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Differentiability and partial Hölder continuity of the solutions of non-linear elliptic systems of order $2m$ with quadratic growth

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Differentiability and Partial Hölder Continuity of the Solutions of Non-Linear Elliptic Systems of Order $2m$ with Quadratic Growth.

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1. – Introduction.

Let Ω be a bounded open set of R^n with points $x = (x_1, \ldots, x_n)$; here m and N are integers $\geqslant 1$, $(\,|\,)_N$ and $\|\;\|_N$ are the scalar product and the norm in R^N. We shall drop the subscript N when there is no danger of confusion.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We denote by $\mathfrak{R}$ the cartesian product $\prod_{|\alpha|\leqslant m} R^N_\alpha$ and by $p = \{p^\alpha\}_{|\alpha|\leqslant m}$, $p^\alpha \in R^N$, a typical point in $\mathfrak{R}$.

If $p \in \mathfrak{R}$ we set

$$
\begin{gathered}
p' = \{p^\alpha\}_{|\alpha|=m}\,, \qquad p'' = \{p^\alpha\}_{|\alpha|<m} \\
\|p\|^2 = \sum_{|\alpha|\leqslant m} \|p^\alpha\|^2_N\,, \qquad \|p'\|^2 = \sum_{|\alpha|=m} \|p^\alpha\|^2_N\,, \qquad \|p''\|^2 = \sum_{|\alpha|<m} \|p^\alpha\|^2_N\,.
\end{gathered} \tag{1.1}
$$

We define, as usual,

$$D_i = \frac{\partial}{\partial x_i}\,, \qquad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$$

and, if $u\colon \Omega \to R^N$, then

$$Du = \{D^\alpha u\}_{|\alpha|\leqslant m}\,, \qquad D' u = \{D^\alpha u\}_{|\alpha|=m}\,, \qquad D'' u = \{D^\alpha u\}_{|\alpha|<m}\,. \tag{1.2}$$

$C^{h,\lambda}(\bar\Omega, R^N)$, h integer $\geqslant 0$ and $0 < \lambda \leqslant 1$, is the space of those vectors $u\colon \bar\Omega \to R^N$ which satisfy a Hölder condition of exponent λ, together with

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all their derivatives $D^\alpha u$, $|\alpha|\leqslant h$; if $u\in C^{h,\lambda}(\bar\Omega, R^N)$, then

$$\|u\|_{C^{h,\lambda}(\bar\Omega,R^N)} = \sup_{\Omega}\sum_{|\alpha|\leqslant h}\|D^\alpha u\| + \sum_{|\alpha|=h}[D^\alpha u]_{\lambda,\bar\Omega} \tag{1.3}$$

where

$$[u]_{\lambda,\bar\Omega} = \sup_{x,y\in\Omega}\frac{\|u(x)-u(y)\|}{\|x-y\|^\lambda}\,. \tag{1.4}$$

$H^{s,p}(\Omega, R^N)$ and $H_0^{s,p}(\Omega, R^N)$, s integer $\geqslant 0$ and $p\geqslant 1$, are the usual Sobolev spaces and, if $1\leqslant p<+\infty$, then

$$\begin{aligned} |u|_{s,p,\Omega} &= \left\{\int_\Omega \sum_{|\alpha|=s}\|D^\alpha u\|^p\,dx\right\}^{1/p}\\ \|u\|_{s,p,\Omega} &= \left\{\sum_{h=0}^{s}|u|^p_{h,p,\Omega}\right\}^{1/p}\end{aligned} \tag{1.5}$$

$H^{0,p}(\Omega, R^N) = L^p(\Omega, R^N)$ and

$$|u|_{0,p,\Omega} = \left\{\int_\Omega \|u\|^p\,dx\right\}^{1/p}.$$

If $p=2$, then we shall simply write H^s, H_0^s, $|\ |_{s,\Omega}$, $\|\ \|_{s,\Omega}$.

Let $a^\alpha(x,p)$, $|\alpha|\leqslant m$, be vectors of R^N, defined in $\Omega\times\mathfrak{R}$, measurable in x and continuous in p; assume that $\forall(x,p)\in\Omega\times\mathfrak{R}$ with $\|p''\|\leqslant K$

$$\|a^\alpha(x,p)\|\leqslant M(K)\{f^\alpha(x)+\|p'\|\} \quad \text{if } |\alpha|=m \tag{1.6}$$

$$\|a^\alpha(x,p)\|\leqslant M(K)\{f^\alpha(x)+\|p'\|^2\} \quad \text{if } |\alpha|<m \tag{1.7}$$

where

$$f^\alpha\in L^2(\Omega) \quad \text{if } |\alpha|=m\,, \qquad f^\alpha\in L^1(\Omega) \quad \text{if } |\alpha|<m\,.$$

Let us consider the differential system of order $2m$

$$\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha a^\alpha(x, Du) = 0 \tag{1.8}$$

which is assumed to be strongly elliptic, i.e. the vector functions $p'\to a^\alpha(x,p)$, $|\alpha|=m$, are differentiable and there exists $\nu(K)>0$ such that

$$\sum_{h,k=1}^{N}\sum_{|\alpha|=|\beta|=m}\frac{\partial a_h^\alpha(x,p)}{\partial p_k^\beta}\xi_h^\alpha\xi_k^\beta\geqslant \nu(K)\sum_{|\alpha|=m}\|\xi^\alpha\|_N^2 \tag{1.9}$$

for every set $\{\xi^\alpha\}_{|\alpha|=m}$ of vectors in R^N and for every $(x, p) \in \Omega \times \mathfrak{R}$ with $\|p''\| \leqslant K$.

A solution of system (1.8) is a vector $u \in H^m \cap H^{m-1,\infty}(\Omega, R^N)$ such that

$$\int_\Omega \sum_{|\alpha| \leqslant m} (a^\alpha(x, Du)|D^\alpha \varphi)\, dx = 0 \tag{1.10}$$

$$\forall \varphi \in H_0^m \cap H^{m-1,\infty}(\Omega, R^N)\,.$$

In this paper we shall investigate the problem of the local differentiability of the solutions of system (1.10): if $0 < \lambda < 1$ and $u \in H^m \cap$ $\cap\, C^{m-1,\lambda}(\bar{\Omega}, R^N)$ is a solution of system (1.10), then under what conditions on the vectors $a^\alpha(x, p)$ can we show that

$$u \in H^{m+1}_{\text{loc}}(\Omega, R^N)\ ? \tag{1.11}$$

We take solutions of class $C^{m-1,\lambda}(\bar{\Omega}, R^N)$ because it is already known that, if $u \in H^m \cap H^{m-1,\infty}(\Omega, R^N)$, problem (1.11) is, in general, answered negatively even if the vectors $a^\alpha(x, p)$ are very smooth.

If $m = 1$ (second order systems), it is well known that the answer to problem (1.11) is positive under the following conditions: denoting by p the vector $(u, p^1, \ldots, p^n)$, where u and p^i are vectors of R^N, then

$$a^i(x, p)\,, \quad i = 1, \ldots, n, \text{ are of class } C^1 \text{ in } \bar{\Omega} \times R^{(n+1)N} \tag{1.12}$$

and $\forall (x, p) \in \Omega \times R^{(n+1)N}$ with $\|u\| \leqslant K$

$$\|a^i(x, p)\| \leqslant M(K)\Big\{1 + \sum_{j=1}^n \|p^j\|\Big\}\,, \qquad i = 1, \ldots, n\,, \tag{1.13}$$

$$\|a^0(x, p)\| \leqslant M(K)\Big\{1 + \sum_{j=1}^n \|p^j\|^2\Big\}\,, \tag{1.14}$$

$$\left\|\frac{\partial a^i(x, p)}{\partial p^i_k}\right\| \leqslant M(K)\,, \qquad i = 1, \ldots, n\,, \tag{1.15}$$

$$\left\|\frac{\partial a^i(x, p)}{\partial u_k}\right\| + \left\|\frac{\partial a^i(x, p)}{\partial x_r}\right\| \leqslant M(K)\Big\{1 + \sum_{j=1}^n \|p^j\|\Big\}\,, \qquad i = 1, \ldots, n\,, \tag{1.16}$$

$$\left\|\frac{\partial a^0(x, p)}{\partial p^i_k}\right\| \leqslant M(K)\Big\{1 + \sum_{j=1}^n \|p^j\|\Big\}\,, \tag{1.17}$$

$$\left\|\frac{\partial a^0(x, p)}{\partial u_k}\right\| + \left\|\frac{\partial a^0(x, p)}{\partial x_r}\right\| \leqslant M(K)\Big\{1 + \sum_{j=1}^n \|p^j\|^2\Big\}\,. \tag{1.18}$$

If $m = N = 1$ see for instance [8]; if $m = 1$, $N > 1$ see for example [9] and also [5], chapter V, n. 3.

In order to get the differentiability result (1.11) for second order systems ($m = 1$), the following inequality is essential: using the notation $B(x^0, \sigma) = \{x\colon \|x - x^0\| < \sigma\}$, *if* $u \in H^1 \cap C^{0,\lambda}(\bar{\Omega}, R^N)$ *is a solution of the strongly elliptic system*

$$\sum_{=1}^{n} D_i a^i(x, Du) = a^0(x, Du)$$

under the hypotheses (1.13) (1.14) *with* $K = \sup_{\Omega}\|u(x)\|$, *then there exists* $\sigma_0 > 0$ *such that* $\forall B(x^0, \sigma) \subset\subset \Omega$ *with* $\sigma \leqslant \sigma_0$ *and* $\forall \varphi \in H_0^1 \cap L^\infty\big(B(x^0, \sigma), R^N\big)$

$$\int_{B(x^0,\sigma)} \|\varphi\|^2 \sum_i \|D_i u\|^2 dx \leqslant o(\sigma) \int_{B(x^0,\sigma)} \sum_i \|D_i \varphi\|^2 dx \tag{1.19}$$

where $o(\sigma)$ *tends to zero with* σ.

For strongly elliptic systems of order $2m > 2$, problem (1.11) remained unsolved because of the difficulty of proving a proper extension of inequality (1.19) (see for instance [10]).

In section **3** of this work we deal with problem (1.11) following a different method.

In addition to the strong ellipticity (1.9), we shall assume that the vectors $a^\alpha(x, p)$ satisfy the following hypotheses:

(1.20) *if* $|\alpha| < m$, *then the vectors* $a^\alpha(x, p)$ *are measurable in* x $\forall p \in \mathfrak{R}$, *continuous in* p $\forall x \in \Omega$, *and for every* $(x, p) \in \Omega \times \mathfrak{R}$ *with* $\|p''\| \leqslant K$

$$\|a^\alpha(x, p)\| \leqslant M(K)\{f^\alpha(x) + \|p'\|^2\}$$

where $f^\alpha \in L^2(\Omega)$;

(1.21) *if* $|\alpha| = m$, *then the vectors* $a^\alpha(x, p)$ *are of class* C^1 *in* $\bar{\Omega} \times \mathfrak{R}$ *and for every* $(x, p) \in \Omega \times \mathfrak{R}$ *with* $\|p''\| \leqslant K$

$$\|a^\alpha\| + \sum_{i=1}^{n} \left\|\frac{\partial a^\alpha}{\partial x_i}\right\| + \sum_{k=1}^{N} \sum_{|\beta|<m} \left\|\frac{\partial a^\alpha}{\partial p_k^\beta}\right\| \leqslant M(K)\{1 + \|p'\|\}$$

$$\sum_{k=1}^{N} \sum_{|\beta|=m} \left\|\frac{\partial a^\alpha}{\partial p_k^\beta}\right\| \leqslant M(K)\,.$$

We remark that hypothesis (1.21) is a formal extension of the hypotheses (1.13) (1.15) (1.16), whereas hypothesis (1.20) is less restrictive than the assumptions we made on a^0 if $m = 1$.

As usual we denote by $H^\theta(\Omega, R^N)$, $0 < \theta < 1$, the space consisting of those vectors $u \in L^2(\Omega, R^N)$ such that

(1.22) $$|u|^2_{\theta,\Omega} = \int_\Omega dx \int_\Omega \frac{\|u(x) - u(y)\|^2}{\|x - y\|^{n+2\theta}} dy < +\infty$$

and by $H^{m+\theta}(\Omega, R^N)$ the subspace of $H^m(\Omega, R^N)$ consisting of those vectors u for which

$$D^\alpha u \in H^\theta(\Omega, R^N)\,, \qquad |\alpha| = m\,.$$

In section **3** we show that, if the conditions (1.9) (1.20) (1.21) are fulfilled and $u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ is a solution of system (1.10), then

(1.23) $$u \in H^{m+\theta}_{\text{loc}}(\Omega, R^N)\,, \qquad \forall 0 < \theta < \lambda/2\,.$$

As $u \in C^{m-1,\lambda}(\bar{\Omega}, R^N)$, from (1.23) and theorem 2.I we get

(1.24) $$u \in H^{m,p}_{\text{loc}}(\Omega, R^N)\,, \qquad \forall p < \frac{2(1+\theta)n}{n - 2\theta\lambda}\,.$$

Now, by an induction process (theorem 3.IV) we show that

(1.25) $$u \in H^{m+\theta}_{\text{loc}}(\Omega, R^N)\,, \qquad \forall 0 < \theta < 1.$$

Then (1.24) implies

(1.26) $$u \in H^{m,4}_{\text{loc}}(\Omega, R^N)\,.$$

Once we have this information we can easily prove the following local differentiability theorem, where

$$Q^0(x, \sigma) = \{x\colon |x_i - x_i^0| < \sigma,\ i = 1, \dots, n\}\,.$$

THEOREM 1.I. *If* $u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ *is a solution of system* (1.10) *under the hypotheses* (1.9) (1.20) (1.21), *then*

(1.27) $$u \in H^{m+1}_{\text{loc}}(\Omega, R^N)$$

and $\forall Q(x^0, 2\sigma) \subset\subset \Omega$

(1.28) $$|u|^2_{m+1, Q(x^0,\sigma)} \leqslant c\Big\{1 + F^2 + \int_{Q(x^0,2\sigma)} \|D'u\|^4\, dx\Big\}$$

where

$$F = \sum_{|\alpha|<m} \|f^\alpha\|_{0,\Omega}$$

and c *depends on* σ *and on the norm* $\|u\|_{C^{m-1,\lambda}(\bar{\Omega},R)}$.

As we show in section **3**, (1.28) can be replaced by the following equivalent inequality

$$|u|^2_{m+1,Q(x^0,\sigma)} \leq c\{1 + F^2 + |u|^2_{m,Q(x_0,2\sigma)}\}\,. \tag{1.29}$$

Moreover, we may assume that the function f^α, which appear in (1.20), satisfy the more general hypothesis

$$\mathcal{F} = \sum_{|\alpha|<m} D^\alpha f^\alpha \in H^{-m+1}(\Omega)\,. \text{ (1)}$$

In this case F^2 has to be replaced by $\|\mathcal{F}\|^2$ in (1.28) and by $\|\mathcal{F}\|^{2+\varepsilon}$, $\forall\varepsilon > 0$, in (1.29).

In section **4** we prove that the differentiability result of theorem 1.I and theorem 3.I of [5] (chap. IV) allow us to obtain the partial Hölder continuity of the derivatives $D^\alpha u$, $|\alpha| = m$.

Assume that

$$\frac{\partial a^\alpha(x, p)}{\partial p_k^\beta}\,, \qquad |\alpha| = |\beta| = m \text{ and } 1 \leq k \leq N$$

are uniformly continuous in $\bar{\Omega}\times\mathfrak{R}$ *and that in condition* (1.20) *we have*

$$f^\alpha \in L^p(\Omega) \qquad \text{with } p > n$$

or, in general,

$$\mathcal{F} = \sum_{|\alpha|<m} D^\alpha f^\alpha \in H^{-m+1,p}(\Omega) \qquad \text{with } p > n\,.$$

Then

THEOREM 1.II. *If* $u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ *is a solution of system* (1.10) *under the hypotheses* (1.9) (1.20) (1.21), *then there exists a set* $\Omega_0 \subset \Omega$, *closed in* Ω, *such that*

$$\mathcal{H}_{n-q}(\Omega_0) = 0 \qquad \text{for a certain } q > 2 \tag{1.30}$$

$$D^\alpha u \in C^{0,\mu}(\Omega\setminus\Omega_0, R^N)\,, \qquad |\alpha| = m,\ \forall 0 < \mu < 1 - n/p \tag{1.31}$$

where $\mathcal{H}_{n-q}$ *is the* $(n - q)$*-dimensional Hausdorff measure.*

It is now easy to prove higher regularity results for the solutions $u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ of system (1.10), using the theory of linear systems.

(1) $H^{-m+1}(\Omega)$ is the dual of $H_0^{m-1}(\Omega)$.

2. – Preliminary results.

In this section we mention a few results that will be used in the sequel of the work.

Here $Q(\sigma) = Q(x^0, \sigma) = \{x\colon |x_i - x_i^0| < \sigma,\ i = 1, \dots, n\}$.

If $u\colon Q(\sigma) \to R^N$, $t \in (0, 1)$, $x \in Q(t\sigma)$ and $|h| < (1 - t)\sigma$, then we define

$$\tau_{i,h} u(x) = u(x + he^i) - u(x)\,, \qquad i = 1, \dots, n$$

where $\{e^i\}_{i=1,\dots,n}$ is the standard base of R^n.

LEMMA 2.I. *If* $u \in L^p(Q(\sigma), R^N)$, $1 < p < +\infty$, *and there exists* $M > 0$ *such that*

$$\|\tau_{i,h} u\|_{0,p,Q(t\sigma)} \leqslant |h|\, M\,, \qquad \forall |h| < (1 - t)\,\sigma,\ i = 1, \dots, n \tag{2.1}$$

then $u \in H^{1,p}(Q(t\sigma), R^N)$ *and*

$$\|D_i u\|_{0,p,Q(t\sigma)} \leqslant M\,, \qquad i = 1, \dots, n\,. \tag{2.2}$$

LEMMA 2.II. *If* $u \in H^{1,p}(Q(\sigma), R^N)$, $1 \leqslant p < +\infty$, *then* $\forall t \in (0, 1)$ *and* $|h| < (1 - t)\,\sigma$

$$\|\tau_{i,h} u\|_{0,p,Q(t\sigma)} \leqslant |h| \cdot \|D_i u\|_{0,p,Q(\sigma)}\,, \qquad i = 1, \dots, n\,. \tag{2.3}$$

The previous lemmas are well known in the mathematical literature (see for instance [5], chap. I).

LEMMA 2.III. *If* $u \in L^2(Q(3\sigma), R^N)$ *and, for* $\theta \in (0, 1)$

$$\sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\theta}} \int_{Q(\sigma)} \|\tau_{i,h} u(x)\|^2\, dx < +\infty \tag{2.4}$$

then $u \in H^\theta(Q(\sigma), R^N)$ *and*

$$|u|^2_{\theta,Q(\sigma)} \leqslant c(n) \sum_{i=1}^{n} \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\theta}} \int_{Q(\sigma)} \|\tau_{i,h} u(x)\|^2\, dx\,. \tag{2.5}$$

See for instance [2], lemma II.3.

If Ω is a bounded open set and $\sigma > 0$, then Ω_σ denotes the set of those points whose distance from $\bar{\Omega}$ is less than σ.

LEMMA 2.IV. *If $\Omega, \Omega_1, \ldots, \Omega_k$ are bounded open sets of R^n and $\Omega = \bigcup_{i=1}^k \Omega_i$, $\sigma > 0$, $0 < \theta < 1$, then* $\forall u \in H^\theta(\Omega, R^N)$

$$|u|^2_{\theta,\Omega} \leq c(k, \theta, \sigma) \left\{ \|u\|^2_{0,\Omega} + \sum_{i=1}^k \int_{\Omega_{i,\sigma} \cap \Omega} dx \int_{\Omega_i} \frac{\|u(x) - u(y)\|^2}{\|x - y\|^{n+2\theta}} dy \right\}. \tag{2.6}$$

See for instance [2], lemma I.3.

We now state a theorem which is interesting in itself and extends theorem 3.III of [3], that deals with the case $\theta = 1$. We set

$$u_E = (\text{meas}\, E)^{-1} \int_E u(x)\, dx\,.$$

THEOREM 2.I. *If Q is a cube of R^n and $u \in H^{1+\theta} \cap C^{0,\lambda}(\bar{Q}, R^N)$, $0 < \theta \leq 1$ and $0 < \lambda \leq 1$, then $\forall t > 0$ and for $i = 1, \ldots, n$*

$$\text{meas}\, \{x \in Q\colon \|D_i u(x) - (D_i u)_Q\| > t\} \leq c^q(n, \theta) \frac{\sum_j |D_j u|^{q/(1+\theta)}_{\theta,Q} \cdot [u]^{q\theta/(1+\theta)}_{\lambda,\bar{Q}}}{t^q} \tag{2.7}$$

where $q = 2(1 + \theta)n/(n - 2\theta\lambda)$.

In particular

$$D_i u \in L^s(Q, R^N)\,, \qquad \forall 1 \leq s < q \tag{2.8}$$

and

$$\int_\Omega \|D_i u - (D_i u)_Q\|^s dx \leq c(n, q, \theta, s)(\text{meas}\, Q)^{1-s/q} \sum_j |D_j u|^{s/(1+\theta)}_{\theta,Q} [u]^{s\theta/(1+\theta)}_{\lambda,\bar{Q}}\,. \tag{2.9}$$

The proof of this theorem is given in the appendix and follows the proof of theorem 3.III of [3]. With formal modifications the previous theorem can be proved also for vectors $u \in H^{1+\theta,p} \cap C^{0,\lambda}(\bar{Q}, R^N)$, $p > 1$.

3. - The local differentiability result.

In this section we prove the differentiability theorem (theorem 1.I). Here Ω is a bounded open set of R^n,

$$Q(x^0, \sigma) = \{x\colon |x_i - x_i^0| < \sigma,\ i = 1, \ldots, n\}$$

and $d(x^0) = \text{dist}\,(x^0, \partial\Omega)$.

$u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ is a solution of the following system

$$\int_\Omega \sum_{|\alpha|\leqslant m} (a^\alpha(x, Du)|D^\alpha \varphi)\, dx = 0 \tag{3.1}$$
$$\forall \varphi \in H_0^m \cap H^{m-1,\infty}(\Omega, R^N)\,.$$

Assume that the hypotheses (1.9) (1.20) (1.21) are fulfilled and let us define

$$K = \sup_\Omega \|D'' u\|\,, \qquad U = \|u\|_{C^{m-1,\lambda}(\bar{\Omega},R^N)}\,, \qquad F = \sum_{|\alpha|<m} \|f^\alpha\|_{0,\Omega}\,. \tag{3.2}$$

If i is an integer, $1\leqslant i\leqslant n$, and $h \in R$, then we set

$$\tau_{i,h} u(x) = u(x + he^i) - u(x)\,.$$

The proof of theorem 1.I is based on theorems 3.I, 3.II, 3.III and 3.IV that we are now going to demonstrate.

THEOREM 3.I. *If* $u \in H^m \cap C^{m-1,\lambda}(\Omega, R^N)$ *is a solution of system* (3.1) *under the hypotheses* (1.9) (1.20) (1.21), *then for every* $Q(x^0, \sigma) \subset\subset \Omega$, *for every* $\psi \in C_0^\infty(\Omega)$ *with* $\psi \geqslant 0$ *and* $\psi = 0$ *in* $\Omega \setminus Q(x^0, \sigma)$, *for* $i = 1, \ldots, n$ *and* $|h| < d(x^0) - \sigma$, *the following inequality holds*:

$$\int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 dx \leqslant c(\nu, U, \Psi)\, h^2\{1 + |u|^2_{m,Q(\sigma+|h|)}\} + \tag{3.3}$$
$$+ c(K, \nu)\int_\Omega \psi^{2m} \|D' u\|^2 \cdot \|\tau_{i,h} D'' u\|^2 dx - \sum_{|\alpha|<m} \int_\Omega \big(a^\alpha(x, D^\alpha u)|\tau_{i,-h} D^\alpha(\psi^{2m} \tau_{i,h} u)\big)\, dx$$

where $\Psi = \sup_\Omega \|D\psi\|$.

PROOF. Let $Q(\sigma) = Q(x^0, \sigma) \subset\subset \Omega$ and $\psi \in C_0^\infty(\Omega)$ with $\psi \geqslant 0$, $\psi = 0$ in $\Omega \setminus Q(\sigma)$. Having fixed i integer, $1\leqslant i\leqslant n$, and h such that $|h| < d(x^0) - \sigma$, let us assume in (3.1)

$$\varphi = \tau_{i,-h}(\psi^{2m} \tau_{i,h} u)\,. \tag{3.4}$$

Then we get

$$\int_\Omega \sum_{|\alpha|=m} \big(\tau_{i,h} a^\alpha(x, Du)|D^\alpha(\psi^{2m} \tau_{i,h} u)\big)\, dx = \tag{3.5}$$
$$= -\int_\Omega \sum_{|\alpha|<m} \big(a^\alpha(x, Du)|\tau_{i,-h} D^\alpha(\psi^{2m} \tau_{i,h} u)\big)\, dx\,.$$

For the sake of simplicity let us set, if $b(x,p)$ is a vector of R^N,

$$\tilde{b} = \int_0^1 b(x + the^i, Du + t\tau_{i,h} Du)\, dt . \tag{3.6}$$

Then

$$\tau_{i,h} a^\alpha(x, Du) = h \frac{\widetilde{\partial a^\alpha}}{\partial x_i} + \sum_{|\beta|\leq m} \sum_{k=1}^N \tau_{i,h} D^\beta u_k \frac{\widetilde{\partial a^\alpha}}{\partial p_k^\beta} . \tag{3.7}$$

Now, if $|\alpha| = m$,

$$D^\alpha(\psi^{2m}\tau_{i,h} u) = \psi^{2m}\tau_{i,h} D^\alpha u + \psi^m \sum_{\gamma<\alpha} C_{\alpha,\gamma}(\psi)\tau_{i,h} D^\gamma u \tag{3.8}$$

with

$$|C_{\alpha,\gamma}(\psi)| \leq c(m, n)\Psi . \tag{3.9}$$

Therefore, from (3.5) we obtain

$$\begin{aligned}
&\int_\Omega \psi^{2m} \sum_{|\alpha|=|\beta|=m} \sum_{k=1}^N \left(\tau_{i,h} D^\beta u_k \frac{\widetilde{\partial a^\alpha}}{\partial p_k^\beta} \middle| \tau_{i,h} D^\alpha u\right) dx = \\
&= - \sum_{|\alpha|=|\beta|=m} \sum_{\gamma<\alpha} \sum_{k=1}^N \int_\Omega \left(\tau_{i,h} D^\beta u_k \frac{\widetilde{\partial a^\alpha}}{\partial p_k^\beta} \middle| \psi^m C_{\alpha,\gamma}\tau_{i,h} D^\gamma u\right) dx + \\
&\quad - \sum_{|\alpha|=m} \sum_{|\beta|<m} \sum_{k=1}^N \int_\Omega \left(\tau_{i,h} D^\beta u_k \frac{\widetilde{\partial a^\alpha}}{\partial p_k^\beta} \middle| D^\alpha(\psi^{2m}\tau_{i,h} u)\right) dx + \\
&\quad - h \sum_{|\alpha|=m} \int_\Omega \left(\frac{\widetilde{\partial a^\alpha}}{\partial x_i} \middle| D^\alpha(\psi^{2m}\tau_{i,h} u)\right) dx + \\
&\quad - \sum_{|\alpha|<m} \int_\Omega (a^\alpha(x, Du) | \tau_{i,-h} D^\alpha(\psi^{2m}\tau_{i,h} u))\, dx = A + B + C + D .
\end{aligned} \tag{3.10}$$

By the hypothesis of strong ellipticity (1.9) we get

$$\int_\Omega \psi^{2m} \sum_{|\alpha|=|\beta|=m} \sum_{k=1}^N \left(\tau_{i,h} D^\beta u_k \frac{\widetilde{\partial a^\alpha}}{\partial p_k^\beta} \middle| \tau_{i,h} D^\alpha u\right) dx \geq \nu \int_\Omega \psi^{2m} \|D'(\tau_{i,h} u)\|^2 dx . \tag{3.11}$$

On the other hand, from hypothesis (1.21) and (3.6) it follows that, if $|\alpha| = m$,

$$\sum_{|\beta|=m} \sum_{k=1}^N \left\| \frac{\widetilde{\partial a^\alpha}}{\partial p_k^\beta} \right\| \leq M(K) , \tag{3.12}$$

$$\left\| \frac{\widetilde{\partial a^\alpha}}{\partial x_i} \right\| + \sum_{|\beta|<m} \sum_{k=1}^N \left\| \frac{\widetilde{\partial a^\alpha}}{\partial p_k^\beta} \right\| \leq M(K)\{1 + \|D'u\| + \|\tau_{i,h} D'u\|\} . \tag{3.13}$$

Then

$$|A| \leqslant c(K, \Psi) \int_\Omega \psi^m \|\tau_{i,h} D' u\| \cdot \|\tau_{i,h} D'' u\| \, dx$$

and $\forall \varepsilon > 0$

$$|A| \leqslant \varepsilon \int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx + c(K, \Psi, \varepsilon) \|\tau_{i,h} u\|^2_{m-1, Q(\sigma)} .$$

And so by lemma 2.II

$$|A| \leqslant \varepsilon \int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx + h^2 c(K, \Psi, \varepsilon) \{1 + |u|^2_{m, Q(\sigma + |h|)}\} . \tag{3.14}$$

Let us now estimate B:

$$|B| \leqslant c(K) \int_\Omega (1 + \|D' u\| + \|\tau_{i,h} D' u\|) \|\tau_{i,h} D'' u\| \cdot$$
$$\cdot \left(\psi^{2m} \|\tau_{i,h} D' u\| + c(\Psi) \psi^m \|\tau_{i,h} D'' u\|\right) dx .$$

By the fact that $u \in C^{m-1,\lambda}(\bar{\Omega}, R^N)$

$$\sup_{Q(\sigma)} \|\tau_{i,h} D'' u\| = o(h) \tag{3.15}$$

where $o(h)$ depends on U and tends to zero with h. Then $\forall \varepsilon > 0$

$$|B| \leqslant \left(\varepsilon + c(K)\, o(h)\right) \int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx +$$
$$+ c(K, \varepsilon) \int_\Omega \psi^{2m} \|D' u\|^2 \cdot \|\tau_{i,h} D'' u\|^2 \, dx + c(K, \Psi, \varepsilon) \|\tau_{i,h} u\|^2_{m-1, Q(\sigma)} .$$

Therefore by lemma 2.II we conclude that

$$|B| \leqslant \left(\varepsilon + c(K)\, o(h)\right) \int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx + \tag{3.16}$$
$$+ c(K, \varepsilon) \int_\Omega \psi^{2m} \|D' u\|^2 \cdot \|\tau_{i,h} D'' u\|^2 \, dx + h^2 c(K, \Psi, \varepsilon) \{1 + |u|^2_{m, Q(\sigma + |h|)}\} .$$

Similarly

$$|C| \leqslant |h|\, c(K) \int_\Omega (1 + \|D' u\| + \|\tau_{i,h} D' u\|) \cdot \left(\psi^{2m} \|\tau_{i,h} D' u\| + c(\Psi) \psi^m \|\tau_{i,h} D'' u\|\right) dx$$

and $\forall \varepsilon > 0$

$$|C| \leqslant \{\varepsilon + c(K)(|h| + h^2)\} \int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx + \tag{3.17}$$
$$+ h^2 c(K, \Psi, \varepsilon, \sigma) \{1 + |u|^2_{m, Q(\sigma + |h|)}\} .$$

From (3.10) (3.11) (3.14) (3.16) (3.17) we obtain inequality (3.3) if we take ε and $|h|$ small enough:

$$\varepsilon < \varepsilon_0(\nu) \qquad \text{and} \qquad |h| < h_0(\nu, U) < d(x^0) - \sigma .$$

However, if $h_0 \leqslant |h| < d(x^0) - \sigma$, then (3.3) is trivial because

$$\int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx \leqslant 2 \frac{h^2}{h^2} |u|^2_{m,Q(\sigma+ \ \ |} .$$

LEMMA 3.I. *If* $u \in H^{m,p} \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$, $2 < p \leqslant 4$, *then for every cube* $Q(2\sigma) = Q(x^0, 2\sigma) \subset\subset \Omega$, *for every* $\psi \in C_0^\infty(\Omega)$ *with* $\psi \geqslant 0$ *and* $\psi = 0$ *in* $\Omega \setminus Q(\frac{3}{2}\sigma)$, *for* $i = 1, \ldots, n$, *for* $|h| < \sigma/4$ *and* $\forall \varepsilon > 0$ *the following inequality holds*:

$$(3.18) \qquad \left| \sum_{|\alpha|<m} \int_\Omega (a^\alpha(x, Du) | \tau_{i,-h} D^\alpha(\psi^{2m} \tau_{i,h} u)) \, dx \right| \leqslant$$
$$\leqslant \varepsilon \int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx + c(\varepsilon, \sigma, \Psi, U) |h|^{p-2+\lambda(2-p/2)} \{1 + F^{p/2} + |u|^p_{m\, p\, Q(2\sigma)}\}$$

where $\Psi = \sup \|D\psi\|$.

PROOF. By hypothesis (1.20) and Hölder's inequality we get $\forall \varepsilon > 0$

$$(3.19) \qquad \left| \sum_{|\alpha|<m} \int_\Omega (a^\alpha(x, Du) | \tau_{i,-h} D^\alpha(\psi^{2m} \tau_{i,h} u)) \, dx \right| \leqslant$$
$$\leqslant c(K) \sum_{\alpha|<m} \int_\Omega (|f^\alpha| + \ \ D' u\|^2) \|\tau_{i,-h} D^\alpha(\psi^{2m} \tau_{i,h} u)\| \, dx \leqslant$$
$$\leqslant \varepsilon |h|^{-2} \int_{Q((7/4)\sigma)} \|\tau_{i,-h} D''(\psi^{2m} \tau_{i,h} u)\|^2 \, dx +$$
$$+ c(\varepsilon, K) |h|^{p-2} \sum_{|\alpha|<m} \int_{Q((7/4)\sigma)} (|f^\alpha| + \|D' u\|^2)^{p/2} \|\tau_{i,-h} D''(\psi^{2m} \tau_{i,h} u))\|^{2-p/2} \, dx .$$

Then, from lemma 2.II, (3.8) and the fact that $u \in C^{m-1,\lambda}(\bar{\Omega}, R^N)$ we conclude that

$$\left| \sum_{|\alpha|<m} \int_\Omega (a^\alpha(x, Du) | \tau_{i,-h} D^\alpha(\psi^{2m} \tau_{i,h} u)) \, dx \right| \leqslant$$
$$\leqslant c(\sigma) \varepsilon |\psi^{2m} \tau_{i,h} u|^2_{m,Q(2\sigma)} + c(\varepsilon, U) |h|^{p-2+\lambda(2-p/2)} \{F^{p/2} + |u|^p_{m,p,Q(2\sigma)}\} \leqslant$$
$$\leqslant c(\sigma) \varepsilon \int_\Omega \psi^{2m} \|\tau_{i,h} D' u\|^2 \, dx + c(\varepsilon, \Psi, U) |h|^{p-2+\lambda(2-p/2)} \{1 + F^{p/2} + |h|_{m,p,Q(2\sigma)}\} .$$

Using theorem 3.I we can easily prove the following fractional differentiability result.

THEOREM 3.II. *If* $u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ *is a solution of system* (3.1) *under the hypotheses* (1.9) (1.20) (1.21), *then*

$$u \in H^{m+\theta}_{\text{loc}}(\Omega, R^N)\,, \qquad 0 < \theta < \lambda/2 \tag{3.20}$$

and for every cube $Q(3\sigma) = Q(x^0, 3\sigma) \subset\subset \Omega$ (²)

$$|D' u|^2_{\theta,Q(\sigma)} \leqslant C(\sigma, \lambda, U)\{1 + F + |u|^2_{m,Q(3\sigma)}\}\,. \tag{3.21}$$

PROOF. Choose $\psi \in C_0^\infty(\Omega)$ with

$$0 \leqslant \psi \leqslant 1\,, \qquad \psi = 1 \quad \text{in } Q(\sigma)\,, \qquad \psi = 0 \quad \text{in } \Omega \setminus Q(2\sigma)\,.$$

From inequality (3.3), hypothesis (1.20) and the fact that $u \in C^{m-1,\lambda}(\bar{\Omega}, R^N)$ we conclude that $\forall |h| < \sigma$

$$\begin{aligned} \sum_{i=1}^n \int_{Q(\sigma)} \|\tau_{i,h} D' u\|^2 dx &\leqslant c(U, \sigma)|h|^{2\lambda}\{1 + |u|^2_{m,Q(3\sigma)}\} + \\ &+ c(K) \sum_{|\alpha|<m} \int_{Q(3\sigma)} (|f^\alpha| + \|D' u\|^2)\, \|\tau_{i,-h} D^\alpha(\psi^{2m} \tau_{i,h} u)\|\, dx \leqslant \\ &\leqslant c(U, \sigma)|h|^\lambda \{1 + F + |u|^2_{m,Q(3\sigma)}\}\,. \text{ (³)} \end{aligned} \tag{3.22}$$

Inequality (3.22) is trivial if $\sigma \leqslant |h| < 2\sigma$ and therefore, if $0 < \theta < \lambda/2$, from (3.22) we easily get

$$\sum_{i=1}^n \int_{-2\sigma}^{2\sigma} \frac{dh}{|h|^{1+2\theta}} \int_{Q(\sigma)} \|\tau_{i,h} D' u\|^2 dx \leqslant c(U, \sigma, \theta, \lambda)\{1 + F + |u|^2_{m,Q(3\sigma)}\}\,. \tag{3.23}$$

(3.20) and (3.21) follow from (3.23) using lemma 2.III.

A more general result is the following:

THEOREM 3.III. *If* $u \in H^{m+\theta} \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$, $0 \leqslant \theta < 1$, *is a solution*

(²) $|D' u|^2_{\theta,Q} = \sum_{|\alpha|=m} |D^\alpha u|^2_{\theta,Q}$:

(³) We note that this theorem is valid even if $f^\alpha \in L^1(\Omega)$, $|\alpha| < m$.

of system (3.1) *under the hypotheses* (1.9) (1.20) (1.21), *then*

$$u \in H^{m+\theta_1}_{\mathrm{loc}}(\Omega, R^N)\,, \qquad \forall \theta_1 < \frac{\lambda}{2} + \theta\left(1 - \frac{\lambda}{2}\right) \tag{3.24}$$

and for every cube $Q(3\sigma) = Q(x^0, 3\sigma) \subset\subset \Omega$

$$|D' u|^2_{\theta_1, Q(\sigma)} \leqslant c(U, \sigma)\{1 + F^{1+\theta} + |u|^2_{m,Q(3\sigma)} + |D' u|^2_{\theta, Q(3\sigma)}\}\,. \tag{3.25}$$

PROOF. Because of theorem 3.II we may assume $0 < \theta < 1$. As $\theta > 0$, by theorem 2.I we get

$$u \in H^{m,p}_{\mathrm{loc}}(\Omega, R^N)\,, \qquad \forall 2 < p < q = \frac{2(1+\theta)n}{n - 2\theta\lambda} \tag{3.26}$$

and for every cube $Q \subset\subset \Omega$

$$\sum_{|\alpha|=m} \int_Q \|D^\alpha u - (D^\alpha u)_Q\|^p\, dx \leqslant C(U)|D' u|^{p/(1+\theta)}_{\theta, Q}\,. \tag{3.27}$$

Now, choose $\psi \in C^\infty_0(\Omega)$ with

$$0 \leqslant \psi \leqslant 1\,, \qquad \psi = 1 \quad \text{in } Q(\sigma)\,, \qquad \psi = 0 \quad \text{in } \Omega \setminus Q(2\sigma)\,.$$

Let $|h| < \sigma/2$. From inequality (3.3) and lemma 3.I, in which we assume ε small enough and $p = 2(1+\theta)$, we conclude that

$$\sum_{i=1}^n \int_{Q(\sigma)} \|\tau_{i,h} D' u\|^2\, dx \leqslant c(U, \sigma)|h|^{2\theta + \lambda(1-\theta)}\{1 + F^{1+\theta} + |u|^{2(1+\theta)}_{m, 2(1+\theta), Q(3\sigma)}\} + \tag{3.28}$$
$$+ \sum_{i=1}^n \int_{Q(2\sigma)} \|D' u\|^2 \cdot \|\tau_{i,h} D'' u\|^2\, dx\,.$$

Having fixed p such that

$$2(1+\theta) < p < q = \frac{2(1+\theta)n}{n - 2\theta\lambda}$$

by Hölder's inequality we get

$$\int_{Q(2\sigma)} \|D' u\|^2 \cdot \|\tau_{i,h} D'' u\|^2\, dx \leqslant$$

$$\leqslant c(K)\, h^2 |u|^2_{m, Q(3\sigma)} + \int_{Q(2\sigma)} \|D' u\|^2 \cdot \sum_{|\alpha| = m-1} \|\tau_{i,h} D^\alpha u\|^2\, dx \leqslant$$

$$\leqslant c(K)\,h^2|u|^2_{m,Q(3\sigma)} + \Big(\int_{Q(2\sigma)}\|D'u\|^p\,dx\Big)^{2/p}\cdot\Big(\int_{Q(2\sigma)}\sum_{|\alpha|=m-1}\|\tau_{i,h}D^\alpha u\|^p dx\Big)^{2\theta/p}\cdot$$

$$\cdot\Big(\int_{Q(2\sigma)}\sum_{|\alpha|=m-1}\|\tau_{i,h}D^\alpha u\|^{2p(1-\theta)/[p-2(1+\theta)]}dx\Big)^{1-2(1+\theta)/p}.$$

Now, by lemma 2.II

$$\Big(\int_{Q(2\sigma)}\sum_{|\alpha|=m-1}\|\tau_{i,h}D^\alpha u\|^p\,dx\Big)^{2\theta/p}\leqslant c|h|^{2\theta}\,|u|^{2\theta}_{m,p,Q(3\sigma)}$$

and by the fact that $u\in C^{(m-1,\lambda)}(\bar{\Omega}, R^N)$

$$\Big(\int_{Q(2\sigma)}\sum_{|\alpha|=m-1}\|\tau_{i,h}D^\alpha u\|^{2p(1-\theta)/[p-2(1+\theta)]}dx\Big)^{1-2(1+\theta)/p}\leqslant c(U)|h|^{2\lambda(1-\theta)}.$$

Therefore

$$(3.29)\qquad \int_{Q(2\sigma)}\|D'u\|^2\cdot\|\tau_{i,h}D''u\|^2\,dx\leqslant c(U,\sigma)|h|^{2\theta+2\lambda(1-\theta)}\{1+|u|^{2(1+\theta)}_{m,p,Q(3\sigma)}\}\,.$$

From (3.28) and (3.29) we deduce that $\forall|h|<\sigma/2$

$$(3.30)\qquad \sum_{i=1}^{n}\int_{Q(\sigma)}\|\tau_{i,h}D'u\|^2\,dx\leqslant c(U,\sigma)|h|^{2\theta+\lambda(1-\theta)}\{1+F^{1+\theta}+|u|^{2(1+\theta)}_{m,p,Q(3\sigma)}\}\,.$$

Furthermore, from (3.27)

$$|u|^{2(1+\theta)}_{m,p,Q(3\sigma)}\leqslant c(U)\{|D'u|^2_{\theta,Q(3\sigma)}+\|(D'u)_{Q(3\sigma)}\|^{2(1+\theta)}\}\leqslant c(U,\sigma)\,\{|D'u|^2_{\theta,Q(3\sigma)}+|u|^{2(1+\theta)}_{m,Q(3\sigma)}\}.$$

Now, by lemma 1 of the appendix

$$|u|^{2(1+\theta)}_{m,Q(3\sigma)}\leqslant c(K)\,\{1+|D'u|^2_{\theta,Q(3\sigma)}\}\,.$$

Hence we conclude that $\forall|h|<\sigma/2$

$$(3.31)\qquad \sum_{i=1}^{n}\int_{Q(\sigma)}\|\tau_{ih}D'u\|^2\,dx\leqslant c(U,\sigma)|h|^{2\theta+\lambda(1-\theta)}\cdot$$

$$\cdot\{1+F^{1+\theta}+|u|^2_{m,Q(3\sigma)}+|D'u|^2_{\theta,Q(3\sigma)}\}\,.$$

The last inequality is trivial if $\sigma/2 \leqslant |h| < 2\sigma$ and so the proof finishes as in theorem 3.II.

By theorems 3.II and 3.III we prove, using an iteration argument, the following

THEOREM 3.IV. *If* $u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ *is a solution of system* (3.1) *under the hypotheses* (1.9) (1.20) (1.21), *then*

$$u \in H^{m+\theta}_{\mathrm{loc}}(\Omega, R^N)\,, \qquad \forall 0 < \theta < 1 \tag{3.32}$$

and $\forall Q(\sigma) \subset\subset Q(\sigma_0) \subset\subset \Omega$ [4]

$$|D' u|^2_{\theta, Q(\sigma)} \leqslant c(\sigma, \sigma_0 - \sigma, U)\{1 + F^{1+\theta} + |u|^2_{m, Q(\sigma_0)}\}\,. \tag{3.33}$$

In particular

$$u \in H^{m,4}_{\mathrm{loc}}(\Omega, R^N)\,. \tag{3.34}$$

PROOF. Let $Q(3\varrho) = Q(x^0, 3\varrho) \subset\subset \Omega$ and choose θ_0 such that $0 < \theta_0 < \lambda/2$. From theorem 3.II we conclude that

$$u \in H^{m+\theta_0}_{\mathrm{loc}} \cap C^{m-1,\lambda}(\overline{Q(\varrho)}, R^N)$$

and

$$|D' u|^2_{\theta_0, Q(\varrho)} \leqslant c(\varrho, U)\{1 + F + |u|^2_{m, Q(3\varrho)}\}\,. \tag{3.35}$$

Then, by theorem 3.III

$$u \in H^{m+\theta_1} \cap C^{m-1,\lambda}(\overline{Q(3^{-1}\varrho)}, R^N)$$

with

$$\theta_1 = \theta_0 + \theta_0(1 - \theta_0) < \theta_0 + \frac{\lambda}{2}(1 - \theta_0)$$

and

$$|D' u|^2_{\theta_1, Q(3^{-1}\varrho)} \leqslant c(\varrho, U)\{1 + F^{1+\theta_0} + |u|^2_{m, Q(3\varrho)}\}\,. \tag{3.36}$$

By induction we obtain that for every integer i

$$u \in H^{m+\theta_i}_{\mathrm{loc}} \cap C^{m-1,\lambda}(\overline{Q(3^{-i}\varrho)}, R^N)$$

[4] $Q(\sigma) = Q(x^0, \sigma)$, $Q(\sigma_0) = Q(x^0, \sigma_0)$.

with

$$\theta_i = \theta_0 \sum_{r=0}^{i} (1-\theta_0)^r \tag{3.37}$$

and

$$|D'u|^2_{\theta_i, Q(3^{-i}\varrho)} \leqslant c(\varrho, U)\{1 + F^{1+\theta_{i-1}} + |u|^2_{m,Q(3\varrho)}\} . \tag{3.38}$$

Now, from lemma 2.IV it follows that $\forall Q(\sigma) \subset\subset Q(\sigma_0) \subset\subset \Omega$

$$|D'u|^2_{\theta_i, Q(\sigma)} \leqslant c(\sigma, \sigma_0 - \sigma, U)\{1 + F^{1+\theta_{i-1}} + |u|^2_{m,Q(\sigma_0)}\} . \tag{3.39}$$

As $\{\theta_i\}$ is an increasing sequence that converges to 1, (3.32) and (3.33) are proved. Using theorem 2.I, (3.34) follows from (3.32) if we fix θ such that

$$\frac{n}{n+4\lambda} < \theta < 1 . \tag{3.40}$$

PROOF OF THEOREM 1.I. The previous results enable us to complete the proof of the differentiability theorem (theorem 1.I).

Let $Q(2\sigma) = Q(x^0, 2\sigma) \subset\subset \Omega$ and $\psi \in C_0^\infty(\Omega)$ with

$$0 \leqslant \psi \leqslant 1 , \quad \psi = 1 \quad \text{in } Q(\sigma) , \quad \psi = 0 \quad \text{in } \Omega \setminus Q(\tfrac{3}{2}\sigma) .$$

Let $|h| < \sigma/2$. From inequality (3.3) and lemma 3.I, in which we assume ε small enough and $p = 4$, we conclude that

$$\begin{aligned}\sum_{i=1}^{n} \int_{Q(\sigma)} \|\tau_{i,h} D'u\|^2 dx \leqslant c(U, \sigma) h^2\{1 + F^2 + |u|^4_{m,4,Q(2\sigma)}\} + \\ + c(K) \sum_{i=1}^{n} \int_{Q(\frac{3}{2}\sigma)} \|D'u\|^2 \cdot \|\tau_{i,h} D''u\|^2 dx .\end{aligned} \tag{3.41}$$

Now, using lemma 2.II

$$\begin{aligned}\int_{Q(\frac{3}{2}\sigma)} \|D'u\|^2 \cdot \|\tau_{i,h} D''u\|^2 dx \leqslant \Big(\int_{Q(\frac{3}{2}\sigma)} \|D'u\|^4 dx\Big)^{\frac{1}{2}} \cdot \Big(\int_{Q(\frac{3}{2}\sigma)} \|\tau_{i,h} D''u\|^4 dx\Big)^{\frac{1}{2}} \leqslant \\ \leqslant ch^2 \Big(\int_{Q(2\sigma)} \|D'u\|^4 dx\Big)^{\frac{1}{2}} \cdot \|u\|^2_{m,4,Q(2\sigma)} \leqslant h^2 c(K)\{1 + |u|^4_{m,4,Q(2\sigma)}\} .\end{aligned} \tag{3.42}$$

Then, for $|h| < \sigma$

$$\sum_{i=1}^{n} \int_{Q(\sigma)} \|\tau_{i,h} D'u\|^2 dx \leqslant c(U, \sigma) h^2\{1 + F^2 + |u|^4_{m,4,Q(2\sigma)}\} . \tag{3.43}$$

Therefore, from lemma 2.I we get that $u \in H^{m+1}(Q(\sigma), R^N)$ and

$$|u|^2_{m+1,Q(\sigma)} \leqslant c(U, \sigma)\{1 + F^2 + |u|^4_{m,4,Q(2\sigma)}\}. \tag{3.44}$$

The proof of theorem 1.I is complete.

REMARK 3.I. In order to prove (1.29) we have only to estimate the term $|u|^4_{m,4,Q(2\sigma)}$.

Let us suppose that $Q(2\sigma) \subset\subset \Omega$ and $0 < \delta < \frac{1}{2}(d(x^0) - 2\sigma)$. Having chosen θ such that

$$\max\left\{\sqrt{2} - 1, \frac{n}{n + 4\lambda}\right\} < \theta < 1 \tag{3.45}$$

we have

$$\frac{2(1+\theta)n}{n - 2\theta\lambda} > 4.$$

Then, by theorem 2.I,

$$\int_{Q(2\sigma)} \|D'u - (D'u)_{Q(2\sigma)}\|^4 dx \leqslant c(\sigma, U)|D'u|^{4/(1+\theta)}_{\theta,Q(2\sigma)}$$

and so

$$\int_{Q(2\sigma)} \|D'u\|^4 dx \leqslant c(\sigma, U)|D'u|^{4/(1+\theta)}_{\theta,Q(2\sigma)} + c(\sigma)|u|^4_{m,Q(2\sigma)}.$$

Now, by lemma 1 of the appendix

$$|u|^4_{m,Q(2\sigma)} \leqslant c(K)\{1 + |D'u|^{4/(1+\theta)}_{\theta,Q(2\sigma)}\}.$$

Thus, recalling (3.33)

$$|u|^4_{m,4,Q(2\sigma)} \leqslant c(\sigma, U)\{|D'u|^{4/(1+\theta)}_{\theta,Q(2\sigma)} + 1\} \leqslant c(\sigma, U)\{1 + F^2 + |u|^{4/(1+\theta)}_{m,Q(2\sigma+\delta)}\}. \tag{3.46}$$

Again, by lemma 1 of the appendix and (3.33)

$$|u|^{4/(1+\theta)}_{m,Q(2\sigma+\delta)} \leqslant c(K)\{1 + |D'u|^{4/(1+\theta)^2}_{\theta,Q(2\sigma+\delta)}\} \leqslant c(\sigma, U)\{1 + F^{2/(1+\theta)} + |u|^{4/(1+\theta)^2}_{m,Q(2(\sigma+\delta))}\}.$$

According to (3.45) we have $2/(1+\theta)^2 < 1$ and then

$$|u|^4_{m,4,Q(2\sigma)} \leqslant c(\sigma, U)\{1 + F^2 + |u|^2_{m,Q(2(\sigma+\delta))}\}. \tag{3.47}$$

Inequality (1.29) follows from (3.44) and (3.47).

REMARK 3.II. The functions f^α that appear in condition (1.20) could be assumed to satisfy the following more general hypothesis:

$$\mathcal{F} = \sum_{|\alpha|<m} D^\alpha f^\alpha \in H^{-m+1}(\Omega) \tag{3.48}$$

which means (see for instance [5], chap. I, n. 4) that $f^\alpha \in L^{q_\alpha}(\Omega)$ with

$$q_\alpha = \max\left\{1, \frac{2n}{n+2(m-1-|\alpha|)}\right\} \qquad \text{if } m-1-|\alpha| \neq \frac{n}{2},$$

$$q_\alpha \in (1, 2) \qquad \text{if } m-1-|\alpha| = \frac{n}{2}.$$

In this case, under the same hypotheses of lemma 3.I, we get instead of (3.18)

$$\left|\sum_{|\alpha|<m} \int_\Omega (a^\alpha(x, Du)|\tau_{i,-h} D^\alpha(\psi^{2m}\tau_{i,h}u))dx\right| \leqslant \varepsilon \int_\Omega \psi^{2m}\|\tau_{i,h}D'u\|^2 dx + \\ + c(\sigma, \Psi, U)|h|^{p-2+\lambda(2-p/2)}\{1 + \|\mathcal{F}\|^2 + |u|^p_{m,p,Q(2\sigma)}\}. \tag{3.49}$$

To do this we note that, if hypothesis (3.48) holds, then $\forall \varepsilon > 0$

$$\sum_{|\alpha|<m} \int_\Omega |f^\alpha| \cdot \|\tau_{i,-h} D^\alpha(\psi^{2m}\tau_{i,h}u)\| dx \leqslant \\ \leqslant c\|\mathcal{F}\| \cdot |\tau_{i,-h}(\psi^{2m}\tau_{i,h}u)|_{m-1,Q((7/4)\sigma)} \leqslant c|h| \cdot \|\mathcal{F}\| \cdot |\psi^{2m}\tau_{i,h}u|_{m,Q(2\sigma)} \leqslant \\ \leqslant c|h| \cdot \|\mathcal{F}\| \cdot \left(\int_\Omega \psi^{2m}\|\tau_{i,h}D'u\|^2 dx\right)^{\frac12} + c(\Psi)|h| \cdot \|\mathcal{F}\| \cdot \|\tau_{i,h}u\|_{m-1,Q(\frac{3}{2}\sigma)} \leqslant \\ \leqslant \varepsilon \int_\Omega \psi^{2m}\|\tau_{i,h}D'u\|^2 dx + c(\Psi, K)h^2\{1 + \|\mathcal{F}\|^2 + |u|^2_{m,Q(2\sigma)}\}.$$

Hence, inequality (3.25) becomes

$$|D'u|^2_{\theta_1,Q(\sigma)} \leqslant c(U, \sigma)\{1 + \|\mathcal{F}\|^2 + |u|^2_{m,Q(3\sigma)} + |D'u|^2_{\theta,Q(3\sigma)}\} \tag{3.50}$$

and inequality (3.33) becomes

$$|D'u|^2_{\theta,Q(\sigma)} \leqslant c(\sigma, \sigma_0 - \sigma, U)\{1 + \|\mathcal{F}\|^2 + |u|^2_{m,Q(\sigma_0)}\}, \qquad \forall 0 < \theta < 1. \tag{3.51}$$

No change is required in the proof of theorem 1.I or inequality (3.44), except from replacing F^2 by $\|\mathcal{F}\|^2$. It is known (see for instance [5]) that

$\|\mathcal{F}\|$ is equivalent to

$$\sum_{|\alpha|<m} \|f^{\alpha}\|_{0,a_\alpha,\Omega} \,.$$

Using (3.51) and repeating the argument of remark 3.I, we can estimate $|u|^4_{m,4,Q(2\sigma)}$ as follows

$$(3.52)\qquad |u|^4_{m,4,Q(2\sigma)} \leq c(\sigma, U)\{1 + \|\mathcal{F}\|^{4/(1+\theta)} + |u|^2_{m,Q(2(\sigma+\delta))}\}\,, \quad \forall 0<\theta<1\,.$$

4. – Partial Hölder continuity of the derivatives $D^\alpha u$, $|\alpha| = m$.

Let $u \in H^m \cap C^{m-1,\lambda}(\bar{\Omega}, R^N)$ be a solution of system

$$(4.1)\qquad \begin{gathered}\int_\Omega \sum_{|\alpha|\leq m} (a^\alpha(x, Du)|D^\alpha\varphi)\, dx = 0 \\ \forall\varphi \in H^m_0 \cap H^{m-1,\infty}(\Omega, R^N)\end{gathered}$$

under the hypotheses (1.9) (1.20) (1.21). We have shown that $u \in H^{m+1}_{\text{loc}}(\Omega, R^N)$ and then, by a standard calculation (see for instance [5], chap. V, n. 4), we obtain that for every open set $\Omega_0 \subset\subset \Omega$

$$u \in H^{m+1} \cap C^{m-1,\lambda}(\bar{\Omega}_0, R^N)$$

is a solution of the quasilinear system of order $2(m+1)$

$$(4.2)\qquad \int_{\Omega_0} \sum_{|\alpha|=|\beta|=m} \sum_{r,s=1}^{n} (B_{\alpha r,\beta s}(x, Du)\, D_s D^\beta u | D_r D^\alpha \varphi)\, dx =$$
$$= \int_\Omega \sum_{|\alpha|=m} \sum_{s=1}^{n} (G^{\alpha,s}(x, Du)| D_s D^\alpha\varphi) + \sum_{|\alpha|<m} \sum_{s=1}^{n} (a^\alpha(x, Du)|D_{ss} D^\alpha \varphi)\, dx$$
$$\forall \varphi \in C^\infty_0(\Omega_0, R^N)$$

where $B_{\alpha r,\beta s} = \{B^{hk}_{\alpha r,\beta s}\}$ are $N\times N$ matrices defined in $\Omega_0 \times \mathfrak{R}$ as follows

$$(4.3)\qquad B^{hk}_{\alpha r,\beta s} = \delta_{rs} \frac{\partial a^\alpha_h}{\partial p^\beta_k}$$

and $G^{\alpha,s}\colon \Omega_0\times\mathfrak{R} \to R^N$ are the following vectors

$$G^{\alpha,s}(x, Du) = -\frac{\partial a^\alpha(x, Du)}{\partial x_s} + \sum_{|\beta|<m} \sum_{k=1}^{N} D_s D^\beta u_k \frac{\partial a^\alpha(x, Du)}{\partial p^\beta_k}\,.$$

System (4.2) is strongly elliptic and, according to (1.21),

$$\|G^{\alpha,s}(x, Du)\| \leqslant c(K)\{1 + \|D'u\|^2\} \tag{4.4}$$

if $x \in \Omega_0$ and $\|D''u\| \leqslant K$.

Assume that

$$\frac{\partial a^\alpha(x, p)}{\partial p_k^\beta}, \qquad |\alpha| = |\beta| = m \text{ and } 1 \leqslant k \leqslant N$$

are uniformly continuous in $\bar{\Omega} \times \mathfrak{R}$ and that in condition (1.20) we have

$$f^\alpha \in L^p(\Omega) \qquad \text{with } p > n$$

or, more generally (see also remark 3.II)

$$\mathcal{F} = \sum_{|\alpha| < m} D^\alpha f^\alpha \in H^{-m+1,p}(\Omega) \qquad \text{with } p > n$$

which means (see for instance [5], chapter I, n. 4) that $f^\alpha \in L^{q_\alpha(p)}(\Omega)$ where

$$q_\alpha(p) = \max\left\{1, \frac{pn}{n + p(m - 1 - |\alpha|)}\right\} \qquad \text{if } m - 1 - |\alpha| \neq \frac{n}{p^*},\ \frac{1}{p} + \frac{1}{p^*} = 1$$

$$q_\alpha(p) \in (1, p) \qquad \text{if } m - 1 - |\alpha| = \frac{n}{p^*}.$$

Then system (4.2) satisfies the hypotheses of theorem 3.I, chap. IV of [5] ([5]), and from this theorem we get the partial Hölder continuity result contained in theorem 1.II of the present work.

Once we obtained the Hölder continuity of the $D^\alpha u$, $|\alpha| = m$, system (4.2) reduces to a linear system with smooth coefficients and right hand side. Therefore, the higher regularity of u is a consequence of the theory of linear systems.

Appendix.

In this appendix we prove theorem 2.I. The proof is completely analogous to that of theorem 3.III of [3], which deals with the case $\theta = 1$. We can therefore suppose $0 < \theta < 1$.

([5]) This theorem extends the result of [4] to systems of order $\geqslant 4$.

We note that (2.8) (2.9) are a trivial consequence of (2.7). To see that, if

$$1 \leqslant s < q = \frac{2(1+\theta)\,n}{n-2\theta\lambda}$$

then $\forall M > 0$

$$\int_Q \|D_i u - (D_i u)_Q\|^s\,dx = s\int_0^{+\infty} t^{s-1}\,\text{meas}\,\{x\in Q\colon \|D_i u(x) - (D_i u)_Q\| > t\}\,dt =$$

$$= s\int_0^M t^{s-1}\,\text{meas}\,\{x\in Q\colon \|D_i u(x) - (D_i u)_Q\| > t\}\,dt +$$

$$+ s\int_M^{+\infty} t^{s-1}\,\text{meas}\,\{x\in Q\colon \|D_i u(x) - (D_i u)_Q\| > t\}\,dt \leqslant$$

$$\leqslant (\text{meas}\,Q)\,M^s + \frac{s}{q-s}\,M^{s-q}A^q$$

where $A^q = c(n,\theta)|D_i\mu|_{\theta,Q}^{q/(1+\theta)}[u]_{\lambda,Q}^{q\theta/(1+\theta)}$.

Inequality (2.9) follows choosing $M = A(\text{meas}\ Q)^{-1/q}$.

As in [3], we derive (2.7) from some interpolation formulas. Let

$$Q(\sigma) = Q(x^0,\sigma) = \{x\colon |x_i - x_i^0| < \sigma\}\,.$$

LEMMA 1. *For every* $u \in H^{1+\theta}(Q(\sigma), R^N)$, $0<\theta<1$, *the following inequality holds*

$$(1)\qquad |u|_{1,Q(\sigma)} \leqslant C(n,\theta)\Big\{\Big(\sum_i |D_i u|^2_{\theta,Q(\sigma)}\Big)^{1/2(1+\theta)}\cdot \|u\|_{0,Q(\sigma)}^{\theta/(1+\theta)} + \sigma^{-1}\|u\|_{0,Q(\sigma)}\Big\}\,.$$

We give a proof for the reader's convenience.

Let $U\in H^{1+\theta}(R^n, R^N)$ be an extension of u such that (see for instance [1], chapters IV and VII)

$$(2)\qquad \begin{aligned} &\|U\|_{0,R^n} \leqslant c\|u\|_{0,Q(\sigma)}\,,\\ &\|U\|_{1,R^n} + \Big(\sum_i |D_i U|^2_{\theta,R^n}\Big)^{\frac12} \leqslant c\,\Big\{\|u\|_{1,Q(\sigma)} + \sum_i |D_i u|^2_{\theta,Q(\sigma)}\Big\}^{\frac12}\,.\end{aligned}$$

Via Fourier transform we get

$$(3)\qquad |U|_{1,R^n} \leqslant c(n,\theta)\,\Big(\sum_i |D_i U|^2_{\theta,R^n}\Big)^{1/2(1+\theta)}\cdot \|U\|_{0,R^n}^{\theta/(1+\theta)}\,.$$

(1) follows from (2) and (3). The way how constants depend on σ may be easily checked by a homothetical argument.

LEMMA 2. *For every* $u \in H^{1+\theta}(Q(\sigma), R^N)$, $0<\theta<1$, *the following inequality holds*

$$(4)\qquad \sum_{i=1}^{n} \|D_i u-(D_i u)_{Q(\sigma)}\|^2_{0,Q(\sigma)} \leqslant C(n,\theta)\Big(\sum_i |D_i u|^2_{\theta,Q(\sigma)}\Big)^{1/1+\theta)} \cdot \|u-u_{Q(\sigma)}\|^{2\theta/(1+\theta)}_{0,Q(\sigma)} .$$

PROOF. Let $\mathcal{T}_1$ be the class of all the polynomials in x with degree $\leqslant 1$ and let $P_0=\sum_i a_i x_i + a_0$ be the polynomial such that

$$\int_{Q(\sigma)} \|u-P_0\|^2 dx = \inf_{P\in\mathcal{T}_1} \int_{Q(\sigma)} \|u-P\|^2 dx .$$

Let us set, also

$$P_1(x)=\sum_{i=1}^{n}(D_i u)_{Q(\sigma)}(x_i-x_i^0)+u_{Q(\sigma)} .$$

According to (1), written for $(u-P_0)$,

$$(5)\qquad |u-P_1|_{1,Q(\sigma)} \leqslant |u-P_0|_{1,Q(\sigma)} \leqslant$$
$$\leqslant c(n,\theta)\Big\{\Big(\sum_i |D_i u|^2_{\theta,Q(\sigma)}\Big)^{1/2(1+\theta)} \cdot \|u-P_0\|^{\theta/(1+\theta)}_{0,Q(\sigma)} + \sigma^{-1}\|u-P_0\|_{0,Q(\sigma)}\Big\} .$$

Now, by Poincaré's inequality

$$\|u-P_0\|_{0,Q(\sigma)} \leqslant \|u-P_1\|_{0,Q(\sigma)} \leqslant c(n)\,\sigma|u-P_1|_{1,Q(\sigma)} \leqslant c(n)\,\sigma^{1+\theta}\Big(\sum_i |D_i u|^2_{\theta,Q(\sigma)}\Big)^{\frac12} .$$

Therefore

$$(6)\qquad \sigma^{-1}\|u-P_0\|_{0,Q(\sigma)} \leqslant c(n)\Big(\sum_i |D_i u|^2_{\theta,Q(\sigma)}\Big)^{1/2(1+\theta)} \cdot \|u-P_0\|^{\theta/(1+\theta)}_{0,Q(\sigma)} .$$

Furthermore

$$(7)\qquad \|u-P_0\|_{\theta,Q(\sigma)} \leqslant \inf_{c\in R}\|u-c\|_{0,Q(\sigma)} = \|u-u_{Q(\sigma)}\|_{0,Q(\sigma)} .$$

Thus, (4) follows from (5) (6) (7).

We recall also the following lemma, due to John-Nirenberg [6]:

LEMMA 3. *Let Q be a cube of R^n and let $Q=\bigcup_k Q_k$ be a subdivision of Q into a denumerable number of cubes Q_k, no two having a common interior point.*

Let u be integrable in Q and assume that for fixed p, $1 < p < +\infty$,

$$K(u) = \left\{ \sup_{\bigcup_k Q_k = Qk} \sum_k (\operatorname{meas} Q_k)^{1-p} \left(\int_{Q_k} \|u - u_{Q_k}\| \, dx \right)^p \right\}^{1/p} < +\infty . \tag{8}$$

Then $\forall t > 0$

$$\operatorname{meas}\{x \in Q \colon \|u(x) - u_Q\| > t\} \leq c(n, p) \left(\frac{K_p(u)}{t} \right)^p . \tag{9}$$

We can now give the proof of theorem 2.I.

Let $Q = \bigcup_k Q_k$ be a subdivision of Q into cubes Q_k, no two having a common interior point. Let $u \in H^{1+\theta} \cap C^{0,\lambda}(\bar{Q}, R^N)$, $0 < \theta < 1$ and $0 < \lambda \leq 1$; let finally $q = 2(1 + \theta) n/(n - 2\theta\lambda)$.

From inequality (4) we conclude that

$$\int_{Q_k} \|D_j u - (D_j u)_{Q_k}\| dx \leq c(n, \theta) (\operatorname{meas} Q_k)^{\frac{1}{2}} \left(\sum_i |D_i u|^2_{0,Q_k} \right)^{1/2(1+\theta)} \cdot \|u - u_{Q_k}\|^{\theta/(1+\theta)}_{0,Q_k} . \tag{10}$$

As

$$\|u - u_{Q_k}\|^{\theta/(1+\theta)}_{0,Q_k} \leq (\text{means}\, Q_k)^{(\theta/(1+\theta))(\frac{1}{2} + \lambda/n)} [u]^{\theta/(1+\theta)}_{\lambda,\bar{Q}}$$

from (10) we get

$$(\operatorname{meas} Q_k)^{1-q} \left(\int_{Q_k} \|D_j u - (D_j u)_{Q_k}\| \, dx \right)^q \leq c^q(n, \theta) \left(\sum_i |D_i u|^2_{\theta,Q_k} \right)^{q/2(1+\theta)} \cdot [u]^{q\theta/1(+\theta)}_{\lambda,\bar{Q}} \leq$$

$$\leq c^q(n, \theta) \left(\sum_i |D_i u|^2_{\theta,Q} \right)^{q/2(1+\theta) - 2} \cdot [u]^{q\theta/(1+\theta)}_{\lambda,\bar{Q}} \cdot \sum_i |D_i u|^2_{\theta,Q_k} .$$

Then

$$K_q(u) \leq c(n, \theta) \left(\sum_i |D_i u|^2_{\theta,Q} \right)^{q/2(1+\theta)} \cdot [u]^{q\theta/(1+\theta)}_{\lambda,\bar{Q}} . \tag{11}$$

The result (2.7) of theorem 2.I follows from (11) by virtue of the John-Nirenberg lemma (lemma 3).

BIBLIOGRAPHY

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York (1975).

[2] S. Campanato, *Sulla regolarità delle soluzioni di equazioni differenziali di tipo ellittico*, Editrice Tecnico Scientifica, Pisa (1963).

[3] S. CAMPANATO, *Maggiorazioni interpolatorie negli spazi* $H_\lambda^{m,p}(\Omega)$, Ann. Mat. Pura Appl., **75** (1967), pp. 261-276.

[4] S. CAMPANATO, *Partial Hölder continuity of the gradient of solutions of some non-linear elliptic systems*, Rend. Sem. Mat. Univ. Padova, **59** (1978), pp. 147-165.

[5] S. CAMPANATO, *Sistemi ellittici in forma di divergenza. Regolarità all'interno*, Quaderni della Scuola Norm. Sup. di Pisa (1980).

[6] M. GIAQUINTA - G. MODICA, *Almost-everywhere regularity results for solutions of non linear elliptic systems*, Manuscripta Math., **28** (1979), pp. 109-158.

[7] F. JOHN - L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), pp. 415-426.

[8] O. LADYZHENSKAYA - N. URAL'TSEVA, *Linear and quasilinear elliptic eqnations*, Academic Press, New York and London (1968) (translation).

[9] C. B. MORREY Jr., *Multiple integrals in the calculus of variations*, Springer-Verlag New York Inc. (1966).

[10] C. B. MORREY Jr., *Partial regularity results for non-linear elliptic systems*, Journal of Math. and Mech., **17**, 7 (1968), pp. 649-670.

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