ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 12, n° 1 (1985), p. 1-42

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Parametrix for a Characteristic Cauchy Problem.

A. BOVE - J. E. LEWIS - C. PARENTI

0. – Introduction, statement of the problem and main results.

In this paper we consider the following second order differential operator with smooth coefficients defined in $R^{n+1} = R_t \times R_x^n$:

$$(0.1)\quad P = t\partial_t^2 - \sum_{i,j=1}^{n} a_{ij}(t,x)\,\partial_{x_i}\partial_{x_j} + (\nu(t,x)+1)\,\partial_t + \sum_{j=1}^{n} b_j(t,x)\,\partial_{x_j} + b_0(t,x)\,.$$

We assume that the functions a_{ij} are real, $a_{ij} = a_{ji}$, $i,j = 1, \ldots, n$ and that for some $\delta > 0$ we have

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geqq \delta|\xi|^2$$

for every $(t,x) \in R^{n+1}$, $\xi \in R^n$. For sake of simplicity we shall suppose that all coefficients in (0.1) are constant outside of a compact set.

We are concerned with the Cauchy problem:

$$(0.2)\qquad \begin{cases} Pu(t,x) = 0\,, & t > 0 \\ u|_{t=0} = g \in \mathcal{E}'(R^n)\,. \end{cases}$$

One can prove that the Cauchy problem (0.2) is C^∞-well posed iff $\nu(0,x) + 1 \notin \{0, -1, -2, \ldots\}$, which we assume from now on.

We propose to construct a parametrix for pb. (0.2), i.e. an operator $E\colon \mathcal{E}'(R_x^n) \to C^\infty([0,T];\mathcal{D}'(R_x^n))$ (for a suitable $T > 0$) such that

$$(0.3)\qquad \begin{cases} PE\colon \mathcal{E}'(R_x^n) \to C^\infty([0,T]\times R_x^n) \\ \gamma E - I\colon \mathcal{E}'(R_x^n) \to C^\infty(R_x^n)\,, \end{cases}$$

where γ denotes the restriction to the hyperplane $t = 0$.

Pervenuto alla Redazione il 13 Aprile 1983.

Actually, under some technical additional conditions, we shall construct a parametrix E with the following properties:

$$WF(\gamma\partial_t^k Eg) \subset WF(g)\,, \qquad g \in \mathcal{E}'(R^n)\,, \quad k = 1, 2, \ldots \tag{0.4}$$

(0.5) For every $g \in \mathcal{E}'(R^n)$ and for every $s \in [0, T[$:

$$WF(Eg|_{t=s}) = (\Lambda^+_{2\sqrt{s}} \cup \Lambda^-_{2\sqrt{s}}) \circ WF(g)\,,$$

where $\Lambda_t^{\pm} \subset (T^*R^n \setminus 0) \times (T^*R^n \setminus 0)$ are the two canonical relations defined in the following way: for every $(y, \eta) \in T^*R^n \setminus 0$ let $(x^{\pm}(t; y, \eta), \xi^{\pm}(t; y, \eta))$ be the integral curve of the Hamiltonian vector field $H_{\pm a(t,x,\xi)}\big(a(t, x, \xi) = \big(\sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j\big)^{\frac{1}{2}}\big)$ issued from (y, η), then

$$\Lambda_t^{\pm} = \big\{\big((x^{\pm}(t; y, \eta), \xi^{\pm}(t; y, \eta), (y, \eta)\big) | (y, \eta) \in T^*R^n \setminus 0\big\}\,. \tag{0.6}$$

We point out that $\Lambda_t^{\pm}$ are the usual canonical relations appearing in the Cauchy problem for the wave operator $\partial_t^2 - \sum_1^n a_{ij}(t, x)\partial_{x_i}\partial_{x_j}$ (see e.g. J. J. Duistermaat [3]).

Then, modulo uniqueness for pb. (0.2), we obtain from (0.5) a precise description of the singularities of the solutions $u \in C^{\infty}(\overline{R_t^+}; \mathcal{D}'(R_x^n))$ of pb. (0.2), while (0.4) implies that singularities do not scatter along the boundary.

The construction of E is quite long and technical since the usual methods of geometrical optics cannot be applied.

To motivate such a construction consider the following particular case of (0.1)

$$P_0 = t\partial_t^2 - \Delta_x + (\nu_0 + 1)\partial_t\,, \qquad \nu_0 \in C\,. \tag{0.1}'$$

To solve (0.2) for P_0 we take the Fourier transform $\hat{u}(t, \xi) = \int \exp[-ix\cdot\xi] \cdot u(t, x)\,dx$ of u and obtain

$$\begin{cases} t\partial_t^2\hat{u}(t, \xi) + (\nu_0 + 1)\partial_t\hat{u}(t, \xi) + |\xi|^2\hat{u}(t, \xi) = 0\,, & t > 0 \\ \hat{u}(0, \xi) = \hat{g}(\xi)\,. \end{cases} \tag{0.7}$$

Putting $z = 2\sqrt{t}\,|\xi|$ and writing $\hat{u}(t, \xi) = t^{-\nu_0/2}v(z, \xi)$, it can be easily seen that v satisfies the Bessel equation

$$z^2\partial_z^2 v(z, \xi) + z\partial_z v(z, \xi) + (z^2 - \nu_0^2)v(z, \xi) = 0\,. \tag{0.8}$$

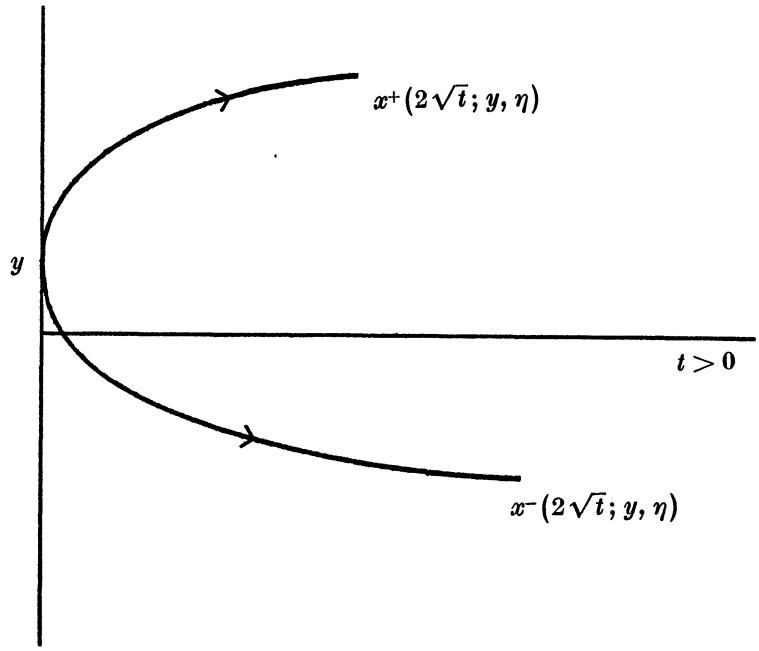

Fig. 1 – The characteristics $x^{\pm}(2\sqrt{t}; y, \eta)$ are tangent to $t = 0$.

Since we are looking for solutions which are smooth in the t variable up to $t = 0$, taking into account the initial condition, we get

$$(0.9) \qquad u(t, x) = E_0 g(t, x) = \int \exp[ix\cdot\xi]\, \tilde{J}_{\nu_0}(2\sqrt{t}\,|\xi|)\hat{g}(\xi)\, \check{d}\xi\,, \qquad (\check{d}\xi = (2\pi)^{-n} d\xi)\,,$$

where

$$(0.10) \qquad \tilde{J}_{\nu_0}(z) = \Gamma(\nu_0 + 1)(z/2)^{-\nu_0} J_{\nu_0}(z)\,, \qquad \nu_0 \notin \{-1, -2, \ldots\}\,,$$

$J_{\nu_0}(z)$ being the usual Bessel function of the first kind.

It can be easily recognized that E_0 extends as a continuous operator from $\mathcal{E}'(R^n)$ into $C^\infty(\overline{R^+_t}; \mathcal{D}'(R^n_x))$ and that $P_0 E_0 g = 0$, $\gamma E_0 g = g$.

Relation (0.4) is trivially verified. To prove (0.5) we split $J_{\nu_0}(z)$ into a sum of the two Hankel functions $J_{\nu_0}(z) = \frac{1}{2}(H^{(1)}_{\nu_0}(z) + H^{(2)}_{\nu_0}(z))$ (see G. Watson [7]).

The functions $H^{(1)}_{\nu_0}(z)$, $H^{(2)}_{\nu_0}(z)$ have the following asymptotic expansion

for $z \to +\infty$:

$$H_{\nu_0}^{(1),(2)}(z) \sim (\pi z/2)^{-\frac{1}{2}} \exp\left[\mp i\left(z - \frac{\pi\nu_0}{2} - \frac{\pi}{4}\right)\right] \sum_{j \geqq 0} c_j (\pm 2iz)^{-j}, \qquad c_0 = 1, \tag{0.11}$$

(see G. Watson [7, Sec. 7.2 (5), (6)]).

By (0.11), for every $t > 0$ the operator E_0 splits into a sum of two independent elliptic Fourier integral operators of the form

$$\int \exp\left[i(x\xi \pm 2\sqrt{t}\,|\xi|)\right] b^{\pm}(2\sqrt{t}\,|\xi|)\,\hat{g}(\xi)\,\check{d}\xi, \tag{0.12}$$

for some $b^{\pm}(z) \in S_{1,0}^{-\mathrm{Re}\,\nu_0 - \frac{1}{2}}(R_z^+)$.

Relation (0.5) is now a straightforward consequence of (0.12) and the calculus of the wave front set (see L. Hörmander [5]).

We remark that the amplitude $\tilde{J}_{\nu_0}(2\sqrt{t}\,|\xi|)$ in (0.9) exhibits a rather different behaviour in the two regions $\sqrt{t}\,|\xi| \lesssim$ const. More precisely, in the region $\sqrt{t}\,|\xi| <$ const. the parametrix E_0 behaves like a pseudo differential operator (with non-classical symbol), while for $\sqrt{t}\,|\xi| \to +\infty$ E_0 is essentially the sum of two elliptic Fourier integral operators whose phases are hidden in the amplitude $\tilde{J}_{\nu_0}$. This remark suggests that, in the general case (0.2), one should perform two different constructions in the regions $\sqrt{t}\,|\xi| <$ const., $\sqrt{t}\,|\xi| >$ const. respectively.

According to this strategy we collect in Ch. 1 all the formal ingredients we need to construct the parametrix: in particular, in Sect.s 1.1-1.3 and Sect.s 1.4-1.6 we construct a formal parametrix for (0.2) in the region $\sqrt{t}\,|\xi| \lesssim$ const. respectively, by using suitable integral representations for Bessel's functions. We point out that such a technique has been already used in the literature (see e.g. S. Alinhac [1]).

In Ch. 2 the two formal parametrices are glued together and a precise operator calculus is developed.

Chapter 1

FORMAL THEORY

1.1. – Formal parametrix in the region $\sqrt{t}\,|\xi| <$ const.

The amplitude for the parametrix E_0 in (0.9) has the homogeneity property $\tilde{J}_{\nu_0}(2\sqrt{t/\lambda}\,|\sqrt{\lambda}\xi|) = \tilde{J}_{\nu_0}(2\sqrt{t}\,|\xi|)$, $\lambda > 0$. This suggests that the right homo-

geneity involved in the problem is of the following type:

$$f(t/\lambda, x, \sqrt{\lambda}\xi) = \lambda^m f(t, x, \xi)\,, \qquad \lambda > 0\,.$$

We are thus led to consider operators of the form

$$\int \exp\,[ix\cdot\xi] q(t, x, \xi)\hat{g}(\xi)\check{d}\xi\,,$$

where the amplitude q is given by an « asymptotic sum » of functions homogeneous in the above sense.

The following definition will be convenient.

DEF. 1.1.1. Let m be a real number.

i) *By* $\mathcal{O}^m$ *we denote the class of the functions*

$$g(x, \xi) \in C^\infty(R^n_x \times \dot{R}^n_\xi) \qquad \textit{such that}$$
$$g(x, \lambda\xi) = \lambda^m g(x, \xi)\,, \qquad \lambda > 0\,.$$

ii) *By* Ψ^m *we denote the class of the functions*

$$f(t, x, \xi) \in C^\infty(\overline{R^+_t} \times R^n_x \times \dot{R}^n_\xi) \qquad \textit{such that}$$
$$f(t/\lambda, x, \sqrt{\lambda}\xi) = \lambda^m f(t, x, \xi)\,, \qquad \lambda > 0\,.$$

It is easy to check that the operator $t^h \partial_t^k \partial_x^\alpha \partial_\xi^\beta$ maps Ψ^m into $\Psi^{m-h+k-|\alpha|/2}$ and that $\mathcal{O}^k \times \Psi^m \ni (g, f) \to gf \in \Psi^{m+k/2}$.

We consider, formally, the following operator:

$$Eg(t, x) = \int \exp\,[ix\cdot\xi] q(t, x, \xi)\hat{g}(\xi)\check{d}\xi\,, \qquad g \in C_0^\infty(R^n_x)\,, \tag{1.1.1}$$

where

$$\begin{cases} q(t, x, \xi) \sim \sum\limits_{j \geqq 0} q_{-j/2}(t, x, \xi) \\ q_{-j/2} \in \Psi^{-j/2}\,, \qquad j = 0, 1, 2, \ldots \end{cases} \tag{1.1.2}$$

Imposing that Eg satisfies (0.2), we obtain

$$\begin{cases} PEg(t, x) = \int \exp\,[ix\cdot\xi]\tilde{q}(t, x, \xi)\hat{g}(\xi)\check{d}\xi \\ \tilde{q}(t, x, \xi) \;\; = \exp\,[-\,ix\cdot\xi] P\big(\exp\,[ix\cdot\xi] q(t, x, \xi)\big) \sim 0 \\ \tilde{q}(0, x, \xi) \sim 1\,. \end{cases} \tag{1.1.3}$$

To implement (1.1.3) we write $\tilde{q} \sim \sum_{j\geqq 0} \tilde{q}_{1-j/2}$, with $\tilde{q}_{1-j/2} \in \Psi^{1-j/2}$, $j \geqq 0$. To compute the formal series $\sum_{j\geqq 0} \tilde{q}_{1-j/2}$ in terms of the q's we replace the coefficients of P by their formal Taylor expansions and collect in $\exp[-ix\cdot\xi] \cdot P(\exp[ix\cdot\xi]q)$ all the terms with the same homogeneity degree in the sense of Def. 1.1.1. It is convenient to introduce the following notation

$$
\text{(1.1.4)} \qquad \begin{cases} A_k(x, \partial_x) = \sum_{i,j=1}^{n} \left(\frac{1}{k!}\, \partial_t^k a_{ji}|_{t=0}\right) \partial_{x_i}\partial_{x_j} \\ B_k(x, \partial_x) = \sum_{j=1}^{n} \left(\frac{1}{k!}\, \partial_t^k b_j|_{t=0}\right) \partial_{x_j}\,, \\ b_{0,k}(x) = \frac{1}{k!}\, \partial_t^k b_0|_{t=0} \qquad\qquad k \geqq 0 \\ \nu_k(x) \;\; = \frac{1}{k!}\, \partial_t^k \nu|_{t=0} \\ a(x, \xi) = \sqrt{A_0(x, \xi)} \\ M_k(x, \xi, \partial_x) = \frac{2}{i} \sum_{i,j=1}^{n} \left(\frac{1}{k!}\, \partial_t^k a_{ij}|_{t=0}\right) \xi_i \partial_{x_j}. \end{cases}
$$

Using (1.1.4) we define the differential operators:

$$
\text{(1.1.5)} \qquad \begin{cases} L_1 \;\; = t\partial_t^2 + (\nu_0(x) + 1)\,\partial_t + A_0(x, \xi) \\ L_{\frac{1}{2}-k} = t^k[M_k(x, \xi, \partial_x) + iB_k(x, \xi)]\,, \qquad k \geqq 0 \\ L_{-k} \;\; = t^{k+1} A_{k+1}(x, \xi) + t^k[-A_k(x, \partial_x) + B_k(x, \partial_x) \\ \qquad\qquad + b_{0,k}(x)] + t^{k+1}\nu_{k+1}(x)\,\partial_t\,, \qquad k \geqq 0\,. \end{cases}
$$

We note that $L_1 \colon \Psi^m \to \Psi^{m+1}$, $L_{\frac{1}{2}-k} \colon \Psi^m \to \Psi^{m+\frac{1}{2}-k}$, $L_{-k} \colon \Psi^m \to \Psi^{m-k}$, $k \geqq 0$.

A straightforward computation yields:

$$
\text{(1.1.6)} \qquad \tilde{q} \sim L_1 q_0 + (L_1 q_{-\frac{1}{2}} + L_{\frac{1}{2}} q_0) + (L_1 q_{-1} + L_{\frac{1}{2}} q_{-\frac{1}{2}} + L_0 q_0) + \ldots = \sum_{j\geqq 0} \tilde{q}_{1-j/2}
$$

with:

$$
\text{(1.1.7)} \qquad \tilde{q}_{1-j/2} = \sum_{h=0}^{j} L_{1-h/2} q_{-j/2+h/2}\,, \qquad j \geqq 0\,.
$$

Conditions (1.1.3) can thus be rewritten as the following sequence of transport equations

$$\text{(1.1.8)} \qquad \begin{cases} L_1 q_0 = 0\,, & t > 0 \\ q_0(0, x, \xi) = 1 \end{cases}$$

$$\text{(1.1.9)}_j \qquad \begin{cases} L_1 q_{-j/2} = -\sum_{h=1}^{j} L_{1-h/2} q_{-j/2+h/2}\,, & t > 0\,, \\ & j \geqq 1 \\ q_{-j/2}(0, x, \xi) = 0\,. \end{cases}$$

1.2. – The first transport equation: $L_1 q_0 = 0$.

To solve the Cauchy problem (1.1.8) we reduce the equation $L_1 q_0 = 0$ to a Bessel equation. For this purpose we change the variables as follows:

$$\text{(1.2.1)} \qquad z = 2\sqrt{t}\, a(x, \xi)$$

$$\text{(1.2.2)} \qquad q_0(t, x, \xi) = t^{-\nu_0(x)/2} w(z; x, \xi)\,.$$

Using the relations

$$\text{(1.2.3)} \qquad \begin{cases} \partial_t = \dfrac{2A_0}{z}\,\partial_z \\ \partial_t^2 = 4A_0^2\left(\dfrac{1}{z^2}\,\partial_z^2 - \dfrac{1}{z^3}\,\partial\right) \\ t\partial_t = \dfrac{1}{2} z \partial_z \end{cases}$$

we obtain the following Bessel equation for $w(z)$:

$$\text{(1.2.4)} \qquad L_1 q_0 = \frac{t^{-\nu_0/2-1}}{4}\left[z^2 \partial_z^2 w(z) + z\partial_z w(z) + (z^2 - \nu_0^2) w(z)\right] = 0\,.$$

The Bessel function

$$\text{(1.2.5)} \qquad J_{\nu_0(x)}(z) = \left(\frac{z}{2}\right)^{\nu_0(x)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu_0(x) + k + 1)} \left(\frac{z}{2}\right)^{2k}, \quad z > 0\,,$$

is a solution of (1.2.4). Taking into account (1.2.1) and (1.2.2) we are led

to define

$$q_0(t, x, \xi) = \Gamma(\nu_0(x) + 1)\left(\frac{z}{2}\right)^{-\nu_0(x)} J_{\nu_0(x)}(z)|_{z=2\sqrt{i}a(x,\xi)}, \tag{1.2.6}$$

which is well defined as an element of Ψ^0 satisfying (1.1.8) provided

$$\nu_0(x) + 1 \notin \{0, -1, -2, \ldots\}, \quad x \in R^n. \tag{1.2.7}$$

From now on we assume that (1.2.7) is satisfied.

1.3. – The other transport equations.

To solve the Cauchy problems $(1.1.9)_j$ we shall use the following integral representation for $J_{\nu_0}(z)$ (see G. Watson [7, p. 163 (1)]):

$$J_{\nu_0}(z) = \frac{\Gamma(\frac{1}{2} - \nu_0)}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu_0}\int_L \exp[iz\sigma](\sigma^2-1)^{\nu_0-\frac{1}{2}}\check{d}\sigma, \left(\check{d}\sigma = \frac{1}{2\pi i}\,d\sigma\right), \tag{1.3.1}$$

where L is the contour shown in fig. 2 and the argument of $\sigma + 1$ and $\sigma - 1$ is chosen to be zero at the point A.

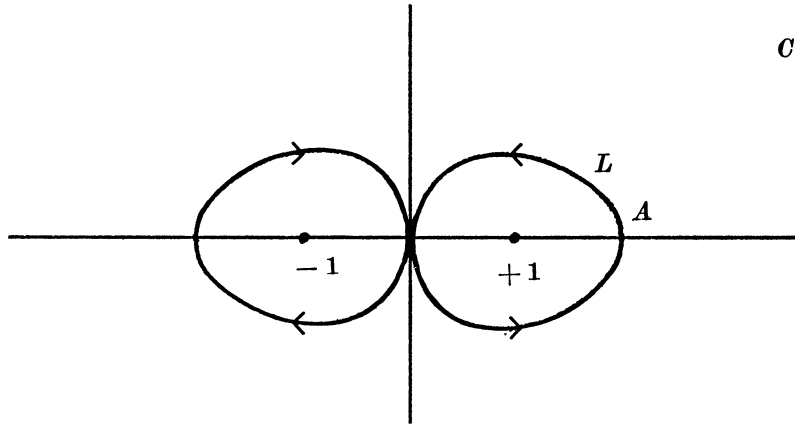

Fig. 2 – L is a contour symmetric with respect to the origin, enclosing the points ± 1.

The above representation makes sense provided

$$\nu_0(x) - \tfrac{1}{2} \notin \{0, 1, 2, \ldots\}, \quad x \in R^n, \tag{1.3.2}$$

which is a technical condition we shall assume from now on.

Putting

$$\hat{q}_0(z;x)=\int_L \exp[iz\sigma](\sigma^2-1)^{\nu_0(x)-\frac{1}{2}}\check{d}\sigma\,, \tag{1.3.3}$$

we have proved in Sect. 1.2 that

$$q_0(t,x,\xi)=\frac{\Gamma(\nu_0(x)+1)\,\Gamma(\frac{1}{2}-\nu_0(x))}{\sqrt{\pi}}\,\hat{q}_0(2\sqrt{t}\,a(x,\xi);x)\,. \tag{1.3.4}$$

To state the main result of this Sect. we need some definitions.

DEF. 1.3.1. Let $p\in Z$, $q\in Z_+$. Define

$$\begin{cases}\varphi_{p,q}(z;x)=\displaystyle\int_L \exp[iz\sigma](\sigma^2-1)^{\nu_0(x)-p-\frac{1}{2}}(\log(\sigma^2-1))^q\check{d}\sigma\\ \tilde{\varphi}_{p,q}(z;x)=\displaystyle\int_L \exp[iz\sigma]\sigma(\sigma^2-1)^{\nu_0(x)-p-\frac{1}{2}}(\log(\sigma^2-1))^q\check{d}\sigma\,.\end{cases} \tag{1.3.5}$$

Let $\alpha,\beta,j\in Z_+$. By $U^{\alpha,\beta}_{-j}$ we denote the class of all functions g of the form

$$g(z;x,\xi)=\sum_{p=0}^{\alpha}\sum_{q=0}^{\beta}c_{p,q}(x,\xi)\varphi_{p,q}(z;x)\,, \tag{1.3.6}$$

where $c_{p,q}\in\mathcal{O}^{-j}$.

Note that the functions $\varphi_{p,q}$ and $\tilde{\varphi}_{p,q}$ are holomorphic with respect to the variable z and C^∞ in x; moreover, because of the symmetry of the contour L, the functions $\varphi_{p,q}$ are even functions of z.

From this remark it follows that given $g(z;x,\xi)\in U^{\alpha,\beta}_{-j}$ then $g(2\sqrt{t}\,a(x,\xi));x,\xi)\in\Psi^{-j/2}$.

THEOREM 1.3.1. *For every $j\geqq 0$ there exists a function $\hat{q}_{-j}(z;x,\xi)\in U^{2j,2j}_{-j}$ such that the functions*

$$q_{-j/2}(t,x,\xi)=\hat{q}_{-j}(2\sqrt{t}\,a(x,\xi);x,\xi)\in\Psi^{-j/2}$$

are solutions of the Cauchy problems (1.1.8), $(1.1.9)_j$.

PROOF. By induction on j. For $j=0$ the assertion follows from the construction in Sect. 1.2 and from (1.3.3), (1.3.4).

Let us suppose that we have already found functions $\hat{q}_0,\hat{q}_{-1},\dots,\hat{q}_{-(j-1)}$, $j\geqslant 1$, with $\hat{q}_{-h}\in U^{2h,2h}_{-h}$, such that $q_{-h/2}(t,x,\xi)=\hat{q}_{-h}(2\sqrt{t}\,a(x,\xi);x,\xi)$ satisfies (1.1.8), if $h=0$, and $(1.1.9)_h$ for $h=1,2,\dots,j-1$.

Equation $(1.1.9)_j$ can be rewritten as

$$L_1 q_{-j/2} = -\sum_{\substack{k\geqq 0\\ 2k+1\leqq j}} L_{\frac{1}{2}-k} q_{-(j-2k-1)/2} - \sum_{\substack{k\geqq 0\\ 2k+2\leqq j}} L_{-k} q_{-(j-2k-2)/2}\,. \tag{1.3.7}$$

Moreover, using (1.1.5) we can write

$$L_{\frac{1}{2}-k} = t^k\Big[\sum_{j=1}^{n} c_{j,k}(x,\xi)\,\partial_{z_j} + c_{0,k}(x,\xi)\Big], \tag{1.3.8}$$

for some $c_{j,k},\ c_{0,k}\in \mathcal{O}^1$.

Analogously:

$$L_{-k} = t^k \sum_{|\alpha|\leqq 2} c_{\alpha,k}(x)\,\partial_x^\alpha + t^{k+1} A_{k+1}(x,\xi) + t^{k+1}\nu_{k+1}(x)\,\partial_t\,, \tag{1.3.9}$$

where $A_{k+1}(x,\xi)$ is defined in (1.1.4) and $c_{\alpha,k}$ are smooth functions of x.

Since $t^k = z^{2k}/(2a(x,\xi))^{2k}$ and $t\partial_t = \frac{1}{2}z\partial_z$, (1.3.8) and the inductive hypothesis imply that

$$\begin{aligned} -\sum_{\substack{k\geqq 0\\ 2k+1\leqq j}} L_{\frac{1}{2}-k} q_{-(j-2k-1)/2} = \sum_{\substack{k\geqq 0\\ 2k+1\leqq j}} \sum_{\substack{0\leqq p\leqq 2(j-2k-1)\\ 0\leqq q\leqq 2(j-2k-1)}} \\ \cdot\Big[c_{p,q,k}(x,\xi) z^{2k}\varphi_{p,q}(z;x) + \sum_{l=1}^{n} d_{p,q,k,l}(x,\xi)\, z^{2k}\partial_{x_l}\varphi_{p,q}(z;x)\Big], \end{aligned} \tag{1.3.10}$$

for some $c_{p,q,k},\ d_{p,q,k,l}\in \mathcal{O}^{-j+2}$.

In the same way, using (1.3.9) we get

$$\begin{aligned} -\sum_{\substack{k\geqq 0\\ 2k+2\leqq j}} L_{-k} q_{-(j-2k-2)/2} = \sum_{\substack{k\geqq 0\\ 2k+2\leqq j}} \sum_{\substack{0\leqq p\leqq 2(j-2k-2)\\ 0\leqq q\leqq 2(j-2k-2)}} \\ \cdot\Big[c_{p,q,k}(x,\xi) z^{2k+2}\varphi_{p,q}(z;x) + \sum_{|\gamma|\leqq 2} d_{p,q,k,\gamma}(x,\xi) z^{2k}\partial_x^\gamma \varphi_{p,q}(z;x) + z^{2k}\{e_{p,q,k}(x,\xi)\varphi_{p,q}(z;x) \\ + f_{p,q,k}(x,\xi)\varphi_{p+1,q}(z;x) + g_{p,q,k}(x,\xi)\varphi_{p,(q-1)_+} + h_{p,q,k}(x,\xi)\varphi_{p+1,(q-1)_+}\}\Big], \end{aligned} \tag{1.3.11}$$

where the coefficients belong to $\mathcal{O}^{-j+2}$.

Taking into account the formula

$$\partial z/\partial x_j = \frac{\partial_{x_j} a(x,\xi)}{a(x,\xi)}\, z\,, \qquad j = 1, \ldots, n\,, \tag{1.3.12}$$

it is easy to recognize that $\partial_{x_j}\varphi_{p,q}$ is a linear combination of $\varphi_{p,q+1}$ and $z\tilde{\varphi}_{p,q}$ with coefficients in $\mathcal{O}^0$, while $\partial_{x_t}\partial_{x_j}\varphi_{p,q}$ is a linear combination, with coefficients in $\mathcal{O}^0$, of $\varphi_{p,q+1}, \varphi_{p,q+2}, z\tilde{\varphi}_{p,q}, z\tilde{\varphi}_{p,q+1}, z^2\varphi_{p,q}, z^2\varphi_{p-1,q}$.

Now we note that the operator L_1 written in the z variable becomes:

$$L_1 = A_0(x,\xi)\left[\partial_z^2 + \frac{2\nu_0(x)+1}{z}\,\partial_z + 1\right]. \tag{1.3.13}$$

Taking into account the preceding remark, formulas (1.3.10), (1.3.11) and (1.3.13), it is easily seen that to solve eq. $(1.1.9)_j$ it is enough to handle the following equations:

$$\left(z\partial_z^2 + (2\nu_0(x)+1)\,\partial_z + z\right)w(z) = \begin{cases} \begin{cases} z^{2k+1}\varphi_{p,q} \\ z^{2k+2}\varphi_{p,q} \end{cases}, & p,\, q \geqq 0 \\ z^{2k+2}\tilde{\varphi}_{p,q}\,, & p,\, q \geqq 0 \\ z^{2k+3}\varphi_{p,q}\,, & p \geqq -1\,,\ q \geqq 0\,. \end{cases} \tag{1.3.14}$$

We look for a solution of (1.3.14) of the form

$$w(z) = \int_L \exp\,[iz\sigma](\sigma^2-1)^{\nu_0(x)-\frac{1}{2}}\theta(\sigma)\,\check{d}\sigma\,. \tag{1.3.15}$$

Since

$$\left(z\partial_z^2 + (2\nu_0(x)+1)\,\partial_z + z\right)w(z) = \frac{1}{i}\int_L \exp\,[iz\sigma]\,(\sigma^2-1)^{\nu_0(x)+\frac{1}{2}}\,\frac{d\theta}{d\sigma}(\sigma)\,\check{d}\sigma\,,$$

we are reduced to solving the equations:

$$(\sigma^2-1)^{\nu_0(x)+\frac{1}{2}}\,\frac{d\theta}{d\sigma}(\sigma) \tag{1.3.16}$$

$$= \begin{cases} \partial_\sigma^{2s+1}(\sigma^2-1)^{\nu_0-p-\frac{1}{2}}(\log(\sigma^2-1))^q\,, & s = k\,,\ k+\tfrac{1}{2}\,,\ p,\, q \geqq 0 \\ \partial_\sigma^{2k+2}\sigma(\sigma^2-1)^{\nu_0-p-\frac{1}{2}}(\log(\sigma^2-1))^q\,, & p\,, q \geqq 0 \\ \partial_\sigma^{2k+3}(\sigma^2-1)^{\nu_0-p-\frac{1}{2}}(\log(\sigma^2-1))^q\,, & p \geqq -1\,,\ q \geqq 0 \end{cases}$$

As a preliminary remark we note that the integration of the second equation in (1.3.16) can be actually reduced to that of the first one in the above formula (with $s = k$ and $p,\, p+1,\, q,\, (q-1)_+$).

Now, by induction, the following formula can be easily proved

$$\partial_\sigma^{2h+1}\left[(\sigma^2-1)^{\nu_0(x)-\frac{1}{2}-l}(\log(\sigma^2-1))^m\right], \tag{1.3.17}$$

$$= \sum_{\substack{h+1\leqq j\leqq 2h+1 \\ 0\leqq i\leqq \min\{m,\,2h+1\}}} c_{ij}(x)\,2\sigma(\sigma^2-1)^{\nu_0(x)-\frac{1}{2}-l-j}(\log(\sigma^2-1))^{m-i} \qquad m,\, h \in Z_+,\quad l \in Z\,,$$

for some smooth functions $c_{ij}(x)$.

Using (1.3.17) we conclude that eq. (1.3.16) can be reduced to the form:

$$\frac{d\theta}{d\sigma}(\sigma) = 2\sigma(\sigma^2-1)^{-r}(\log(\sigma^2-1))^s, \qquad r, s \in Z_+, \quad r \geqq 1. \tag{1.3.18}$$

Equation (1.3.18) is immediately solved:

$$\theta(\sigma) = \begin{cases} (\sigma^2-1)^{1-r}\sum_{j=0}^{s} c_j(\log(\sigma^2-1))^{s-j}, & \text{if } r>1 \\ \dfrac{(\log(\sigma^2-1))^{s+1}}{s+1}, & \text{if } r=1, \end{cases} \tag{1.3.19}$$

where the c_j are suitable constants.

From the above results it follows that eq. (1.3.7) has a solution which, when written in the z-variable, is a linear combination with coefficients in $\mathcal{O}^{-j}$ of functions $\varphi_{p,q}(z; x)$ with $p, q \leqq 2j$. Therefore we have proved that there exists a function $Q_{-j}(z; x, \xi) \in U_{-j}^{2j,2j}$ for which

$$L_1 Q_{-j}(2\sqrt{t}\,a(x,\xi); x, \xi) = -\sum_{h=1}^{j} L_{1-h/2} q_{-j/2+h/2}(t, x, \xi), \qquad t>0.$$

Since $L_1\varphi_{0,0}(2\sqrt{t}\,a(x,\xi); x) = 0$ and $Q_{-j}(0; x, \xi) \in \mathcal{O}^{-j}$, it is enough to put

$$\hat{q}_{-j}(z; x, \xi) = Q_{-j}(z; x, \xi) - \frac{Q_{-j}(0; x, \xi)}{\varphi_{0,0}(0; x)}\varphi_{0,0}(z; x), \tag{1.3.20}$$

which proves the theorem. q.e.d.

REMARK. It is worthwhile to point out that if the coefficients a_{ij} in (0.1) are constants then Theorem 1.3.1 holds with $\hat{q}_{-j}(z; x, \xi) \in U_{-j}^{j,2j}$, $j \geqq 0$. This is a consequence of two remarks:

a) in L_{-k} (see (1.1.5)) there is not the term $t^{k+1}A_{k+1}(x, \xi)$;

b) $\partial_{x_j}\varphi_{p,q} = (\partial_{x_j}\nu_0)\varphi_{p,q+1}$, $\partial^2_{x_i x_j}\varphi_{p,q} = (\partial^2_{x_i x_j}\nu_0)\varphi_{p,q+1} + (\partial_{x_i}\nu_0)(\partial_{x_j}\nu_0)\varphi_{p,q+2}$.

Then only the first equation in (1.3.14) must be solved.

1.4. – Formal parametrix in the region $\sqrt{t}\,|\xi| > \text{const}$.

We shall use the notation

$$\begin{cases} A(t, x, \xi) = \sum_{i,j=1}^{n} a_{i,j}(t, x)\xi_i\xi_j \\ B(t, x, \xi) = \sum_{j=1}^{n} b_j(t, x)\xi_j. \end{cases} \tag{1.4.1}$$

Denote by $\Psi^{\pm}(s, x, \xi)$ the solution of the non-linear Cauchy problem

$$\text{(1.4.2)} \qquad \begin{cases} \dfrac{\partial \Psi^{\pm}}{\partial s}(s, x, \xi) = \pm \sqrt{A(s^2/4, x, d_x \Psi^{\pm}(s, x, \xi))} \\ \Psi^{\pm}(0, x, \xi) = x \cdot \xi \,. \end{cases}$$

It is well known that pb. (1.4.2) has a unique solution $\Psi^{\pm}(s, x, \xi) \in C^{\infty}([0, 2\sqrt{T}] \times R^n_x \times \dot{R}^n_{\xi})$ for a suitable $T > 0$.

Define

$$\text{(1.4.3)} \qquad \varphi^{\pm}(t, x, \xi) = \Psi^{\pm}(2\sqrt{t}, x, \xi)\,, \qquad t \in [0, T]\,.$$

Thus $\varphi^{\pm}$ solves the eikonal equation:

$$\text{(1.4.4)} \qquad \begin{cases} \sqrt{t}\, \dfrac{\partial \varphi^{\pm}}{\partial t}(t, x, \xi) = \pm \sqrt{A(t, x, d_x \varphi^{\pm}(t, x, \xi))}\,, \qquad 0 < t \leqq T \\ \varphi^{\pm}(0, x, \xi) = x \cdot \xi \,. \end{cases}$$

We explicitly note that $\varphi^{\pm}$ is not a smooth function of t at $t = 0$.

Writing the formal Taylor series of $\Psi^{\pm}$ with respect to the s variable and putting as in (1.2.1) $z = 2\sqrt{t}\, a(x, \xi)$, we can write

$$\text{(1.4.5)} \qquad \varphi^{\pm}(t, x, \xi) \sim x \cdot \xi \pm z + R^{\pm}(z; x, \xi)\,,$$

with

$$\text{(1.4.6)} \qquad R^{\pm}(z; x, \xi) \sim \sum_{k \geqq 2} \alpha^{\pm}_{1-k}(x, \xi) z^k\,, \qquad \alpha^{\pm}_{1-k} \in \mathcal{O}^{1-k}\,, \quad k \geqq 2\,.$$

The following definition will be convenient.

DEFINITION 1.4.1. *Let m be a real number.*

i) *By $\dot{\Psi}^m$ we denote the class of the functions $f(t, x, \xi) \in C^{\infty}(R^+_t \times R^n_x \times \dot{R}^n_{\xi})$ such that*

$$f(t/\lambda, x, \sqrt{\lambda}\xi) = \lambda^m f(t, x, \xi)\,, \qquad \lambda > 0\,.$$

ii) *By Φ^m we denote the class of the functions $g(z, x, \xi) \in C^{\infty}(R^+_z \times R^n_x \times \dot{R}^n_{\xi})$ such that*

$$g(z, x, \lambda\xi) = \lambda^m g(z, x, \xi)\,, \qquad \lambda > 0\,.$$

We note that the map

$$\text{(1.4.7)} \qquad \dot{\Psi}^m \ni f(t, x, \xi) \to \tilde{f}(z, x, \xi) = f(z/4a(x, \xi)^2, x, \xi) \in \Phi^{2m}$$

is a bijection.

From now on we shall denote by $\varrho_{-k}(x, \xi)$ (or $\varrho^{\pm}_{-k}(x, \xi)$) elements of $\mathcal{O}^{-k}$ which we do not need to specify.

We are looking for an operator formally defined by

$$Eg(t, x) = \int \{\exp[i\varphi^+(t, x, \xi)]p^+(t, x, \xi) + \exp[i\varphi^-(t, x, \xi)]p^-(t, x, \xi)\} \cdot \hat{g}(\xi)\, \check{d}\xi\,, \tag{1.4.8}$$

where $p^{\pm}(t, x, \xi) \sim \sum_{j \geqq 0} p^{\pm}_{-j/2}(t, x, \xi)$, $p^{\pm}_{-j/2} \in \Psi^{-j/2}$ $j \geqq 0$, and such that $PEg = 0$ for $t > 0$.

As usual we require that

$$\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \sim 0\,. \tag{1.4.9}$$

Using (1.4.4), (1.4.5), by a computation we obtain

$$\begin{aligned} &\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \\ &\sim [t\partial_t^2 + (\nu(t, x) + 1)\partial_t + 2i(t\partial_t\varphi^{\pm})\partial_t]p^{\pm} \pm i\frac{a(x, \xi)^2(2\nu(t, x) + 1)}{z}p^{\pm} \\ &+ i\{[t\partial_t^2 + (\nu(t, x) + 1)\partial_t]R^{\pm} - A(t, x, \partial_x)\varphi^{\pm} + B(t, x, \partial_x)\varphi^{\pm}\}p^{\pm} \\ &+ 2i\sum_{i,j=1}^{1} a_{ij}(t, x)\partial_{x_i}\varphi^{\pm}\partial_{x_j}p^{\pm} + [-A(t, x, \partial_x) + B(t, x, \partial_x) + b_0(t, x)]p^{\pm}\,, \end{aligned} \tag{1.4.10}$$

where z is given by (1.2.1).

We can write

$$\begin{cases} \nu(t, x) \sim \nu_0(x) + \sum_{k \geqq 1} \varrho_{-2k} z^{2k} \\ 2it\partial_t\varphi^{\pm} \sim \pm iz + iz\partial_z R^{\pm} \sim \pm iz + \sum_{k \geqq 2} \varrho^{\pm}_{1-k}(x, \xi) z^k \\ \pm i\dfrac{a(x, \xi)^2(2\nu(t, x) + 1)}{z} \sim \pm i\dfrac{a(x, \xi)^2(2\nu_0(x) + 1)}{z} \\ \qquad\qquad \pm \sum_{k \geqq 1} \varrho_{2-2k}(x, \xi) z^{2k-1} \\ i[t\partial_t^2 + (\nu(t, x) + 1)\partial_t]R^{\pm} \sim \sum_{k \geqq 1} \varrho^{\pm}_{1-k}(x, \xi) z^k \\ -iA(t, x, \partial_x)\varphi^{\pm} \sim \sum_{k \geqq 1} \varrho^{\pm}_{1-k}(x, \xi) z^k \end{cases} \tag{1.4.11}$$

$$
(1.4.11)\qquad \begin{cases} iB(t,x,\partial_x)\varphi^{\pm} \sim \sum\limits_{k\geqq 1} \varrho^{\pm}_{1-k}(x,\xi) z^k \\ 2i \sum\limits_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_i}\varphi^{\pm}\partial_{x_j} \sim \sum\limits_{j=1}^{n}\sum\limits_{k\geqq 0} \varrho^{\pm}_{1-k}(x,\xi) z^k \partial_{x_j} \\ A(t,x,\partial_x) \sim \sum\limits_{i,j=1}^{n}\sum\limits_{k\geqq 0} \varrho_{-2k}(x,\xi) z^{2k}\partial^2_{x_i x_j} \\ B(t,x,\partial_x) \sim \sum\limits_{j=1}^{n}\sum\limits_{k\geqq 0} \varrho_{-2k}(x,\xi) z^{2k}\partial_{x_j} \\ b_0(t,x) \sim \sum\limits_{k\geqq 0} \varrho_{-2k}(x,\xi) z^{2k}\,. \end{cases}
$$

Substituting (1.4.11) into (1.4.10) and replacing $p^{\pm}(t,x,\xi)$ by $\tilde{p}^{\pm}(z,x,\xi)$ according to (1.4.7), yields

$$
\begin{aligned}
(1.4.12)\qquad & \exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \\
&\sim a(x,\xi)^2\left[\partial_z^2 + \frac{2\nu_0(x)+1 \pm iz}{z}\partial_z \pm i\frac{2\nu_0(x)+1}{z}\right]\tilde{p}^{\pm} \\
&+ \sum_{k\geqq 0}\varrho_{-2k}(x,\xi)\, z^{2k+1}\,\partial_z\,\tilde{p}^{\pm} + \sum_{k\geqq 0}\varrho^{\pm}_{1-k}(x,\xi)\, z^{k+1}\,\partial_z\,\tilde{p}^{\pm} \\
&+ \sum_{k\geqq 0}\varrho^{\pm}_{1-k}(x,\xi) z^k \tilde{p}^{\pm} + \sum_{k\geqq 0}\varrho_{-2k}(x,\xi) z^{2k}\tilde{p}^{\pm} + \sum_{k\geqq 0}\varrho^{\pm}_{-k}(x,\xi) z^{k+1}\tilde{p}^{\pm} \\
&+ \sum_{k\geqq 0}\Big(\sum_{l=1}^{n}\varrho^{\pm,(l)}_{1-k}(x,\xi) z^k \partial_{x_l}\Big)\tilde{p}^{\pm} + \sum_{k\geqq 0}\Big(\sum_{l=1}^{n}\varrho^{(l)}_{-2k}(x,\xi) z^{2k}\partial_{x_l}\Big)\tilde{p}^{\pm} \\
&+ \sum_{k\geqq 0}\varrho_{-2k}(x,\xi) z^{2k+2}\partial_z^2\tilde{p}^{\pm} + \sum_{k\geqq 0}\Big(\sum_{l=1}^{n}\varrho^{(l)}_{-2k}(x,\xi) z^{2k+1}\partial^1_{x_l z}\Big)\tilde{p}^{\pm} \\
&+ \sum_{k\geqq 0}\Big(\sum_{i,j=1}^{n}\varrho^{(i,j)}_{-2k}(x,\xi) z^{2k}\partial^2_{x_i x_j}\Big)\tilde{p}^{\pm}\,.
\end{aligned}
$$

Let us define the following operators:

$$
(1.4.13)\qquad \mathfrak{L}^{\pm}_1(z,x,\xi;\partial_z) = a(x,\xi)^2\left\{\partial_z^2 + \frac{2\nu_0(x)+1\pm 2iz}{z}\partial_z \pm i\frac{2\nu_0(x)+1}{z}\right\}
$$

$$
\begin{aligned}
(1.4.14)\qquad \mathfrak{L}^{(k),\pm}_{1/2}(z,x,\xi;\partial_z,\partial_x) &= \varrho^{\pm}_{1-k}(x,\xi)z^{k+1}\partial_z + \sum_{l=1}^{n}\varrho^{\pm,(l)}_{1-k}(x,\xi)z^k\partial_{x_l} + \\
&\quad + \varrho^{\pm}_{1-k}(x,\xi)z^k\,, \qquad k=0,1,\ldots
\end{aligned}
$$

$$
\begin{aligned}
(1.4.15)\qquad \mathfrak{L}^{(k),\pm}_0(z,x,\xi;\partial_x,\partial_z) &= \varrho_{-k}(x,\xi)z^{k+2}\partial_z^2 + \sum_{l=1}^{n}\varrho^{(l)}_{-k}(x,\xi)z^{k+1}\partial^2_{x_l z} \\
&+ \sum_{i,j=1}^{n}\varrho^{(i,j)}_{-k}(x,\xi)z^k\partial^2_{x_i x_j} + \varrho_{-k}(x,\xi)z^{k+1}\partial_z \\
&+ \sum_{l=1}^{n}\varrho^{(l)}_{-k}(x,\xi)z^k\partial_{x_l} + \varrho^{\pm}_{-k}(x,\xi)z^k + \varrho^{\pm}_{-k}(x,\xi)z^{k+1}\,, \qquad k=0,1,2,\ldots\,.
\end{aligned}
$$

We explicitly note that when k is odd, in $\mathfrak{L}_0^{(k),\pm}$ all the coefficients vanish except for $\varrho_{-k}^{\pm}(x,\xi)z^{k+1}$.

It is worth remarking that

$$\mathfrak{L}_1^{\pm}\colon \Phi^m \to \Phi^{m+2}, \quad \mathfrak{L}_{\frac{1}{2}}^{(k),\pm}\colon \Phi^m \to \Phi^{m+1-k}, \quad \mathfrak{L}_0^{(k),\pm}\colon \Phi^m \to \Phi^{m-k}, \qquad k \geqq 0.$$

Using (1.4.13)-(1.4.15) and writing $\tilde{p}^{\pm} \sim \sum_{j\geqq 0} \tilde{p}_{-j/2}^{\pm}$, we can put (1.4.12) in the final form

$$\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \sim \sum_{k\geqq 0}\Big[\mathfrak{L}_1^{\pm}\tilde{p}_{-k/2}^{\pm} + \sum_{\substack{j,h\geqq 0\\ j+h=k-1}} \mathfrak{L}_{\frac{1}{2}}^{(j),\pm}\tilde{p}_{-h/2}^{\pm} + \sum_{\substack{j,h\geqq 0\\ j+h=k-2}} \mathfrak{L}_0^{(j),\pm}\tilde{p}_{-h/2}^{\pm}\Big]. \tag{1.4.16}$$

In (1.4.16) we use the convention that a sum over negative integers is zero.

To implement (1.4.9) we are forced to solve the following sequence of transport equations:

$$\mathfrak{L}_1^{\pm}\tilde{p}_0^{\pm} = 0, \quad z>0, \tag{1.4.17}$$

$$\mathfrak{L}_1^{\pm}\tilde{p}_{-\frac{1}{2}}^{\pm} = -\mathfrak{L}_{\frac{1}{2}}^{(0),\pm}\tilde{p}_0^{\pm}, \quad z>0 \tag{1.4.18}$$

$$\mathfrak{L}_1^{\pm}\tilde{p}_{-k/2}^{\pm} = -\Big(\sum_{\substack{j,h\geqq 0\\ j+h=k+1}} \mathfrak{L}_{\frac{1}{2}}^{(j),\pm}\tilde{p}_{-h/2}^{\pm} + \sum_{\substack{j,h\geqq 0\\ j+h=k-2}} \mathfrak{L}_0^{(j),\pm}\tilde{p}_{-h/2}^{\pm}\Big), \quad z>0, \quad k\geqq 2. \tag{1.4.19}_k$$

1.5. – The first transport equation $\mathfrak{L}_1^{\pm}\tilde{p}_0^{\pm} = 0$.

The following transmutation formula will play a crucial role in the sequel:

$$\exp[\pm iz]\mathfrak{L}_1^{\pm}(\exp[\mp iz]G(z)) = \frac{a(x,\xi)^2}{z}M(z,x;\partial_z)G(z), \tag{1.5.1}$$

where

$$M(z,x;\partial_z) = z\partial_z^2 + (2\nu_0(x)+1)\partial_z + z. \tag{1.5.2}$$

Under the hypothesis (1.3.2) the equation $MG(z)=0$, $z>0$, has two independent solutions given by:

$$I_{\nu_0}^{\pm}(x;z) = \frac{\Gamma(\frac{1}{2}-\nu_0(x))}{\sqrt{\pi}}\int_{L^{\pm}} \exp[iz\sigma](\sigma^2-1)^{\nu_0(x)-\frac{1}{2}}\check{d}\sigma, \tag{1.5.3}$$

where $L^{\pm}$ are the contours shown in fig. 3.

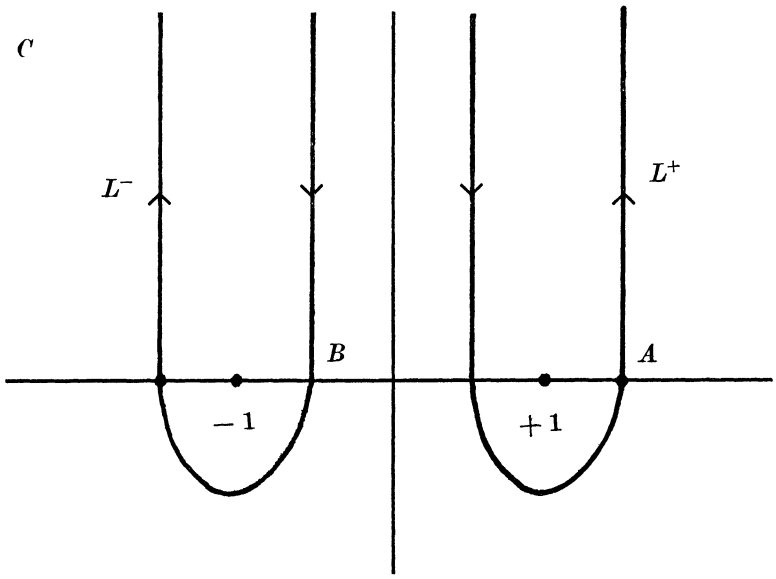

Fig. 3 – The contours $L^{\pm}$; $\arg(\sigma+1)$ and $\arg(\sigma-1)$ is chosen to be 0 in A and $-\pi$ in B.

From G. Watson [7, p. 167, (6), (7)] it follows that

$$\text{(1.5.4)}\qquad \begin{cases} I^{+}_{\nu_0}(x;z) = (z/2)^{-\nu_0(x)} H^{(1)}_{\nu_0(x)}(z) \\ I^{-}_{\nu_0}(x;z) = (z/2)^{-\nu_0(x)} H^{(2)}_{\nu_0(x)}(z) \\ I^{+}_{\nu_0} + I^{-}_{\nu_0} = (z/2)^{-\nu_0(x)} J(z)_{\nu_0(x)}\,, \end{cases}$$

where $H^{(1)}_{\nu_0}, H^{(2)}_{\nu_0}$ are the Hankel functions and $J_{\nu_0(x)}(z)$ is given by (1.2.5).

Using (1.5.1) we solve the first transport equation (1.4.17) putting:

$$\text{(1.5.5)}\qquad \begin{cases} \tilde{p}^{+}_0(z,x,\xi) = \Gamma(\nu_0(x)+1)\exp[-iz] I^{+}_{\nu_0}(x;z) \\ \tilde{p}^{-}_0(z,x,\xi) = \Gamma(\nu_0(x)+1)\exp[iz] I^{-}_{\nu_0}(x;z)\,. \end{cases}$$

We point out that $\tilde{p}^{\pm}_0 \in \Phi^0$, i.e. $p^{\pm}_0(t,x,\xi) = \tilde{p}^{\pm}_0(2\sqrt{t}\,a(x,\xi),x,\xi) \in \Psi^0$, and $\tilde{p}^{+}(0,x,\xi) + \tilde{p}^{-}(0,x,\xi) = 1$.

1.6. – The other transport equations.

To solve eq. (1.4.18), $(1.4.19)_k$, we shall need the following definitions.

DEFINITION 1.6.1. *Let*

$$S^0 = \{\sigma \in C| \ |\mathrm{Re}\,\sigma| < 2, \sigma \neq iy, y \geqq 0\}$$

$$S^{\pm} = S_0 \pm 1$$

$$\hat{S}_0 = \{\sigma \in C| \ |\mathrm{Re}\,\sigma| < 2, \sigma \neq 0\} .$$

By $\mathcal{A}^0_{-k,j}$, $j, k \in Z_+$, *we denote the class of functions* $\psi(\sigma, x, \xi) \in C^\infty(S^0 \times R^n_x \times \dot{R}^n_\xi)$ *such that*:

i) $\psi(\sigma, x, \lambda\xi) = \lambda^{-k}\psi(\sigma, x, \xi)$, $\lambda > 0$.

ii) *For every* $\alpha, \beta \in Z^n_+$ $\partial^\alpha_x \partial^\beta_\xi \psi$ *can be written in the form*

$$\partial^\alpha_x \partial^\beta_\xi \psi = \sum_{l=0}^{m_{\alpha\beta}} \psi^{\alpha\beta}_l(\sigma, x, \xi)(\log \sigma)^l ,$$

where the $\psi^{\alpha\beta}_l$ *are holomorphic functions of* σ *in* $\hat{S}_0$ *having a pole of order at most* j *at* $\sigma = 0$. *Furthermore, for every* $K \subset\subset R^n_x$ *and for every* $\delta \in]0, 1[$ *there exists a non-negative integer* $N = N(\alpha, \beta, l, K, \delta)$ *such that*

$$\sup_{\substack{x \in K, |\xi| = 1 \\ |\mathrm{Re}\,\sigma| \leqq 1-\delta \\ |\mathrm{Im}\,\sigma| \geqq 1}} |\mathrm{Im}\,\sigma|^{-N} |\psi^{\alpha\beta}_l(\sigma, x, \xi)| < +\infty \tag{1.6.1}$$

(*here* $\log \sigma$ *is defined cutting* C *along the positive imaginary axis*). *By* $\mathcal{A}^{\pm}_{-k,j}$ *we denote che class of functions* $\psi^{\pm}(\sigma, x, \xi)$ *defined in* $S^{\pm} \times R^n_x \times \dot{R}^n_\xi$ *such that* $\psi^{\pm}(\sigma \pm 1, x, \xi) \in \mathcal{A}^0_{-k,j}$.

DEFINITION 1.6.2. *Let* $\psi^{\pm}(\sigma, x, \xi) \in \mathcal{A}^{\pm}_{-k,j}$. *We define*

$$I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm}) = \int_{L^{\pm}} \exp[iz\sigma]\,(\sigma \mp 1)^{\nu_0(x) - \frac{1}{2}} \psi^{\pm}(\sigma; x, \xi)\, \check{d}\sigma , \tag{1.6.2}$$

where $L^{\pm}$ *are the contours described in fig.* 3.

We remark that, for $\psi^{\pm} \in \mathcal{A}^{\pm}_{-k,j}$, $I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm}) \in \Phi^{-k}$.

Furthermore (1.5.3) can be rewritten as $I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm})$ with $\psi^{\pm}(\sigma, x, \xi) = \Gamma(\frac{1}{2} - \nu_0(x))\pi^{-\frac{1}{2}}(\sigma \pm 1)^{\nu_0(x) - \frac{1}{2}} \in \mathcal{A}^{\pm}_{0,0}$.

We have the following result.

THEOREM 1.6.1. *Let $\tilde{p}_0^{\pm}$ be defined as in* (1.5.5). *Then for every $k \geqq 1$ the transport equations* (1.4.18), $(1.4.19)_k$ *have a solution of the form*

$$\tilde{p}^{\pm}_{-k/2}(z, x, \xi) = \exp[\mp iz] I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm}_{-k}) \tag{1.6.3}$$

for suitable $\psi^{\pm}_{-k} \in \mathcal{A}^{\pm}_{-k,k}$.

PROOF. As we noted above the first transport equation (1.4.17) has already been solved by a function of the form (1.6.3); thus we can proceed by induction on k. Suppose we have already constructed $\tilde{p}_0^{\pm}, \ldots, \tilde{p}^{\pm}_{-(k-1)/2}$ of the form (1.6.3); let us now try to find $\tilde{p}^{+}_{-k/2}$ (the case $\tilde{p}^{-}_{-k/2}$ is quite analogous).

We look for $\tilde{p}^{+}_{-k/2}$ of the form $\exp[- iz] G(z, x, \xi)$. Using (1.5.1) we obtain

$$\begin{aligned} \mathfrak{L}_1^{+}(\exp[- iz] G) = a(x, \xi)^2 \exp[- iz] \frac{1}{z} M(z, x, \partial_z) G(z) = \\ = - \sum_{\substack{j,h \geqq 0 \\ j+h=k-1}} \mathfrak{L}_{1/2}^{(j),+}(\exp[- iz] I^{+}(z, x, \xi; \nu_0, \psi^{\pm}_{-h})) \\ - \sum_{\substack{j,h \geqq 0 \\ j+h=k-2}} \mathfrak{L}_{0}^{(j),+}(\exp[- iz] I^{+}(z, x, \xi; \nu_0, \psi^{\pm}_{-h})) , \end{aligned} \tag{1.6.4}$$

where $\psi^{+}_{-h} \in \mathcal{A}^{+}_{-h,h}$ are the functions appearing in $\tilde{p}^{+}_{-h/2}$, $h = 0, 1, \ldots, k-1$.

The last sum in (1.6.4) vanishes if $k = 1$.

A straightforward computation shows that the r.h.s. in (1.6.4) can be written in the form $\exp[- iz](I^{+}(z, x, \xi; \nu_0, \chi) + I^{+}(z, x, \xi; \nu_0, \mu))$ for some $\chi \in \mathcal{A}^{+}_{2-k,k-1}$, $\mu \in \mathcal{A}^{+}_{2-k,k-2} \subset \mathcal{A}^{+}_{2-k,k-1}$.

We are thus reduced to solve the equation

$$M(z, x; \partial_z) G(z) = I^{+}(z, x, \xi; \nu_0, \psi) , \tag{1.6.5}$$

where $\psi \in \mathcal{A}^{+}_{-k,k}$.

We look for a G in the form

$$G(z, x, \xi) = \int_{L^{+}} \exp[iz\sigma](\sigma^2 - 1)^{\nu_0(x) - \frac{1}{2}} \Phi(\sigma, x, \xi) \check{d}\sigma . \tag{1.6.6}$$

For Φ we obtain the equation

$$\begin{aligned} \frac{d\Phi}{d\sigma}(\sigma, x, \xi) &= i(\sigma^2 - 1)^{-\nu_0(x) - \frac{1}{2}} (\sigma - 1)^{\nu_0(x) - \frac{1}{2}} \psi(\sigma, x, \xi) \\ &= f(\sigma, x, \xi) \in \mathcal{A}^{+}_{-k,k+1} . \end{aligned} \tag{1.6.7}$$

By Def. 1.6.1, $f(\sigma, x, \xi) = g(\sigma - 1, x, \xi)$ with $g \in \mathcal{A}^0_{-k,+1}$; thus we can write

$$(1.6.8) \qquad g(\sigma-1, x, \xi) = \sum_{l=0}^{m} g_l(\sigma-1, x, \xi)(\log(\sigma-1))^l =$$

$$= \sum_{l=0}^{m} \sum_{j=0}^{k+1} \frac{c_{jl}(x, \xi)}{(\sigma-1)^j} (\log(\sigma-1))^l + \sum_{l=0}^{m} \theta_l(\sigma-1, x, \xi)(\log(\sigma-1))^l,$$

for some $c_{jl} \in \mathcal{O}^{-k}$ and some functions $\theta_l(\zeta, x, \xi)$ holomorphic in the strip $|\mathrm{Re}\,\zeta| < 2$ and vanishing at $\zeta = 0$.

It is now a trivial fact to recognize that eq. (1.6.7) can be solved within the class $\mathcal{A}^+_{-k,k}$. q.e.d.

1.7. – Asymptotic expansions of some integrals.

In this Sect. we study the asymptotic expansion for $z \to +\infty$ of integrals of the following type:

$$(1.7.1) \qquad I(z, x, \xi; \nu_0, \psi) = \int_{L_0} \exp[iz\sigma]\sigma^{\nu_0(x)-\frac{1}{2}}\psi(\sigma, x, \xi)\,d\sigma\,, \qquad z > 0\,,$$

where $\psi \in \mathcal{A}^0_{-k,j}$ and L_0 is the contour shown in fig. 4.

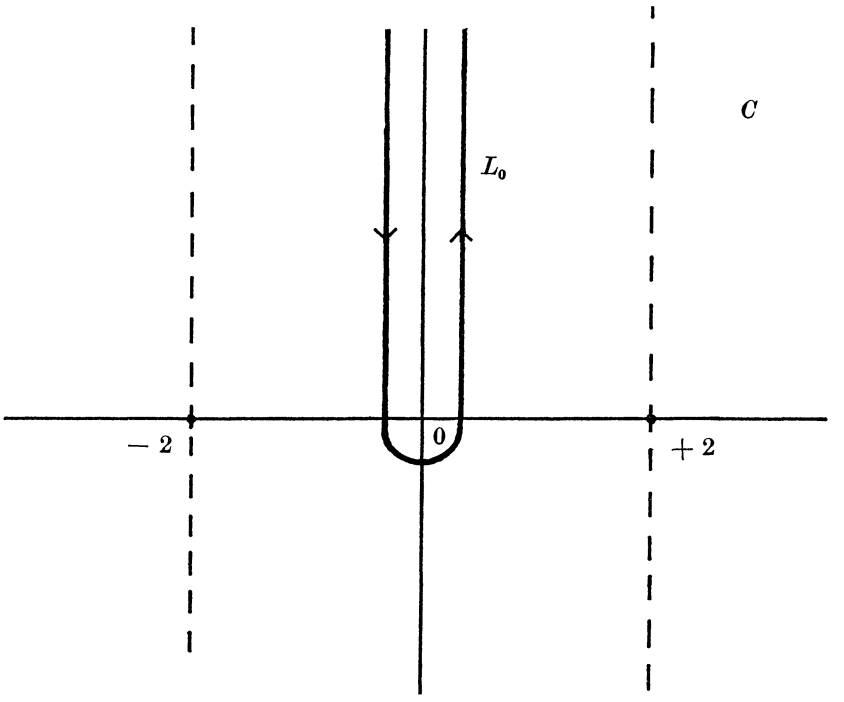

Fig. 4 – The contour L_0.

Here and in the sequel we shall always suppose that condition (1.3.2) is satisfied.

Performing the change of variables $\sigma = iu/z$, $d\sigma = i/z\,du$, since for $\sigma \in \mathcal{S}^0$ $\log \sigma = \log(iu) - \log z$, we obtain

$$(1.7.2) \qquad I(z, x, \xi; \nu_0, \psi) = iz^{-\nu_0(x)-\frac{1}{2}} \int_\gamma \exp[-u](iu)^{\nu_0(x)-\frac{1}{2}} \psi\left(\frac{iu}{z}, x, \xi\right) \check{d}u\,,$$

where γ is the contour shown in fig. 5.

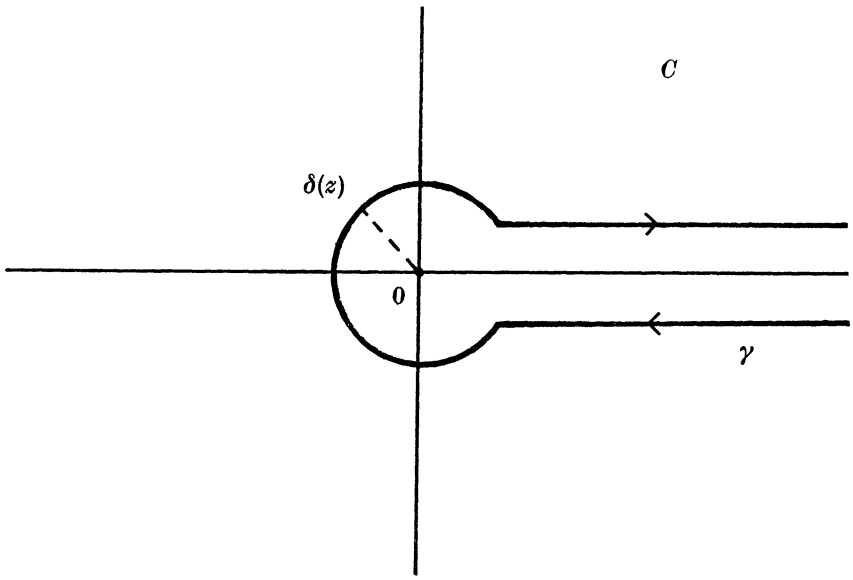

Fig. 5 – The contour γ; the radius $\delta(z)$ is chosen such that $0 < \delta(z) < z/2$.

Let us prove the following lemmas.

LEMMA 1.7.1. *Let $\psi \in \mathcal{A}^0_{-k,0}$. Then, for every $K \subset\subset R^n_x$, $M \in Z_+$, $\varepsilon > 0$, there exists a constant $C = C(M, \varepsilon, K, \psi) > 0$ such that*

$$(1.7.3) \qquad |I(z, x, \xi; \nu_0, \sigma^M \psi(\sigma, x, \xi))| \leqq C z^{-\operatorname{Re}\nu_0(x)-\frac{1}{2}-M+\varepsilon}, \quad (z, x, \xi) \in [1, +\infty[\times K \times \mathcal{S}^{n-1}.$$

PROOF. – For $z \geqq 1$ we choose $\delta = \delta(z) = \frac{1}{2}$; then, by Def. 1.6.1, $I(z, x, \xi; \nu_0, \sigma^M \psi)$ is linear combination of integrals of the type

$$iz^{-\nu_0(x)-\frac{1}{2}-M} \int_\gamma \exp[-u](iu)^{\nu_0(x)+M-\frac{1}{2}} \chi\left(\frac{iu}{z}, x, \xi\right)\left(\log\left(\frac{iu}{z}\right)\right)^l \check{d}u\,,$$

where $\chi(\sigma, \cdot)$ is holomorphic in $\mathcal{S}^0$.

From the estimates

$$|\chi(iu/z, x, \xi)| \leqq \text{const}\,(1 + |u/z|)^N$$
$$|\log(iu/z)|^l \leqq \text{const.}\,(\log|z|)^l(\left|\log|iu|\right| + 1)^l\,,$$

which hold with a suitable N for $z \geqq 1$, $(x, \xi) \in K \times S^{n-1}$, it follows

$$|z|^{-\varepsilon}|\log(iu/z)|^l \leqq \text{const.}\,(\left|\log|iu|\right| + 1)^l\,.$$

Hence the lemma is proved. q.e.d.

LEMMA 1.7.2. *For every $\nu \in C$ and every $j \in Z^+$ we have*

$$\int_{L_0} \exp[iz\sigma]\sigma^{\nu-\frac{1}{2}-j}\check{d}\sigma = z^{-\nu-\frac{1}{2}+j}C(\nu - j)\,. \qquad z > 0\,,$$

where

$$C(\zeta) = \frac{1}{2\pi i}\exp\left[i(\zeta - \tfrac{1}{2})\frac{\pi}{2}\right]\Gamma(\zeta + \tfrac{1}{2})[1 - \exp[-2\pi i(\zeta - \tfrac{1}{2})]\,.$$

PROOF. Both sides of the above relation are entire functions of $\zeta = \nu - j \in C$. The equality is trivially proved when $\operatorname{Re}\zeta - \frac{1}{2} > -1$; hence the lemma. q.e.d.

LEMMA 1.7.3. *For every $\nu \in C$ and $l \in Z_+$ we have*

$$\int_{L_0} \exp[iz\sigma]\sigma^{\nu-\frac{1}{2}}(\log\sigma)^l\check{d}\sigma = z^{-\nu-\frac{1}{2}}(\partial_\nu - \log z)^l C(\nu) =$$
$$= z^{-\nu-\frac{1}{2}}\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(\log z)^{l-j}\partial_\nu^j C(\nu)\,,$$

with the same $C(\nu)$ as in the preceding lemma.

PROOF. Straightforward. q.e.d.

We state now the main results of this Sections.

THEOREM 1.7.1. *Let $\psi \in \mathcal{A}^0_{-k,j}$ containing powers of $\log\sigma$ of order at most L. For every pair of integers (m, l), $m \geqq -j$, $l \in \{0, \ldots, L\}$, there exist functions $\varrho^{(m,l)}_{-k}(x, \xi) \in \mathcal{O}^{-k}$ such that for every $K \subset\subset R^n_x$, $M \geqq 0$, $\varepsilon > 0$, there is a constant $C > 0$ for which:*

$$|I(z, x, \xi; \nu_0, \psi) - \sum_{m=-j}^{M}\sum_{l=0}^{L}\varrho^{(m,l)}_{-k}(x, \xi)z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l|$$
$$\leqq Cz^{-\operatorname{Re}\nu_0(x)-\frac{1}{2}-M-1+\varepsilon}\,, \qquad z \geqq 1\,, \quad (x, \xi) \in K \times S^{n-1}\,.$$

PROOF. – Using the representation

$$\psi(\sigma, x, \xi) = \sum_{l=0}^{L} \sum_{m=-j}^{M} a_{-k}^{(m,l)}(x, \xi)\sigma^m(\log\sigma)^l + \\ + \sigma^{M+1}\sum_{l=0}^{L} b_{-k}^{(l)}(\sigma, x, \xi)(\log\sigma)^l, \qquad a_{-k}^{(m,l)} \in \mathcal{O}^{-k}, \quad b_{-k}^{(l)} \in \mathcal{A}_{-k,0}^{0},$$

The Theorem easily follows from Lemmas 1.7.1.-1.7.3. q.e.d.

THEOREM 1.7.2. *Let* $\tilde{p}_{-j/2}^{\pm}(z, x, \xi) \in \Phi^{-j}$ *be the functions constructed in Theorem* 1.6.1. *and* $L^{\pm}(j)$ *the maximum order of powers of* $\log\sigma$ *appearing in the integral representation of* $\tilde{p}_{-j/2}^{\pm}$. *Then for every pair of integers* (m, l), $m \geqq -j$, $0 \leqq l \leqq L^{\pm}(j)$, *there exist functions* $\varrho_{-j}^{(m,l),\pm}(x, \xi) \in \mathcal{O}^{-j}$ *such that for every* $K \subset\subset R_x^n$, $M \geqq 0$, $\varepsilon > 0$, $\alpha, \beta \in Z_+^n$, $r \in Z_+$, *there is a constant* $C_{\alpha,\beta,K,r} > 0$ *for which*

$$(1.7.4) \qquad \left|\partial_x^\alpha \partial_\xi^\beta \partial_z^r \tilde{p}_{-j/2}^{\pm}(z, x, \xi) - \sum_{m=-j}^{M}\sum_{l=0}^{L^{\pm}(j)} z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l \varrho_{-j}^{(m,l),\pm}(x, \xi)\right| \\ \leqq C_{\alpha,\beta,K,r} z^{-\operatorname{Re}\nu_0(x)-\frac{1}{2}-r-M-1+\varepsilon}|\xi|^{-j-|\beta|},$$

for $z \geqq 1$, $x \in K$, $\xi \neq 0$.

(1.7.4) *will be written briefly*

$$(1.7.5) \qquad \tilde{p}_{-j/2}^{\pm}(z, x, \xi) \sim \sum_{m \geqq -j}\sum_{l=0}^{L^{\pm}(j)} z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l \varrho_{-j}^{(m,l),\pm}(x, \xi).$$

PROOF A trivial consequence of Theorems 1.6.1. 1.7.1. q.e.d.

In the next theorem we prove some kind of converse of the preceding result.

THEOREM 1.7.3. *Let* $\tilde{p}_{-j/2}^{\pm}(z, x, \xi) \in \Phi^{-j}$, $j \geqq 0$, *be such that*:

i) $\tilde{p}_0^{\pm}(z, x, \xi) = I^{\pm}(z, x, \xi; \nu_0, \zeta)$, *for some* $\zeta \in \mathcal{A}_{0,0}^{\pm}$.

ii) *For* $j \geqq 1$, $\tilde{p}_{-j/2}^{\pm}(z, x, \xi)$ *are solutions of the equations* (1.4.18), $(1.4.19)_j$:

iii) *For every* $j \geqq 0$ *there exist* $J^{\pm}(j) \in Z$, $L^{\pm}(j) \in Z^+$ *and a sequence of functions* $\varrho_{-j}^{(m,l),\pm}(x, \xi) \in \mathcal{O}^{-j}$, $m \geqq -J^{\pm}(j)$, $0 \leqq l \leqq L^{\pm}(j)$, *such that*

$$(1.7.6)_j \qquad \tilde{p}_{-j/2}^{\pm}(z, x, \xi) \sim \sum_{m \geqq -J^{\pm}(j)}\sum_{l=0}^{L^{\pm}(j)} z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l \varrho_{-j}^{(m,l),\pm}(x, \xi),$$

where the $\sim$ *has the same meaning as in* (1.7.5).

Then there exist functions $\psi_j^{\pm}(\sigma, x, \xi) \in \mathcal{A}^{\pm}_{-j,j}$, $j \geqq 0$, *for which*

$$(1.7.7)_j \qquad \tilde{p}^{\pm}_{-j/2}(z, x, \xi) = \exp[\mp iz] I^{\pm}(z, x, \xi; \nu_0, \psi_j^{\pm}) ,$$

where $I^{\pm}$ *are the integrals defined in* (1.6.2).

PROOF. – By induction on j. When $j = 0$ there is nothing to prove since by assumption i),

$$\tilde{p}_0^{\pm}(z, x, \xi) = \exp[\pm iz] I^{\pm}\Big(z, x, \xi; \nu_0, \psi_0^{\pm}\Big) = \frac{\Gamma(\nu_0(x) + 1)\Gamma(\frac{1}{2} - \nu_0(x))}{\sqrt{\pi}} (\sigma \pm 1)^{\nu_0(x) - \frac{1}{2}}.$$

Suppose that the assertion holds up to $j - 1$, $j \geqslant 1$, and let us prove it for $\tilde{p}^{\pm}_{-j/2}$ (the case $\tilde{p}^{-}_{-j/2}$ is quite analogous).

Write $\tilde{p}^{+}_{-j/2}(z, x, \xi) = \exp[- iz] G(z, x, \xi)$. By the inductive hypothesis we have

$$(1.7.8) \qquad MG(z) = \Big(z\partial_z^2 + (2\nu_0(x) + 1)\partial_z + z\Big) G(z, x, \xi) = I^{+}(z, x, \xi; \nu_0, \chi) ,$$

with a suitable $\chi \in \mathcal{A}^{+}_{-j,j}$.

Since two independent solutions of the homogeneous equation $MG(z) = 0$, $z > 0$, are given by $I^{\pm}_{\nu_0}(x; z)$ (see (1.5.4)), by the proof of Theorem 1.6.1 there exists a function $\psi^{+}(\sigma, x, \xi) \in \mathcal{A}^{+}_{-j,j}$ such that

$$(1.7.9) \qquad G(z, x, \xi) = I^{+}(z, x, \xi; \nu_0, \psi^{+}) + c^{+}_{-j}(x, \xi) I^{+}_{\nu_0}(x; z) + c^{-}_{-j}(x, \xi) I^{-}_{\nu_0}(x; z) ,$$

for some functions $c^{\pm}_{-j} \in \mathcal{O}^{-j}$.

From $(1.7.6)_j$ and Lemma 1.7.1 we get

$$(1.7.10) \qquad \exp[- iz] I^{+}(z, x, \xi; \nu_0, \psi^{+}) + c^{+}_{-j}(x, \xi) I^{+}_{\nu_0}(x; z) \sim \sum_{m \geqq \min(0, -J^{+}(j))} \sum_{l=0}^{L^{+}(j)} z^{-\nu_0(x) - \frac{1}{2} - m} (\log z)^l \tilde{\varrho}^{(m,l),+}_{-j}(x, \xi) ,$$

with some new $\tilde{\varrho}^{(m,l),+}_{-j} \in \mathcal{O}^{-j}$.

On the other hand

$$(1.7.11) \qquad \exp[iz] c^{-}_{-j}(x, \xi) I^{-}_{\nu_0}(x; z) \sim \sum_{m \geqq 0} z^{-\nu_0(x) - \frac{1}{2} - m} c^{-}_{-j}(x, \xi) b_m(\nu_0(x)) ,$$

for some suitable functions b_m, with $b_0(\nu_0(x)) \neq 0$ (see W. Magnus - F. Oberhettinger - R. P. Soni [6], p. 139).

Comparing (1.7.10), (1.7.11) with $(1.7.6)_j$ we conclude that $c_{-j}^- = 0$. Choosing in (1.7.9) $\psi_j^+ = \psi^+ + c_{-j}^+ \Gamma(\frac{1}{2} - \nu_0(x))\pi^{-\frac{1}{2}}(\sigma+1)^{\nu_0(x)-\frac{1}{2}}$ we prove the theorem. q.e.d.

CHAPTER 2

THE RIGOROUS DISCUSSION

2.1. – Symbol classes and oscillatory integrals.

To put the formal series $\Sigma q_{-j/2}$ and $\Sigma p_{-j/2}^{\pm}$ constructed in Ch. 1 in a rigorous framework we need to define some classes of symbols which are closely connected with those considered by L. Boutet de Monvel [2]. In the sequel by a cutoff function we mean any function $\chi \in C_0^\infty(R)$ which is identically 1 in a neighborhood of the origin.

DEFINITION 2.1.1. *By $S^{m,k}(0, T)$, $m, k, T \in R$, $0 < T \leqq +\infty$, we denote the class of all functions $p(t, x, \xi) \in C^\infty([0, T] \times R_x^n \times R_\xi^n)$ such that for every $K \subset\subset R_x^n$, $\alpha, \beta \in Z_+^n$, $r \in Z_+$, there exists a constant $C_{\alpha,\beta,r,K} > 0$ for which*

$$|\partial_t^r \partial_x^\alpha \partial_\xi^\beta p(t, x, \xi)| \leqq C_{\alpha,\beta,r,K} |\xi|^{m-|\beta|} \left(\sqrt{t} + \frac{1}{|\xi|}\right)^{k-2r}, \tag{2.1.1}$$

for $|\xi| \geqq 2$, $x \in K$, $0 \leqq t < \min\{\frac{1}{2}, T\}$.

We put $S^{-\infty,k}(0, T) = \bigcap_m S^{m,k}(0, T)$, $S^{m,\infty}(0, T) = \bigcup_k S^{m,k}(0, T)$.

By $\tilde{S}^{m,k}(0, T)$ we denote the intersection $\bigcap_{\varepsilon>0} S^{m+\varepsilon,k+\varepsilon}(0, T)$.

$S_0^{m,k}(0, T)$ will denote the space of all symbols $p(t, x, \xi)$ such that $\chi(t|\xi|^2)p(t, x, \xi) \in S^{m,k}$ for every cutoff function χ. $\tilde{S}_\infty^{m,k}(0, T)$ will denote the space of all symbols $p(t, x, \xi)$ such that $\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi) \in \tilde{S}^{m,k}$ for every cutoff function χ. All these spaces are equipped with their natural topology.

EXAMPLES. 1) Let $q(z, x, \xi) \in C^\infty(R_z \times R_x^n \times \dot{R}_\xi^n)$ satisfy:

i) $q(z, x, \lambda\xi) = \lambda^m q(z, x, \xi)$, $\lambda > 0$;

ii) $z \to q(z, \cdot)$ in an even analytic function of z. Then for every fixed cutoff χ_0 the symbol

$$q_1(t, x, \xi) = \big(1 - \chi_0(|\xi|^2)\big) q(2\sqrt{t}\, a(x, \xi), x, \xi) \in S_0^{m,0}(0, +\infty).$$

2) Let $p(z, x, \xi) \in C^\infty(R_z^+ \times R_x^n \times \dot{R}_\xi^n)$ satisfy:

i) $p(z, x, \lambda\xi) = \lambda^m p(z, x, \xi)$, $\lambda > 0$;

ii) for some $\mu \in R$ and for every $K \subset\subset R_x^n$, $\alpha, \beta \in Z_+^n$, $r \in Z_+$, $\varepsilon > 0$ there is a constant $C > 0$ for which $|(z\partial_z)^r \partial_x^\alpha \partial_\xi^\beta p(z, x, \xi)| \leqq c|\xi|^{m-|\beta|} z^{\mu+\varepsilon}$ for $x \in K$, $\xi \neq 0$, $z \geqq 1$. Then for every fixed cutoff χ_0 the symbol $p_1(t, x, \xi) = \big(1 - \chi_0(|\xi|^2)\big) p\big(2\sqrt{t}\, a(x, \xi), x, \xi\big) \in \tilde{S}_\infty^{m+\mu, \mu}(0, +\infty)$.

In the next lemma some properties of the classes of symbols defined above are collected. The proof, which follows along standard arguments will be omitted (see e.g. L. Boutet de Monvel [2]).

LEMMA 2.1.1.

i) $S^{m,k}(0, T) \hookrightarrow S^{m',k'}(0, T)$ *iff* $m \leqq m'$ *and* $m - k \leqq m' - k'$.

ii) *Let* χ *be a cutoff function and* $\lambda \geqq 1$. *Define*

$$\varphi_\lambda^1(t, x, \xi) = 1 - \chi\big(|\xi|/\lambda\big), \qquad \varphi_\lambda^2(t, x, \xi) = \varphi_\lambda^1(t, x, \xi)\, \chi(\lambda^2 t).$$

Then:

a) $\varphi_\lambda^1 \in S^{0,0}$, $1 - \varphi_\lambda^1 \in S^{-\infty,0}$, $\{\lambda\varphi_\lambda^1 | \lambda \geqq 1\}$ *is a bonnded subset of* $S^{1,0}$.

b) $\varphi_\lambda^2 \in S^{0,0}$, $1 - \varphi_\lambda^2 \in S^{0,\infty}$, $\{\lambda\varphi_\lambda^2 | \lambda \geqq 1\}$ *is a bounded subset of* $S^{0,-1}$.

iii) *Let* $p_j \in S_0^{m-j,k}(0, T)$, $j = 0, 1, \ldots$; *then there exists a symbol* $p \in S_0^{m,k}(0, T)$ *such that* $p \sim \sum_{j \geqq 0} p_j$, *i.e.*

$$p - \sum_{j=0}^{M-1} p_j \in S_0^{m-M,k}(0, T), \qquad \forall M \geqq 1.$$

iv) *Let* $p_j \in \tilde{S}_\infty^{m,k+j}(0, T)$, $j = 0, 1, \ldots$; *then there exists a symbol* $p \in \tilde{S}_\infty^{m,k}(0, T)$ *such that* $p \sim \sum_{j \geqq 0} p_j$, *i.e.*

$$p - \sum_{j=0}^{M-1} p_j \in \tilde{S}_\infty^{m,k+M}(0, T), \qquad \forall M \geqq 1.$$

True symbols can be recovered from the «formal» symbols of Chp. 1 using the following lemma.

LEMMA 2.1.2.

i) *Let* $f(t, x, \xi) \in \Psi^{-j/2}$, *then for every cutoff* χ

$$\big(1 - \chi(|\xi|)\big) f(t, x, \xi) \in S_0^{-j,0}(0, +\infty)$$

ii) *Let* $\psi^{\pm}(\sigma, x, \xi) \in \mathcal{A}^{\pm}_{\pm k,j}$ *and define*

$$p^{\pm}(t, x, \xi) = \exp[\mp iz] I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm})|_{z=2\sqrt{t}a(x,\xi)}$$

Then for every cutoff χ

$$\big(1 - \chi(|\xi|)\big) p^{\pm}(t, x, \xi) \in \tilde{S}^{\mu+j-k,\mu+j}_{\infty}(0, +\infty),$$

where $\mu = \sup_{x \in R^n} (-\operatorname{Re} \nu_0(x) - \frac{1}{2})$.

iii) *Let* $f \in S^{m,k}_0(0, T)$ *and* χ *any cutoff. Then, for every* $l \in Z_+$

$$\partial_t^l \big(\chi(t|\xi|^2) f(t, x, \xi)\big) \in C^0([0, T]; S^{m-k+2l+2}_{1,0}(R^n_x \times R^n_\xi)) .$$

iv) *Let* $f \in \tilde{S}^{m,k}_{\infty}(0, T)$ *and* χ *any cutoff. Then, for every* $l \in Z_+$

$$\partial_t^l \big((1 - \chi(t|\xi|^2)) f(t, x, \xi)\big) \in \begin{cases} C^{(k-2l+\varepsilon)/2}([0, T]; S^{m+\varepsilon}_{1,0}(R^n_x \times R^n_\xi)), \\ \qquad\qquad \varepsilon > 0, \quad \text{if } k - 2l \geqq 0 \\ C^{\frac{1}{2}}([0, T]; S^{m+2l-k+1}_{1,0}(R^n_x \times R^n_\xi)), \\ \qquad\qquad \text{if } k - 2l < 0 . \end{cases}$$

PROOF. i) By Def. 1.1.1 $f(t, x, \xi) = |\xi|^{-j} f(t|\xi|^2, x, \xi/|\xi|)$, thus the conclusion follows taking into account that $\sqrt{t} + 1/|\xi| \sim 1/|\xi|$ on the support of $\chi(t|\xi|^2)$.

ii) Is a trivial consequence of Theorem 1.7.1, of Example 2) and Lemma 2.1.1 iv).

iii) Since $\partial_t \colon S^{m,k}_0 \to S^{m+2,k}_0$ it is enough to prove the assertion in the case $l = 0$. Now, locally in x we have

$$|\partial_x^\alpha \partial_\xi^\beta [\chi(t|\xi|^2) f(t, x, \xi)]| \leqq C|\xi|^{m-|\beta|} \left(\sqrt{t} + \frac{1}{|\xi|}\right)^k \leqq C|\xi|^{m-k-|\beta|}, \qquad |\xi| \geqq 2, \ t \in [0, T] .$$

Moreover, locally in x we have

$$\left|\partial_x^\alpha \partial_\xi^\beta \big[\partial_t \big(\chi(t|\xi|^2) f(t, x, \xi)\big)\big]\right| \leqq C|\xi|^{m-k+2-|\beta|}, \qquad |\xi| \geqq 2, \qquad t \in [0, T] .$$

Thus the claim follows.

iv) Suppose first $k-2l \geqq 0$. Then for $j=0,1$ locally in x we have

$$\left|\partial_t^{j+l}\partial_x^\alpha\partial_\xi^\beta\left[\left(1-\chi(t|\xi|^2)\right)f(t,x,\xi)\right)\right| \leqq$$
$$\leqq C|\xi|^{m-|\beta|+\varepsilon}(\sqrt{t})^{k-2l+\varepsilon-2j}, \qquad \forall\varepsilon>0, \quad |\xi|\geqq 2, \quad t\in[0,T].$$

Hence the first assertion follows. If $k-2l<0$, take $\varepsilon>0$ such that $1+2l-k-\varepsilon>0$. Then for $j=0,1$ locally in x we have

$$\left|\partial_t^{j+l}\partial_x^\alpha\partial_\xi^\beta\left[\left(1-\chi(t|\xi|^2)\right)f(t,x,\xi)\right)\right| \leqq$$
$$\leqq C|\xi|^{m-|\beta|+\varepsilon}t^{\frac{1}{2}-j}t^{(k-2l-1+\varepsilon)/2} \leqq C|\xi|^{m+2l-k+1-|\beta|}t^{\frac{1}{2}-j}, \qquad |\xi|\geqq 2, \quad t\in[0,T].$$

Hence the second assertion follows. q.e.d.

We now turn to the discussion of some oscillatory integrals. Let $q(t,x,\xi)\in S_0^{m,k}(0,T)$ and let χ be any cutoff function. We consider the following operator:

$$(2.1.2)\qquad \begin{cases} E\colon C_0^\infty(R^n)\to C^\infty([0,T]\times R^n) \\ Eg(t,x)=\int\exp[ix\cdot\xi]\chi(t|\xi|^2)q(t,x,\xi)\hat{g}(\xi)\check{d}\xi\,. \end{cases}$$

The continuity of E follows from Lemma 2.1.2 iii). We now show that E can be continuously extended to an operator, still denoted by E, from $\mathcal{E}'(R^n)$ into $C^\infty([0,T];\mathcal{D}'(R^n))$.

Take $g\in C_0^\infty(R^n)$ and $f(t,x)\in C_0^\infty([0,T]\times R^n)$; then

$$\int_0^T\!\!\int(Eg)(t,x)f(t,x)\,dt\,dx=\int\hat{g}(\xi)(\mathcal{E}f)(\xi)\check{d}\xi\,,$$

with

$$(\mathcal{E}f)(\xi)=\int_0^T\!\!\int\exp[ix\cdot\xi]\chi(t|\xi|^2)q(t,x,\xi)f(t,x)\,dt\,dx\,.$$

Integration by parts with respect to x shows that $\mathcal{E}f(\xi)$ is C^∞ and rapidly decreasing for $\xi\to\infty$.

Therefore, by the Paley-Wiener theorem we can define $\mathcal{E}'(R^n)\ni g\to Eg$ by the relation $\langle Eg,f\rangle=\int\hat{g}(\xi)(\mathcal{E}f)(\xi)\,d\xi$.

By an application of Lemma 2.1.2 iii) it follows that $Eg\in C^\infty([0,T];\mathcal{D}'(R^n))$ and the map $g\to Eg$ is continuous.

It is worthwhile to observe that for every $j \geqq 0$ and for every $s \in [0, T]$ the operator

$$\mathcal{E}'(R^n) \ni g \to \partial_t^j E g|_{t=s} \in \mathcal{D}'(R^n)$$

is a pseudo differential operator of order $m - k + 2j$.

Operators of the form (2.1.2) will take care of the formal parametrix in the region $t|\xi|^2 \leqq$ const. constructed in Sects. 1.1-1.3.

To give meaning to the formal operators introduced in Sect. 1.4 let $\varphi(t, x, \xi)$ denote any one of the two phase functions $\varphi^{\pm}(t, x, \xi)$ defined in (1.4.3).

Let $p(t, x, \xi) \in \tilde{S}_{\infty}^{m,k}(0, T)$ and let χ be any cutoff function.

Consider the operator

$$(2.1.3) \qquad Eg(t, x) = \int \exp[i\varphi(t, x, \xi)]\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\,, \qquad g \in C_0^{\infty}(R^n)\,.$$

We now show that E maps continuously $C_0^{\infty}(R^n)$ into $C_{\text{flat}}^{\infty}([0, T] \times R^n)$ where the latter denotes the subspace of $C^{\infty}([0, T] \times R^n)$ whose elements are flat functions at $t = 0$.

It is easy to recognize that $\partial_t^j \partial_x^{\alpha} Eg(t, x)$ can be written as an sum of integrals like (2.1.3) with new amplitudes in $\tilde{S}_{\infty}^{m+2j,k}(0, T)$ and new cutoffs. This proves that $Eg \in C^{\infty}([0, T] \times R^n)$. To show that Eg is flat at $t = 0$ consider $t^{-N} Eg(t, x)$, $N \geqq 0$. Locally in x we have the estimate

$$\begin{aligned}\left|t^{-N}\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi)\right| &\leqq Ct^{-N+k/2+\varepsilon/2}(1 + |\xi|)^{m+\varepsilon} \\ &\leqq C(t|\xi|^2)^{-N+k/2+\varepsilon/2}(1 + |\xi|)^{m+2N-k} \leqq C(1 + |\xi|)^{m+2N-k}\,,\end{aligned}$$

if $2N - k > 0$.

Therefore $t^{-N} Eg(t, x) \to 0$, $t \to 0+$, for N large enough.

Let us now show that the operator (2.1.3) can be continuously extended as an operator from $\mathcal{E}'(R^n)$ into $C_{\text{flat}}^{\infty}([0, T]; \mathcal{D}'(R^n))$.

Take $g \in C_0^{\infty}(R^n)$ and $f \in C_0^{\infty}([0, T] \times R^n)$; then

$$\int_0^T\!\!\int Eg(t, x) f(t, x)\,dt\,dx = \int \hat{g}(\xi)(\mathcal{E}f)(\xi)\,\check{d}\xi\,,$$

where

$$(\mathcal{E}f)(\xi) = \int_0^T\!\!\int \exp[i\varphi(t, x, \xi)]\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi) f(t, x)\,dt\,dx\,.$$

As a consequence of (1.4.3) the following estimate holds for x in a compact set:

$$|d_x\varphi(t, x, \xi)| \geqq C|\xi|\,, \qquad t \in [0, T]\,, \quad \xi \neq 0\,.$$

Consider the operator

$$L = \sum_{j=1}^{n} \frac{1}{i}\,(1 - \theta(\xi))\,\frac{d_{x_j}\varphi(t, x, \xi)}{|d_x\varphi(t, x, \xi)|^2}\,d_{x_j} + \theta(\xi)\,,$$

where θ is a cutoff function.

Integrating by parts we get, for every $N \geqq 0$:

$$(\mathcal{E}f)(\xi) = \int_0^T\!\!\int \exp\,[i\varphi(t, x, \xi)]\,\big(1 - \chi(t|\xi|^2)\big)^t L^N[p(t, x, \xi)\,f(t, x)]\,dt\,dx\,.$$

It is easily verified that, for every $\varepsilon > 0$ we have the estimate

$$\big|\big(1 - \chi(t|\xi|^2)\big)^t L^N[p(t, x, \xi)\,f(t, x)]\big| \leqq C_N(1 + |\xi|)^{m+\varepsilon-N+\max(0,-k-\varepsilon)}\,.$$

The rapid decrease of $(\mathcal{E}f)(\xi)$ allows to define Eg, when $g \in \mathcal{E}'(R^n)$, according to the formula

$$\langle Eg, f\rangle = \int \hat{g}(\xi)(\mathcal{E}f)(\xi)\,\check{d}\xi\,.$$

One can easily see that $Eg \in C^\infty([0, T]; \mathcal{D}'(R^n))$. Moreover, arguing as above, one can verify that $\partial_t^j Eg|_{t=0} = 0$, $\forall j \geqq 0$. Of course, for every $j \geqq 0$ and for every $s \in]0, T[$, the operator

$$\mathcal{E}'(R^n) \ni g \to \partial_t^j Eg|_{t=s} \in \mathcal{D}'(R^n)$$

is a Fourier integral operator with phase $\varphi(s, x, \xi)$ and amplitude in $S^{m+2j-k+\varepsilon}_{1,0}$, $\forall \varepsilon > 0$.

2.2. – Construction of the true parametrix.

Our first attempt to construct a parametrix for pb. (0.2) will be to consider an operator of the form:

$$\begin{aligned}(2.2.1)\qquad Eg(t, x) = &\int \exp\,[ix\cdot\xi]\,\chi(t|\xi|^2)\,q(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi + \\ &+ \int \exp\,[i\varphi^+(t, x, \xi)]\big(1 - \chi(t|\xi|^2)\big)\,p^+(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi + \\ &+ \int \exp\,[i\varphi^-(t, x, \xi)]\,\big(1 - \chi(t|\xi|^2)\big)\,p^-(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\,, \qquad g \in C_0^\infty(R^n)\,,\end{aligned}$$

where:

i) χ is any cutoff function.

ii) $q \in S_0^{0,0}(0, +\infty)$ with $q \sim \sum_{j \geqq 0} q_{-j/2}$ and the $q_{-j/2} \in \Psi^{-j/2}$, $j \geqq 0$, are the functions constructed in Sects. 1.2, 1.3. We recall that, for any cutoff χ_0, $\big(1 - \chi_0(|\xi|)\big) q_{-j/2}(t, x, \xi) \in S_0^{-j,0}(0, +\infty)$ according to Lemma 1.1.2 i).

iii) $p^{\pm} \in \tilde{S}_{\infty}^{\mu;\mu}(0, T)$, $\mu = \sup_{x \in R^n} (-\operatorname{Re} \nu_0(x) - \frac{1}{2})$, with $p^{\pm} \sim \sum_{j \geqq 0} p_{-j/2}^{\pm}$ and the $p_{-j/2}^{\pm}$ are those constructed in Sects. 1.5, 1.6.

We recall that, for any cutoff χ_0, $\big(1 - \chi_0(|\xi|)\big) p_{-j/2}^{\pm} \in \tilde{S}_{\infty}^{\mu,\mu+j}(0, T)$ according to Lemma 2.1.2 ii).

iv) $\varphi^{\pm}(t, x, \xi)$ are the phases (defined for $t \in [0, T]$) constructed in (1.4.3).

First we observe that

$$Eg|_{t=0} - g = \int \exp[ix \cdot \xi](1 - q(0, x, \xi)) \hat{g}(\xi) \check{d}\xi .$$

Since $q(0, x, \xi) \in S_{1,0}^0(R^n)$ and

$$q_{-j/2}(0, x, \xi) = \begin{cases} 1, & j = 0 \\ 0, & j > 0 , \end{cases}$$

we conclude that $1 - q(0, x, \xi) \in S_{1,0}^{-\infty}$ and thus the second condition in (0.3) is fulfilled.

Now

$$\begin{aligned} (2.2.2) \qquad PEg(t, x) = & \int \exp[ix \cdot \xi] \chi(t|\xi|^2) [\exp[-ix \cdot \xi] P(\exp[ix \cdot \xi] q)] \hat{g}(\xi) \check{d}\xi \\ & + \int \exp[i\varphi^+(t, x, \xi)] \big(1 - \chi(t|\xi|^2)\big) [\exp[-i\varphi^+] P(\exp[i\varphi^+] p^+)] \hat{g}(\xi) \check{d}\xi \\ & + \int \exp[i\varphi^-(t, x, \xi)] \big(1 - \chi(t|\xi|^2)\big) [\exp[-i\varphi^-] P(\exp[i\varphi^-] p^-)] \hat{g}(\xi) \check{d}\xi \\ & + \int [P, \chi(t|\xi|^2)] \{\exp[ix \cdot \xi] q(t, x, \xi)\} \hat{g}(\xi) \check{d}\xi \\ & - \int [P, \chi(t|\xi|^2)] \{\exp[i\varphi^+(t, x, \xi)] p^+(t, x, \xi) + \exp[i\varphi^-(t, x, \xi)] p^-(t, x, \xi)\} \hat{g}(\xi) \check{d}\xi . \end{aligned}$$

Now the following crucial remarks are in order:

I) By construction $q - \sum_{j=0}^{N-1} q_{-j/2} \in S_0^{-N,0}$, for large ξ, for every $N \geqq 1$.

Moreover, by the construction performed in Sect. 1.3 $\exp[-ix\cdot\xi]P\cdot(\exp[ix\cdot\xi]q) \in S^{1,0}_0$ has, for large ξ, the asymptotic expansion

$$\exp[-ix\cdot\xi]P(\exp[ix\cdot\xi]q) \sim \sum_{j\geqq 0} \tilde{q}_{-j/2}\,, \qquad \text{with}$$

$$\tilde{q}_{-j/2} = \sum_{h=0}^{j} L_{1-h/2} q_{-j/2+h/2}\,, \qquad j \geqq 0\,,$$

and the operators $L_{1-h/2}$ are defined in (1.1.5).

From Theorem 1.3.1 it follows that $\exp[-ix\cdot\xi]P(\exp[ix\cdot\xi]q) \sim 0$. As a consequence the operator

$$g \to \int \exp[ix\cdot\xi]\chi(t|\xi|^2)[\exp[-ix\cdot\xi]P(\exp[ix\cdot\xi]q)]\hat{g}(\xi)\,\check{d}\xi$$

is smoothing.

II) By construction $p^{\pm} - \sum_{j=0}^{N-1} p^{\pm}_{-j/2} \in \tilde{S}^{\mu,\mu+N}_{\infty}(0, T)$, for large ξ, for every $N \geqq 1$.

Now we claim that $\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \in \tilde{S}^{\mu+1,\mu}_{\infty}(0, T)$ with asymptotic expansion (1.4.15), for large ξ, computed for $z = 2\sqrt{t}\,a(x, \xi)$.

To prove our claim, i.e. to show that the formal computations performed in Sect. 1.4 have a meaning within the classes $\tilde{S}_{\infty}$ we only need to show that for large ξ, $\varphi^{\pm}(t, x, \xi) \in \tilde{S}^{1,0}_{\infty}(0, T)$ with the asymptotic expansion (1.4.4) (computed for $z = 2\sqrt{t}\,a(x, \xi)$).

To prove this fact we recall (1.4.2); from the Taylor expansion

$$\psi^{\pm}(s, x, \xi) \sim x\cdot\xi \pm a(x, \xi)s + \sum_{k\geqq 2} \frac{1}{k!}(\partial_s^k \psi^{\pm})(0, x, \xi)s^k$$

we get the estimate:

$$\partial_t^l \partial_x^{\alpha} \partial_{\xi}^{\beta} \left[\psi^{\pm}(s, x, \xi) - (x\cdot\xi \pm a(x, \xi)s + \sum_{k\geqq 2}^{N} \frac{\partial_s^k \psi^{\pm}}{k!}(0, x, \xi)s^k\right]$$

$$= 0\big(|\xi|^{1-|\beta|}s^{N+1-l}\big)\,, \qquad |\xi| \geqq 1\,, \quad 0 \leqq s < \min\{\tfrac{1}{2}, 2\sqrt{T}\}\,.$$

Hence the estimate:

$$\partial_t^l \partial_x^{\alpha} \partial_{\xi}^{\beta} \left[\varphi^{\pm}(t, x, \xi) - \Big(x\cdot\xi \pm 2\sqrt{t}\,a(x, \xi) + \sum_{k=2}^{N} \alpha_{1-k}(x, \xi)(2\sqrt{t}\,a(x, \xi))^k\Big)\right]$$

$$= O\big(|\xi|^{1-|\beta|}(\sqrt{t})^{N+1-2l}\big) = O\big(|\xi|^{1-|\beta|}(\sqrt{t} + 1/|\xi|)^{N+1-2l}\big)\,,$$

in any region $t|\xi|^2 \geqq \text{const.}$, $|\xi| \geqq 1$, $0 \leqq t < \min\{\tfrac{1}{2}, T\}$ and locally in x.

The claim on $\varphi^{\pm}$ being proved, from the construction performed in Theorem 1.6.1 it follows that $\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \in \tilde{S}^{\mu+1,\infty}_{\infty}(0, T)$.

As a consequence, it is easily verified that

$$(2.2.3) \qquad \big(1-\chi(t|\xi|^2)\big)\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) = b^{\pm}(t,x,\xi) \in C^{\infty}_{\text{flat}}([0,T]; S^{\mu+1}_{1,0}(R^n_x \times R^n_{\xi}))\,.$$

III) To control the symbol

$$[P, \chi(t|\xi|^2)]\{\exp[ix\cdot\xi]q(t,x,\xi) - \exp[i\varphi^{+}(t,x,\xi)]p^{+}(t,x,\xi) - \exp[i\varphi^{-}(t,x,\xi)]p^{-}(t,x,\xi)\}$$

we need to prove the following assertion:

If $R(t,x,\xi) \in S^{1,2}_{\infty}(0,T)$ and $\tilde{\chi} \in C^{\infty}_0(R^+)$, $\tilde{\chi} \equiv 1$ on some interval, then:

i) $\tilde{\chi}(t|\xi|^2)R(t,x,\xi) \in S^{-1,0}(0,T)$.

ii) For every $N \geqslant 1$

$$\tilde{\chi}(t|\xi|^2)\left[\exp[iR(t,x,\xi)] - \sum_{j=0}^{N-1}\frac{(iR(t,x,\xi))^j}{j!}\right] \in S^{-N,0}(0,T)\,.$$

The proof of i) is obvious since $\sqrt{t} \sim 1/|\xi|$ on the support of χ. To prove ii) we write

$$\exp[iR] - \sum_{j=0}^{N-1}\frac{(iR)^j}{j!} = \frac{(iR)^N}{(N-1)!}\int_0^1 (1-\sigma)^{N-1}\exp[i\sigma R]\,d\sigma\,,$$

and note that on the support of $\tilde{\chi}$ and locally in x we have the estimates:

$$\partial_t^l\partial_x^{\alpha}\partial_{\xi}^{\beta}R = O(|\xi|)^{-1-|\beta|-2l}, \qquad \partial_t^l\partial_x^{\alpha}\partial_{\xi}^{\beta}\exp[i\sigma R] = O(1)$$

if $|\alpha| + |\beta| + 1 = 0$ and $= O(|\xi|^{-1-|\beta|-2l})$ if $|\alpha| + |\beta| + l > 0$.

Hence $\tilde{\chi}R^N \in S^{-N,0}(0,T)$ and $\tilde{\chi}\int_0^1 (1-\sigma)^{N-1}\exp[i\sigma R]\,d\sigma \in S^{0,0}(0,T)$. This proves our assertion.

The commutator $[P, \chi(t|\xi|^2)]$ can be written as

$$\tilde{\chi}_1(t|\xi|^2)\alpha(t,x)\partial_t + \tilde{\chi}_2(t|\xi|^2)\beta(t,x) \quad \text{for suitable functions}$$
$$\tilde{\chi}_1, \tilde{\chi}_2 \in C^{\infty}_0(R^+) \quad \text{and } \alpha, \beta \in C^{\infty}\,.$$

Let us consider the symbol

$$(2.2.4)\qquad \tilde{\chi}_2(t|\xi|^2)[\exp[ix\cdot\xi]q(t,x,\xi) - \exp[i\varphi^+(t,x,\xi)]p^+(t,x,\xi) - \exp[i\varphi^-(t,x,\xi)]p^-(t,x,\xi)] .$$

We recall that in Theorem 1.3.1 the symbol $q_{-j/2}(t,x,\xi)$ was obtained as $q_{-j/2}(t,x,\xi) = \hat{q}_{-j}(z;x,\xi)|_{z=2\sqrt{t}a(x,\xi)}$, where $\hat{q}_{-j}$ has the form

$$\hat{q}_{-j}(z,x,\xi) = \sum_{h,k\leqq 2j} \varrho_{-j}^{(h,k)}(x,\xi)\varphi_{h,k}(z;x) , \qquad \varrho_{-j}^{(h,k)} \in \mathcal{O}^{-j} ,$$

with the $\varphi_{h,k}$ given in (1.3.5).

Putting

$$\varphi_{h,k}^{\pm}(z;x) = \int_{L^{\pm}} \exp[iz(\sigma \mp 1)](\sigma^2-1)^{\nu_0(x)-\frac{1}{2}-h}(\log(\sigma^2-1))^k \check{d}\sigma ,$$

the contours $L^{\pm}$ being those of fig. 3, we define accordingly

$$\begin{cases} \hat{q}_{-j}^{\pm}(z;x,\xi) = \sum_{h,k\leqq 2j} \varrho_{-j}^{(h,k)}(x,\xi)\varphi_{h,k}^{\pm}(z;x) \\ q_{-j/2}^{\pm}(t,x,\xi) = \hat{q}_{-j}^{\pm}(z;x,\xi)|_{z=2\sqrt{t}a(x,\xi)} . \end{cases}$$

Thus

$$q_{-j/2}(t,x,\xi) = \exp[iz]q_{-j/2}^{+}(t,x,\xi) + \exp[-iz]q_{-j/2}^{-}(t,x,\xi) , \qquad z = 2\sqrt{t}a(x,\xi) .$$

Let us fix a function $\tilde{\chi} \in C_0^{\infty}(R^+)$ with $\tilde{\chi} \equiv 1$ on $\operatorname{supp}\tilde{\chi}_1 \cup \operatorname{supp}\tilde{\chi}_2$. By Lemma 2.1.1 iii) we can construct two symbols $q^{\pm}(t,x,\xi) \in S_0^{0,0}(0,+\infty)$ with $q^{\pm} \sim \sum_{j\geqq 0} \tilde{\chi}(t|\xi|^2)q_{-j/2}^{\pm}$.

It follows that (2.2.4) can be rewritten as

$$(2.2.5)\qquad \tilde{\chi}_2(t|\xi|^2)[\exp[i(x\cdot\xi+z)]q^+ + \exp[i(x\cdot\xi-z)]q^- - \exp[i\varphi^+]p^+ - \exp[i\varphi^-]p^-] = \exp[i\varphi^+]\tilde{\chi}_2(t|\xi|^2) \cdot\{\exp[-iR^+]q^+ - p^+\} + \exp[i\varphi^-]\tilde{\chi}_2(t|\xi|^2)\{\exp[-iR^-]q^- - p^-\} ,$$

where $R^{\pm} \in S_{\infty}^{1,2}(0,T)$ are defined in (1.4.4).

Now we claim that we can modify the symbols $p_{-j/2}^{\pm}$, $j \geqq 1$, constructed in Theorem 1.6.1. in such a way that the new $p_{-j/2}^{\pm}$, $j \geqq 1$, satisfy the transport equations (1.4.17), (1.4.18), keep the structure (1.6.3) and be such that

$$(2.2.6)\qquad \tilde{\chi}_2(t|\xi|^2)\{\exp[-iR^{\pm}]q^{\pm} - p^{\pm}\} \in S^{-\infty,0}(0,T) .$$

The proof of our claim is based on the remark that while $p_0^+ + p_0^-$ satisfies the initial condition $p_0^+ + p_0^-|_{t=0} = 1$, no initial condition has ever been imposed until now on $p_{-j/2}^+ + p_{-j/2}^-$ for $j \geqq 1$.

We deal only with the sign $+$ and make a preliminary formal computation. From the definition of $q_{-j/2}^{\pm}$ and the construction of Sect. 1.3, we have

$$\exp[-ix\cdot\xi]P\Big(\exp[ix\cdot\xi]\exp[iz]\sum_{j\geqq 0} q_{-j/2}^+\Big) \sim 0\,,$$

so that

$$\exp[-i\varphi^+]P\Big[\exp[i\varphi^+]\Big(\exp[-iR^+]\sum_{j\geqq 0} q_{-j/2}^+\Big)\Big] \sim 0\,. \tag{2.2.7}$$

On the other hand the construction in Sect. 1.6 yields

$$\exp[-i\varphi^+]P\Big[\exp[i\varphi^+]\sum_{j\geqq 0} p_{-j/2}^+\Big] \sim 0\,. \tag{2.2.8}$$

Hence

$$\exp[-i\varphi^+]P\Big[\exp[i\varphi^+]\Big(\exp[-iR^+]\sum_{j\geqq 0} q_{-j/2}^+ - \sum_{j\geqq 0} p_{-j/2}^+\Big)\Big] \sim 0\,. \tag{2.2.9}$$

Expanding $\exp[-iR^+]$ as a sum of terms of decreasing homogeneity, we write

$$\exp[-iR^+]\sum_{j\geqq 0} q_{-j/2}^+ - \sum_{j\geqq 0} p_{-j/2}^+ = \sum_{j\geqq 0} \Phi_{-j/2},\ \Phi_{-j/2} \in \Psi^{-j/2}\,,\quad j \geqq 0\,. \tag{2.2.10}$$

The symbols $\Phi_{-j/2}$ satisfy the transport equations (1.4.17), (1.4.18) and have an asymptotic expansion as in Theorem 1.7.3; moreover, $\Phi_0 = 0$, so that we can apply Theorem 1.7.3 and conclude that there exists a function $\psi_j^+ \in \mathcal{A}_{-j,j}^+$ for which

$$\Phi_{-j/2} = I^+(z, x, \xi; \nu_0, \psi_j^+)|_{z=2\sqrt{ia}(x,\xi)}\,,\qquad j \geqq 1\,. \tag{2.2.11}$$

We define new symbols $p_{-j/2}^{\#,+}$ by

$$p_{-j/2}^{\#,+} = p_{-j/2}^+ + \Phi_{-j/2}\,,\quad j \geqq 1\,. \tag{2.2.12}$$

For convenience we shall continue to denote by $p_{-j/2}^+$ the modified symbols $p_{-j/2}^{\#,+}$. We emphasize that no modification is needed for p_0^+.

Let us now turn to the claim (2.2.6); we have

$$(2.2.13)\quad \tilde{\chi}_2(t|\xi|^2)\{\exp[-iR^+]q^+ - p^+\} = \tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]\cdot\Big[q^+ - \sum_{j=0}^{N-1} q^+_{-j/2}\Big] - \Big[p^+ - \sum_{j=0}^{N-1} p^+_{-j/2}\Big]\Big\} + \tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]\sum_{j=0}^{N-1} q^+_{-j/2} - \sum_{j=0}^{N-1} p^+_{-j/2}\Big\}.$$

Now

$$(2.2.14)\quad \begin{cases} \tilde{\chi}(t|\xi|^2)\Big[q^+ - \sum_{j=0}^{N-1} q^+_{-j/2}\Big] \in S^{-N,0}(0, +\infty) \\ \tilde{\chi}_2(t|\xi|)^2 \exp[-iR^+] \in S^{0,0}(0, T)\,. \end{cases}$$

Hence, since $\tilde{\chi}\tilde{\chi}_2 = \tilde{\chi}_2$,

$$(2.2.15)\quad \tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]\Big[q^+ - \sum_{j=0}^{N-1} q^+_{-j/2}\Big]\Big\} \in S^{-N,0}(0, T)\,.$$

Furthermore, by definition:

$$(2.2.16)\quad p^+ - \sum_{j=0}^{N-1} p^+_{-j/2} \in \tilde{S}^{\mu,\mu+N}_{\infty}(0, T)\,.$$

Hence, since $\sqrt{t} \sim 1/|\xi|$ on supp $\tilde{\chi}_2$,

$$(2.2.17)\quad \tilde{\chi}_2(t|\xi|^2)\Big[p^+ - \sum_{j=0}^{N-1} p^+_{-j/2}\Big] \in S^{-N,0}(0, T)\,.$$

Now

$$\tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]\sum_{j=0}^{N-1} q^+_{-j/2} - \sum_{j=0}^{N-1} p^+_{-j/2}\Big\}$$
$$= \tilde{\chi}_2(t|\xi|^2)\Big\{\Big(\sum_{l=0}^{N-1} \frac{(-iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big) - \sum_{j=0}^{N-1} p^+_{-j/2}\Big\}$$
$$+ \tilde{\chi}_2(t|\xi|^2)\Big(\exp[-iR^+] - \sum_{l=0}^{N-1} \frac{(iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big)\,.$$

By Remark III, ii)

$$(2.2.18)\quad \tilde{\chi}_2(t|\xi|^2)\Big(\exp[-iR^+] - \sum_{j=0}^{N-1} \frac{(-iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big) \in S^{-N,0}(0, T)\,.$$

Since $R^+ \in S^{1,2}_\infty$ with asymptotic expansion $R^+ \sim \sum_{k\geqq 2} \alpha^+_{1-k}(x,\xi)z^k$ (see (1.4.5)), we have

$$\tilde{\chi}_2(t|\xi|^2)\Big(R^+ - \sum_{k=2}^{N} \alpha^+_{1-k} z^k\Big) \in S^{-N,0}(0,T)\,. \tag{2.2.19}$$

Then

$$\begin{aligned}
&\tilde{\chi}_2(t|\xi|^2)\Big(\sum_{l=0}^{N-1}\frac{(-iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big)\\
&\quad = \sum_{l=0}^{N-1}\frac{(-i)^l}{l!}\Big[\tilde{\chi}_2(t|\xi|^2)\Big(\sum_{k=2}^{N}\alpha^+_{1-k}z^k\Big)^l + \text{a symbol of } S^{-N,0}\Big]\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big)\\
&\quad = (\text{mod. } S^{-N,0}(0,T)) = \tilde{\chi}_2(t|\xi|^2)\sum_{j=0}^{N-1}\Big(\exp[-iR^+]\sum_{k\geqq 0} q^+_{-j/2}\Big)_{-j/2}\,.
\end{aligned}$$

By the modification of the $p^+_{-j/2}$ performed above, we can conclude that

$$\tilde{\chi}_2(t|\xi|^2)\left\{\Big(\sum_{l=0}^{N-1}\frac{(-iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big) - \sum_{j=0}^{N-1} p^+_{-j/2}\right\} \in S^{-N,0}(0,T)\,. \tag{2.2.20}$$

As a consequence we have that claim (2.2.6) is proved. In the same way one can prove that

$$\tilde{\chi}_1(t|\xi|^2)\,\partial_t\{\exp[-iR^\pm]q^\pm - p^\pm\} \in S^{-\infty,0}(0,T)\,. \tag{2.2.21}$$

We summarize all the preceding remarks in the following theorem.

THEOREM 2.2.1. *There exist a symbol* $q(t,x,\xi) \in \tilde{S}^{0,0}_0(0,+\infty)$ *and two symbols* $p^\pm(t,x,\xi) \in \tilde{S}^{\mu,\mu}_\infty(0,T)$, $\mu = \sup_{x\in R^n}(-\operatorname{Re} v_0(x) - \frac{1}{2})$, *such that the operator* E *defined in* (2.2.1) *has the following properties*:

i) $PE = C + B^+ + B^-$,

where

$$Cg(t,x) = \int \exp[ix\cdot\xi]c(t,x,\xi)\hat{g}(\xi)\,\check{d}\xi\,, \tag{2.2.22}$$

with a symbol $c(t,x,\xi) \in C^\infty(\overline{R^+_t};\, S^{-\infty}_{1,0}(R^n_x \times R^n_\xi))$, *and*

$$B^\pm g(t,x) = \int \exp[i\varphi^\pm(t,x,\xi)]b^\pm(t,x,\xi)\hat{g}(\xi)\,\check{d}\xi\,, \tag{2.2.23}$$

with symbols $b^\pm(t,x,\xi) \in C^\infty_{\text{flat}}([0,T];\, S^{\mu+1}_{1,0}(R^n_x \times R^n_\xi))$.

ii) $\gamma E - I$ *is a smoothing operator.*

The operator C is obviously a smoothing operator, precisely C: $\mathcal{E}'(R^n) \to C^\infty(\overline{R_t^+} \times R_x^n)$. Therefore, to obtain a parametrix we need to exorcise the terms $B^\pm$. This will be done in the following theorem.

THEOREM 2.2.2. *There exist two symbols*

$$r^\pm(t, x, \xi) \in C^\infty_{\text{flat}}([0, T]; S^\mu_{1,0}(R^n_x \times R^n_\xi))$$

such that

$$\exp[-i\varphi^\pm]P(\exp[i\varphi^\pm]r^\pm) + b^\pm \in C^\infty_{\text{flat}}([0, T]; S^{-\infty}_{1,0}(R^n_x \times R^n_\xi)) \,. \tag{2.2.24}$$

PROOF. We prove the theorem in the case of the sign $+$, dropping for simplicity the superscript. Putting $s = 2\sqrt{t}$ and $T' = 2\sqrt{T}$ we need to prove that for a given symbol $\tilde{b}(s, x, \xi) = b(s^2/4, x, \xi) \in C^\infty_{\text{flat}}([0, T']; S^{\mu+1}_{1,0})$ there exists a symbol $\tilde{r}(s, x, \xi)$ belonging to $C^\infty_{\text{flat}}([0, T']; S^\mu_{1,0})$ for which

$$\exp[-i\psi(s, x, \xi)]\tilde{P}[\exp[i\psi(s, x, \xi)]r(s, x, \xi)] + \tilde{b}(s, x, \xi) \in C^\infty_{\text{flat}}([0, T']; S^{-\infty}_{1,0}) \,, \tag{2.2.25}$$

where $\psi = \psi^+(s, x, \xi)$ has been defined in (1.4.1) and $\tilde{P}$ is the operator (0.1) written in the new variables (s, x), i.e.

$$\tilde{P} = \partial_s^2 + \frac{2\nu(s^2/4, x) + 1}{s}\partial_s - A(s^2/4, x, \partial_x) + B(s^2/4, x, \partial_x) + b_0(s^2/4, x) \,. \tag{2.2.26}$$

A computation yields:

$$\begin{aligned} \exp[-i\psi]\tilde{P}(\exp[i\psi]\tilde{r}) = {} & 2i\Big[\partial_s\psi\partial_s + \sum_{j,i=1}^n a_{ij}(s^2/4, x)\partial_{x_i}\psi\partial_{x_j}\Big]\tilde{r} \\ & + i\left(\partial_s^2\psi - A(s^2/4, x, \partial_x)\psi + \frac{2\nu(s^2/4, x) + 1}{s}\partial_s\psi\right)\tilde{r} \\ & + \tilde{P}\tilde{r} + \frac{2\nu(s^2/4, x) + 1}{s}\partial_s\tilde{r} \,. \end{aligned} \tag{2.2.27}$$

Since $\partial_s\psi = \sqrt{A(s^2/4, x, d_x\psi)} \neq 0$ we divide both sides of (2.2.27) by $(2i\partial_s\psi)/s$ and obtain the condition

$$\begin{aligned} & s\Big(\partial_s + \sum_{j=1}^n c_{j0}(x, x; \xi)\partial_{x_j}\Big)\tilde{r} + \\ & + d_0(s, x; \xi)\tilde{r} + Q(s, x, \xi; \partial_s, \partial_x)\tilde{r} + \tilde{g} \in C^\infty_{\text{flat}}([0, T']; S^{-\infty}_{1,0}) \,, \end{aligned} \tag{2.2.28}$$

where $\tilde{g}(s, x, \xi) = (s/2i\partial_s\psi)\tilde{b} \in C^\infty_{\text{flat}}([0, T']; S^\mu_{1,0})$, $c_{j0}(s)$, $d_0(s) \in \mathcal{O}^0$ and Q is a second order operator with smooth coefficients homogeneous of degree -1 with respect to ξ; note that

$$c_{j0}(s, x; \xi) = \sum_{i=1}^{n} a_{ij}(s^2/4, x)\,\partial_{x_i}\psi(s, x, \xi)(2i\partial_s\psi(s, x, \xi))^{-1}\,, \qquad j = 1, \ldots, n.$$

As a consequence, the following system:

$$\text{(2.2.29)} \qquad \begin{cases} \dfrac{d}{ds}\, x_j(s; y) = c_{j0}(s, x(s; y), \xi)\,, & j = 1, \ldots, n \\ x_j(0; y) = y_j \end{cases}$$

is a part of the Hamiltonian system for ψ so that the map $[0, T'] \times R^n \ni (s, y) \to (s, x(s; y))$ is a global diffeomorphism. Writing (2.2.28) in the new variables (s, y) we obtain:

$$\text{(2.2.30)} \qquad [s\partial_s + d_0(s, y; \xi) + Q(s, y, \xi; \partial_s, \partial_y)]\tilde{r}(s, y, \xi) + \tilde{g}(s, y, \xi) \in C^\infty_{\text{flat}}([0, T']; S^{-\infty}_{1,0})\,,$$

where, for simplicity, we continue to denote with the same notation the functions written in the new variables.

Since $Q(s, y, \xi; \partial_s, \partial_y)$ maps $C^\infty_{\text{flat}}([0, T']; S^\mu_{1,0})$ into $C^\infty_{\text{flat}}([0, T']; S^{\mu-1}_{1,0})$, to prove the existence of $\tilde{r} \in C^\infty_{\text{flat}}([0, T']; S^\mu_{1,0})$ satisfying (2.2.30) it will be enough to show that for every $m \in R$ and every $G(s, y, \xi) \in C^\infty_{\text{flat}}([0, T']; S^m_{1,0})$ there exists a symbol $h(s, y, \xi) \in C^\infty_{\text{flat}}([0, T']; S^m_{1,0})$ such that

$$\text{(2.2.31)} \qquad (s\partial_s + d_0(s, y, \xi))h = G\,, \qquad s \in [0, T']\,.$$

To prove this assertion consider the operator

$$HG(s, y, \xi) = \int_0^1 G(\sigma, y, \xi)\,\frac{d\sigma}{\sigma}\,,$$

which maps C^∞_{flat} into itself and satisfies the equation $s\partial_s HG = G$. To solve (2.2.31) we take $h = H\Phi$ and obtain the equation $\Phi + d_0(s, x, \xi)H\Phi = G$ which can be solved in $C^\infty_{\text{flat}}([0, T']; S^m_{1,0})$ by the standard Picard's approximation procedure.

Let us now turn to (2.2.30). Using the preceding result we can construct

a formal series $\sum_{j\geqq 0}\tilde{r}_j$ with:

i) $\tilde{r}_j(s, y, \xi)\in C^\infty_{\text{flat}}([0, T']; S^{\mu-j}_{1,0})$, $j\geqq 0$.

ii) $[s\partial_s + d_0 + Q]\Big(\sum_{j=0}^{N-1}\tilde{r}_j\Big) + \tilde{g}\in C^\infty_{\text{flat}}([0, T']; S^{\mu-N}_{1,0})$, $\forall N\geqq 1$.

By a standard argument one can find $\tilde{r}\in C^\infty_{\text{flat}}([0, T']; S^{\mu}_{1,0})$ such that $\tilde{r} - \sum_{j=0}^{N-1}\tilde{r}_j\in C^\infty_{\text{flat}}([0, T']; S^{\mu-N}_{1,0})$, $\forall N\geqq 1$.

This completes the proof of the theorem. q.e.d.

As a final consequence of Theorems 2.2.1, 2.2.2 we have

GOROLLARY 2.2.1. *A parametrix for the Cauchy problem* (0.2) *is given by* $E + \mathcal{R}^+ + \mathcal{R}^-$ *where* E *is given by* (2.2.1) *and*

$$\mathcal{R}^\pm g(t, x) = \int \exp[i\varphi^\pm(t, x, \xi)] r^\pm(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\,, \qquad t\in[0, T]\,, \tag{2.2.32}$$

with the symbols $r^\pm$ *given by Theorem* 2.2.2.

In the next theorem we list some microlocal properties of the constructed parameterix $E + \mathcal{R}^+ + \mathcal{R}^- = \mathcal{Q}$.

THEOREM 2.2.3. *For every* $g\in \mathcal{E}'(R^n_x)$ *we have*:

i) $WF(\partial^k_t \mathcal{Q}g|_{t=0})\subset WF(g)$, $k = 0, 1, \ldots$

ii) *For every* $s\in[0, T[$:

$$WF(\mathcal{Q}g|_{t=s}) = (\Lambda^+_{2\sqrt{s}}\bigcup\Lambda^-_{2\sqrt{s}})\circ WF(g)\,, \tag{2.2.33}$$

where $\Lambda^\pm_t$ *have been defined in* (0.6).

PROOF. We split E into a sum $E = E_0 + E^+ + E^-$ corresponding to the three terms in (2.2.1). Then $\partial^k_t \mathcal{Q}g|_{t=0} = \partial^k_t E_0 g|_{t=0}$. As we have already remarked the operator $g\to\partial^k_t E_0 g|_{t=0}$ is a pseudo-differential operator, and this proves i).

To prove ii) we observe that (2.2.33) is obvious when $s = 0$ since $\Lambda^\pm_0 = \Delta(T^*R^n\setminus 0)$, the diagonal of $T^*R^n\setminus 0\times T^*R^n\setminus 0$. For $s > 0$ we have $WF(\mathcal{Q}g|_{t=s}) = WF[(E^+ + \mathcal{R}^+)g|_{t=s} + (E^- + \mathcal{R}^-)g|_{t=s}]$ since the operator $g\to E_0 g|_{t=s}$ is smoothing. As we have already remarked $g\to(E^\pm + \mathcal{R}^\pm)g|_{t=s}$ are Fourier integral operators with phases $\varphi^\pm(s, x, \xi)$ and amplitudes $p^\pm(s, x, \xi) + r^\pm(s, x, \xi)\in S^\mu_{1,0}(R^n_x\times R^n_\xi)$; therefore by well known results on the calculus of WF (see L. Hörmander [5]) we have

$$WF((E^\pm + \mathcal{R}^\pm)g|_{t=s})\subset\Lambda^\pm_{2\sqrt{s}}\circ WF(g)\,.$$

To prove the converse inclusion we recall that for every s $\Lambda^{\pm}_{2\sqrt{s}}$ are the graphs of the symplectomorphism

$$T^*R^n \setminus 0 \ni (y, \eta) \to \left(x^{\pm}(2s; y, \eta), \xi^{\pm}(2s; y, \eta)\right) \in T^*R^n \setminus 0$$

(see (0.6)). Moreover, $p^{\pm}(s, x, \xi) + r^{\pm}(s, x, \xi) = p_0^{\pm}(s, x, \xi) + r^{\pm}(s, x, \xi)$ modulo $S^{\mu-1}_{1,0}$. Now $p_0^{\pm}(s, x, \xi)$ is an elliptic symbol as follows from (1.5.6), i.e. $p_0^{\pm}(s, x, \xi) = \Gamma(\nu_0(x)+1) \exp[iz_n](z/2)^{-\nu_0(x)} H^{(1),(2)}_{\nu_0(x)}(z)|_{z=2\sqrt{s}a(x,\xi)}$. Since $r^{\pm}(s, x, \xi)$ is flat at $s = 0$, we can conclude that for some $\varepsilon > 0$ the symbol $p^{\pm}(s, x, \xi) + r^{\pm}(s, x, \xi)$ is invertible in $S^{-\mu}_{1,0}$ for $s \leqq \varepsilon$. As a consequence we obtain

$$WF\left((E^{\pm} + \mathcal{R}^{\pm}) g_{t=s}\right) = \Lambda^{\pm}_{2\sqrt{s}} \circ WF(g), \qquad 0 \leqq s \leqq \varepsilon .$$

To finish we observe that

$$\left(\Lambda^{+}_{2\sqrt{s}} \circ WF(g)\right) \cap \left(\Lambda^{-}_{2\sqrt{s}} \circ WF(g)\right) = \emptyset , \qquad s > 0 ,$$

so that

$$WF(\mathcal{Q}g|_{t=s}) = WF\left((E^{+} + \mathcal{R}^{+}) g|_{t=s}\right) \cup WF\left((E^{-} + \mathcal{R}^{-}) g|_{t=s}\right), \qquad 0 \leqq s \leqq \varepsilon .$$

The above equality holds then for all $s \in [0, T[$. To see this we observe that $WF(\partial_t \mathcal{Q}g|_{t=s}) \subset WF(\mathcal{Q}g|_{t=s})$; and that for $t \geqslant s > 0$, $g \to \mathcal{Q}g$ solves a Cauchy problem for the strictly hyperbolic operator P, with $P\mathcal{Q}^{+} \in C^{\infty}$.

Known results on the propagation of singularities for strictly hyperbolic Cauchy problems yield our thesis (see e.g. J. J. Duistermaat [3]). q.e.d.

REMARKS. 1) The construction of the parametrix $\mathcal{Q}$ for pb. (0.2) has been performed under the hypotheses $\nu_0(x) + 1 \notin \{0, -1, -2, \ldots\}$, $\nu_0(x) - \frac{1}{2} \notin \{0, 1, 2, \ldots\}$. While the first condition on ν_0 is natural because of its necessity for C^{∞}-well posedness of the Cauchy problem (0.2), the second one is, in our opinion, only technical. We believe that by changing the integral representation for Bessel functions one should provide a way to drop the condition $\nu_0(x) - \frac{1}{2} \notin \{0, 1, 2, \ldots\}$.

2) According to Theorem 2.2.3, the parametrix $\mathcal{Q}$ allows to describe the singularities of solutions of the equation $Pu \in C^{\infty}(\overline{R^+_t} \times R^n_x)$ which are normally regular, i.e. $u \in C^{\infty}(\overline{R^+_t}; \mathcal{D}'(R^n_x))$ (at least when $\nu_0(x)$ satisfies condition (1.2.7)). However, since $t = 0$ is characteristic for P, one can find solutions of the equation $Pu = 0$, $t > 0$, which are not normally regular

distributions and with $WF(u|_{t=s}) = \Lambda^{+}_{2\sqrt{s}} \circ WF(g)$ or $WF(u|_{t=s}) = \Lambda^{-}_{2\sqrt{s}} \circ WF(g)$. Typical examples are the following ones;

$$u^{\pm}(t, x) = \frac{1}{\sqrt{\pi/2}} \int \exp\left[i[x\cdot\xi \pm 2\sqrt{t}\,|\xi|\right] \frac{1}{2\sqrt{t}\,|\xi|}\, \hat{g}(\xi)\, \check{d}\xi$$

which solve $(t\partial_t^2 - \frac{1}{2}\partial_t - \Delta)u^{\pm}(t, x) = 0,\ t > 0$.

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