad \text{if } t \geqslant \eta_0^2$$
$$g' \geqslant 0$$

We put

$$\Sigma_\varepsilon := \{x \in A : d(x, \partial A) = \varepsilon g(d^2(x, \Gamma))\},$$

i.e. we move ∂A a bit into the interior of A, at least away from Γ.

If $\varepsilon_0 > 0$ is sufficiently small and $0 < \varepsilon \leqslant \varepsilon_0$, Σ_ε is a graph over ∂A and has positive mean curvature and intersects ∂K at an angle less than $\pi/2$.

Furthermore, since M already meets Γ transversally, from Sard's lemma we conclude that we can also achieve that M intersects Σ_ε transversally for some $\varepsilon \in (0, \varepsilon_0]$.

Thus, by replacing A by

$$\{x \in A : d(x, \partial A) \geqslant \varepsilon g(d^2(x, \Gamma))\}$$

if necessary, we can assume w.l.o.g. that M intersects ∂A transversally.

If we extend the exterior unit normal of ∂A into $A_{\delta_0} \setminus A$ as being constant on geodesic rays normal to ∂A we obtain a function ν^* in $A_{\delta_0} \setminus A$ with $|\nu^*| = 1$ and $\operatorname{div} \nu^* \geqslant 0$ (since ∂A_δ has nonnegative mean curvature for $0 \leqslant \delta \leqslant \delta_0$). Furthermore, if ν_K is the outward exterior normal of K, then

$$\nu^* \cdot \nu_K \geqslant 0$$

in $A_{\delta_0} \setminus A \cap \partial K$.

If N now is a surface in $A_{\delta_0} \cap X \setminus K$ meeting ∂K transversally and intersecting ∂A in a collection of closed Jordan curves and arcs with endpoints on ∂K, if $E \subset \partial A$ and $F \subset \partial K \cap (A_{\delta_0} \setminus A)$ are surfaces with

$$\partial N \cap \partial K = \partial F \cap (A_{\delta_0} \setminus A)$$
$$\partial N \cap \partial A = \partial E \cap (X \setminus K)$$

and

$$E \cap F \cup N = \partial G \qquad \text{for some open } G \subset A_{\delta_0} \setminus A,$$

then applying the divergence theorem on $\bar{G}$, we get

$$\int_E \nu^* \cdot \nu^* \, d\mathcal{H}^2 + \int_F \nu^* \cdot \nu_K \, d\mathcal{H}^2 \leqslant \int_N \nu^* \cdot \nu_G \, d\mathcal{H}^2$$

where ν_G in the outward unit normal of ∂G.

Hence

$$|E| < |N| .$$

(Here and in the sequel, we denote by $|M|$ the area of the surface M.)

Thus, we can apply the replacement argument of [AS, Thms. 1 and 9] to M to obtain a new surface contained in A with area not exceeding the area of M. We may have decreased the topological type by this procedure, but later on we shall supply conditions that exclude this possibility and for the moment we can restore the original topological type by adding handles or cross-caps with arbitrarily small area.

Therefore, we can assume w.l.o.g. that our area minimizing sequence is contained in A. After selection of a subsequence we get a varifold

$$V = \lim_{k\to\infty} v(M_k)$$

if we denote the sequence by (M_k).

V is stationary in $X \setminus \Gamma$, since

$$\|\psi_\# V\|(U) \geqslant \|V\|(U)$$

for any open subset U of A_{δ_0} and any diffeomorphism ψ of X that leaves the complement of U and a neighborhood of Γ fixed and maps ∂K into itself. Furthermore, by construction

$$\operatorname{spt} \|V\| \subset A .$$

2. – The principle of rescaling.

In this section, we describe a general argument how to extend Euclidean considerations to the context of a Riemannian manifold X of bounded geometry, i.e. having a bound for the sectional curvature as well as a positive lower bound for the injectivity radius.

Let $U(0,1)=\{x\in\mathbb{R}^n\colon |x|<1\}$ be a coordinate chart for (some subset of) X with metric tensor $g_{ij}(x)$.

We define a new metric on $U(0,1)$ via

$$g_{ij,R}(x)=g_{ij}(Rx)\qquad R\in\mathbb{R}\,,\quad R\geqslant 1\,.$$

LEMMA 2.1. *Each $p\in X$ is contained in some ball $B(p,r)$ where $r>0$ can be estimated from below in terms of a bound for the sectional curvature of X, a lower bound for the injectivity radius, and the dimension of X only, with the following property:*

$B(p,r)$ is contained in (the image of) the coordinate chart $U(0,1)$ with metric g_{ij} of class $C^{1,\alpha}$ and $g_{ij,R}$ converges to the Euclidean metric δ_{ij} on $U(0,1)$ in the $C^{1,\alpha}$-norm for any $\alpha\in(0,1)$.

PROOF. We introduce harmonic coordinates on $B(p,r)$ which is possible if $r\leqslant r_0$ where $r_0>0$ can be estimated from below in terms of the quantities mentioned in the statement by [JK]. The claim then follows from the estimates of [JK] for harmonic coordinates. q.e.d.

This rescaling process of course amounts to replacing $B(p,r)$ by $B(p,r/R)$ and multiplying the metric by the factor R.

COR. 2.1. *Suppose $\varrho(g)$ is any expression defined in $U(0,1)$ involving a metric g and its first derivatives.*

If

$$\varrho\big((\delta_{ij})\big)>0\qquad \big((\delta_{ij})\ \textit{is the Euclidean metric}\big)$$

then also

$$\varrho(g_{ij,R})>0$$

if R is chosen large enough.

In particular, if v is a vectorfield on $U(0,1)$ with positive divergence with respect to the Euclidean metric, then also its divergence with respect to $g_{ij,R}$ is positive, provided R is sufficiently large.

As a consequence, all constructions performed in a fixed Euclidean ball can be extended to the Riemannian case after suitable rescaling, provided the corresponding Riemannian expressions involve only derivatives of the metric of first order and we always make sure that we have strict inequalities in our constructions.

Therefore, we shall carry out most constructions only in the Euclidean case and make sure that these conditions are always satisfied.

3. – Regularity in the interior and at the fixed boundary.

In this section, we want to show that the varifold limit V of § 1 is represented by an embedded minimal surface M with $\partial M = \Gamma$, at least in $X \setminus K$. We shall prove the regularity of M at the free boundary ∂K in § 5. Also, by construction $M \subset A$.

We make the following assumption that will be justified later on by additional hypotheses

(A) There exist $r > 0$ and $\alpha < \infty$ with the property that if any M_K intersects an open set U, diffeomorphic to the unit ball, with diameter $\leqslant r$ and $d(U, \partial K) \geqslant \alpha |\partial U|_{\frac{1}{2}}$ transversally, then for each component γ of $M_K \cap \partial U$ there is an embedded disk $N \subset M_K$ with $\partial N = \gamma$.

With this assumption, we can prove the interior regularity of V as in [AS] (with the modifications of [MSY, § 4] since we treat the general Riemannian case). Furthermore, by [JK], $M \setminus \partial M$ is of class $C_{2,\alpha}$ for any $\alpha \in (0, 1)$ $(M = \operatorname{spt} \|V\|)$. Thus we are only concerned with boundary regularity.

We assume $\Gamma \in C^2$.

Let $x_0 \in \Gamma \subset \partial A$. We can normalize the situation so that $x_0 = 0$ and the tangent plane of ∂A at x_0 is the $x_2 x_3$-plane and the interior normal points into the direction of the positive x_1-axis.

As in [AS, § 6], we let $V' = V \llcorner (X \setminus \Gamma) \times G(3, 2)$ and see that

$$\Gamma \subset \operatorname{spt} \|V'\| . \tag{3.1}$$

We let L be the tangent of Γ at 0, and C_+ be any varifold tangent of V' at 0, $C_+ = \lim_{k \to \infty} \mu_{r_k} V'$ with $r_k \to \infty$ as $k \to \infty$.

Then (cf. [AS, § 7])

$$L \subset \operatorname{spt} \|C_+\| \tag{3.2}$$

$$\|C_+\|(L) = 0 \tag{3.3}$$

$$r^{-2} \|C_+\|\big(B(0, r)\big) = \theta\big(\|V'\|, 0\big) \qquad \forall r > 0 \tag{3.6}$$

We put $U^+(0, R) = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1 > 0,\ |x| < R\}$ and construct a sequence (N_k) of embedded disks out of (M_k) with the following properties

$$N_k \subset U^+(0, 2) \cup L \tag{3.5}$$

$$\partial N_k \cap U(0, 1) \subset L \cap U(0, 1) \tag{3.6}$$

$$(3.7)\qquad |N_k| \leqslant |N| + 1/k \quad \text{for all disks } N \text{ with } \partial N = \partial N_k$$

$$(3.8)\quad \lim v(N_k) \llcorner \left(\overline{U}^+(0,2)\setminus L\right)\times G(3,2) = C_+ \llcorner \left(\overline{U}(0,2)\setminus L\right)\times G(3,2)\,.$$

Finally,

$$(3.9)\qquad C_+ = \sum_{i=1}^{m} v(H_i)$$

where $H_1, \ldots, H_m$ are half-planes with common boundary L, contained in $\mathbb{R}^3 \cap \{x\colon x_1 \geqslant 0\}$.

We flatten U^+ away from $\cup H_i$ a bit so that we obtain a convex set $\overline{U}^+$ with

$$\overline{U}^+ \cap H_i = U^+ \cap H_i \quad \text{for } i = 1, \ldots, m$$

$$(3.10)\qquad |\partial \overline{U}^+ \cap \{x\colon x_1 > 0\}| < 2\pi - \alpha \quad \text{for some } \alpha > 0$$

$$\overline{U}^+ \cap \{x\colon x_1 = 0\} = U^+ \cap \{x\colon x_1 = 0\}\,.$$

By Sard's lemma we can assume w.l.o.g. (after performing a suitable homothetic expansion with dilation factor arbitrarily close to 1) that N_k intersects $\partial\overline{U}^+(0,1)$ transversally in a collection $\Gamma_1^k, \Gamma_2^k, \ldots, \Gamma_{n_k}^k$ of Jordan curves.

Furthermore, given $\varepsilon > 0$, we can find a sufficiently large k so that by using the coarea formula and possibly again performing a homothetic dilation of N_k with factor $\in (3/4, 1]$, say, we can assume

$$(3.11)\qquad \mathcal{H}^1\left(\Gamma_j^k \setminus \left\{x\colon \operatorname{dist}\left(x, \bigcup_{i=1}^m H_i\right) > \varepsilon\right\}\right) < \varepsilon \quad \text{for } j = 1, \ldots, n_k\,.$$

Then, after renumeration, Γ_1^k consists of $L \cap U(0,1)$ and a Jordan arc in $\overline{U}^+(0,1) \cap \{x\colon x_1 > 0\}$, $\Gamma_2^k, \ldots, \Gamma_{p_k}^k$ for some $p_k \leqslant n_k$ are curves in $\partial\overline{U}^+(0,1) \cap \{x\colon x_1 > 0\}$ that approach some of the semicircles $\partial\overline{U}^+ \cap H_i$, while $\Gamma_{p_k+1}^k, \ldots, \Gamma_{n_k}^k$ enclose an area A_j^k in $\partial\overline{U}^+(0,1)$ with

$$|A_j^k| < c\varepsilon$$

for some fixed c.

As in [AS, § 7], we see that the curves $\Gamma_{p_k+1}^k \ldots, \Gamma_{n_k}$ can be discarded without changing the varifold limit.

For each curve $\Gamma_2^k, \ldots, \Gamma_{n_k}^u$ we have two possibilities:

a) It is a boundary curve of a component of $N_k \cap \overline{U}^+(0,1)$ which is disjoint to L.

b) It is a boundary curve of the component containing $L \cap \overline{U}^+(0,1)$.

To any sequence of components of case a) we can apply the interior regularity arguments of [AS, § 5f.] and conclude that in the limit we get an embedded minimal surface M. The tangent plane of M at 0 has to coincide with the x_2x_3-plane, i.e. the tangent plane of ∂A.

Moreover, M is contained in A. This contradicts the maximum principle, however, since ∂A has positive mean curvature. Hence case a) cannot occur for sufficiently large k.

The curves of case b) can be deleted with the help of the replacement argument of [AS], using assumption (A) again.

Thus, only Γ_1^k is left. By (3.10), it divides $\partial\overline{U}^+(0,1)$ into two components one of which has area less than $\frac{1}{2}(3\pi-\alpha)$.

From (3.7) and (3.8) we can therefore infer that

$$\|C_+\|\big(U(0,1)\big) < \frac{3\pi}{2}.$$

Hence by (3.4)

$$\theta(\|V'\|, 0) < \tfrac{3}{2}.$$

Since this density has to be odd on the other hand (see [AS, § 7] again),

$$\theta(\|V'\|, 0) = \tfrac{1}{2} \tag{3.12}$$

and we conclude from Allard's boundary regularity theorem ([AW2]) and the arguments of [AS] that

$$V = v(M)$$

where M is an embedded minimal surface contained in A. By the maximum principle again, the interior of M is contained in the interior of A; also $\partial M = \Gamma$.

Altogether, we have proved

THEOREM 3.1. *Suppose (A) holds, $\Gamma \in C^2$, and (M_k) is an area minimizing sequence in $\mathcal{M}(g)$ or $\mathcal{M}(g,k)$ and*

$$V = \lim_{k\to\infty} v(M_k)$$

exists in the varifold sense.

Then

$$\operatorname{spt}\|V\| \cap X\setminus K = M$$

where M is an embedded minimal surface of class $C^{1,\alpha}$, with $\Gamma \subset \partial M$. M may be disconnected, but the component of M containing Γ has multiplicity precisely 1. $M\setminus\Gamma \cap X\setminus K$ is of class $C^{2,\alpha}$, for any $\alpha \in (0,1)$.

4. – Area comparison and replacement arguments at free boundaries.

We first want to derive an area comparison lemma. Two points are essential. First of all, according to the rescaling principle of section 2, we want to perform comparisons only between areas contained in some fixed bounded set. Moreover, if we minimize in the presence of a free boundary ∂K among surfaces in $\mathcal{M}(g)$ we don't have (a priori) any control over the number of boundary curves on ∂K. Thus, our comparison arguments have to include not only disks but also disks with holes on ∂K. Our comparison results will be similar to Lemma 3 of [MSY] (the statements as well as the proofs).

If $U \subset X$ is open, we define

$$d_U(x) := d(x, U) \qquad \text{for } x \in X$$

and

$$U(t) := \{x \in X : d_U(t) < t\} \qquad \text{for } t > 0 .$$

The following considerations will of course always contain $K = \emptyset$ as a special case, and hence are suited for regularity in the interior or at the fixed boundary as well as at the free boundary.

LEMMA 4.1 (Area comparison). *Suppose $U \subset X$ is open, bounded, of class C^2 and that ∂U and $\partial U \cap X \setminus K$ are simply connected, and that the following assumptions hold:*

a) If $\gamma \colon [0, T] \to X$ is a geodesic arc parametrized by arclength with

$$\gamma(0) \in \partial U \cap X \setminus K \tag{4.1}$$

and

$$\dot\gamma(0) = \nu_0(\gamma(0)) , \qquad \textit{where } \nu_0 \textit{ is the exterior unit normal of } \partial U \tag{4.2}$$

then

$$\gamma(t) \notin K \quad \textit{if } 0 \leqslant t \leqslant T \tag{4.3}$$

b)

$$\Delta d_u > 0 \qquad \textit{in } U(T) \setminus U \tag{4.4}$$

(this is for example the case if $\partial U(t)$ has positive mean curvature for $0 \leqslant t \leqslant T$,

c) The following isoperimetric inequality holds: If $R(t) := \{\gamma(t)\colon \gamma$ as in a)$\}$ and λ is a system of Jordan curves in $R(t)$ dividing $R(t)$ into two components E_1, E_2 (not necessarily connected), then

$$\min\big(|E_1|, |E_2|\big) \leqslant \beta(\text{length } \lambda)^2 \tag{4.5}$$

for some $\beta \geqslant 1$ and all $t \in [0, T]$.

(The existence of such a β is readily checked in applications, for example if U is a geodesic ball)

d) $$T > 2\beta^{\frac{1}{2}}|\partial U \cap X\setminus K|^{\frac{1}{2}}.$$

Then, if M is a C^2-surface (with boundary) in $X\setminus K$ intersecting ∂U transversally and satisfying

$$\partial M \cap X\setminus K \cap U = \emptyset$$

and if Λ is a component of $M\setminus(U \cap M)$ with

$$\partial M \cap X\setminus K \cap \Lambda = \emptyset$$

then there exists a (not necessarily connected) surface $F \subset \partial U \cap X\setminus K$ with $\partial F \cap X\setminus K = \Lambda \cap \partial U \cap X\setminus K$ and

$$|F| < |\Lambda \cap U(T)|. \tag{4.6}$$

PROOF. We put

$$U'(t) := \{\gamma(s)\colon \gamma \text{ as in } a),\ 0 \leqslant s < t\} \subset U(t) \cap X\setminus K$$

and recall

$$R(t) = \{\gamma(t)\colon \gamma \text{ as in } a)\} \subset \partial U(t) \cap X\setminus K$$

We note that

$$R(0) = \partial U \cap X\setminus K$$

and that by applying the divergence theorem to grad d_U on $U'(t)\setminus U$

$$|R(t)| \text{ is monotonically increasing for } 0 \leqslant t \leqslant T \tag{4.7}$$

W.l.o.g. we replace Λ by a connected component of $\Lambda \cap U'(T)$, and we can assume

$$|\Lambda| \leqslant \tfrac{1}{2}|\partial U \cap X\setminus K| \tag{4.8}$$

because otherwise the claim is trivial.

If

$$\Lambda \cap R(t) \neq \emptyset\,, \tag{4.9}$$

and if Λ intersects $R(t)$ transversally (which is the case for almost all t by Sard's lemma) then is divides $R(t)$ into two (not necessarily connected) sets $F^1(t), F^2(t)$. We label these sets with the index $i \in \{1, 2\}$ in such a way that they depend continuously on t.

We claim, that there exists $t_0 \in [T/2, T]$ and $i \in \{1, 2\}$ with

$$|F^i(t_0)| \leqslant \tfrac{1}{4}\,|\partial U \cap X \setminus K| \tag{4.10}$$

Otherwise, for each $t \in [T/2, T]$ we obtain from the isoperimetric inequality (4.5) and (4.7)

$$\text{length }\big(\Lambda \cap R(t)\big) = \text{length }\big(\partial F^i(t) \cap U'(T)\big) \geqslant \frac{1}{2\beta^{\frac{1}{2}}}\,|\partial U \cap \setminus K|^{\frac{1}{2}}\,. \tag{4.11}$$

On the other hand, putting

$$E(t) = \Lambda \cap U'(t)$$

the coarea formula yields for almost all t

$$|E(t)| = \int_0^t \text{length }\big(\Lambda \cap R(s)\big)\,ds \tag{4.12}$$

(4.11) and (4.12) give (w.l.o.g. (4.12) holds for $t = T$)

$$|E(T)| \geqslant \frac{T}{4\beta^{\frac{1}{2}}}\,|\partial U \cap X \setminus K|^{\frac{1}{2}}\,. \tag{4.13}$$

On the other hand, $|E(T)| \leqslant |\Lambda|$, and (4.8) and (4.13) imply

$$T \leqslant 2\beta^{\frac{1}{2}} |\partial U \cap X \setminus K|^{\frac{1}{2}}$$

contradicting the choice of T.

This proves (4.10).

We choose i as in (4.10) and then drop the index i, i.e. write $F(t)$ instead of $F^i(t)$.

By (4.4) we can apply the divergence theorem to the vector field grad d_u to obtain for $0 \leqslant t_1 < t_2 \leqslant T$

$$|F(t_1)| - |F(t_2)| \leqslant \int_{\overline{E}(t_2)/E(t_1)} |\langle \nu, \operatorname{grad} d_u \rangle| \tag{4.14}$$

where ν is the unit normal vector field of Λ (Note that grad d_u is tangential to $\partial U'(t)\setminus(R(t) \cup R(0))$ so that these boundary components don't give a contribution) (4.14) implies that if $E(t_2) \neq E(t_1)$

$$|F(t_1)| - |F(t_2)| < |\bar{E}(t_2)| - |E(t_1)| \tag{4.15}$$

and in general at least

$$|F(t_1)| - |F(t_2)| \leqslant |\bar{E}(t_2)| - |E(t_1) \qquad (0 \leqslant t_1 < t_2 \leqslant T)\,. \tag{4.16}$$

If $|F(t_2)| = 0$ for some $t_2 \in [T/2, T]$, (4.15) and (4.8) (note $|\bar{E}(t_2)| \leqslant |\Lambda|$) give

$$|F(0)| < |\Lambda|$$

which proves the lemma, putting $F = F(0)$.

In general, we have at least from (4.15) and (4.16)

$$|F(0)| - |\bar{E}(t)| < |F(t)| \qquad (0 < t \leqslant T) \tag{4.17}$$

Since $|\bar{E}(t_2)| \leqslant |\Lambda|$, (4.16), (4.8), and (4.10) (choosing $t_2 = t_0$) imply

$$|F(t)| \leqslant \tfrac{3}{4}\,|\partial U \cap X\setminus K| \leqslant \tfrac{3}{4}\,|R(t)| \tag{4.18}$$

for $0 \leqslant t \leqslant T/2$ (using (4.7) for the last inequality).

Hence the isoperimetric inequality (4.5) gives

$$|F(t)| \leqslant 4\beta|\Lambda \cap R(t)|^2 = 4\beta\left(\frac{d}{dt}\,|E(t)|\right)^2 \tag{4.19}$$

for almost all $t \in [0, T/2]$.

(4.9) and (4.19) imply

$$|F(0)| - |E(t)| \leqslant 4\beta\left(\frac{d}{dt}\left(|F(0)| - |E(t)|\right)\right)^2 \tag{4.20}$$

for almost all $t \in [0, T/2]$, and after integration, since $|F(0)| - |E(0)|$ is monotonically decreasing

$$\left(|F(0)| - |E(0)|\right)^{\frac{1}{2}} - \left(|F(0)| - |E(T/2)|\right)^{\frac{1}{2}} \geqslant \frac{T}{2\beta^{\frac{1}{2}}} \tag{4.21}$$

if

$$|F(0)| > |E(T/2)|\,. \tag{4.22}$$

In this case again (from (4.21))

$$T \leqslant 2\beta^{\frac{1}{2}}|\partial U \cap X \setminus K|^{\frac{1}{2}}$$

contradicting the choice of T. Therefore, (4.22) cannot hold, and

$$|F(0)| \leqslant |E(T/2)| < |E(T)| \quad \text{(otherwise the argument after (4.16) applies)}$$

and since $|E(T)| \leqslant |\Lambda|$, the lemma is proved with $F = F(0)$. q.e.d.

If we want to use Lemma 4.1 in a replacement argument, the following difficulty might arise: We cannot immediately replace Λ by F because the fixed boundary $\partial M \cap X \setminus K$ is «trapped» between Λ and F. We have to consider only the case where at the same time

$$|\Lambda| < |\partial U \cap X \setminus K| \tag{4.23}$$

and

$$|F| < \tfrac{1}{2}|\partial U \cap X \setminus K| \tag{4.24}$$

because otherwise we can replace F by its complement in $\partial U \cap X \setminus K$.

Since we assume that Λ intersects ∂U transversally, Λ and F yield a surface N which (after smoothing out the coners) can be assumed to be of class C^2, with $\partial N \subset \partial K$ and

$$|N| < \tfrac{3}{2}|\partial U \cap X \setminus K| \; . \tag{4.25}$$

In all applications, we may assume w.l.o.g. that $|\partial U \cap X \setminus K|$ is arbitrarily small. Hence we can use the following lemma of [MSY]

LEMMA 4.2. *There exist $r_0 > 0$ and $\delta \in (0, 1)$ (depending only on X and K) with the property that if N is a C^2-surface in $X \setminus K$ with $\partial N \subset \partial K$ and*

$$|N \cap B(x, r_0)| < \delta^2 r_0^2 \quad \textit{for each } x \in X \tag{4.26}$$

then there exists a unique compact $C(N) \subset X \setminus K$ which is «bounded by N modulo ∂K». i.e. $\partial C(N) \cap X \setminus K = N$ and

$$\operatorname{vol}\big(C(N) \cap B(x, r_0)\big) \leqslant \delta^2 r_0^2 \quad \textit{for each } x \in X \tag{4.27}$$

as well as

$$\operatorname{vol}\big(C(N)\big) \leqslant c_0|N|^{\frac{3}{2}} \tag{4.28}$$

where c_0 again depends (only) on X and K.

PROOF. The proof of Lemma 1 of [MSY], given for $K = \emptyset$, easily covers the case of Lemma 4.2 as well. q.e.d.

If we apply Lemma 4.2 to the surface N constructed before (4.25), than either

$$C(N) \cap U = \emptyset$$

or

$$U \cap X \setminus K \subset C(N)\,.$$

In the latter case, we consider $C'(N) = C(N) \setminus U$ instead. It is then bounded by Λ and the complement F' of F in $\partial U \cap X \setminus K$. We want to show that in this case also F' satisfies the conclusion of Lemma 4.1.

LEMMA 4.3 (Area comparison). *Suppose that in addition to the assumptions of Lemma* 4.1

$$|\partial U \cap X \setminus K| \leqslant \tfrac{1}{2}\delta^2 r_0^2 \tag{4.29}$$

where $\delta \in (0, 1)$ *and* $r_0 > 0$ *are given from Lemma* 4.2.

Suppose that (*in addition to d*) *of Lemma* 4.1) *also*

$$T > 12(\tfrac{3}{2})^{\frac{1}{3}} c_0 |\partial U \cap X \setminus K|^{\frac{1}{2}}$$

(c_0 *again from Lemma* 4.2).

If Λ *as in Lemma* 4.1 *saatisfies*

$$|\Lambda| < |\partial U \cap X \setminus K| \tag{4.30}$$

(*which we can assume w.l.o.g. for replacement arguments*), *then there exist* $F' \subset \partial U \cap X \setminus K$ *and a compact set* $C' \subset X \setminus (\mathring{K} \cup U)$ *with*

$$\partial F' \cap X \setminus K = \Lambda \cap \partial U \cap X \setminus K \tag{4.31}$$

$$\partial C' \cap X \setminus K = \Lambda \cup F' \tag{4.32}$$

$$\operatorname{vol} C' \leqslant c_0 (\tfrac{3}{2})^{\frac{3}{2}} |\partial U \cap X \setminus K|^{\frac{3}{2}} \tag{4.33}$$

$$|F'| < |\Lambda \cap U(T)| \tag{4.34}$$

PROOF. Take F as in Lemma 4.1. If (4.24) does not hold, take F' as the complement of F in $\partial U \cap X \setminus K$ and aply Lemma 4.2 to get C with (4.33) and take $C' = C \setminus U$. If (4.24) holds, then F and Λ bound a set C with (4.33) by Lemma 4.2.

If $C \cap U = \emptyset$, we take $F' = F$ and $C' = C$.

Otherwise, we choose F' as the complement of F in $\partial U \cap X \setminus K$ and $C' = C \setminus U$. C' then also satisfies (4.33).

In the proof of Lemma 4.1 we choose the sets $F^1(t)$, $F^2(t)$ (defined after (4.9)) such that

$$F^1(0) = F'$$

By the coarea formula

$$\int_{T/2}^{T} |F^1(t)| \leqslant \operatorname{vol} C' \leqslant c_0(\tfrac{3}{2})|^{\frac{3}{2}}\partial U \cap X \setminus K|^{\frac{3}{2}}$$

(by (4.33)). Therefore by the assumption on T, there exists $t_0 \in [T/2, T]$ with

$$|F^1(t_0)| \leqslant \tfrac{1}{4}|\partial U \cap X \setminus K| . \tag{4.35}$$

Then the proof proceeds as the one of Lemma 4.1, using (4.35) instead of (4.10). q.e.d.

With the help of Lemma 4.3, it is now straightforward to extend the replacement arguments of Thms. 1 and 9 of [AS] (cf. also [MSY; p. 638]) and to obtain

LEMMA 4.4 (Replacement lemma). *Suppose that U satisfies the assumptions of the Lemmata* 4.1 *and* 4.3, *and*

$$\theta > 0 .$$

Let $M \in \mathcal{M}(g, k)$ intersect ∂U transversally.

Suppose that $\partial M \cap X \setminus K$ is not contained in any set C', where $C' \subset X \setminus (\mathring{K} \cup U)$ satisfies (4.32) *and* (4.33) *and Λ is a component of M.*

Then there exists $\tilde{M} \in \mathcal{M}(g', k')$ with $g' \leqslant g$, $k' \leqslant k$, satisfying the following properties:

$$\partial \tilde{M} \cap X \setminus K = \partial M \cap X \setminus K \tag{4.36}$$

$$\tilde{M} \setminus (\tilde{M} \cap U) \cap M \setminus (M \cap U) \tag{4.37}$$

$$\tilde{M} \cap U_\theta \subset M \cap U_\theta \quad \textit{with } U_\theta := \{x \in U : d(x, \partial U) \geqslant \theta\} \tag{4.38}$$

$\tilde{M}$ intersects ∂U transversally

$$|\tilde{M}| + |M \setminus \tilde{M} \cap U_\theta| \leqslant |M| \tag{4.39}$$

If for each component σ of $M \cap \partial U \cap X \setminus K$ there is $\Lambda \subset M$ with $\partial\Lambda \cap \partial U \cap X\setminus K = \sigma$ which is topologically a disk with holes on ∂K, i.e. $\Lambda \in \mathcal{M}(0)$, then

$$\tilde{M} \cap \overline{U} = \bigcup_{j=1}^{k} N_j \quad \text{with } N_j \in \mathcal{M}(0) \quad (j = 1, \ldots, k) \tag{4.40}$$

If in addition

$$|M| \leqslant |P| + \varepsilon \tag{4.41}$$

for every $P \in \mathcal{M}(g, k)$ with $\partial P \cap X\setminus K = \partial M \cap X\setminus K$, then ther e are $\varepsilon_1, \ldots, \varepsilon_k \geqslant 0$ with $\sum_{j=1}^{j} \varepsilon_j \leqslant \varepsilon$ and

$$|N_j| \leqslant |P_j| + \varepsilon_j \tag{4.42}$$

for any $P_j \in \mathcal{M}(0)$ with $\partial P_j \cap X\setminus K = \partial N_j \cap X\setminus K$ and P_j having at most as many boundary components on ∂K as N_j.

If (4.41) *holds for $P \in \mathcal{M}(g)$, then P_j in* (4.42) *may have arbitrarily many boundary components on ∂K.*

LEMMA 4.5 (Boundary filigree). *Suppose $\{Y_s\}_{s\in[0,1]}$ is an increasing family of convex sets, where each Y_s satisfies the assumptions of the set U in Lemmata* 4.1 *and* 4.3 *(with β and T independant of s).*

Assume furthermore that Y_s is given as

$$Y_s = \{x \in X : f(x) < s\},$$

where $f: X \to \mathbb{R}^+$ is of class C^2 on $Y_1 \setminus Y_0$, $Df \neq 0$ on $Y_1\setminus Y_0$,

$$\sup_{Y_1/Y_0} |Df| \leqslant c_1$$

Suppose $M \in \mathcal{M}(0, k)$ and that $\partial M \cap X\setminus K$ is not contained in any set $C' \subset X\setminus(\mathring{K} \cup Y_s)$ satisfying (4.32) *and* (4.33) *with Y_s instead of U (for any s).*

Finally, we assume that for some $\varepsilon > 0$

$$|M| \leqslant |N| + \varepsilon \quad \text{for all } N \in \mathcal{M}(0, k') \quad \text{with } k' \leqslant k$$

and

$$\partial N \cap X\setminus K = \partial M \cap X\setminus K.$$

Then

$$|M \cap Y_{t_0}| \leqslant 2\varepsilon$$

if

$$t_0 = 1 - 2c_1 c_2^{\frac{1}{2}} |M \cap Y_1|^{\frac{1}{2}} > 0$$

PROOF. The proof which is basically on easy modification of the proof of Lemma 3 of [AS] can be bound in [GJ2]. Of course, we have to perform the modifications mentioned in [MSY; p. 639]. q.e.d.

5. – Regularity at the free boundary.

THEOREM 5.1. *Let* (N_k) *be an area minimizing sequence in* $\mathcal{M}(0)$, *and suppose*

$$W = \lim_{k\to\infty} v(N_k)$$

exists in the varifold sense. Then for each point $x_0 \in \operatorname{spt} \|W\| \cap \partial K$ *there are* $n \in \mathbf{N}$, $\varrho > 0$ *(both depending on* x_0*) and a minimal surface* M *meeting* ∂K *orthogonally with*

$$W \llcorner B(x_0, \varrho) \times G(3, 2) = nv(M)$$

PROOF. The first part of the proof is a modification of [AS; § 5 f.]. As in [AS, p. 463], we see using the boundary filigree lemma 4.3, that W is stationary, rectifiable and there is some $c > 0$ with

$$\theta^2_*(\|W\|, x) \geqslant c \tag{5.1}$$

for all $x \in \operatorname{spt} \|W\|$.

Also W is integral.

Let $x_0 \in \operatorname{spt} \|W\| \cap \partial K$.

We assume for a moment that W has a varifold tangent C at x_0 with $\operatorname{spt} \|C\|$ contained in a half plane H.

Since W is also stationary w.r.t. to variations of its boundary on ∂K, C has to contain the interior normal of ∂K at x_0.

W.l.o.g. $x_0 = 0$ and $(0, 0, 1)$ is the exterior normal of ∂K (using a suitable coordinate chart.).

Let

$$\tilde{K}_{\varrho,\sigma} := \big(D_\varrho \sim \partial D_\varrho) \times (-\sigma, \sigma)\big) \cap X \setminus K .$$

By rescaling, tilting the « top » $D_\varrho \times \{\sigma\}$ and the « bottom » $D_\varrho X\{-\sigma\}$ slightly agains ∂K, and smoothing out the corners $\partial D_\varrho \times \{\sigma\}$ and $\partial D_\varrho X\{-\sigma\}$, we obtain a set $K_{\varrho,\sigma}$ satisfying the assumptions of Lemma 4.4. Let $\gamma_1(\varrho, \sigma)$,

$\gamma_2(\varrho, \sigma)$ be curves on $K_{\varrho,\sigma}$ that are close (e.g. obtained by nearest point projection) to the corners $\partial D_\varrho \times \{\sigma\} \cap X \setminus K$ and $\partial D_\varrho \times \{-\sigma\} \cap X \setminus K$, resp. These curves $\gamma_1(\varrho, \sigma)$ and $\gamma_2(\varrho, \sigma)$, then divide $\partial K_{\varrho,\sigma} \cap X \setminus K$ into a cylinder $(C \cap X \setminus K) Z_{\varrho,\sigma}$ (corresponding to $\partial D_\varrho \times (-\sigma, \sigma) \cap X \setminus K$) and a union $E_{\varrho,\sigma}$ of two half disks (corresponding to $D_\varrho \times \{\sigma, -\sigma\}$).

$$\mu_{r_k\#} W \to C \quad \text{by definition of } C . \tag{5.2}$$

for some sequence $(r_k) \to \infty$ as $k \to \infty$.

Let $\sigma_0 \in (0, 1)$ be given.

(5.1) implies that we can find $r \in (r_k)$ with

$$K_{1,1} \cap \operatorname{spt} \|\mu_{r\#} W\| \subset K_{1,\sigma_0/2} . \tag{5.3}$$

W.l.o.g. also

$$\begin{aligned} &|\operatorname{spt} \|\mu_{r\#} W\| \cap (\partial D_{\frac{1}{2}} \times \mathbb{R})| = \emptyset \\ &|\operatorname{spt} \|\mu_{r\#} W\| \cap C_r| = \emptyset, \qquad \text{where } C_r = \partial \mu_r(X \setminus K), \end{aligned} \tag{5.4}$$

By assumption

$$|\mu_r(N_k)| \leqslant |N| + r^2 \varepsilon \tag{5.5}$$

for all $N \in M$ with $\partial N \cap C_r = \partial \mu_r(N_k) \cap C_r$.

Since $v(\mu_r(N_k)) \to \mu_r W$, (5.3) and the coarea formula yield for almost all $\sigma \in (\sigma_{0,2}, 1)$ and $k \to \infty$

$$\mathcal{H}^1(\mu_r(N_k) \cap (E_{1,\sigma})) \to 0 .$$

Thus for sufficiently large k, we can find $\sigma_k \in (\frac{3}{4}\sigma_0, \sigma_0)$ and $\varrho_k \in (\frac{3}{4}, 1)$ with

$$\mu_r(N_k) \cap (\gamma_1(\varrho_k, \sigma_k) \cup \gamma_2(\varrho_k, \sigma_k) \cup (C_r \cap E_{\varrho_k,\sigma_k}) = \emptyset . \tag{5.6}$$

From now on, we shall write E_k instead of E_{ϱ_k,σ_k}, etc.

Furthermore, by Sard's Lemma, we can assume that $\mu_r(N_k)$ intersects E_k and Z_k transversally. Moreover $\mu_r(N_k)$ intersects C_r transversally by assumption.

We now want to apply Lemma 4.4 for $M = \mu_r(N_k)$ and $U = K_k$.

We find integers $0 < R_k^1 \leqslant R_k^2 \leqslant R_k^3$ and discs (with holes) $P_k^1, \ldots, P_k^{R_k^3} \in \mathcal{M}(0)$ with

$$\begin{aligned} &\partial P_k^1, \ldots, \partial P_k^{R_k^2} \subset Z_k \cup (C_r \cap K_k) =: B_k \\ &\partial P_k^{R_k^2+1}, \ldots, \partial P_k^{R_k^3} \cap E_k \qquad ((\text{note } (5.6)) \end{aligned}$$

and $\partial P_k^1, \dots, \partial P_k^{R_k^1}$ are homotopically nontrivial in B_k, while $\partial P_k^{R_k^1+1}, \dots, \partial P_k^{R_k^2}$ bound discs in B_k:

Moreover,

$$(5.7) \qquad |P_k^l| \leqslant |P| + \varepsilon_{k,l} \qquad \forall P \in \mathcal{M} \text{ with } \partial P = P_k^l \qquad l = 1, \dots, R_k^3$$

and

$$\sum_{l+1}^{R_k^3} \varepsilon_{k,l} \leqslant r^2 \varepsilon_k$$

and using (5.4) and [AW1; 2.6 (2) (d)],

$$(5.8) \qquad \mu_{r\#} W \llcorner K_{\frac{1}{2},1} \times G(3,2) = \lim_{k\to\infty} \sum_{l=1}^{R_k^3} \boldsymbol{v}(P_k^l \cap K_{\frac{1}{2},1}).$$

Then, first of all, $P_k^{R_k^2+1}, \dots, P_k^{R_k^3}$ can be discarded as in [AS], p. 465 f., without changing the varifold limit in (5.8).

We now want to delete $P_k^{R_k^1+1}, \dots, P_k^{R_k^2}$.

Let $\Delta_{k,l}$ be the intersection of the disc bounded by P_k^l B_k with Z_k ($l = R_k^1 + 1, \dots, R_k^2$).

Clearly

$$|\Delta_{k,l}| \leqslant 2\pi \varrho_k \sigma_0$$

Choosing $P = \Delta_{k,l}$ in (5.7) and k sufficiently large, hence

$$(5.9) \qquad |P_k^l| < 2\pi\sigma_0 + \varepsilon_{k,i} \qquad (l = R_k^1 + 1, \dots, R_k^2).$$

Choosing σ_0 sufficiently small and using the boundary filigree lemma 4.5 for the family of cylinders (again after tilting top and bottom slightly)

$$\begin{aligned} Y_t &= \left\{x = (x_1, x_2, x_3) : \sqrt{x_1^2 + x_2^2} < t\varrho_k;\ |x_3| \leqslant 10\right\} \\ f(x) &= \frac{1}{\varrho_k}\sqrt{x_1^2 + x_2^2} \qquad \text{for } |x_3| \leqslant 5 \\ c_1 &= \varrho_k^{-1} \end{aligned}$$

we get

$$|P_k^l \cap K_{\frac{1}{2},1}| \leqslant 3\varepsilon_{k,i}, \qquad l = R_k^1 + 1, \dots, R_k^2$$

and thus also these P_k^l can be discarded without changing the varifold limit in (5.8).

Thus

$$(5.10) \qquad \mu_{r\#} W \llcorner K_{\frac{1}{2},1} \times G(3,2) = \lim_{k\to\infty} \sum_{l=1}^{R_k^1} \boldsymbol{v}(P_k^1 \cap K_{\frac{1}{2},1}).$$

For $l = 1, \ldots, R_k^1$, we have

$$\tfrac{1}{2}\pi\varrho^2(1-\delta(r)) \leqslant |P_k^l \cap K_{\varrho,1}| \qquad (\varrho \in (0, \varrho_k]) , \tag{5.11}$$

where $\delta(r) \to 0$ as $r \to \infty$, since $K \in C^2$.

Furthermore, comparing P_k^l with either of the parts into which ∂P_k^l divides $\partial K_{\varrho_k,\sigma_0}$, and using (5.7)

$$|P_k^l| \leqslant \tfrac{1}{2}\pi\varrho_k^2 + 2\pi\varrho_k\sigma_0 + \varepsilon_{k,l} .$$

We now choose k so large that $\varepsilon_k < \pi\sigma_0$: (Note that the σ_0 employed here can be chosen independantly from the one leading to the deletion of P_k^l for $l = R_k^1 + 1, \ldots, R_k^2$).

Thus

$$|P_k^l| \leqslant \tfrac{1}{2}\pi\varrho_k^2 + 3\pi\sigma_0 \tag{5.12}$$

(5.10) and (5.11) imply that R_k^1 is bounded independantly of k.

After selection of a subsequence, we find a positive integer n and

$$\varrho_k \to \varrho_0 \in [\tfrac{3}{4}, 1] \quad \text{as } k \to \infty$$

and for $l = 1, \ldots, n$

$$v(\mu_{\varrho_k^{-1}} P_k^l) \text{ converges to a varifold } W_l$$

with (using (5.11), (5.12), (5.3))

$$\tfrac{1}{2}\pi\varrho^2(1-\delta(r)) \leqslant \|W_l\|(K_{\varrho,1}) \text{ for each } \varrho \in [0,1] \tag{5.13}$$

$$\|W_l\|(K_{1,1}) \leqslant \frac{\pi}{2} + 3\pi\sigma_0\varrho_0^{-2} \leqslant \frac{\pi}{2} + 20\sigma_0 \tag{5.14}$$

$$\operatorname{spt}\|W_l\| \subset K_1 \tag{5.15}$$

$$(\mu_{\varrho_0^{-1}} W) \llcorner K_{\frac{1}{2},1} \times G(3,2) = \sum_{l=1}^{n} W_l \llcorner K_{\frac{1}{2},1} \times G(3,2) . \tag{5.16}$$

Since $K_{1-\sigma_0,\sigma_0} \subset U(0,1)$, (5.13) and (5.15) imply

$$\|W_l\|\, U(0,1) \geqslant \|W_l\| K_{1-\sigma_0,\sigma_0} = \|W_l\| K_{1-\sigma_0,1} \geqslant \frac{\pi}{2}(1-\delta(r)\big((1-\sigma_0)^2 . \tag{5.17}$$

Since $U(0,1) \cap X \setminus K \subset K_{1,1}$, (5.14) yields

$$\|W_l\| \, U(0,1) \leqslant \frac{\pi}{2} + 20\sigma_0 . \tag{5.18}$$

Since we can make σ_0 and $\delta(r)$ as small as we want by choosing r sufficiently large (satisfying (5.3)), we obtain, using the monotonicity at the free boundary of [$GJ1$],

$$\theta(\|W\|, x_0) = \frac{n}{2} \tag{5.19}$$

We now apply the first part of the proof to $\left(\mu_{\varrho_q^{-1}}(P_k^l)\right)$ instead of (N_k) $(l = 1, \ldots, n)$. This, together with interior regularity, implies that each W_l is a stationary integral varifold with density 1 $\|W_l\|$-almost everywhere. Taking σ_0 in (5.18) sufficiently small, the free boundary regularity of [GJ1] implies

$$W_l \llcorner K_{\frac{1}{2},1} \times G(3,2) = \boldsymbol{v}(M_l) , \qquad l = 1, \ldots, n$$

where M_l is a minimal surface which can be represented as a graph over $D_{\frac{1}{2}} \cap \mu_{r\sigma_0^{-1}}(X \setminus K)$

$$M_1 = \{(x_1, x_2, x_3) \colon x_3 = u_1(x_1, x_2),\, x \in D_{\frac{1}{2}} \cap \mu_{r\varrho_0^{-1}}(X \setminus K)\}$$

By (5.19) (remembering $x_0 = 0$) and (5.16),

$$u_l(0) = 0 \qquad (l = 1, \ldots, n) .$$

Since for $l, m \in \{1, \ldots, n\}$ either $u_l \leqslant u_m$ or $u_l \geqslant u_m$ by construction of W_l, and since we can apply the strong maximum principle to the difference of two solutions of the minimal surface equation also at the free boundary points, $u_l \equiv u_m$.

Hence

$$\mu_{r\sigma_0^{-1}\#} W \llcorner \tilde{K}_{\frac{1}{2},1} \times G(3,2) = n\boldsymbol{v}(M_l) .$$

In order to finish the proof, we have to show that at each $x_0 \in \partial K \cap \operatorname{spt} \|W\|$, there is a varifold tangent C of V of the form $n\boldsymbol{v}(M)$ with M a half plane and $n \in \mathbf{N}$.

W.l.o.g. $x_0 = 0$ again.

Let

$$C \in \operatorname{Var\,Tan}(W, x_0)$$

$$C = \lim_{q \to \infty} \mu_{t_k \#} W \qquad \text{for some sequence } (t_k) \to \infty$$

We choose a sequence (M_k) in $\mathcal{M}(0)$ with

$$M_k = \mu_{r_k}(N_k) \tag{5.20}$$

$$v(M_k) \to C \quad \text{as } k \to \infty \tag{5.21}$$

$$|M_k| \leqslant |N| + \tilde{\varepsilon}_k \qquad \forall N \in \mathcal{M}(0) \tag{5.22}$$

with $\partial N \cap \mu_{r_k}(X \setminus K) = \partial M_k \cap \mu_{r_\#}(X \setminus K)$

$$\tilde{\varepsilon}_k \to 0 \quad \text{as } l \to \infty.$$

Therefore, C is stationary. By the reflection principle of [GJ1; 4.11], we can reflect C across $T_{x_0}(\partial K)$ to obtain a stationary $\tilde{C}$. We then apply the interior arguments of [AS; § 6] to $\tilde{C}$ and deduce that it is contained in a plane. Hence C either is a halfplane containing the normal of ∂K at x_0, or it is the tangent plane $T_{x_0}(\partial K)$. The first case was already treated above.

Therefore, we only have to exclude that

$$\operatorname{spt} \|C\| = T_{x_0}(\partial K) . \tag{5.23}$$

We put (for any given small $\eta > 0$)

$$D_k = B\big((0, 0, -\eta), \sqrt{1 + \eta^2}\big) \cap \mu_{r_k}(X \setminus K) . \tag{5.24}$$

Recalling the normalization that $x_0 = (0, 0, 0)$ and $(0, 0, 1)$ is the exterior normal of ∂K at x_0, we see that after suitable rescaling, D_k satisfies the assumptions of Lemma 4.4 Therefore, we can assume that M_k intersects D_k in a collection of disks with holes. Also, D_k intersects C (assuming that (5.23) holds) in a unit disk.

Moreover, given $\varepsilon > 0$, we can find a sufficiently large k (using (5.21) and the coarea formula and possibly suitably rescaling with a controlled factor $\varrho \in [\frac{1}{2}, 1]$, say) with

$$\mathcal{H}^1\big(M_k \cap \partial D_k \cap \{x\colon d(x, C) < \varepsilon\}\big) < \varepsilon \tag{5.25}$$

Hence with the help of the ioperimetric inequality we see that there is an annalus $A_k \subset \partial D_k \cap X \setminus K$ with

$$\partial A_k \subset (M_k \cap \partial D_k) \cup \partial K$$

and

$$|A_k| \leqslant c\varepsilon \tag{5.26}$$

where c is a fixed constant.

Choosing $N = A_k$ in (5.22) we see that

$$|M_k| \to 0 \quad \text{as } k \to \infty$$

and hence from (5.21)

$$c = 0 .$$

Thus, (5.23) is excluded, and the proof is complete, since Var Tan $(W, x_0) \neq \emptyset$. q.e.d.

THEOREM 5.2. *Suppose that ∂K has positive mean curvature with respect to the exterior normal of K.*

Let (N_k) be an area minimizing sequence in $\mathcal{M}(0, h)$ and suppose that

$$W = \lim_{k \to \infty} \boldsymbol{v}(N_k)$$

exists in the varifold sense.

Then the conclusion of Thm. 5.1 *holds.*

PROOF. The proof is the same as the one of Thm. 5.1, except that we have to exclude (5.23) this time in a different way since (5.22) only holds for comparison surfaces $N \in \mathcal{M}(0, h)$ (The other replacement arguments based on Lemma 4.4 never increased the number of holes, and in case it was decreased the original number can always be restored by adding arbitrarily thin tubes with holes that will then disappear in the limit).

We choose D_k as in (5.24). Applying Lemma 4.4 again, we can assume that M_k intersects D_k in a number of disks with holes. Each component that has a free boundary inside D_k can be replaced by a region on ∂D_k with arbitrarily small area and is hence excluded as in the proof of Thm. 5.1.

This process is not possible for a component P_k with $P_k \cap D_k \cap \partial\mu_{r_k}(X \setminus K) = \emptyset$, since then such a replacement would increase the number of free boundary curves.

In this case, we argue as follows.

We define

$$g_\varepsilon(x) := \begin{cases} \varepsilon^5 \exp\left(\dfrac{\varepsilon^2}{d^2(x, x_0) - \varepsilon^2}\right) & \text{if } d(x, x_0) < \varepsilon \\ 0 & \text{otherwise} . \end{cases}$$

If $0 < \varepsilon \leqslant \varepsilon_0$ and ε_0 is sufficiently small, the C^2-norm of g_ε becomes arbitrarily small. We let $\pi_k\colon \mu_{r_k}(X \setminus K) \to \partial\mu_{r_k}(X \setminus K)$ be the nearest point projection and define

$$X_{k,\varepsilon} := \{x \in \mu_{r_k}(X \setminus \mathring{K})\colon d(x, \partial\mu_{r_k}(X \setminus K)) \geqslant g_\varepsilon(\pi(x))\} .$$

Since $X \setminus K$ has positive mean curvature with respect to the interior normal, we can find $\varepsilon_0 > 0$ so that for $0 < \varepsilon \leqslant \varepsilon_0$ and fixed k (ε_0 of course depends on k) $X_{k,\varepsilon}$ likewise (is of class C^2 and) has positive mean curvature with respect to the interior normal of $\partial X_{k,\varepsilon}$.

As in § 1, by projection and replacement we then construct a minimizing sequence $\tilde{M}_k$ in $X_{k,\varepsilon}$ (of course, it creates no difficulties if there are several P_k's, since we can move their projections onto $X_{k,\varepsilon}$ a bit apart so that we still have an embedded sequence.

By interior regularity, $\tilde{M}_k$ converges to an embedded minimal surface in $X_{k,\varepsilon}$.

On the other hand, also $M_k \setminus (M_k \cap D_k)$ converges in the interior to an embedded minimal surface. By construction, these surfaces have to coincide in $X_{k,2\varepsilon}$. Hence, by unique continuat they have to coincide everywhere. This implies that X_0, since not contained in $X_{k,\varepsilon}$, does not lie in the support of W, because spt $\|W\|$ has no isolated points.

Therefore, (5.23) again is excluded, and the proof is complete. q.e.d.

REMARK. The considerations of Thms. 5.1 and 5.2 also apply if we minimize in $\mathcal{M}(g)$ or $\mathcal{M}(g, k)$, resp. provided the following hypothesis (which will be justified later on in § 6) is satisfied

(*B*) *There exists $r > 0$ with the property that if any N_k (where (N_k) is a minimizing sequence in $\mathcal{M}(g)$ or $\mathcal{M}(g, k)$) intersects an open set U, diffeomorphic to the unit ball, with diameter not exceeding r, transversally, then for each component γ of $N_k \cap \partial U$ there is a disk N with holes on ∂K, $N \subset N_k$, with $\partial N = \gamma$.*

6. – Existence theorems.

We make the following assumptions about the geometric setting:

i) *X is a threedimensional manifold of bounded geometry, i.e. the sectional curvature is bounded and the injectivity radius is bounded from below by a positive constant.*

ii) *∂A has nonnegative mean curvature in the sense of Meeks-Yau ([MY2]), namely is consists of a finite number of C^2-surfaces $H_1, \ldots, H_m$ with*

a) H_i has nonnegative mean curvature with respect to the interior normal

b) H_i is a compact subset of a smooth surface $\bar{H}_i$ in X with

$$\bar{H}_i \cap A = H_i \qquad \partial \bar{H}_i \subset X \setminus A \qquad (i = 1, \ldots, m)$$

iii) *$K \subset X$ is a closed subset. ∂K consists of a finite number of twodimensional pieces with bounded C^2-norm.* (K *may be empty.*)

iv) *Γ is a Jordan curve on ∂A, $\Gamma \cap K = \emptyset$.*

v) *If ν_A and ν_K denote the resp. unit normal vectors*

$$\nu_A \cdot \nu_K \geqslant 0 \quad \text{in } \partial A \cap \partial K\,. \tag{6.1}$$

At points where two or more pieces of ∂A or ∂K come together, (6.1) *is required to hold for the normal vectors of all these pieces.*

THEOREM 6.1. *Suppose $\mathcal{M}(0) \neq \emptyset$. There exists an embedded minimal surface M in $X \setminus K$ which is continuous up to the boundary, having Γ as a fixed boundary curve and possibly free boundary curves on ∂K. At all points of $M \cap \partial K$ where a unique exterior normal vector of ∂K exists, ∂K is met orthogonally by M. M is of class $C^{2,\alpha}$ in the interior and as regular at the boundary as Γ and ∂K permit.*

M topologically is a disk with holes corresponding to the free boundaries on ∂K, i.e. $M \in \mathcal{M}(0)$. Furthermore, M minimizes area in this class. In particular, it is stable.

PROOF. We first assume that ∂A and ∂K are of class C^2, ∂A has positive mean curvature with respect to the interior normal, and $\Gamma \in C^2$.

We minimize the area in $\mathcal{M}(0)$. Using Lemma 4.4, we can satisfy (A) of § 3. By the results of § 3 and § 5, after selection of a subsequence, a minimizing sequence converges to a varifold V whose support is represented by an embedded minimal surface. We take M as that component of this minimal surface that contains Γ. Using (3.12) and the constancy theorem, we infer that the multiplicity of this component of spt $\|V\|$ is one at interior points.

From the arguments of [MSY: § 3, in particular Remark (3.27)], we deduce that M topologically is a disk with holes (in particular, M is orientable, because otherwise it would have multiplicity 2 at interior points).

M is bounded, because otherwise, using $\Gamma \subset M$, there would be infinitely many disjoint convex balls $B(x, \delta)$ with $x \in M$, where $\delta > 0$ can be chosen uniformly because X is of bounded geometry, and hence the area of M would be infinite by the monotonicity formula.

Moreover, the number of boundary curves on ∂K is finite (it may be zero, of course). Namely, otherwise, there would exist distinct boundary curves γ_i, $i = 1, 2, \ldots$, and points $x_i \in \gamma_i$ with $x_i \to x_0$ as $i \to \infty$.

Since $\partial K \in C^2$, we can find a ball $B(x_0, \delta)$ with $\delta > 0$ which satisfies the assumptions of Thm. 4.13 of [GJ1]. W.l.o.g. $x_i \in B(x_0, \delta/2)$ for all i. Let M be the component of $M \cap B(x_0, \delta)$ containing $x_0 (x_0 \in M$, since M is closed).

For small enough δ

$$|M_0| \leqslant \tfrac{1}{2}\omega_n^{-1}(\delta/2)^{-n}(1+\varepsilon) \tag{6.2}$$

(ε as in [GJ1; 4.13]) and the result of [GJ1; 4.13] applies which implies in particular

$$M_0 \cap \gamma_i = \emptyset \quad \text{for all } i\,.$$

On the other hand, also $(M \cap B(x_0, \delta)) \setminus M_0 =: M'$ represents a nontrivial stationary varifold V'. The support of this varifold is closed, hence

$$x_0 \in \operatorname{spt} \|V'\|\,.$$

Althogether, the multiplicity of M at x_0 has to be $n \cdot \frac{1}{2}$, with $n \geqslant 2$. This is not possible, however, since M is a minimal surface of multiplicity 1.

We now treat the general case by approximation, making use of arguments of [MY1] and [MY3]. First, we pass from $\partial A \in C^2$ with positive mean curvature to ∂A satisfying only ii). Since we have already supplied an argument yielding an a-priori bound for $d(x, \Gamma)$ where $x \in M$ and M is an area minimizing surface with $\Gamma \subset \partial M$, we can assume w.l.o.g. that A is compact. In [MY3; § 1] it is proved that A can be approximated by a sequence of compact manifolds A_k with boundary ∂A_k of positive mean curvature with respect to the interior normal that converge to A in the sense that the metrics and their derivatives and the boundaries ∂A_k converge to the corresponding objects of A uniformly. Also, Γ is approximated by smooth curves $\Gamma_k \subset \partial A_k$. A_k and ∂A_k can be chosen as smooth as desired.

We minimize the area in A_k among surfaces from $\mathcal{M}(0)$ with fixed boundary Γ_k. As shown above, we obtain an embedded minimal surface M_k, minimizing area in its class. We want to show that (M_k) converges (after selection of a subsequence) to an embedded minimal surface M in A that satisfies the conclusions of the theorem. In order to achieve this, we have to get uniform estimates for (M_k). M_k is conformally equivalent to a domain S_k, the unit disc in the plane with $m_k \geqslant 0$ interior disks removed, cf. [J1; § 3]. Therefore, we can consider this minimal surface as an injective conformal map

$$f_k \colon S_k \to A_k$$

mapping the outer boundary $\{x \in \mathbb{R}^2 \colon |x| = 1\}$ monotonically onto Γ_k. Furthermore, f_k can be normalized by a three-point condition on the outer boundary. Of course, f_k also satisfies the minimal surface equation for the metric of A_k.

Since the metrics converge it does not matter whether we measure area with respect to the metric of A or of A_k. In particular, the area of M_k and hence the Dirichlet integral of f_k can be assumed to be uniformly bounded,

$$\int |\nabla f_k|^2 \leqslant D\,, \quad \text{say}\,. \tag{6.3}$$

Given $\delta > 0$ and $x_0 \in \bar{S}_k$, by the Courant-Lebesgue Lemma, we can always find $r \in (\delta, \sqrt{\delta})$ with

$$d\big(f_k(x_1), f_k(x_2)\big) \leqslant 2\pi D^{\frac{1}{2}}(\log(1/\delta))^{-\frac{1}{2}} \tag{6.4}$$

for all $x_1, x_2 \in \partial B(x_0, r) \cap \bar{S}_k$ (cf. e.g. [J1; Lemma 3.1]). If $\delta > 0$ is chosen small enough, $\gamma_k := f_k\big(\partial B(x_0, r) \cap \bar{S}_k\big)$ is therefore contained in a set U satisfying the assumptions of Lemma 4.4 (for the metric of A_k; δ can be chosen uniformly in k, however, since also the derivatives of the metrics converge, see [MY3; p. 154]; cf. also § 2).

First assume $|x_0| + \sqrt{\delta} < 1$.

Let σ_k be a curve in ∂U that is homotopic to γ_k in $M_k \cap \bar{U}$. It bounds a disk with holes on M_k (or S_k). By Lemmata 4.3 and 4.4 this disk also is contained in $\bar{U}$, since M_k is minimizing. Because of the three-point condition on the outer boundary of S_k, a similar argument applies at Γ. Therefore, (f_k) is equicontinuous.

It is then standard to derive uniform $C^{1,\alpha}$-estimates for (f_k) at least away from the free boundary. Therefore, (f_k) converges uniformly to some $C^{1,\alpha}$ map $f\colon S \to A$, at least after selection of a subsequence. Here, S is the limiting domain of (S_k). So far, it may be degenerate in the sense that some of the interior boundary circles of S_k have shrunk to a point or become tangent to each other in the limit. Since (f_k) equicontinuous, however, and $\Gamma \cap K = \emptyset$, no interior boundary circle can approach the outer boundary circle in the limit. Moreover, f is of class $C^{1,\alpha}$ and a weak solution of the minimal surface equation in interior subdomains of S. Hence it is also a strong solution and of class $C^{2,\alpha}$ in the interior (for a proof, that i) is sufficient for $f \in C^{2,\alpha}$, cf. [JK]).

Moreover, it is regular in a neighborhood of Γ. It is also monotonic on the outer boundary by the argument of [HH].

On the other hand, after slight modifications near Γ, (M_k) is also a minimizing sequence for the area (computed with respect to the metric of A) among disks with holes on ∂K and fixed boundary Γ. Therefore, by the arguments of [AS] and § 5, it converges to an embedded minimal surface M in A, possibly with free boundaries on ∂K, at least away from Γ. But near Γ, we know regularity already.

M is topologically a disk with holes by the same argument as above.

Therefore, we have produced the desired minimal surface. How to pass from a smooth Jordan curve on ∂A to a general Jordan curve σ on ∂A is demonstrated in [MY1; p. 426] (Note that it is important that σ lies on some surface; this implies in particular that σ bounds a surface with finite area).

Likewise, if ∂K is as in iii), we take an approximation K_i with $\partial K_i \in C^2$. In order to get the equicontinuity of the corresponding sequence M_i also at the edges of ∂K, we only have to note that we can still find sets U that satisfy the assumptions of Lemma 4.4, for example by taking the intersection of $X \setminus K$ with suitable balls having their center in the interior of K.

Finally, it is trivial how to pass from strict to weak inequality in (6.1) via approximation.

For higher regularity at the boundary, we refer to [HH] and [GHN].

That M meets ∂K orthogonally follows because M is stationary with respect to variations of its trace on ∂K. q.e.d.

REMARKS. i) Similar results can be obtained when Γ is empty. We need an hypothesis to guarantee that the limit of a minimizing sequence cannot disappear at infinity. We could assume that $X \setminus K$ is compact, or that we have again a barrier ∂A of nonnegative mean curvature, where A is compact. Also, in general one cannot exclude anymore that the limit of a minimizing sequence is a nonorientable surface with multiplicity 2, cf. [MSY; Remark (3.27)].

ii) As mentioned in the introduction, if we drop the requirement of embeddedness and look for a parametric solution, then Thm. 5.1 is contained in the result of Tolksdorf [Td]. Since he is not looking for embedded solutions, he does not need a barrier like ∂A. However, in his approach branch points are not excluded.

THEOREM 6.2. *Assume in addition that K has nonnegative mean curvature with respect to the exterior normal.*

Let $h \geqslant 0$ be the smallest integer with the property that there exists an embedded surface N of genus 0 with $\Gamma \subset \partial N$ and $\partial N \setminus \Gamma \subset \partial K$.

Then there exists an embedded minimal surface in $\mathcal{M}(0, h)$ satisfying the conclusion of Thm. 5.1.

PROOF. Again, we first treat the case $\partial A \in C^2$, $\Gamma \in C^2$ and ∂A and ∂K having positive mean curvature (with respect to the interior resp. exterior normal). Then we get an embedded minimal surface in $\mathcal{M}(0, h)$ as before, using Thm. 5.2 instead of Thm. 5.1. The general case again follows by approximation.

REMARKS. i) A similar result should be accessible to the methods of Meeks-Yau ([MY1] and [MY2]).

ii) We can also prove a corresponding result if h is not topologically least possible, provided the following Douglas condition holds

$$\inf\{|M|\colon M \in \mathcal{M}(0, h)\} < \inf\{|M|\colon M \in \mathcal{M}(0, h-1)\} \, . \tag{6.5}$$

We shall be concerned with a Douglas type criterion in more detail in the next theorem.

In order to avoid additional technical complications we shall assume for the rest of this section that K is empty.

THEOREM 6.3. *Let X, A, Γ be as before, in particular assume that ∂A has nonnegative mean curvature*

$$\mathcal{M}(g) := \{M\colon M \subset A \text{ compact oriented embedded surface of genus } g \text{ with } \partial M = \Gamma\} \, ,$$

$$\alpha_g := \inf\{|M|\colon M \in \mathcal{M}(g)\} \, .$$

If $\mathcal{M}(g) = \emptyset$, we put $\alpha_g = \infty$
If

$$\alpha_g < \alpha_{g-1} \quad (\text{we assume in particular } \alpha_g < \infty) \, , \tag{6.6}$$

then there is an embedded minimal surface $M \subset A$ of genus g with $\partial M = \Gamma$ and

$$|M| = \alpha_g \, . \tag{6.7}$$

For the proof of Thm. 6.3, we need the following result of Almgren-Simon (cf. [AS; Lemma 5]).

LEMMA 6.1. *Suppose $M \in \mathcal{M}(g)$, $|M| < \alpha_{g-1}$, $\varrho < \frac{1}{2}\big((\alpha_{g-1} - |M|)\diagdown(8\pi + 3)\big)^{\frac{1}{2}}$, U is a convex open set of class C^2 with diameter $\leqslant \varrho$.*

If $\partial M \cap U = \emptyset$ and M intersects ∂U transversally, then for each component σ of $M \cap \partial U$ there exists an embedded disk $N \subset M$ with $\partial N = \sigma$.

(Actually, in the Riemannian context, we might have to rescale in order that the above restriction on ϱ is sufficient; cf. § 2).

PROOF of Thm 6.3. We first assume again that ∂A is C^2 with positive mean curvature. Then the result follows as in [AS; § 10], using Thm. 3.1 for boundary regularity. (Of course, Lemma 6.1 implies that (A) is satisfied).

Again, cf. [MSY; Remark (3.27)] the produced minimal surface has multiplicity 1 and is hence in particular orientable and of genus at most g. Lower genus than g of course is ruled out by (6.6).

For the general case, we let A_k, Γ_k be as in the proof of Thm. 6.1 and $M_k \in \mathcal{M}(g)$ be the corresponding embedded minimal surfaces in A_k with $\partial M_k = \Gamma_k$.

We shall now use some of the arguments presented in [J2].

Again, we can consider the parametric representation $f_k\colon S_k \to A_k$, $f_k(S_k) = M_k$, S_k is a surface of genus g and a metric of constant curvature. ∂S_k is geodesic, f_k conformal and harmonic (for the metric of A_k, since M_k is a minimal surface), cf. [J2; Thm. 1].

W.l.o.g. we consider the case $g \geqslant 2$, where S_k has a hyperbolic metric, because this is the most difficult one. We want to apply Mumford's compactness theorem [Mu] to (S_k). We have to exclude that the lengths of closed geodesics can tend to zero, as $k \to \infty$. In that case, however, we can use the arguments of [J2; § 2] to find homotopically nontrivial curves γ_k in S_k for which the image curve $f_k(\gamma_k)$ become arbitrarily short as $k \to \infty$.

On the other hand, since the metrics converge also the area of M_k with respect to the metric of A approaches α_g. Hence, w.l.o.g.

$$
\begin{aligned}
|M_k| &\leqslant \alpha_g - \tfrac{1}{2}(\alpha_g - \alpha_{g-1}) \quad \text{for all } k \\
&< \alpha_{g-1} \quad \text{by (6.6)}
\end{aligned}
$$

and we see that the conclusion we have derived from the existence of arbitrarily short closed geodesics on S_k (with respect to the hyperbolic metric) is not compatible with the assertion of Lemma 6.1.

Therefore, by Mumford's theorem, (S_k) converges to a hyperbolic surface S of genus g. W.l.o.g. $S_k = S$ for all k.

$f_k\colon S \to A$ then satisfies uniform $C^{1,\alpha}$-estimates as in the proof of Thm. 6.1 and hence (after selection of a subsequence) converges to a $C^{1,\alpha}$ – map $f\colon S \to A$ which is weakly harmonic and therefore of class $C^{2,\alpha}$. It is also conformal and hence a parametric minimal surface. $f|\partial S$ again is monotonic by [HH].

$M := f(S)$ on the other hand also is the varifold limit of (M_k) and hence is embedded in the interior by the arguments of [AS], since (M_k) is an area minimizing sequence in $\mathcal{M}(g)$ (after making slight modifications near Γ again).

Thus, M is an embedded minimal surface of multiplicity 1 and oriented and of genus at most g as before (cf. [MSY; § 3]). On the other hand

$$|M| \leqslant \alpha_g$$

by lower semicontinuity. By (6.6), M then is precisely of genus g and (6.7) holds.

That we need no regularity assumptions for Γ follows again from [MY1]. q.e.d.

REMARKS. i) For the case where ∂A is strictly convex and $\Gamma \in C^2$, this result is already proved in [AS].

ii) A not necessarily embedded minimal surface of genus g was produced in [J2], provided a Douglas condition like (6.6) is satisfied. Of course, if one does not require that the solution is embedded, a barrier like ∂A is not needed.

COROLLARY 6.1. *Let X, A, Γ be as before, Let $\Gamma \subset \partial A$ bound an embedded oriented surface $N \subset A$ of genus g.*

Assume that the induced map on the fundamental groups

$$i_*: \pi_1(N) \to \pi_1(A) \tag{6.8}$$

is injective.

Then Γ bounds an embedded oriented minimal surface M of genus g in A which minimizes the area among all such surfaces.

PROOF. Here, (6.8) guarantees that the genus of the limit of a minimizing sequence cannot drop and that (A) in § 3 is satisfied. The rest follows as before. q.e.d.

REMARK. If we do not require that the minimal surface is embedded then this result is due to Tomi and Tromba [TT2] (at least for $X = \mathbb{R}^3$). Topological arguments to show the embeddedness of an area minimizing surface were indicated by Freedman-Hass-Scott ([FHS, § 7]), generalizing the work of Meeks-Yau ([MY1], [MY2]) on the genus 0 case. In the case where $\Gamma = \emptyset$, a minimal surface of genus g was produced by Schoen-Yau [SY] and shown to be embedded in [FHS], in case this is topologically possible.

Furthermore, Tomi and Tromba also treated the following important special case, again without showing that their solution is embedded.

COROLLARY 6.2. *Suppose A is a compact solid body in $\mathbb{R}^3$ with boundary ∂A of nonnegative mean curvature. Suppose ∂A is an oriented surface of genus $2g$. Then $\pi_1(A)$ is generated by $2g$ generators $A_1, \ldots, 2g$.*

Suppose Z is in $\pi_1(A)$ homotopic to

$$A_1 A_2 A_1^{-1} A_2^{-1} \ldots A_{2g-1} A_{2g} A_{2g-1}^{-1} \ldots A_{2g}^{-1}.$$

Then Γ bounds an embedded minimal surface of genus g.

Proof. Tomi and Tromba observed, based on a result of Zieschang, that Γ bounds a surface M' of genus g for which the induced map

$$i_*:= \pi_1(M') \to \pi_1(A)$$

is injective. Hence Cor. 6.1 applies. q.e.d.

The following result is again based on the work of Tomi and Tromba.

Corollary 6.3. *Suppose $A \subset \mathbb{R}^3$ is a solid torus and ∂A again has non-negative mean curvature, and $\Gamma \subset \partial A$, a Jordan curve, is homotopic in A to $2a$ where a is the generator of $\pi_1(A)$. Then Γ bounds an embedded minimal Möbius strip in A.*

Proof. If we minimize the area among embedded Möbius strips then we get an embedded minimal surface $M \subset A$ with $\partial M = \Gamma$ as before. Γ cannot bound a disk in A because otherwise a would be homotopically trivial in A.

On the other hand, as in [AS, § 10], for each $\delta > 0$, we can find an embedded Möbius strip N_δ in a δ-neighborhood of M. In particular, we can choose δ so small that the projection π: $\{x\colon \text{dist}\,(x, M) \leqslant \delta\} \to M$ is continuous. If γ is any curve on M, we can lift it to N_δ and denote the lifted curve by γ'. Then $2\gamma'$ is homotopic to zero in N_δ. Hence also $\pi(2\gamma')$ is homotopic to zero in M. Thus, the first Betti number of M vanishes, and M has to be a Möbius strip. q.e.d.

7. – Concluding remarks.

Of course, one can think of more general situations than the ones considered in § 6, for example nonorientable surfaces of higher genus, surfaces of higher genus in the presence of free boundaries, several fixed boundary curves, etc.

We tried to present the technical arguments of § 1 and § 3-§ 5 in as much generality as possible so that they can also cover such more general situations. On the other hand, since it seems possible to invent situations of arbitrary complexity, we did not strive to achieve utmost generality in the final results presented in § 6, but rather restricted ourselves to some particularly interesting cases.

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