ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Piecewise atone interpolation of monotone operators

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 9, nº 4 (1982), p. 497-528

<http://www.numdam.org/item?id=ASNSP_1982_4_9_4_497_0>

Piecewise Atone Interpolation of Monotone Operators.

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1. – The problem.

The inequality

$$\langle x_1 - x_2,\ Mx_1 - Mx_2 \rangle \geqslant 0$$

characterizes the class of monotone operators, whereas the equation

$$\langle x_1 - x_2,\ Mx_1 - Mx_2 \rangle = 0$$

characterizes its « boundary ». The points on this boundary, being operators at once monotone increasing and monotone decreasing, are the so called « atone operators ».

Since often the elements of a class can be constructed out of boundary elements, the above picture prompts the idea that monotone operators can be built out of atone ones. An indication of how such a construction can be effected is gained from the one dimensional case where any increasing function can be interpolated at a finite number of points, and thus approximated, by means of an increasing step function, that is, by means of a « piecewise atone monotone operator ». Nothing in this statement being necessarily one-dimensional it is natural to turn it into a conjecture:

« *Any monotone operator $M: X \to 2^Y$ from a space into its dual can be interpolated at a given finite set of points of its graph by a piecewise atone maximal monotone operator* ».

This paper is devoted to the proof of this conjecture and to the effective construction of the interpolating operators. A few comments are in order to get a good grasping of the problem and of the ideas that led to its solution. The interpolation theory has a marked algebro-topological flavor,

Pervenuto alla Redazione il 18 Dicembre 1980.

its objects—the piecewise atone operators—may be conceived as manifolds composed of « atone elements »—polyhedra carrying atone operators—joined together according to rules that guarantee monotonicity across the common boundaries. The construction of an interpolating operator calls for the determination of the two distinct elements entering into the make up of piecewise atone operators: the partition of the space into polyhedra, and the atone operators acting on them. On the line there is little to choose from, the polyhedra are intervals, and the atone operators constants, but in higher dimensions the choices in either category grow rapidly with the dimension, and make the problem increasingly complex. However, it is not the abondance of choices that creates the difficulty but the fact that, unlike in one dimension where any partition separating the interpolating points serves the purpose, in higher dimensions—in fact, already in two—the two aspects cannot be dealt with separately. Indeed, there is no a priori way of telling which partitions carry interpolating operators, and the problem has to be taken as a whole: geometry and operators have to be constructed at once. All this gives the question a rather elusive quality, and points to the need of a unifying point of view to bring all this diversity together. Such a point of view is found by lifting the problem out into larger spaces where the sought partition becomes that of a simplex into its relatively open faces, and where all atone operators are embodied in a single one; the procedure is thoroughly constructive.

In Hilbert space piecewise atone interpolation of monotone operators is equivalent to interpolation of contractions by means of piecewise unitary mappings, problem which, therefore, also finds a solution here.

2. – Notation.

All through this paper we shall be working with two dual locally convex Hausdorff topological real vector spaces X and Y paired by a bilinear form $\langle x, y\rangle$, and endowed with topologies compatible with the duality set up by the bilinear form. Most of what we have to say is indifferent to topology, and so in general it does not matter which topologies are used.

For a set S in a linear space we set

$$\operatorname{co} S = \left\{\sum_1^n \alpha_i s_i \middle| \forall \alpha_i \geqslant 0,\ s_i \in S,\ i = 1, 2, \ldots, n,\ \sum_1^n \alpha_i = 1,\ \forall n\right\};$$

$$\operatorname{aff} S = \left\{\sum_1^n \alpha_i s_i \middle| \forall \alpha_i,\ s_i \in S,\ i = 1, 2, \ldots, n,\ \sum_1^n \alpha_i = 1,\ \forall n\right\}.$$

These sets are the convex and affine hull of S respectively.

If N is an affine space

$$N^\tau = \{n_1 - n_2 | \forall n_1, n_2 \in N\}$$

is the linear space of all translations acting on N.

If $C \subset X$ is a cone with vertex at the origin its dual is the cone

$$C^\perp = \{v \in Y | \langle u, v \rangle \leqslant 0, \ \forall u \in C\},$$

and similarly for a cone in Y; the dual of a cone is always closed and convex. Let us recall that $C^{\perp\perp} = \overline{\mathrm{Co}\, C}$. In particular, if C is a linear space $C^\perp$ is its anihilator; in Hilbert space $C^\perp$ can be identified with the orthogonal complement of $\bar{C}$. It is convenient to extend this language and say that two affine spaces $N \subset X$ and $P \subset Y$ are orthogonal any time that their translation spaces are contained in each other duals, and are complementary when their translation spaces are each other duals.

The letter K is used to denote a generic closed convex set in either X or Y; the class of such sets does not depend on the topology. The interior of K as a subset of its closed affine hull is called its *relative interior* and is denoted $\mathring{K}$. If $\overline{\mathrm{aff}\, K}$ is finite dimensional $\mathring{K}$ is nonempty and is independent of the topology. A convex polyhedron is the intersection of a finite number of closed half spaces; a relative convex polyhedron is the intersection of a convex polyhedron with a closed affine space. However, to simplify the language, we shall use the term convex polyhedron even when speaking of a relative one, the context will always make it clear on which closed affine manifold the set is lying. Convex polyhedra have nonempty relative interiors. The indicator function of K is denoted ψ_K.

For multimappings $M: X \to 2^Y$ we write as usual

$$G(M) = \{(x, y) \in X \times Y | y \in Mx\}, \quad D(M) = \{x \in X | Mx \neq \emptyset\},$$

$$R(M) = \{y \in Y | \exists x \in X, \ y \in Mx\}.$$

The inverse of M is the multimapping $M^{-1}: Y \to 2^X$ defined by

$$M^{-1}y = \{x \in X | y \in Mx\};$$

naturally $D(M^{-1}) = R(M)$, $R(M^{-1}) = D(M)$. Often we shall have to deal with subdifferentials of indicator functions; note that

$$\partial\psi_K(x) = \{v \in Y | \langle x - x', v \rangle \geqslant 0, \ \forall x' \in K\}$$

is the cone of normals to K at x, and that it is the dual of the support cone of K at x: $\overline{\bigcup_{t\geqslant 0} t(K-x)}$. This geometrical interpretation holds even when K is not closed.

3. – Atone operators.

The main notion in this paper is that of « atone operator ».

DEFINITION 3.1. An atone operator is a multimapping $A\colon X\to 2^Y$ both monotone increasing and monotone decreasing, that is, a multimapping satisfying

$$\langle x_1-x_2, y_1-y_2\rangle = 0\,, \qquad \forall (x_1, y_1), (x_2, y_2)\in G(A)\,. \tag{3.1}$$

A maximal atone operator is an atone operator having no proper atone extensions.

It is not known if maximal atone operators are necessarily maximal monotone, but it seems unlikely. However, in certain spaces such as the finite dimensional spaces or Hilbert space maximal atonicity implies maximal monotonicity. Even in general spaces this holds for atone operators with either closed domain or closed range (cf. Lemma 3.3, Cor. 1).

The following lemma furnishes precise information concerning maximal atone operators.

LEMMA 3.1. *If $A\colon X\to 2^Y$ is maximal atone, then*

a) $D(A)$ and $R(A)$ are affine spaces and

$$(R(A)^\tau)^\perp\subset D(A)^\tau\,, \qquad (D(A)^\tau)^\perp\subset R(A)^\tau; \tag{3.2}$$

b) A is constant on any coclass of X modulo $(R(A)^\tau)^\perp$, and its value is a coclass of Y modulo $(D(A)^\tau)^\perp$;

c) The bilinear form $\langle\tilde{x},\tilde{y}\rangle = \langle x, y\rangle$, where the symbol $\sim$ indicates coclass in either space, sets up a duality between the quotient spaces $D(A)^\tau/(R(A)^\tau)^\perp$ and $R(A)^\tau/(D(A)^\tau)^\perp$, and the application $\tilde{A}\colon D(A)/ /(R(A)^\tau)^\perp\to R(A)/(D(A)^\tau)^\perp$ defined by

$$\tilde{A}\tilde{x} = \widetilde{Ax} \tag{3.3}$$

is a maximal atone affine bijection.

Conversely, any operator satisfying a), b) and c) is maximal atone.

PROOF. We give the proof of the direct proposition, that of the converse offers no difficulty and is left to the reader.

If $(x, y) \in G(A)$ we may write on use of atonicity,

$$\langle x_i - x, y_j - y\rangle + \langle x_j - x, y_i - y\rangle = 0\,, \qquad \forall (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in G(A)\,,$$

whence, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers with $\sum_1^n \alpha_i = 1$,

$$\begin{aligned}\langle \sum_i \alpha_i x_i - x, \sum_j \alpha_j y_j - y\rangle &= \sum_i \sum_j \alpha_i \alpha_j \langle x_i - x, y_j - y\rangle \\ &= \tfrac{1}{2} \sum_i \sum_j \alpha_i \alpha_j [\langle x_i - x, y_j - y\rangle + \langle x_j - x, y_i - y\rangle] = 0\,,\end{aligned}$$

which shows that the value $\sum_j \alpha_j y_j$ assigned to $\sum_i \alpha_i x_i$ is consistent with atonicity, and hence—since A is maximal atone—that $\sum_i \alpha_i x_i \in D(A)$, $\sum_i \alpha_i y_i \in$ $\in R(A)$, and $\sum_i \alpha_i y_i \in A\big(\sum_i \alpha_i x_i\big)$. Thus $D(A)$ and $R(A)$ are affine spaces. Now if $y_1 \in A x_1$, $x_2 - x_1 \in \big((R(A)^\tau\big)^\perp$,

$$\langle x_2 - x, y_1 - y\rangle = \langle x_1 - x, y_1 - y\rangle + \langle x_2 - x_1, y_1 - y\rangle = 0\,, \qquad \forall (x, y) \in G(A)\,,$$

and again by atone maximality $x_2 \in D(A)$ and $y_1 \in A x_2$. It follows that $A x_1 \subset A x_2$, and by symmetry, that $A x_2 \subset A x_1$, and in consequence $A x_1 =$ $= A x_2$. Thus we have shown that $\big(R(A)^\tau\big)^\perp \subset D(A)^\tau$, and that A is constant along the coclasses modulo $\big(R(A)^\tau\big)^\perp$. Similarly, arguing on A^{-1} one deduces that $\big(D(A)^\tau\big)^\perp \subset R(A)^\tau$, and that A^{-1} remains constant on any coclass modulo $\big(D(A)^\tau\big)^\perp$. This last property amounts to saying that Ax is the union of coclasses modulo $\big(D(A)^\tau\big)^\perp$. Let us see that in fact there is at most one coclass. If y and y' belong to Ax then for $\forall (x_1, y_1)$, $(x_2, y_2) \in G(A)$,

$$\begin{aligned}\langle x_1 - x_2, y - y'\rangle &= \langle x_1 - x, y - y'\rangle + \langle x - x_2, y - y'\rangle \\ &= -\langle x_1 - x, y_1 - y\rangle + \langle x_1 - x, y_1 - y'\rangle + \langle x - x_2, y - y_2\rangle \\ &\quad - \langle x - x_2, y' - y_2\rangle = 0\,,\end{aligned}$$

which, x_1 and x_2 being arbitrary in $D(A)$, indicates that $y - y' \in \big(D(A)^\tau\big)^\perp$, that is, that y and y' belong to the same coclass modulo $\big(D(A)^\tau\big)^\perp$, and in consequence that if not empty Ax consists of just one class modulo $(D(A)^{\mathrm{r}})^{\perp}$; the same argument applied to A^{-1} shows that when defined $A^{-1}y$ is made up of a single class modulo $(R(A)^{\mathrm{r}})^{\perp}$. From these facts it follows that the operator $\tilde{A}$ appearing in the statement of the lemma is a well defined bijection between $D(A)/(R(A)^{\mathrm{r}})^{\perp}$ and $R(A)/(D(A)^{\mathrm{r}})^{\perp}$. $\tilde{A}$ receives atonicity from A, and also maximality because any extension of $\tilde{A}$ induces an extension of A. By passing to coclasses in the relation $\sum_i \alpha_i A x_i \subset A\left(\sum_i \alpha_i x_i\right)$ proved above it follows that $\tilde{A}$ is affine.

An important particular case of atone operators is that where $\tilde{A}$ takes a constant value. In this case A assigns to each point of a closed affine space a fixed complementary affine space.

The operator $\tilde{A}$ is called the *kernel of* A, and the common dimension of the quotient spaces $D(A)^{\mathrm{r}}/(R(A)^{\mathrm{r}})^{\perp}$ and $R(A)^{\mathrm{r}}/(D(A)^{\mathrm{r}})^{\perp}$ its *rank*. It is essentially with atone operators of finite rank that we are concerned in this paper; note that they have closed domain and range. The underlying linear operator associated with the kernel $\tilde{A}$ is antisymmetric but not necessarily antiselfadjoint ($\tilde{A}^* + \tilde{A} = 0$); this is not even true in Hilbert space, but it holds whenever the operator is of finite rank.

LEMMA 3.2. *Any atone operator $A: X \to 2^Y$ admits a maximal atone extension $\hat{A}$ such that $D(A) \subset \overline{D(\hat{A})} = \overline{\operatorname{aff} D(A)}$.*

PROOF. We start out by making an extension of A to the affine hull of $D(A)$, and pose for $x \in \operatorname{aff} D(A)$,

$$A_1 x = \Big\{\sum_i \alpha_i y_i \,|\, \forall \{x_i\}_1^k \subset D(A),\ \forall \{\alpha_i\}_1^k,\ \sum_i \alpha_i = 1,\ x = \sum_i \alpha_i x_i,\ \forall y_i \in Ax_i,\ i = 1, 2, \dots, k\Big\}.$$

The multimapping thus defined is an extension of A having $\operatorname{aff} D(A)$ as domain and $\operatorname{aff} R(A)$ as range. To check that it is atone we take two points (x, y) and (x', y') in its graph, expressed in the form

$$x = \sum_1^k \alpha_i x_i,\ y = \sum_1^k \alpha_i y_i,\ x' = \sum_1^k \alpha' x_i,\ y' = \sum_1^k \alpha_i' y_i'$$
$$\{x_i\}_1^k \subset D(A),\ y_i, y_i' \in Ax_i,\ i = 1, 2, \dots, k,\ 1 = \sum_1^k \alpha_i = \sum_1^k \alpha_i',$$

(obviously there is no loss of generality in assuming that the points in $D(A)$ used to represent x and x' are the same) and proceed to the following calculations:

$$\begin{aligned}
\langle x-x', y-y'\rangle &= \langle \sum_i (\alpha_i-\alpha_i')x_i,\ \sum_j \alpha_j y_j - \sum_j \alpha_j' y_j'\rangle = \\
&= \langle \sum_i (\alpha_i-\alpha_i')x_i,\ \sum_j (\alpha_j-\alpha')y_j\rangle + \langle \sum_i (\alpha_i-\alpha_i')x_i,\ \sum_j \alpha_j'(y_j-y_j')\rangle \\
&= \sum_{i,j} (\alpha_i-\alpha_i')(\alpha_j-\alpha_j')\langle x_i, y_j\rangle + \sum_{j,i} (\alpha_i-\alpha_i')\alpha_j'\langle x_i, y_j-y_j'\rangle \\
&= \tfrac{1}{2}\sum_{i,j} (\alpha_i-\alpha_i')(\alpha_j-\alpha')[\langle x_i, y_j\rangle + \langle x_j, y_i\rangle] + \\
&\qquad + \sum_{i,j} (\alpha_i-\alpha_i')\alpha_j'\langle x_i-x_j, y_j-y_j'\rangle + \sum_{i,j} (\alpha_i-\alpha_i')\alpha_j'\langle x_j, y_i-y_j'\rangle \\
&= \tfrac{1}{2}\sum_{i,j} (\alpha_i-\alpha_i')(\alpha_j-\alpha_j')[\langle x_i, y_i\rangle + \langle x_j, y_j\rangle - \langle x_i-x_j, y_i-y_j\rangle] + \\
&\qquad + \sum_{i,j} (\alpha_i-\alpha_i')\alpha_j'[\langle x_i-x_j, y_j-y_j'\rangle - \langle x_i-x_j, y_i-y_j\rangle] \\
&\qquad + \sum_{i,j} (\alpha_i-\alpha_i')\alpha_j'\langle x_j, y_j-y_j'\rangle .
\end{aligned}$$

An inspection of the last term on the right shows that, either by atonicity or because of the equations $\sum_i (\alpha_i-\alpha_i') = \sum_j (\alpha_j-\alpha_j') = 0$, all the sums vanish, and hence that A_1 is atone.

Next a new extension is made by assigning to any $x \in \operatorname{aff} D(A)$ the set

$$A_2 x = A_1 x + \big((\operatorname{aff} D(A))^\tau\big)^\perp .$$

It is clear that A_2 is atone and that $D(A_2) = \operatorname{aff} D(A)$, $R(A_2) = \operatorname{aff} R(A) + \big((\operatorname{aff} D(A))^\tau\big)^\perp$. Zorn's Lemma applied to the ordered family of all atone extensions of A_2 furnishes a maximal extension $\hat{A}$. For this $\hat{A}$ we have

$$D(\hat{A}) \supset \operatorname{aff} D(A) , \qquad R(\hat{A}) \supset \operatorname{aff} R(A) + \big((\operatorname{aff} D(A)^\tau\big)^\perp .$$

Thus the lemma will be proved as soon as we check that $D(\hat{A}) \subset \overline{\operatorname{aff} D(A)}$. If $\hat{x} \in D(\hat{A})$, then by atonicity

$$\langle \hat{x}-x, \hat{y}-y\rangle = 0 , \quad \text{for } \hat{y} \in \hat{A}\hat{x} \quad \text{and} \quad \forall (x, y) \in G(\hat{A}) .$$

But since $(x, y) \in G(\hat{A})$ implies $(x, y+u) \in G(\hat{A})$ for $\forall u \in \big((\operatorname{aff} D(A))^\tau\big)^\perp$ we must also have

$$\langle \hat{x}-x, \hat{y}-y-u\rangle = 0 ,$$

and so,

$$\langle \hat{x}-x, u\rangle = 0 .$$

Hence, by the arbitrariness of u, $\hat{x} - x \in ((\text{aff}\, D(A)^\tau)^{\perp\perp} = \overline{(\text{aff}\, D(A))^\tau}$, and $\hat{x} \in x + \overline{(\text{aff}\, D(A))^\tau} \subset \overline{\text{aff}\, D(A)}$. Thus the proof concludes.

The extension $\hat{A}$ of A whose existence we just proved is not unique in general, but it is so if aff $D(A)$ is closed.

Now we turn our attention to the relations between atone and monotone operators. Let us first recall the notion of support half-space of a set.

DEFINITION 3.2. A closed half space Π is said to support a set S at one of its points s if $S \subset \Pi$ and s belongs to the boundary of Π.

LEMMA 3.3. *The sum $A + M$ of an atone operator A and a maximal monotone operator M is maximal monotone any time that A is defined on the intersection of all support half spaces of $D(M)$, the intersection being taken as the whole space if there are no support half spaces.*

PROOF. One must prove that

$$\{(x_0, y_0) \in X \times Y,\ \langle x - x_0, u + v - y_0\rangle \geqslant 0,\ \forall x \in D(M),\ \forall u \in Ax,\ \forall v \in Mx\} \Leftrightarrow$$
$$\Leftrightarrow \{y_0 \in (A + M)x_0\}.$$

Assume that the condition on the left is satisfied. If Π is a support half space of $D(M)$ at x, and n is its normal, then by maximal monotonicity $v \in Mx$ implies $v + tn \in Mx$, $\forall t \geqslant 0$, and therefore

$$\langle x - x_0, u + v + tn - y_0\rangle \geqslant 0\,, \qquad \forall t > 0\,,$$

whence dividing by t and passing to the limit $t \to \infty$, $\langle x - x_0, n\rangle \geqslant 0$, which amounts to $x_0 \in \Pi$. This being true for all support half spaces, $x_0 \in D(A)$. If $u_0 \in Ax_0$ atonicity allows us to replace $\langle x - x_0, u\rangle$ by $\langle x - x_0, u_0\rangle$ and obtain

$$\langle x - x_0, u_0 + v - y_0\rangle \geqslant 0\,, \qquad \forall (x, v) \in G(M)\,.$$

By M's maximal monotonicity these inequalities imply $y_0 - u_0 \in Mx_0$, that is, $y_0 \in u_0 + Mx_0 \subset (A + M)x_0$, as we set out to prove.

COROLLARY 1. *Any maximal atone operator with either closed domain or closed range is maximal monotone.*

PROOF. If the domain is closed apply the theorem with $\partial\psi_{D(A)}$ in place of M; if it is the range that is closed apply the previous case to A^{-1}.

COROLLARY 2. *If K is the intersection of its supporting halfspaces, $\partial\psi_K$ is maximal monotone, and $K \subset D(A)$, then $A + \partial\psi_K$ is maximal monotone.*

Monotone extensions of atone operators, when not atone, give rise to a curious phenomenom.

LEMMA 3.4. *If $M: X \to 2^Y$ is a monotone extension of a maximal atone operator A and $(x_0, y_0) \in G(M)$, then $\langle x_0 - x, y_0 - y \rangle$ is independent of $(x, y) \in G(A)$.*

PROOF. If (x_1, y_1), $(x_2, y_2) \in G(A)$ then $(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in G(A)$, and since $G(A) \subset G(M)$,

$$\begin{aligned}0 \leqslant \langle x_0 - (tx_1 + (1-t)x_2), y_0 - (ty_1 + (1-t)y_2) \rangle \\ &= \langle x_0 - x_2 + t(x_2 - x_1), y_0 - y_2 + t(y_2 - y_1) \rangle \\ &= \langle x_0 - x_2, y_0 - y_2 \rangle + t[\langle x_2 - x_1, y_0 - y_2 \rangle + \langle x_0 - x_2, y_2 - y_1 \rangle] \\ &+ t^2 \langle x_2 - x_1, y_2 - y_1 \rangle .\end{aligned}$$

The last term on the right vanishes by atonicity, and the rest, by the arbitrariness of t, yields

$$\begin{aligned}0 = \langle x_2 - x_1, y_0 - y_2 \rangle + \langle x_0 - x_2, y_2 - y_1 \rangle = \langle x_0 - x_1, y_0 - y_1 \rangle \\ - \langle x_0 - x_2, y_0 - y_2 \rangle - \langle x_1 - x_2, y_1 - y_2 \rangle = \langle x_0 - x_1, y_0 - y_1 \rangle \\ - \langle x_0 - x_2, y_0 - y_2 \rangle ,\end{aligned}$$

which is precisely the lemma's assertion.

4. – Piecewise atone operators.

The simplest and the most aesthetically pleasing definition of piecewise atone monotone operator is perhaps the following:

« A monotone operator $M: X \to 2^Y$ is said to be piecewise atone subordinated to a family of atone operators $A_1, A_2, \ldots, A_r$ if, for any x, Mx is contained in at least one of the sets $A_1x, A_2x, \ldots, A_rx$. »

However, it is difficult if not impossible to obtain from it, without additional topological conditions, satisfactory information as to the nature of the sets on which the individual atone operators act and on the way these sets are put together. For this reason we have adopted a more particularized definition, equivalent to the above in most cases occurring in practice, and sufficient for the purposes of the interpolation theory.

The building blocks out of which piecewise atone monotone operators are constructed are the « atone elements »; these are couples $\{\Pi, A\}$ consisting of a convex polyedron Π and a maximal atone operator A acting on it.

DEFINITION 4.1. A paving of a convex polyhedron Π is a family $\{\Pi_i\}_1^r$ of convex polyhedra contained in Π, with nonempty disjoint interiors with regard to Π, whose union is Π. The Π_i's are called the « tiles » of the paving.

With this we are now ready for our definition of piecewice atone monotone operator:

DEFINITION 4.2. A monotone operator $M: X \to 2^Y$ is said to be piecewise atone if its domain is a convex polyhedron Π and there is a finite family of atone elements $\{\Pi_i, A_i\}_1^r$ such that $\{\Pi_i\}_1^r$ is a paving of Π, and

$$(4.1) \qquad Mx = \mathrm{co}\Big(\bigcup_{i\in I(x)} A_i x\Big) + \partial\psi_\Pi(x), \qquad I(x) = \{i | x \in \Pi_i\}, \qquad \forall x \in X.$$

In such a case one says that M is subordinated to the family $\{\Pi_i, A_i\}_1^r$ of atone elements.

An important subclass of piecewise atone monotone operators is that of « step operators », obtained by taking constant operators for the A_i's.

The lemma below tells us that it is enough to know the behaviour of the operator at the interior of the tiles to have the rest.

LEMMA 4.1. *Let $\{\Pi_i, A_i\}_1^n$ be a family of atone elements such that $\{\Pi_i\}_1^n$ is a paving of a convex polyhedron Π. If the operator $M: X \to 2^Y$ defined on $\bigcup_i \mathring{\Pi}_i$ by*

$$Mx = A_i x, \qquad \forall x \in \mathring{\Pi}_i, \qquad i = 1, 2, \ldots, n$$

is monotone, then it admits a unique maximal monotone extension with domain Π, namely, the operator $\tilde{M}$ defined by

$$\tilde{M}x = \mathrm{co}\Big(\bigcup_{i\in I(x)} A_i x\Big) + \partial\psi_\Pi(x), \qquad I(x) = \{i | x \in \Pi_i\}, \qquad \forall x \in X.$$

PROOF. The first step is to derive the matching conditions imposed by monotonicity on contiguous atone elements. Let $x_0 \in \Pi_i \cap \Pi_j$, and let V_i and V_j the supporting cones at x_0 of Π_i and Π_j respectively. If $v_i \in \mathring{V}_i$, $v_j \in \mathring{V}_j$, then for $t > 0$ sufficiently small $x_0 + tv_i \in \mathring{\Pi}_i$, $x_0 + tv_j \in \mathring{\Pi}_j$, and by monotonicity,

$$0 \leqslant \langle (x_0 + tv_i) - (x_0 + tv_j), M(x_0 + tv_i) - M(x_0 + tv_j) \rangle =$$
$$= t\langle v_i - v_j, A_i(x_0 + tv_i) - A_j(x_0 + tv_j) \rangle .$$

In spite of the possible multivaluedness of the operators there is no ambiguity in these formulas because by Lemma 3.1 the angular brackets have but one value, value which is an affine, hence continuous function of t. Dividing by t and letting $t\to 0$,

$$0\leqslant\langle v_i-v_j,\ A_ix_0-A_jx_0\rangle\,,\qquad \forall v_i\in\mathring{V}_i,\ \forall v_j\in\mathring{V}_j\,,$$

whence

$$A_ix_0-A_jx_0\subset(\mathring{V}_j-\mathring{V}_j)^\perp=\mathring{V}_j^\perp\cap(-\mathring{V}_i)^\perp=V_j^\perp\cap(-V_i^\perp)\,,$$

relations that are equivalent to the apparently simpler ones,

$$A_ix_0-A_jx_0\in V_j^\perp=\partial\psi_{\Pi_i}(x_0)\,. \tag{4.2}$$

These are the « matching conditions ».

Now let us try to see which could be the behaviour at a point $x_0\in\Pi$ of a maximal monotone extension $\tilde{M}$ of M with domain Π. We begin by renumbering the tiles so that $\Pi_1,\Pi_2,\dots,\Pi_r$ are those concurring at x_0; let, as before, $V_1,V_2,\dots,V_r$ be the corresponding support cones. Note that $V_1\cup V_2\cup\dots\cup V_r=V$ is the support cone of Π at x_0. If $y_0\in\tilde{M}x_0$ and if $u_i\in\mathring{V}_i$, then for any sufficiently small positive t, $x_0+tu_i\in\mathring{\Pi}_i$, and by monotonicity

$$0\leqslant\langle(x_0+tu_i)-x_0\,,\ M(x_0+tu_i)-y_0\rangle=t\langle u_i,\ A_i(x_0+tu_i)-y_0\rangle\,.$$

As in the above discussion one deduces,

$$y_0-A_ix_0\in V_i^\perp\,,\qquad i=1,2,\dots,r\,. \tag{4.3}$$

Therefore, if $e_i\in y_0-A_ix_0$, we can write (4.2) and (4.3) in the form

$$e_i\in V_i^\perp\,,\qquad e_i-e_j\in V_j^\perp\,,\qquad i,j=1,2,\dots,r\,. \tag{4.4}$$

We now look at the situation from the point of view of the space $X_1=\overline{(\operatorname{aff}\Pi)}^\tau$ generated by V, and its dual $Y_1=Y/X_1^\perp$. If $\tilde{e}_1,\tilde{e}_2,\dots,\tilde{e}_r$ are the classes modulo $X_1^\perp$ containing $e_1,e_2,\dots,e_r$ respectively one sees from (4.4) that the cones $V_1,V_2,\dots,V_r$ in X_1 and the vectors $\tilde{e}_1,\tilde{e}_2,\dots,\tilde{e}_r$ in Y_1 are in the position described in the lemma below. In consequence there exist nonnegative real numbers $\alpha_1,\alpha_2,\dots,\alpha_r$ with $\sum_i\alpha_i=1$ such that $\alpha_1\tilde{e}_1+$ $+\alpha_2\tilde{e}_2+\dots+\alpha_r\tilde{e}_r=(\widetilde{\alpha_1e_1+\alpha_2e_2+\dots+\alpha_re_r})$ belongs to the dual of V in the duality (X_1,Y_1). Such a cone being $(\widetilde{V^\perp})$ one concludes first that

$\sum_i \alpha_i e_i \in V^\perp = \partial\psi_\Pi(x_0)$, and then that $y_0 \in \operatorname{co}\Big(\bigcup_{i\in I(x_0)} A_i x_0\Big) + \partial\psi_\Pi(x_0)$. Thus we have shown that

$$\tilde{M}x_0 \subset \operatorname{co}\Big(\bigcup_{i\in I(x_0)} A_i x_0\Big) + \partial\psi_\Pi(x_0)\,.$$

On the other hand by $\tilde{M}$'s maximality $A_i x_0 = \lim_{t\downarrow 0} A(x_0 + tv_i)$ (limit via the spaces X_1 and Y_1) belongs to $\tilde{M}x_0$, as well as the sets $A_i x_0 + \partial\psi_\Pi(x_0)$, and along with them the convex hull of their union. From this and the previous opposite inclusion one concludes that

$$\tilde{M}x_0 = \operatorname{co}\Big(\bigcup_{i\in I(x_0)} A_i x_0\Big) + \partial\psi_\Pi(x_0)\,,$$

which shows that there is at most one possible choice for $\tilde{M}$. That this choice is in fact monotone, and hence maximal monotone, will be seen in Lemma 4.3.

LEMMA 4.2. *Let $V_1, V_2, \ldots, V_r$ be r closed convex cones with vertex at 0 in X, having nonempty disjoint interiors, and whose union is a closed convex cone V, then*

$$\{e_i \in V_i^\perp,\ e_i - e_j \in V_i^\perp,\ i, j = 1, 2, \ldots, r\} \Rightarrow$$
$$\Rightarrow \Big\{\exists \alpha_1, \alpha_2, \ldots, \alpha_r | \alpha_i \geqslant 0,\ \sum_i \alpha_i = 1,\ \sum_i \alpha_i e_i \in V^\perp\Big\}.$$

PROOF. Let us consider the closed convex set $\Delta + V^\perp \subset Y$, where $\Delta = \operatorname{co}(-e_1, -e_2, \ldots, -e_r)$. We claim that $V_i = \partial\psi_{\Delta + V^\perp}(-e_i)$, $i = 1, 2, \ldots, r$. If $v_i \in V_i$, then for any j and any $v^\perp \in V^\perp$,

$$\langle v_i, (-e_i) - (-e_j + v^\perp)\rangle = \langle v_i, e_j - e_i\rangle - \langle v_i, v^\perp\rangle\,.$$

Both terms on the right are nonnegative: the first because $e_i - e_j \in V_i^\perp$, and the second because $V^\perp \subset V_i^\perp$. Then, taking convex combinations,

$$\Big\langle v_i, (-e_i) - \Big(\sum_j \alpha_j(-e_j) + v^\perp\Big)\Big\rangle \geqslant 0\,,$$

that is, $v_i \in \partial\psi_{\Delta + V^\perp}(-e_i)$, and hence $V_i \in \partial_{\Delta + V^\perp}(-e_i)$. On the other hand if $u_i \in \partial\psi_{\Delta + V^\perp}(-e_i)$,

$$0 \leqslant \langle u_i, (-e_i) - (-e_i + v^\perp)\rangle = -\langle u_i, v^\perp\rangle\,, \quad \forall v^\perp \in V^\perp\,,$$

and therefore $u_i \in V^{\perp\perp} = V$. Thus we have shown

$$V_i \in \partial\psi_{\Delta+V^\perp}(-e_i) \subset V .$$

Suppose now that for some i, V_i is properly contained in $\partial\psi_{\Delta+V^\perp}(-e_i)$. In such a case, since V_i is closed, there would be a $x_i \in \operatorname{int}\partial\psi_{\Delta+V^\perp}(-e_i)$ not belonging to V_i. Moreover, since $\operatorname{int}\partial\psi_{\Delta+V^\perp}(-e_i) \cap \partial\psi_{\Delta+V^\perp}(-e_j) = \emptyset$, $\forall j \neq i$, x_i would not belong to any of the other V_i's either, and therefore would be outside of $V_1 \cup V_2 \cup \ldots \cup V_r = V$, in contradiction with the fact $x_i \in \partial\psi_{\Delta+V^\perp}(-e_i) \subset V$. We have thus substantiated our claim. Now on use of $V_i = \partial\psi_{\Delta+V^\perp}(-e_i)$ we can write

$$0-(-e_i) = e_i \in V_i^\perp = \left(\partial\psi_{\Delta+V^\perp}(-e_i)\right)^\perp ,$$

whence

$$0 \in \bigcap_i \left[(-e_i) + \left(\partial\psi_{\Delta+V^\perp}(-e_i)\right)^\perp\right] .$$

The sets in square brackets are the supporting cones of $\Delta + V^\perp$ at $-e_1$, $-e_2, \ldots, -e_r$ translated to these points; their intersection is $\Delta + V^\perp$. Henec $0 \in \Delta + V^\perp$, which is the sought result.

The matching conditions (4.2) found in the course of the proof of Lemma 4.1 as necessary for a family of atone operators to compose a piecewise atone monotone operator are also sufficient:

LEMMA 4.3. *A family of atone elements $\{\Pi_i, A_i\}_1^r$ is the family associated to a piecewise atone monotone operator if and only if $\{\Pi_i\}_r^1$ is a paving of a polyhedron Π, and*

$$A_i x - A_j x \subset \partial\psi_{\Pi_j}(x) , \quad \forall x \in \Pi_i \cap \Pi_j,\ \forall i, j . \tag{4.5}$$

In such a case the subordinated operator is

$$Mx = \operatorname{co}\Big(\bigcup_{i \in I(x)} A_i x\Big) + \partial\psi_\Pi(x) , \quad I(x) = \{i | x \in \Pi_i\} . \tag{4.6}$$

PROOF. Necessity has already been established while proving Lemma 4.1. By the same lemma, to demonstrate sufficiency, all we need is to show that the operator $M_1 \colon X \to 2^Y$ defined by

$$M_1 x = A_i x , \quad \forall x \in \mathring{\Pi}_i,\ \forall i ,$$

is monotone. In other terms, we must prove that

$$\langle x''-x', y''-y'\rangle \geqslant 0 , \quad \forall x', x'' \in \bigcup_i \mathring{\Pi}_i,\ \forall y' \in Mx',\ \forall y'' \in Mx''.$$

In the interior of each tile $\langle x''-x', y''-y'\rangle$ does not depend on the choice of y' and y'' in M_1x' and M_1x'', and by the A_i's atonicity it is an affine function of x' and x'', and as such is continuous along any straight line. This remark allows one to confine attention only to the case where the union of the tile's boundaries intersects the segment $[x', x''] = \{tx' + (1-t)\,x''\}_{0\leqslant t\leqslant 1}$ in a finite number of points. Let these points be $z_1, z_2, \ldots, z_k$, ordered from x' to x'' so that each of the segments $[x', z_1]$, $[z_1, z_2]$, ..., $[z_{k-1}, z_k]$, $[z_k, x'']$ is entirely contained in one tile. Now letting $u = x''-x'$ we write with the help of a positive ε,

$$\begin{aligned}
&\langle x''-x', M_1x''-M_1x'\rangle = \\
&\quad = [\langle u, M_1(z_1-\varepsilon u)-M_1x'\rangle + \langle u, M_1(z_2-\varepsilon u)-M_1(z_1+\varepsilon u)\rangle + \ldots + \\
&\quad + \langle u, M_1(z_k-\varepsilon u)-M_1(z_{k-1}+\varepsilon u)\rangle + \langle u, M_1x''-M_1(z_k+\varepsilon u)\rangle] + \\
&\quad + [\langle u, M_1(z_1+\varepsilon u)-M_1(z_1-\varepsilon u)\rangle + \langle u, M_1(z_2+\varepsilon u)-M_1(z_2-\varepsilon u)\rangle + \ldots + \\
&\quad + \ldots + \langle u, M_1(z_k+\varepsilon u)-M_1(z_k-\varepsilon u)\rangle].
\end{aligned}$$

If ε is sufficiently small each segment $[x', z_1-\varepsilon u]$, $[z_1+\varepsilon u, z_2-\varepsilon u]$, ..., ..., $[z_k+\varepsilon u, x'']$ is contained in the interior of the respective tile, and all the terms in the first sum on the right vanish because of M's atonicity at the interior of the tiles. As to the rest, assuming the tiles involved to be $\Pi_{i_0}, \Pi_{i_1}, \ldots, \Pi_{i_{k-1}}, \Pi_{i_k}$, it can be given the form

$$\begin{aligned}
\langle x''-x', M_1x''-M_1x'\rangle &= \langle u, A_{i_1}(z_1+\varepsilon u)-A_{i_0}(z_1-\varepsilon u)\rangle + \\
&+ \langle u, A_{i_2}(z_2+\varepsilon u)-A_{i_1}(z_2-\varepsilon u)\rangle + \ldots + \langle u, A_{i_k}(z_k+\varepsilon u)-A_{i_{k-1}}(z_k-\varepsilon u)\rangle,
\end{aligned}$$

whence letting $\varepsilon\downarrow 0$,

$$\begin{aligned}
\langle x''-x', M_1x''-M_1x'\rangle &= \langle u, A_{i_1}z_1-A_{i_0}z_1\rangle + \langle u, A_{i_2}z_2-A_{i_1}z_2\rangle + \ldots + \\
&+ \langle u, A_{i_k}z_k-A_{i_{k-1}}z_k\rangle.
\end{aligned}$$

By hypothesis $A_{i_{j-1}}z_j-A_{i_j}z_j$ is normal to Π_{i_j} at z_j, whereas u points in the direction of the segment crossing the common boundary from $\Pi_{i_{j-1}}$ to Π_i and hence belongs to the support cone of Π_{i_j}. Therefore all the terms in the above sum are nonnegative, and M_1 is monotone.

COROLLARY. *Piecewise atone monotone operators are maximal monotone.*

5. – Piecewise atone interpolation of monotone operators.

THEOREM 5.1. *Any monotone operator $M: X \to 2^Y$ defined on a finite number of affinely independent points admits an extension of the form $A + \partial\psi_\Delta$, where $\Delta = \operatorname{co} D(M)$, and A is a maximal atone operator with domain $\operatorname{aff}\Delta$; if M is cyclically monotone A can be taken to be a constant operator.*

PROOF. If $D(M) = \{x_0, x_1, \dots, x_n\}$ then Δ is the n-dimensional simplex having the x_i's as vertices. By M's monotonicity,

$$\langle x_i - x_j, y_i\rangle \geqslant \langle x_i - x_j, y_j\rangle\,, \qquad \forall y_i \in Mx_i,\ \forall y_j \in Mx_j,\ i, j = 0, 1, \dots, n\,,$$

whence for every ordered couple of indices one deduces the existence of a separating quantity k_{ij}:

$$\langle x_i - x_j, y_i\rangle \geqslant k_{ij} \geqslant \langle x_i - x_j, y_j\rangle\,, \qquad \forall y_i \in Mx_i,\ \forall y_j \in Mx_j,\ i \neq j\,.$$

These inequalities being equivalent to

$$\langle x_j - x_i, y_j\rangle \geqslant -k_{ij} \geqslant \langle x_j - x_i, y_i\rangle\,, \qquad \forall y_i \in Mx_i,\ \forall y_j \in Mx_j,\ i \neq j\,,$$

the k_{ij} can be chosen so that $k_{ij} + k_{ji} = 0$, $\forall i \neq j$. According to this construction the closed halfspace in Y,

$$\Pi_{ij} = \{y \mid \langle x_i - x_j, y\rangle \geqslant k_{ij}\}\,,$$

bounded by the hyperplane orthogonal to $x_i - x_j$,

$$\pi_{ij} = \{y \mid \langle x_i - x_j, y\rangle = k_{ij}\}\,,$$

contains Mx_i. If i is kept fixed and j is allowed to vary over the other indices one obtains n hyperplanes $\{\pi_{ij}\}_{j\neq i}$, one for each edge issuing from x_i and perpendicular to it. Due to the affine independence of the x_k's these edges are linearly independent, and the hyperplanes converge on an affine space Z_i of codimension n, which is thus assigned to the vertex x_i'. From the above,

$$\langle x_i - x_j, z_i\rangle = k_{ij}\,, \qquad \langle x_j - x_i, z_j\rangle = k_{ji}\,, \qquad i \neq j,\ z_i \in Z_i,\ z_j \in Z_j\,,$$

whence

$$\langle x_i - x_j, z_i - z_j \rangle = k_{ij} + k_{ji} = 0 , \quad \forall i, j ,$$

which simply says that the multimapping $x_i \to Z_i$, $i = 0, 1, \ldots, n$, is atone. An appeal to Lemma 3.2 then yields a maximal atone operator A defined on aff Δ such that

$$Ax_i \supset Z_i , \quad i = 0, 1, \ldots, n .$$

Let us now see which are the relations between A and the original operator M. By definition of Π_{ij} and Z_i,

$$\Pi_{ij} - z_i = \{v | \langle x_i - x_j, v \rangle \geqslant 0\} , \quad \forall z_i \in Z_i ,$$

hence

$$\bigcap_{j \neq i} \Pi_{ij} - z_i = \bigcap_{j \neq i} (\Pi_{ij} - z_i) = \{v | \langle x_i - x_j, v \rangle \geqslant 0, \ j \neq i\} .$$

The set on the right is the cone of normals to Δ at x_i, that is, $\partial\psi_\Delta(x_i)$; thus we can write,

$$\bigcap_{j \neq i} \Pi_{ij} = z_i + \partial\psi_\Delta(x_i) \subset Ax_i + \partial\psi_\Delta(x_i) = (A + \partial\psi_\Delta)(x_i) .$$

But, as we have seen $Mx_i \subset \Pi_{ij}$, $j \neq i$, and so

$$Mx_i \subset \bigcap_{j \neq i} \Pi_{ij} \subset (A + \partial\psi_\Delta)x_i , \quad i = 0, 1, \ldots, n ,$$

and $A + \partial\psi_\Delta$ is an extension of M.

Now we treat the cyclically monotone case. It is clear that for A to be a constant operator it is necessary that M be cyclically monotone because, in such a case, $A + \partial\psi_\Delta$ is cyclically monotone. The proof of sufficiency is only slightly more complicated. Let M be cyclically monotone. It is well known, and besides very easy to check, that the function

$$g(x) = \sup \{\langle x - x_{i_k}, y_{i_k} \rangle + \langle x_{i_k} - x_{i_{k-1}}, y_{i_{k-1}} \rangle + \ldots + \langle x_{i_1} - x_{i_0}, y_{i_0} \rangle\}$$
$$\{(x_{i_k}, y_{i_k}), (x_{i_{k-1}}, y_{i_{k-1}}), \ldots, (x_{i_0}, y_{i_0})\} \subset G(M) , \quad x_{i_0} = x_0, \ \forall k ,$$

is lower semicontinuos and convex, and

$$Mx_i \subset \partial g(x_i) , \quad i = 0, 1, \ldots, n .$$

The new function

$$f(x) = \begin{cases} \sum_0^n \alpha_i g(x_i)\,, & x = \sum_0^n \alpha_i x_i,\ \ \sum_0^n \alpha_i = 1 \\ +\infty, & x \notin \operatorname{aff} D(M)\,, \end{cases}$$

is also lower semicontinuous and convex; it majorates g and coincides with it on $D(M)$. Hence

$$\partial g(x_i) \subset \partial f(x_i)\,,$$

and

$$Mx_i \in \partial f(x_i)\,, \qquad i = 0, 1, \ldots, n\,.$$

A correct interpretation of this result leads to the sought conclusion. The mapping $x = \sum_0^n \alpha_i x_i \to \sum_0^n \alpha_i g(x_i)$, defined on $\operatorname{aff} D(M)$, being affine and continuous, admits an extension of the same nature to the whole space. Since such an extension is necessarily of the form $\langle x, z\rangle + \text{const}$, $z \in Y$,

$$f(x) = \langle x, z\rangle + \text{const} + \psi_\Delta\,,$$
$$\partial f(x) = z + \partial\psi_\Delta(x) = Z + \partial\psi_\Delta(x)\,,$$

where Z is the coclass modulo $\big((\operatorname{aff} D(M)^\tau)^\perp$ containing z. Hence, if A is the operator that to any point of $\operatorname{aff} D(M)$ assigns Z,

$$Mx_i \subset \partial f(x_i) = (A + \partial\psi_\Delta)x_i\,, \qquad i = 0, 1, \ldots, n\,,$$

and sufficiency is proved.

LEMMA 5.1. *If $\Delta \subset X$ is a finite dimensional simplex and $A\colon X \to 2^Y$ a maximal atone operator with domain* $\operatorname{aff}\Delta$, *then $A + \partial\psi_\Delta$ and $(A + \partial\psi_\Delta)^{-1}$ are piecewise atone monotone operators, and $D(A + \partial\psi_\Delta) = \Delta$, $D\big((A + \partial\psi_\Delta)^{-1}\big) = Y$.*

PROOF. By translation things can be arranged so that $\operatorname{aff}\Delta$ is a vector space, more precisely, a finite dimensional vector space $X_1 \subset X$. According to the hypothesis the value of $A + \partial\psi_\Delta$ at any point of its domain is composed of coclasses modulo $X_1^\perp$. Consequently, if $\pi\colon Y \to Y/X_1^\perp = Y_1$ is the canonical mapping of Y on the quotient space, it will be sufficient to prove the lemma for the restriction to X_1 of the operator $\pi(A + \partial\psi_\Delta)$. But, since $\pi(A + \partial\psi_\Delta) = \pi A + \pi\,\partial\psi_\Delta$, and because the restriction to X_1

of πA is a maximal atone operator $A_1\colon X_1 \to 2^{Y_1}$, and the restriction of $\pi\, \partial\psi_\Delta$ is $\partial_1 \psi_\Delta$, where ∂_1 is the subdifferentiation associated with the duality (X_1, Y_1), the resulting operator $A_1 + \partial_1\psi_\Delta\colon X_1 \to 2^{Y_1}$ is of the same nature as the original one, the only difference being that now Δ generates the whole space. Therefore, as far as the proof is concerned, we may assume that the simplex has a nonempty interior and, in consequence, that the atone operator is singlevalued and everywhere defined.

Having gained this new additional hypothesis we proceed to the proof of the lemma. We need not discuss the operator $A + \partial\psi_\Delta$ because by definition it is piecewise atone. As for its inverse we write $T = (A + \partial\psi_\Delta)^{-1}$ and observe that, as the inverse of a maximal monotone operator, it is maximal monotone. Let us denote Δ_j, $j = 1, 2, \ldots, 2^{m+1} - 1$, $m = \dim \Delta$, the various faces of Δ, among which we count Δ itself as its only m-dimensional face. From $\Delta = \bigcup_j \mathring{\Delta}_j$, we get

$$D(T) = R(A + \partial\psi_\Delta) = (A + \partial\psi_\Delta)\Delta = \bigcup_j (A + \partial\psi_\Delta)\,\mathring{\Delta}_j\,.$$

For points in $\mathring{\Delta}_j$, $\partial\psi_\Delta(x)$ is a closed convex cone V_j independent of x, in fact, it is the smallest cone $\partial\psi_\Delta(x')$, $x' \in \Delta_j$. Thus,

$$(A + \partial\psi_\Delta)\,\mathring{\Delta}_j = A(\mathring{\Delta}_j) + \partial\psi_\Delta(\mathring{\Delta}_j) \subset A(\Delta_j) + \partial\psi_\Delta(\mathring{\Delta}_j) \subset (A + \partial\psi_\Delta)\Delta_j \subset D(T)\,,$$

whence on account of the previous equation,

$$D(T) = \bigcup_j [A(\Delta_j) + \partial\psi_\Delta(\mathring{\Delta}_j)] = \bigcup_j (A(\Delta_j) + V_j)$$

V_j is the dual of the supporting cone of Δ at any point in the relative interior of Δ_j, and as such is a polyhedron of codimension equal to the dimension of Δ_j. As to $A(\Delta_j)$, it is a polyhedron $\tilde{\Delta}_j$ whose dimension does not exceed that of Δ_j, and whose vertices are the images by A of some of the vertices of Δ_j. Hence,

$$D(T) = \bigcup_j (\tilde{\Delta}_j + V_j)\,.$$

$\tilde{\Delta}_j + V_j$—the sum of a compact convex set and a closed convex set— is closed and convex, and $D(T)$—the finite union of closed sets—is closed. Moreover, $D(T)$, being the finite dimensional domain of a maximal monotone operator, is also convex. From $D(T) = (A + \partial\psi_\Delta)\Delta$ it follows that $D(T)$ contains at least one halfline in the direction of any normal to Δ, that is, in any direction, property which is consistent with the fact that

$D(T)$ is closed and convex only if $D(T) = Y$. Thus we can write

$$Y = D(T) = \bigcup_j [\tilde{\Delta}_j + V_j].$$

Out of this decomposition we shall be extracting a paving for Y. We simply select among the $\tilde{\Delta}_j + V_j$'s those having a nonempty interior, and call them $\Pi_1, \Pi_2, \ldots, \Pi_n$. Since the union of all other sets is closed and nowhere dense, $\bigcup_i \Pi_i$ must be dense in Y, and since it is closed, $Y = \bigcup_i \Pi_i$.

The Π_i's are sets of the form $\tilde{\Delta}_j + V_j$ and therefore are convex polyhedra; to show that they form a paving it only remains to check that their interiors do not overlap. If $\Pi_i = A(\Delta_{j_i}) + V_{j_i} = \tilde{\Delta}_{j_i} + V_{j_i}$, then—arguing from the facts that the sum of the dimensions of $\tilde{\Delta}_{j_i}$ and V_{j_i} does not exceed the dimension of the space, and that Π_i has a nonempty interior—we deduce, on one hand, that $\tilde{\Delta}_{j_i}$ and Δ_{j_i} have the same dimension and thus that A_{j_i} sets up an affine bijection between these two sets, and on the other, that any $y \in Y$ admits a unique expression of the form

$$y = \tilde{y}_{j_i} + v_{j_i}, \qquad \tilde{y}_{j_i} \in \operatorname{aff} \tilde{\Delta}_{j_i}, \qquad v_{j_i} \in \big((\operatorname{aff} \Delta_{j_i})^\tau\big)^\perp .$$

It follows that any $y \in Y$ can be uniquely represented in the form,

$$y = Ax_{j_i} + v_{j_i}, \qquad x_{j_i} \in \operatorname{aff} \Delta_{j_i}, \qquad v_{j_i} \in \big((\operatorname{aff} \Delta_{j_i})^\tau\big)^\perp .$$

The resulting mapping $y \to x_{j_i}$, being the composition of the oblique projection $y \to y_{j_i}$—which is open—and of the affine bijection A^{-1}—which is also open—is open. Therefore, since it maps $A(\Delta_{j_i}) + V_{j_i}$ onto Δ_{j_i}, it must map $\operatorname{int}\big(A(\Delta_{j_i}) + V_{j_i}\big)$ onto $\mathring{\Delta}_{j_i}$, and

$$\mathring{\Pi}_i = \operatorname{int}\big(A(\Delta_{j_i}) + V_{j_i}\big) = A(\mathring{\Delta}_{j_i}) + V_{j_i} = (A + \partial\psi_{j_i})\, \mathring{\Delta}_{j_i} .$$

Any y belonging to both $\tilde{\Delta}_j + V_j$ and $\tilde{\Delta}_k + V_k$ admits a double representation:

$$y = y_j + v_j = y_k + v_k, \quad y_j \in Ax_j,\ y_k \in Ax_k,\ x_j \in \Delta_j, \quad x_k \in \Delta_k,\ v_j \in V_j,\ v_k \in V_k,$$

whence

$$0 = \langle x_j - x_k, y_j - y_k \rangle + \langle x_j - x_k, v_j - v_k \rangle = \langle x_j - x_k, v_j - v_k \rangle .$$

It follows that x_i and x_j belong to the same face, the face common to both Δ_j and Δ_k. Since this cannot happen if either x_i or x_j is in the relative interior of its face, we can conclude

$$(A(\mathring{\Delta}_j) + V_j) \cap (A(\Delta_k) + V_k) = \emptyset\,, \quad j \neq k\,.$$

In particular,

$$\mathring{\Pi}_i \cap \Pi_k = \emptyset\,, \quad i \neq k\,.$$

We draw two important consequences: first, the $\mathring{\Pi}_i$'s are disjoint and $\{\Pi_i\}_1^n$ is a paving for Y, and second, no point in $\mathring{\Pi}_i$ can be the image by $A + \partial\psi_\Delta$ of a point not belonging to $\mathring{\Delta}_{j_i}$. This last fact implies

$$T(\mathring{\Pi}_i) = (A + \partial\psi_\Delta)^{-1}\mathring{\Pi}_i = \mathring{\Delta}_{j_i}\,.$$

Since $A + \partial\psi_\Delta$ is atone on $\mathring{\Delta}_j$, $T = (A + \partial\psi_\Delta)^{-1}$ is atone on $\mathring{\Pi}_i$, and T is piecewise atone, by Lemma 4.1. Thus we have come to the end of the proof.

THEOREM 5.2. *Any monotone operator $M\colon X \to 2^Y$ defined on a finite set admits a piecewise atone monotone extension $\tilde{M}$ whose inverse is also piecewise atone, such that $D(\tilde{M}) = \operatorname{co} D(M)$, $R(\tilde{M}) = Y$; if M is cyclically monotone $\tilde{M}$ can be chosen so as to be cyclically monotone.*

PROOF. The idea of the proof is to shift the points of the domain orthogonaly to the range so as to place them in affinely independent positions, thus making the application of Theorem 5.1 possible. In general there is no room in the ground spaces X and Y for this type of maneuver and one is constrained to move out into larger auxiliary spaces. In a particular case it suffices to add a finite number of dimensions, number that varies from case to case. To avoid all discussions concerning dimensions and to be in conditions to deal with all cases at once it is better to make, once and for all, an infinite dimensional extension.

We take two infinite dimensional dual real vector spaces U and V and build with them the auxiliary spaces $\hat{X} = X \oplus U$ and $\hat{Y} = Y \oplus V$; a duality is established between $\hat{X}$ and $\hat{Y}$ by means of the bilinear form $\langle (x, u), (y, v)\rangle = \langle x, y\rangle + \langle u, v\rangle$. As usual we identify X and Y with $X \oplus \{0\}$ and $Y \oplus \{0\}$ respectively, and likewise for U and V; this permits us to conceive X and Y as imbedded in $\hat{X}$ and $\hat{Y}$, and to think of V and U as their respective orthogonal complements: $X^\perp = V$, $Y^\perp = U$. We denote P_X and P_Y the projections $x + u \to x$ and $y + v \to y$ respectively; observe that they are open continuous mappings.

Let $D(M) = \{x_1, x_2, \ldots, x_k\}$, and in correspondence with these points choose k affinely independent points in U: $\{u_1, u_2, \ldots, u_k\}$. With this choice

$\{x_1 + u_1, x_2 + u_2, \ldots, x_k + u_k\}$ is an affinely independent system in $\hat{X}$, and the operator $\hat{M}(x_i + u_i) = Mx_i$, $i = 1, 2, \ldots, k$, is monotone from $\hat{X}$ into $\hat{Y}$. By Theorem 5.1, $\hat{M}$ admits an extension of the form $\hat{A} + \hat{\partial}\psi_{\hat{\Delta}}$, where $\hat{\Delta} = \mathrm{co}\,\{x_i + u_i\}_1^k$, $\hat{A}:\hat{X} \to 2^{\hat{Y}}$ is maximal atone with domain aff $\hat{\Delta}$, and $\hat{\partial}$ is the subdifferential operator associated with the dual spaces $\hat{X}$ and $\hat{Y}$. We claim that the operator $\tilde{M}$: $X \to 2^Y$ defined by

$$\tilde{M}x = [(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})\, P_X^{-1} x] \cap Y$$

meets all the theorem's requirements. Obviously $\tilde{M}$ is a monotone extension of M; moreover, $P_X\hat{\Delta} = \Delta = \mathrm{co}\, D(M)$, $R(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}}) = \hat{Y}$, since

$$D(M) \subset D(\tilde{M}) \subset \mathrm{co}\, D(M)\,, \qquad R(\tilde{M}) = Y\,.$$

We discuss $\tilde{M}^{-1}$ first. This operator is the trace on Y of $P_X(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})^{-1}$, and for this reason its domain of definition is the whole of Y. By Lemma 5.1, $(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})^{-1}$ is piecewise atone and monotone, and defined everywhere in $\hat{Y}$. Let $\{\hat{\Pi}_j, \hat{A}_j\}_1^r$ be the associated atone elements, and let us assume that the paving $\{\hat{\Pi}_j\}_1^r$ is such that the intersection of any number of tiles is either empty or coincides with a face common to them all. This is no restriction because by subdivision one can always obtain from a given paving another having this property. Our next step is to produce a paving for Y from $\{\hat{\Pi}_j\}_1^r$. We do this by taking the sets of the form $Y \cap \hat{\Pi}_j$ having a nonempty interior relatively to Y; let them be $\{\Pi_j\}_1^s$. To begin with note that the Π_i's are convex polyhedra, and that their union is Y because it is everywhere dense and closed. If $Y \cap \hat{\Pi}_j \neq 0$ there is a minimal face $\hat{\Phi}_j$ of $\hat{\Pi}_j$ such that $Y \cap \hat{\Pi}_j = Y \cap \hat{\Phi}_j$. Because of its minimality $\hat{\Phi}_j$ is intersected by Y accross its relative interior, and $(Y \mathring{\cap} \hat{\Pi}_j) = Y \cap \mathring{\hat{\Phi}}_j$. It follows that the Π_i's are disjoint by way of the fact that the relative interiors of all faces of the $\hat{\Pi}_j$'s have this property. Therefore $\{\Pi_i\}_s^1$ is a paving for Y.

As we have just seen if $y \in \Pi_i$ then $y \in Y \cap \hat{\Phi}$, where $\hat{\Phi}$ is a face of the paving $\{\hat{\Pi}_j\}_1^r$, and therefore a face common to a group of tiles, say, $\hat{\Pi}_{j_1}$, $\hat{\Pi}_{j_2}, \ldots, \hat{\Pi}_{j_l}$. So, since $(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})^{-1}$ is piecewise atone,

$$(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})^{-1} y = \mathrm{co}\,\{\hat{A}_{j_h} y\}_1^l\,.$$

By the matching conditions (4.5) the difference between two points in $\mathrm{co}\,\{\hat{A}y_{j_h}\}_1^l$ is orthogonal to aff $\hat{\Phi}$, and since this space contains Y, orthogonal to Y, and thus belongs to U. But then,

$$\tilde{M}^{-1} y = P_X(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})^{-1} y = P_X \hat{A}_{j_h} y\,, \quad h = 1, 2, \ldots\,, \quad \forall y \in \mathring{\hat{\Pi}}_i\,,$$

and since obviously the restriction to Y of $P_X\hat{A}_{j_h}$ is an atone mapping $A_i\colon Y\to 2^X$, we have proved

$$\tilde{M}^{-1}y = A_i y\,, \qquad \forall y \in \mathring{\Pi}_i,\ i = 1, 2, \ldots, s\,.$$

Now we examine the behaviour of $\tilde{M}^{-1}$ at a point y where several tiles: $\Pi_i, \Pi_j, \ldots, \Pi_l$ meet; let $\{\hat{\Pi}_{i_1}, \hat{\Pi}_{i_2}, \ldots\}$, $\{\hat{\Pi}_{j_1}, \hat{\Pi}_{j_2}, \ldots\}, \ldots, \{\hat{\Pi}_{l_1}, \hat{\Pi}_{l_2}, \ldots\}$ be the tiles of $\{\hat{\Pi}_j\}_1^r$ concurring on $\Pi_i, \Pi_j, \ldots, \Pi_l$ respectively. Then,

$$(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})^{-1}y = \operatorname{co}[(\hat{A}_{i_1}y \cup \hat{A}_{i_2}y \cup \ldots) \cup (\hat{A}_{j_1}y \cup \hat{A}_{j_2}y \cup \ldots) \cup \cup \ldots \cup (\hat{A}_{l_1}y \cup \hat{A}_{l_2}y \ldots)]\,,$$

and since P_X is linear,

$$\tilde{M}^{-1}y = P_X(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})^{-1}y = \operatorname{co}[(P_X\hat{A}_{i_1}y \cup P_X\hat{A}_{i_2}y \cup \ldots) \cup \cup (P_X\hat{A}_{j_1}y \cup P_X\hat{A}_{j_2}y \cup \ldots) \cup \ldots \cup (P_X\hat{A}_{l_1}y \cup P_X\hat{A}_{l_2}y \cup \ldots)]\,.$$

Because of the continuity of the operators involved, the equations

$$P_X\hat{A}_{i_1}y = P_X\hat{A}_{i_2}y = \ldots = A_i y$$

proved above as valid for $y \in \mathring{\Pi}_i$ hold also for any y in Π_i, and similarly for the tiles $\Pi_j, \ldots, \Pi_l$. Hence,

$$\tilde{M}^{-1}y = \operatorname{co}[A_i y \cup A_j y \cup \ldots \cup A_l y]\,, \qquad \forall y \subset \Pi_i \cap \Pi_j \cap \ldots \cap \Pi_l\,,$$

and $\tilde{M}^{-1}$ is piecewise atone.

This done we come now to the discussion of $\tilde{M}$. $\tilde{M}$ is maximal monotone because it is the inverse of a maximal monotone operator; moreover, since $D(M) \subset D(\tilde{M}) \subset \operatorname{co} D(M)$, $(\operatorname{co} \mathring{D}(M)) \subset D(\tilde{M})$ (cf. [2, proposition 2.9]; for the application of this result remark that $\tilde{M}$ is defined on a finite dimensional space). Since $\hat{\Delta}$ is the union of the relative interiors of its faces and P_X is an open mapping, $P_X^{-1}x = x + U$ cuts $\hat{\Delta}$ accross the relative interiors of a family of faces $\{\hat{\Delta}_i, \hat{\Delta}_j, \ldots\}$, and so

$$(\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})P_X^{-1}x = (\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})(P_X^{-1}x \cap \mathring{\hat{\Delta}}_i) \cup (\hat{A} + \hat{\partial}\psi_{\hat{\Delta}})(P_X^{-1}x \cap \mathring{\hat{\Delta}}_j) \cup \ldots = \\ = [\hat{A}(P_X^{-1}x \cap \mathring{\hat{\Delta}}_i) + \hat{\partial}\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_i)] \cup [\hat{A}(P_X^{-1}x \cap \mathring{\hat{\Delta}}_j) + \hat{\partial}\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_j)] \cup \ldots\,,$$

where in writing the last equation we have made use of the fact that the cone of normals is the same for all points at the relative interior of a face. Then,

$$\tilde{M}x = [(\hat{A} + \partial\psi_{\hat{\Delta}})P_X^{-1}x] \cap Y = \{[\hat{A}(P_X^{-1}x \cap \mathring{\hat{\Delta}}_i) + \partial\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_i)] \cap Y\} \cup \{[\hat{A}(P_X^{-1}x \cap \mathring{\hat{\Delta}}_j) + \partial\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_j)] \cap Y\} \dots .$$

The significant point is that in the above expression for $\tilde{M}x$ the faces making an effective contribution are those of a family consisting of one face and all its subfaces. To see this we take two points y_i and y_j in $\tilde{M}x$ coming from two different faces $\hat{\Delta}_i$ and $\hat{\Delta}_j$, and write them in the form

$$y_i = z_i + w_i \in \tilde{M}x \subset Y, \quad z_i \in \hat{A}(x + u_i), \quad w_i \in \partial\psi_{\hat{\Delta}}(x + u_i), \quad u_i \in U, \quad x + u_i \in \mathring{\hat{\Delta}}_i,$$

$$y_j = z_j + w_j \in \tilde{M}x \subset Y, \quad z_j \in \hat{A}(x + u_j), \quad w_j \in \partial\psi_{\hat{\Delta}}(x + y_j), \quad u_j \in U, \quad x + u_j \in \mathring{\hat{\Delta}}_j.$$

Since U and Y are orthogonal the above yields

$$0 = \langle u_i - u_j, y_i - y_j\rangle = \langle (x + u_i) - (x + u_j), z_i - z_j\rangle + \langle (x + u_i) - (x + u_j), w_i - w_j\rangle .$$

The first term on the right vanishes because $\hat{A}$ is atone, so

$$0 = \langle (x + u_i) - (x + u_j), w_i - w_j\rangle ,$$

which says that both w_i and w_j are normals of $\hat{\Delta}$ simultaneously, at $x + u_i$ and $x + u_j$ and hence are normals at any point of the open segment $\{t(x + u_i) + (1 - t)(x + u_j)\}_{0<t<1}$. This segment is contained in the relative interior of the smallest face containing $\hat{\Delta}_i$ and $\hat{\Delta}_j$; if the face is called $\hat{\Delta}_k$ then, bearing in mind that $\tilde{M}x$ is convex and $\hat{A}$ affine, one derives

$$\begin{aligned}
&\tilde{M}x \ni ty_i + (1-t)y_j = \big(tz_i + (1-t)z_j\big) + \big(tw_i + (1-t)w_j\big) ,\\
&tz_i + (1-t)z_j \in \hat{A}\Big(x + \big(tu_i + (1-t)u_j\big)\Big) ,\\
&tw_i + (1-t)w_j \in \partial\psi_{\hat{\Delta}} \Big(x + \big(tu_i + (1-t)u_j\big)\Big) ,\\
&tu_i + (1-t)u_j \in U , \quad x + \big(tu_i + (1-t)u_j\big) \in \mathring{\hat{\Delta}}_k ,
\end{aligned}$$

whence

$$y_k = ty_i + (1-t)y_j \in \tilde{M}x \cap \{[\hat{A}(P_X^{-1}x \cap \mathring{\hat{\Delta}}_k) + \partial\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_k)] \cap Y\},$$

and we have shown that $\hat{\Delta}_k$ makes an effective contribution to Mx. Therefore, since any two effective faces are contained in a third, there is a maximal face containing them all. The above argument applied to any effective face $\hat{\Delta}_j$ and the maximal effective face $\hat{\Delta}_i$ indicates that the normals effectively appearing in $\partial\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_j)$ are normals at points in $\mathring{\hat{\Delta}}_i$. In consequence,

$$Mx = [\hat{A}(P_X^{-1}x \cap \mathring{\hat{\Delta}}_i) + \partial\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_i)] \cap Y,$$

where—it is important to notice—the operator appearing on the right:

$$B_i x = [\hat{A}(P_X^{-1}x \cap \mathring{\hat{\Delta}}_i) + \partial\psi_{\hat{\Delta}}(\mathring{\hat{\Delta}}_i)] \cap Y$$

is atone. Moreover, since any v in $\big((\operatorname{aff} D(M))^\tau\big)^\perp$, when considered as a vector in $\hat{Y}$, belongs to $((\operatorname{aff}\hat{\Delta}_i)^\tau)^\perp$, $y \in B_i x$ implies $y + v \in B_i x$. This means $B_i x$ is made up of cosets in Y modulo $\big((\operatorname{aff} D(M))^\tau\big)^\perp$, just like $\tilde{M}x$.

The upshot of the foregoing discussion is that for any $x \in D(\tilde{M})$ there is a maximal face $\hat{\Delta}_i$ of $\hat{\Delta}$ and an associated atone operator B_i such that

$$\tilde{M}x = B_i x .$$

Of these B_i's let $B_{i_1}, B_{i_2}, \ldots, B_{i_l}$ be those having domains dense in aff $D(M)$; the class is nonempty because otherwise $\tilde{M}$ could not be defined in $(\operatorname{co} \mathring{D}(M))$, The B_{i_k}'s take values consisting of a single coclass modulo $\big((\operatorname{aff} D(M))^\tau\big)^\perp$, and by Lemma 3.3 the operators $M - B_{i_k}$ are maximal monotone. It follows that the sets $\Sigma_k = \{x | (M - B_{i_k})x = 0\} = \{x | B_{i_k}x \subset Mx\}$ are closed and convex; in addition, because of the maximality of the faces involved, they have nonoverlapping relative interiors. The remaining points of $D(M)$ are distributed over a finite number of closed affine spaces—the closure of the domains of the corresponding B_i's—nowhere dense in $\overline{\operatorname{aff} D(M)}$, and hence, nowhere dense in $(\operatorname{co} \mathring{D}(M))$. Thus $\bigcup_k \Sigma_k$ must be dense in $(\operatorname{co} \mathring{D}(M))$, and, therefore, in $D(M)$; but, as it is closed and contained therein, $\bigcup_k \Sigma_k = D(M)$. It is now clear that $D(M)$ is a closed convex polyhedron, that $\{\Sigma_k\}_1^l$ is a paving for $D(M)$, and that $\tilde{M}x = B_{i_k}x$, $x \in \mathring{\Sigma}_k$. Thus $\tilde{M}$ is piecewise atone by Lemma 4.1.

As to the cyclically monotone case it suffices to note that in such a case $\hat{A}$ can be chosen to be a constant operator, choice that makes $\tilde{M}$ cyclically monotone.

6. – Piecewise atone approximation of monotone operators.

Up to this point, engaged in the discussion of piecewise atone interpolation of monotone operators—a finite dimensional geometrical question—we had no need of a topology for the operators, it is to give a precise meaning to the approximation furnished by the interpolating operators that now we need one. The « graph topology » defined below appears suitable for this purpose (1).

DEFINITION 6.1. The graph topology for the class of all multimappings $T: X \to 2^Y$ is the topology having as open basis the subclasses

$$\{T | G(T) \cap (O_i^{(X)} \times O_i^{(Y)}) \neq \emptyset,\ i = 1, 2, \ldots, n\},$$

where $O_j^{(X)}$ and $O_i^{(Y)}$ are open sets in X and Y respectively, and n any positive integer.

According to this definition $T \in \overline{\mathcal{S}}$, where $\mathcal{S}$ is a set of multimappings, if and only if for any finite class of open sets in the cartesian product $X \times Y$ intersecting $G(T)$ there is an $S \subset \mathcal{S}$ such that $G(S)$ is intersected by all sets in the class. Since any open set contains the multimapping assigning to each $x \in X$ the whole space Y, the graph topology does not separate points; the closure of a singleton S is the class of multimappings whose graphs are contained in $\overline{G(S)}$, and hence the only closed « points » are the multimappings whose graphs are singletons. Alternately, a generalized (directed, filtrating) sequence $\{T_i\}_{\in iI}$ converges to a limit T if any $(x, y) \in G(T)$ is the limit of a sequence $(x_i, y_i) \in G(T_i)$, $i \in I$; in general the limit is not unique. It is useful to remark that *if X and Y have countable bases so does the space of multimappings under the graph topology*, a countable base for the latter being obtained by choosing the sets $O_i^{(X)}$ and $O_i^{(Y)}$ within countable bases in X and Y respectively. An observation of similar nature of which we shall make use in the next section is that *the restriction of the graph topology to an equicontinuous class of mappings coincides with the pointwise convergence topology.* In fact, since this last topology has the sets $\{T | G(T) \cap (\{x_i\} \times O_i^{(Y)}) \neq \emptyset,\ i = 1, 2, \ldots, n\}$ as open basis it is stronger than the graph topology; on the other hand, if $\mathcal{E}$ is an equicontinuous family, any set $\{T \subset \mathcal{E} | G(T) \cap (\{x_i\} \times O_i^{(Y)}) \neq \emptyset\} = \{T \in \mathcal{E} | T(x_i) \subset O_i^{(Y)}\}$ contains the set $\{T \in \mathcal{E} | G(T) \cap (O_i^{(X)} \times O_i^{(Y)}) \neq \emptyset\}$ as soon as $O_i^{(X)}$ is a sufficiently small

(1) The idea of this topology derives from a notion of convergence for a sequence of operators introduced by H. Attouch [1].

neighborhood of x_i, and the pointwise convergence topology is weaker than the graph topology. Thus the two topologies coincide.

One sees without difficulty that $T \to T^{-1}$ and $T \to \lambda T$, where λ is a non-vanishing real number, are continuous operations in the graph topology, and hence, since their inverses are of the same nature, that they are homeomorphisms. It is less evident that for a continuous single valued mapping T_0, $T \to T + T_0$ is continuous, yet it is only a matter of remarking that if $\mathcal{O}$ is open in $X \times Y$ then $\{(x, y - T_0 x)\}_{(x,y) \in \mathcal{O}}$ is also open. By the same argument $T \to T - T_0$ is continuous, and in consequence, $T \to T + T_0$ is a homeomorphism.

The graph topology becomes much tighter when applied to the class $\mathcal{M}$ of maximal monotone operators. This is due to the fact that no inclusion relations are possible between the graphs of maximal monotone operators. Yet, in general one cannot assert that singletons are closed in $\mathcal{M}$. That would be the case if, for instance, the bilinear form $\langle x, y \rangle$ were such that $\{(x, y) \in X \times Y \mid \langle x, y \rangle \geqslant 0\}$ is a closed set in $X \times Y$, because then the graphs of maximal monotone operator would be closed; naturally, in this case the limits in $\mathcal{M}$ are unique.

Among maximal monotone operators the cyclically monotone ones form a very important class, we denote it $\mathcal{M}_0$; $\mathcal{M}_0$ is in fact the class of all maximal monotone subdifferentials. In general $\mathcal{M}_0$ is not stable under passage to the limit, that is, $\mathcal{M}_0$ is not closed in $\mathcal{M}$, even if the above condition on $\langle x, y \rangle$ is satisfied. To be able to guarantee stability we need something stronger still, namely that $\langle x, y \rangle$ be a continuous function of $(x, y) \in X \times Y$. This happens, for instance, when X and Y are reflexive Banach spaces endowed with the norm topology. It is important to bear in mind that while the class of maximal monotone operators does not depend on the topologies of the spaces in duality, the graph topology depends entirely on them, and hence that there is a large choice of topologies for $\mathcal{M}$, from the weakest—corresponding to the weak topologies in X and Y—to the strongest—built out of the strongest topologies compatible with the duality.

The graph topology offers an adequate language to express the approximation furnished by the piecewise atone operators interpolating a given maximal monotone operator.

THEOREM 6.1. *Piecewise atone monotone operators are dense in the class of maximal monotone operators relatively to the graph topology; likewise, piecewise atone cyclically monotone operators are dense in the class of maximal cyclically monotone ones.*

PROOF. Let $M: X \to 2^Y$ be a maximal monotone operator, and, for any finite set $F \in G(M)$, let M_F be a piecewise atone monotone operator interpolating M at the points of F, that is, such that $F \subset G(M_F)$, whose existence is assured by Theorem 5.2; if M is cyclically monotone choose M_F so as to be cyclically monotone too.

The family M_F ordered according to the inclusions of the F's: $M_{F_1} < < F_{F_2} \doteq F_1 \subset F_2$, is a directed set in $\mathcal{M}$; let us see that it converges to M. For any $(x, y) \in G(M)$ build the sequence (x_F, y_F) as follows:

$$(x_F, y_F) = \begin{cases} (x, y) \text{ if } (x, y) \in F\,, \\ \text{any point in } F \text{ if } (x, y) \notin F\,. \end{cases}$$

Obviously $(x_F, y_F) \subset G(M_F)$, and $(x_F, y_F) \to (x, y)$. Therefore, since (x, y) is any point in $G(M)$, $M_F \to M$, and the theorem is proved.

7. – Hilbert space. The classes $\mathcal{M}$, $\mathcal{T}$ and $\mathcal{C}$.

Maximal monotone operators in Hilbert space have a special significance because of their close relation with other types of operator, particularly with contracting mappings, in terms of which the interpolation theorem acquires an interesting meaning. In the discussion that follows we place ourselves in the position where, having identified Hilbert space with its dual, we take for X and Y the same real Hilbert space $\mathcal{H}$, and give $\langle x, y\rangle$ the meaning of the scalar product. To the effects of the graph topology $\mathcal{H}$—both as space of departure and arrival—is endowed with the norm topology, thus assuring the continuity of the pairing form.

We start out by recalling the basic identities relating a multimapping $M\colon \mathcal{H} \to 2^{\mathcal{H}}$ with the multimappings $P = (I + M)^{-1}$ and $C = 2(I + M)^{-1} - I$ (where I is the identity mapping in $\mathcal{H}$):

$$(7.1)\qquad \langle x_1 - x_2, m_1 - m_2\rangle = \langle p_1 - p_2, (z_1 - p_1) - (z_2 - p_2)\rangle = \\ = \tfrac{1}{4}[\|z_1 - z_2\|^2 - \|q_1 - q_2\|^2]\,,$$

$$\forall x_1, x_2 \in D(M),\ \forall m_i \in Mx_i,\quad z_i = x_i + m_i = (I + M)x_i,\quad p_i \in Pz_i,\ q_i \in Cz_i\,.$$

Their verification is immediate. They show that if M is monotone then P and C are singlevalued and satisfy the inequalities

$$(7.2)\quad \langle Pz_1 - Pz_2, (I - P)z_1 - (I - P)z_2\rangle \geqslant 0\,,\qquad \|Cz_1 - Cz_2\| \leqslant \|z_1 - z_2\|\,,$$

respectively, for all z_1 and z_2 in their common domain of definition. It is also apparent that any extension of any of the three operators respecting the corresponding characteristic inequality produces an extension of the other two, and hence that these operators are simultaneously maximal in their classes. But, as the class of the C's—that of the contracting mappings—has as maximal elements the contracting mappings everywhere defined (a theorem due to Kirzbraum [4] asserts that any contraction in $\mathcal{H}$ admits an extension to the whole space), maximal monotone operators are associated with mappings P and C everywhere defined, and conversely. From this brief discussion it follows that the operation $M \to (I + M)^{-1} = P$ is a bijection of the class $\mathcal{M}$ of all maximal monotone operators onto the class $\mathcal{P}$ of all mappings $P\colon \mathcal{H} \to \mathcal{H}$ satisfying the inequality

$$(7.3) \qquad \langle Px_1 - Px_2, (I - P)x_1 - (I - P)x_2\rangle \geqslant 0 \,, \quad \forall x_1, x_2 \in \mathcal{H} \,,$$

whereas $P \to 2P - I$ maps $\mathcal{P}$ bijectively onto $\mathcal{C}$—the class of all contractions C of $\mathcal{H}$ into itself—characterized by

$$(7.4) \qquad \|Cx_1 - Cx_2\| \leqslant \|x_1 - x_2\| \,, \quad \forall x_1, x_2 \in \mathcal{H};$$

as a composition of these two bijections $M \to 2(I + M)^{-1} - I = C$ is a bijection of $\mathcal{M}$ onto $\mathcal{C}$. *These bijections become homeomorphisms the moment the classes $\mathcal{M}$, $\mathcal{P}$ and $\mathcal{C}$ are given the graph topology*, because, according to the discussion in the preceeding section, they are obtained by composition of homeomorphisms. A simple application of Schwarz' inequality to (7.3) shows that the operators P in class $\mathcal{P}$ are contracting, just like those C's in class $\mathcal{C}$. Hence $\mathcal{P}$ and $\mathcal{C}$ being equicontinuous classes, on them the graph topology coincides with the pointwise convergence topology. If $\mathcal{H}$ is separable, pointwise convergence in $\mathcal{C}$ is equivalent to convergence on a dense countable set $\{x_i\}_1^\infty$, which in turn amounts to convergence according to the metric

$$\varrho(C_1, C_2) = \sum_1^\infty \frac{1}{2^i} \frac{\|C_1x_i - C_2x_i\|}{1 + \|C_1x_i - C_2x_i\|} \,.$$

It is a simple excercise to check that any Cauchy sequence of contractions under ϱ converges to a contraction, and hence that $\mathcal{M}$, $\mathcal{P}$ *and* $\mathcal{C}$ *are complete metrizable spaces when* $\mathcal{H}$ *is separable.* This proposition is due to H. Attouch [1, Proposition 1.1]. If, in addition, $\mathcal{H}$ *is finite dimensional then these classes, handled modulo constant mappings, are compact spaces*, by Arzelà's Theorem.

The operation $M \to (I + M)^{-1} = P$ as well as its inverse $P \to P^{-1} - I = M$ preserve cyclical monotonicity, and hence *the class $\mathcal{M}_0$ of maximal cyclically monotone operators is homeomorphic with the class $\mathcal{P}_0$ of all cyclically* *monotone mappings belonging to* $\mathcal{T}$. Any $P_0 \in \mathcal{T}_0$ is a continuous subdifferential, hence is a gradient mapping, that is, is of the form $P_0 x = \nabla p_0(x)$, where p_0 is a real valued differentiable function everywhere defined. Through this remark $\mathcal{T}_0$ may be described as the class of all gradient mappings satisfying (7.3), and thus be recognized as the class of operators studied by J. J. Moreau [3] under the name of *proximal mappings* (abbreviated to prox maps). Since the transition formulas between $\mathcal{T}$ and $\mathcal{C}$: $P \to 2P - I = C$, $C \to \frac{1}{2}(I + C) = P$, both transform gradient mappings into gradient mappings, *the class* $\mathcal{C}_0$ *corresponding to* $\mathcal{T}_0$, *and hence to* $\mathcal{M}_0$, *is simply that of all contractive gradients*. In conclusion, $\mathcal{M}_0$, $\mathcal{T}_0$ *and* $\mathcal{C}_0$ *are homeomorphic spaces*. Very important are the following operators associated with a closed convex set K: the subdifferential of the indicator function $\partial\psi_K$, belonging to $\mathcal{M}_0$, the projector onto K, $P_K = (I + \partial\psi_K)^{-1}$, a member of $\mathcal{T}_0$, and the symmetry with respect to $S_K = 2P_K - I$, lying in $\mathcal{C}_0$. Geometrically P_K is the nearest point mapping, and S_K the mapping assigning to each point in space its mirror image with respect to its projection on K. Naturally, all remarks concerning separability, metrizability and compactness of $\mathcal{M}$, $\mathcal{T}$, and $\mathcal{C}$ when $\mathcal{H}$ is either separable or finite dimensional apply equally well to $\mathcal{M}_0$, $\mathcal{T}_0$, and $\mathcal{C}_0$. It is worth noticing that the classes $\mathcal{M}$, $\mathcal{T}$, and $\mathcal{C}$, as well as $\mathcal{M}_0$, $\mathcal{T}_0$, and $\mathcal{C}_0$ are invariant under the operations $M \to M^{-1}$, $P \to I - P$, and $C \to -C$ respectively, operations which correspond with each other under the transition formulas. As a final remark let us mention that the direct sum is respected by the passage formulas, that is, that to the direct sum of operators in one class correspond the direct sum in the others.

8. – Interpolation of contractions by piecewise unitary mappings.

In order to give an interpretation of the interpolation Theorem 5.2 in terms of the classes $\mathcal{T}$ and $\mathcal{C}$ we must first find out what the various elements appearing in its statement correspond to in these classes. From equation (7.1) it follows that if M is atone then $2(I + M)^{-1} - I$ is an isometry, and conversely. Therefore, since any isometry can be extended to the whole space, *if* $A: \mathcal{H} \to 2^{\mathcal{H}}$ *is maximal atone* $U = 2(I + A)^{-1} - I$ *is an isometry of* $\mathcal{H}$ *into itself, and viceversa*. In particular, *maximal atone operators in Hilbert space are maximal monotone*.

To simplify the discussion suppose now that $0 \in D(A)$; this way both $D(A)$ and $R(A)$ are linear spaces. From Lemma 3.1 one sees that a maximal atone operator A can be constructed out of maximal atone operator from $\overline{D(A)}$ into $\overline{D(A)}$ by enlarging its values to the cosets modulo $R(A)^{\perp}$ containing them. This idea can be expressed by saying that A is the direct sum of a maximal skewsymmetric operator $A_1 = P_{\mathcal{K}_1} A$ acting on $\mathcal{K}_1 = \overline{D(A)}$, and the operator A_2 acting on $\mathcal{K}_2 = D(A)^{\perp}$ that assigns the whole space to the origin. By the remark made at the end of § 7 the isometry$_1$ $U = 2(I + A)^{-1} - I$ is the direct sum of the isometry $U_1 = 2(I_1 + A_1)^{-1} - I$ on $\mathcal{K}_1$, and the isometry $-I_2 = 2(I_2 + A_2)^{-1} - I_2$ on $\mathcal{K}_2$, where I_1 and I_2 are the identity mappings in $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Since both A_1 and U_1 are densely defined and linear, their adjoints are defined, and related by $U_1^* = 2(I_1 + A_1^*)^{-1} - I_1$. In particular, if A_1 is antiselfadjoint, then $U_1^* = 2(I_1 - A_1)^{-1} - I$, and U_1^*, being the contraction associated with the maximal atone operator $-A_1$, is an everywhere defined isometry in $\mathcal{K}_1$. Recalling then that an isometry whose adjoint is an isometry is a unitary mapping, we deduce that $U_1: \mathcal{K}_1 \to \mathcal{K}_1$ is unitary, and along with it that so is $U = U_1 \oplus (-I_2)$. In conclusion, *if A is a maximal atone operator whose underlying linear part is antiselfadjoint then, up to a translation, $U = 2(I + A)^{-1} - I$ is unitary* (the translation comes in as the correcting term required to bring about the normalizing condition $0 \in D(A)$). This result applies specially to atone operators of finite rank, the only ones appearing in the interpolation theory.

Next we investigate the nature of the contraction C associated with a piecewise atone monotone operator M subordinated to a family of atone elements of finite rank $\{\Pi_i, A_i\}_1^r$. In doing this we assume, as we have done before, that the intersection of any number of tiles is either empty or equal to a common face. In other words, the family $\mathcal{F} = \{\Delta\}$ of the tiles' faces contains any nonempty intersection of its members. Moreover the $\mathring{\Delta}$'s are nonempty, nonoverlapping, and $D(M) = \bigcup_{\Delta \in \mathcal{F}} \mathring{\Delta}$. Therefore, $\mathcal{K} = R(I + M) = \bigcup_{\Delta \in \mathcal{F}} (I + M)\mathring{\Delta}$, and since for any x there is a unique $y \in D(M)$ such that $x \in (I + M)y$, $\{(I + M)\mathring{\Delta}\}_{\Delta \in \mathcal{F}}$ is a partition of $\mathcal{K}$ into disjoint sets. Thus C is entirely determined by its behaviour on the individual pieces $(I + M)\mathring{\Delta}$. If $\Pi_{i_1}, \Pi_{i_2}, \ldots, \Pi_{i_l}$ are the tiles meeting on a face Δ, then $Mx = \operatorname{co}\{A_{i_k}x\}_1^l$, $\forall x \in \mathring{\Delta}$. By the matching conditions (4.5) the difference between two vectors in Mx is orthogonal to aff Δ, and in consequence the restriction of M to $\mathring{\Delta}$ is atone. It follows that for any $\Delta \in \mathcal{F}$ there is a maximal atone operator A_Δ with domain dense in aff Δ such that $Mx \subset A_\Delta x$, $\forall x \in \mathring{\Delta}$. Then we have $(I + M)x \subset (I + A_\Delta)x$, and

$$(I + A_\Delta)^{-1}(I + M)x \subset (I + A_\Delta)^{-1}(I + A_\Delta)x = x = (I + M)^{-1}(I + M)x, \quad \forall x \in \mathring{\Delta}.$$

The above inclusion cannot be proper because $(I + A_\Delta)^{-1}$ is everywhere defined and $(I + M)x$ is not empty. Hence,

$$(I + A_\Delta)^{-1}y = (I + M)^{-1}y, \quad \forall y \in (I + M)\mathring{\Delta},$$

whence,

$$(2(I + A_\Delta)^{-1} - I)y = (2(I + M)^{-1} - I)y\,, \qquad \forall y \in (I + M)\mathring{\Delta}\,.$$

In other words, C and the isometry $U_\Delta = 2(I + A_\Delta)^{-1} - I$ coincide on $(I + M)\mathring{\Delta}$. The A_Δ's, like the A_i's, are of finite rank, and therefore the U_Δ's are unitary up to a constant. The coincidence of C and U_Δ takes place in a closed convex set containing $(I + M)\Delta$. In fact,

$$\{x\,|\,Cx = U_\Delta x\} = \{x\,|\,U_\Delta^{-1}Cx = x\} = \left\{x \,\middle|\, \frac{I - U_\Delta^{-1}C}{2}x = 0\right\} = \left(\frac{I - U_\Delta^{-1}C}{2}\right)^{-1} 0\,,$$

which proves our assertion, because, $U_\Delta^{-1}C$ being a contraction, $(I - U_\Delta^{-1}C)/2$ is a maximal monotone operator, and $((I - U^{-1}C)/2)^{-1}0$ a closed convex set. Among these sets those having a nonempty interior produce a partition of $\mathcal{H}$ into closed convex sets with non overlapping interiors. Under these circumstances it is not difficult to see that such a partition is a paving for the space. Since on each tile C coincides with a unitary mapping, we have demonstrated the lemma:

LEMMA 8.1. *If M is piecewise atone of finite rank and monotone, $C = 2(I + M)^{-1} - I$ is piecewise unitary.*

With this Lemma at our disposal we are now in a position to give an immediate answer to the problem:

PROBLEM. « *Construct a piecewise unitary contraction coinciding with a given contraction at a finite number of given points* ».

SOLUTION. Let C be the given contraction, M the corresponding maximal monotone operator and $x_1, x_2, \dots, x_n$ the given points. If $\tilde{M}$ is a piecewise atone monotone operator such that $\tilde{M}u_i = Mu_i$, $u_i \supset \frac{1}{2}(I + C)x_i$, $i = 1, 2, \dots, n$, then $\tilde{C} = 2(I + \tilde{M})^{-1} - I$ is a piecewise unitary contraction satisfying $\tilde{C}x_i = Cx_i$, $i = 1, 2, \dots, n$.

We leave the verification to the reader. Thus we can state the theorem:

THEOREM 8.1. *Any contraction in Hilbert space can be interpolated at a finite number of given points by means of a piecewise unitary contraction. If the given contraction is a gradient mapping the interpolating contraction can be chosen so as to be a gradient mapping too.*

Naturally, there is a similar theorem for operators of the class $\mathcal{T}$. An interesting final remark is that, in spite of their apparent diversity, *piecewise unitary mappings, when viewed locally, are all of the type $S_\Pi^A = 2(I + A + \partial\psi_\Pi)^{-1} - I$, where Π is a convex polyhedron with a nonempty interior, and A*

a maximal atone operator of finite rank defined on Π. In fact, because of the matching conditions, $Mx = A_i x + \partial\psi_{\Pi_i} x$, for all x in Π_i; as in the preceeding discussion it follows

$$Cy = \big(2(I+M)^{-1} - I\big)y = \big(2(I + A_i + \partial\psi_{\Pi_i})^{-1} - I\big)y\,, \quad \forall y \in (I+M)\Pi_i\,.$$

By replacing all tiles associated with a fixed atone operator by their union, which must be convex by Lemma 3.3, we may assume that the A_i's are different. This condition secured, $\{(\overset{\circ}{(I + M)\Pi_i})\}_1^r$ is an open covering for $\mathcal{H}$, and the assertion follows from the above equation. S_Π^A is a kind of twisted symmetry with regard to Π.

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