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Global Existence for the Hamilton-Jacobi Equations in Hilbert Space.

V. BARBU - G. DA PRATO

1. - Introduction.

In this paper we are concerned with the Hamilton-Jacobi equation

$$(1.1) \quad \begin{cases} \varphi_t(t, x) + F(\varphi_x(t, x)) + (Ax, \varphi_x(t, x)) = g(t, x); & x \in D(A), t \in [0, T] \\ \varphi(0, x) = \varphi_0(x) \end{cases}$$

in a Hilbert space H . Here F is a convex Fréchet differentiable function on H and $-A$ is the infinitesimal generator of a strongly continuous semigroup of linear continuous operators on H . The subscripts t and x denote the partial differentiation with respect to t and x and g, φ_0 are given real valued functions on $[0, T] \times H$ and H , respectively.

The contents of this paper are outlined below.

In section 2 we shall exhibit several properties of the operator $\varphi \rightarrow F(\varphi_x)$. In particular it is shown that it arises as the generator of a semigroup of contractions on an appropriate subset K of the space $C(H)$ defined below.

In section 3 it is studied equation (1.1) with $A \equiv 0$. An explicit form of the solution in term of the semigroup $S(t)$ is given for the homogeneous equation and it is proved the existence and uniqueness of a weak solution in the class of continuous convex functions φ satisfying the conditions

$$(F'(\partial\varphi(x)), x) \geq 0, \quad x \in H; \quad 0 \in \partial\varphi(0).$$

Section 4 is concerned with equation

$$(1.2) \quad \begin{cases} \varphi_t(t, x) + \frac{1}{2}|\varphi_x(t, x)|^2 + (Ax, \varphi_x(t, x)) = g(t, x), & t \in [0, T]; x \in D(A) \\ \varphi(0, x) = \varphi_0(x). \end{cases}$$

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The main results of this section, Theorems 3 and 4 give existence and uniqueness of weak and classical solutions in the class of continuous convex functions on H . In particular, some existence results for the operator equation

$$(1.3) \quad \begin{cases} E_t + E_x A + A^* E + E_x E = G, & t \in [0, T] \\ E(0) = E_0 \end{cases}$$

are derived. Particular cases of equation (1.3) have been previously studied in [4], [5]. The situation in which G is a linear continuous self-adjoint operator on H (the Riccati equation) has been extensively studied in the past decade and we refer the reader to [12] and [15] for significant results and complete references.

In section 5 the relevance of eq. (1.1) in control theory and calculus of variations is explained. In section 6 we give some regularity properties for equation (1.3).

As far as we know, the present paper is the first attempt to study the Hamilton-Jacobi equations in Hilbert spaces. As regards the study of these equations in $\mathbb{R}^n$ the fundamental works of Kruzkov [16], Douglis [13], Fleming [14] must be cited. In [10] Crandall proposed a new method in the study of hyperbolic conservation laws equations based on the theory of nonlinear semigroup of contractions in Banach spaces (see also [6], [11]). The semigroup approach has been subsequently used in the study of Hamilton Jacobi equations in $\mathbb{R}^n$ by Aizawa [1], Burch [7], Burch and Goldstein [8] and other authors.

We conclude this section by listing briefly some definitions and notations that will be in effect throughout this paper. Let H be a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$.

Given a lower semicontinuous convex function $\varphi: H \rightarrow \mathbb{R} = \mathbb{R}^1 \cup +\{\infty\}$ we shall denote by $\partial\varphi: H \rightarrow H$ the subdifferential of φ , i.e.,

$$(1.4) \quad \partial\varphi(x) = \{x^* \in H; \varphi(x) \leq \varphi(y) + (x^*, x - y), \text{ for all } y \in H\}$$

and by φ^* the conjugate of φ ,

$$\varphi^*(y) = \sup \{(x, y) - \varphi(x); x \in H\}.$$

If φ is Fréchet (or more generally Gâteaux) differentiable at x then $\partial\varphi(x)$ consists of a single element, namely the gradient of φ . In the sequel we shall use either the symbol φ' or φ_x for the gradient of φ instead of the more conventional symbol $\nabla\varphi$.

For each $R > 0$ we shall denote by Σ_R the closed ball

$$\Sigma_R = \{x \in H; |x| \leq R\}.$$

$C(\Sigma_R)$ will denote the Banach space of all continuous and bounded functions $\varphi: \Sigma_R \rightarrow \mathbb{R}^1 =]-\infty, \infty[$ endowed with the norm

$$(1.5) \quad |\varphi|_R = \sup \{|\varphi(x)|; x \in \Sigma_R\}.$$

Let $C(H)$ be the space of all continuous functions $\varphi: H \rightarrow \mathbb{R}^1$ which are bounded on bounded subsets, topologized with the family of seminorms $\{|\varphi|_R; R > 0\}$. By $C^1(H)$ we shall denote the space of all Fréchet differentiable functions φ on H with Fréchet differential φ_x continuous, bounded on every bounded subset of H and with $\varphi_x(0) = 0$. $C^1(H)$ is a locally convex space endowed with the family of seminorms

$$(1.6) \quad |\varphi|_{1,R} = \sup \{|\varphi_x(x)|; x \in \Sigma_R\}.$$

By $\text{Lip}(H)$ we shall denote the space of all functions $\varphi: H \rightarrow \mathbb{R}^1$ such that

$$(1.7) \quad |\varphi|_{\text{Lip},R} = \sup \left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|}; x \neq y; x, y \in \Sigma_R \right\}, \quad \forall R > 0.$$

Further, we shall denote by $C^k(H, H)$, k a natural number or zero, the space of all continuously k times differentiable mappings $f: H \rightarrow H$ such that $f^{(j)}(0) = 0$ for $j = 0, 1 \dots k-1$ and

$$(1.8) \quad \|f\|_{k,R} = \sup \{ \|f^{(k)}(x)\|_{L(H,H)}; x \in \Sigma_R \}, \quad \forall R > 0.$$

Here $f^{(j)}$ denotes the Fréchet differential of order j of f and $\|\cdot\|_{L(H,H)}$ the norm in the space $L(H, H)$ of linear continuous operators from H into itself.

We shall denote by $C_{\text{Lip}}^0(H, H)$ the space of all continuous mappings $f: H \rightarrow H$ which are Lipschitzian on every bounded subset, endowed with the family of seminorms

$$(1.9) \quad \|f\|_{0,R} = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|}; x, y \in \Sigma_R, x \neq y \right\}.$$

By $C_{\text{Lip}}^k(H, H)$, where k is a natural number, we shall denote the space of all $f \in C^k(H, H)$ such that

$$(1.10) \quad \|f\|_{k,R} = \sup \left\{ \frac{\|f^{(k)}(x) - f^{(k)}(y)\|_{L(H,H)}}{|x - y|}; x, y \in \Sigma_R, x \neq y \right\}, \quad \forall R > 0.$$

Given a Banach space Z we shall denote by $C([0, T]; Z)$ the space of all continuous functions from $[0, T]$ to Z . If Z is one of the spaces $C(H)$, $C^1(H)$, $C^k(H, H)$ or $C_{\text{Lip}}^k(H, H)$ we set

$$(1.11) \quad C([0, T]; Z) = \left\{ f \in C([0, T] \times H); f(t, \cdot) \in Z \text{ for } t \in [0, T] \right. \\ \left. \text{and } |f|_{Z, T} = \sup_{t \in [0, T]} |f(t, \cdot)|_{Z, R} < \infty \right\}$$

(here $|\cdot|_{Z, R}$ is one of the seminorms (1.5), (1.6), (1.7), (1.8), (1.9) or (1.10)) and

$$(1.12) \quad C^1([0, T]; Z) = \{ \varphi \in C([0, T]; Z); \varphi_t \in C([0, T]; Z) \}.$$

By $L^1(0, T; C(H))$ we shall denote the space of functions $f: [0, T] \rightarrow C(H)$ having the property that $f(t, x) \in L^1(0, T)$ for every $x \in H$ and

$$(1.13) \quad \sup \left\{ \int_0^T |f(t, x)| dt; x \in \Sigma_R \right\} < \infty$$

for every $R > 0$.

2. - Assumptions and auxiliary results.

To begin with let us set forth the assumptions which will be in effect throughout this paper.

(a) H is a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$.

(b) The function F is convex and belongs to $C^1(H)$. $F' \in C_{\text{Lip}}^0(H, H)$ and

$$(2.1) \quad \lim_{|x| \rightarrow \infty} F(x)/|x| = \infty.$$

(c) For every $R > 0$ there exists $\omega_R > 0$ such that

$$(2.2) \quad (F'(x) - F'(y), x - y) \geq \omega_R |x - y|^2 \quad \forall x, y \in \Sigma_R.$$

(d) The linear operator $-A$ is the infinitesimal generator of a strongly continuous semigroup of contractions $\exp(-At)$ on H . By A^* we shall denote the dual operator and by $D(A)$ the domain of A endowed with the graph norm.

We shall denote by K the set of all convex functions $\varphi \in C(H)$ satisfying the following two conditions

$$(2.3) \quad (F'(y), x) \geq 0 \quad \forall [x, y] \in \partial\varphi; \quad 0 \in \partial\varphi(0).$$

LEMMA 1. *Let F satisfy assumptions (b) and (c) and let $\varphi \in K$ be a given function. Then for every $t > 0$ the equation*

$$(2.4) \quad \partial\varphi(y) - (F')^{-1}(t^{-1}(x - y)) \ni 0$$

has a unique solution $y = y_t(x)$ satisfying

$$(2.5) \quad |y_t(x)| \leq |x|.$$

Moreover, for each $t > 0$ the mapping $x \rightarrow y_t(x)$ belongs to $C_{\text{Lip}}^0(H, H)$.

PROOF. According to a well-known perturbation result due to Browder (see e.g. [2], p. 46) the operator $Fy = \partial\varphi(y) - (F')^{-1}(t^{-1}(x - y))$ is maximal monotone on H . Since $F' \in C^0(H, H)$, $(F')^{-1}$ is coercive and therefore F is onto H . Hence eq. (2.4) has for each $t > 0$ at least one solution $y = y_t(x)$. Writing (2.4) as

$$y + tF'(\partial\varphi(y)) \ni x$$

and using condition (2.3) it follows (2.5). The uniqueness of y as well as the Lipschitzian dependence of $y(x)$ with respect to x follows by assumption (c).

In particular if $\varphi \in C^1(H)$ then by (2.4) it follows that

$$(2.6) \quad \varphi'(y_t(x)) = (F')^{-1}(t^{-1}(x - y_t(x))).$$

For every $\varphi \in K$ and $t \geq 0$, define

$$(2.7) \quad (S(t)\varphi)(x) = (\varphi^* + tF^*)(x).$$

Since $\lim_{|x| \rightarrow \infty} (\varphi^*(x) + tF(x))/|x| = +\infty$, we may infer that for each $\varphi \in K$ and $t > 0$, $S(t)\varphi$ is a continuous convex function on H as well. Moreover, by Fenchel's duality theorem (see e.g. [3], p. 188), $S(t)$ can be equivalently defined for $t > 0$, as

$$(2.8) \quad \begin{aligned} (S(t)\varphi)(x) &= \inf \{ \varphi(y) + tF^*(t^{-1}(x - y)); y \in H \} = \\ &= \varphi(y_t(x)) + tF^*(t^{-1}(x - y_t(x))), \quad x \in H. \end{aligned}$$

It turns out that $\{S(t); t \geq 0\}$ is a semigroup of contractions on $C(H)$. More precisely, one has

LEMMA 2. *Let $\{S(t); t \geq 0\}$ be the family of nonlinear operators on $C(H)$ defined by formula (2.7). Then*

$$(2.9) \quad S(t)K \subset K \quad \text{for all } t \geq 0$$

$$(2.10) \quad S(t+s)\varphi = S(t)S(s)\varphi \quad \text{for all } t, s \geq 0, \varphi \in K$$

$$(2.11) \quad |S(t)\varphi - S(t)\psi|_R \leq |\varphi - \psi|_R \quad \text{for all } \varphi, \psi \in K, t \geq 0 \text{ and } R > 0.$$

Moreover, for every $t > 0$, $S(t)$ maps K into $C^1(H)$ and

$$(2.12) \quad \lim_{t \downarrow 0} t^{-1}(\varphi - S(t)\varphi) = F(\varphi') \quad \text{in } C(H)$$

for each $\varphi \in K \cap C^1(H)$ such that φ' is uniformly continuous on every bounded subset of H .

PROOF. Let φ be fixed in K . As observed earlier, $S(t)\varphi$ is a continuous real valued convex function on H . By Lemma 1 and formula (2.8) it follows that $S(t)\varphi$ is bounded on every bounded subset of H (because by assumption (2.1) F^* is bounded on bounded subsets). Moreover, by (2.7) it follows that (see e.g. [3], p. 100)

$$(2.13) \quad \partial(S(t)\varphi)(x) = (\partial\varphi^* + tF')^{-1}(x), \quad \forall x \in H, t > 0.$$

Since F' is strictly monotone on H , for each $t > 0$, the map $(\partial\varphi^* + tF')^{-1}$ is single valued and Lipschitzian on every bounded subset as well. Hence $S(t)\varphi \in C^1(H)$ for all $t > 0$ and by (2.6), (2.13) reduces to

$$(2.14) \quad \begin{aligned} (S(t)\varphi)_x(x) &= (\partial\varphi^* + tF')^{-1}(x) = \\ &= (F')^{-1}(t^{-1}(x - y_t(x))) ; \quad \forall x \in H, t > 0. \end{aligned}$$

Hence

$$(F'(S(t)\varphi)_x(x), x) = t^{-1}(x - y_t(x), x) \geq 0 ; \quad x \in H, t > 0$$

and again by Lemma 1 it follows that $(S(t)\varphi)_x(0) = 0$. Thus we have shown that $S(t)\varphi \in K$ for every $t > 0$.

Let φ, ψ be two elements of K . Since in virtue of Lemma 1, $y_t(x) \in \Sigma_R$

whenever $|x| \leq R$, it follows by (2.8),

$$\begin{aligned} (S(t)\varphi)(x) - (S(t)\varphi)(x) &\leq \\ &\leq \psi(y_t(x)) - \varphi(y_t(x)) \leq |\psi - \varphi|_R \quad \text{for all } x \in \Sigma_R \text{ and } t > 0. \end{aligned}$$

(Here $y_t(x)$ is defined by Lemma 1). The latter implies (2.11).

Let us now prove the semigroup property (2.10). Using again the Fenchel theorem we have by (2.7) and (2.8)

$$\begin{aligned} (S(t+s)\varphi)(x) &= \inf \{(\varphi^* + sF)^*(y) + tF^*(t^{-1}(x-y)); y \in H\} = \\ &= \inf \{(S(s)\varphi)(y) + tF^*(t^{-1}(x-y)); y \in H\} = \\ &= (S(t)S(s)\varphi)(x). \end{aligned}$$

It remains to prove equality (2.12). To this end we fix $\varphi \in K \cap C^1(H)$ and observe that by (2.8) we have

$$\begin{aligned} (2.15) \quad \varphi(x) - (S(t)\varphi)(x) &= \\ &= \varphi(x) - \varphi(y_t(x)) - tF^*(t^{-1}(x - y_t(x))) \leq (\varphi'(x), x - y_t(x)) - tF^*(t^{-1}(x - y_t(x))). \end{aligned}$$

Along with the well known conjugacy formula, (see e.g. [3], p. 91)

$$(2.16) \quad F^*(F'(y)) + F(y) = (y, F'(y)), \quad \forall y \in H,$$

relations (2.6) and (2.4) lead to

$$(2.17) \quad t^{-1}(\varphi(x) - (S(t)\varphi)(x)) \leq (\varphi'(x) - \varphi'(y_t(x)), F'(\varphi'(y_t(x))) + F(\varphi'(y_t(x))).$$

On the other hand, since F is convex, one has the inequality

$$F'(\varphi'(y_t(x))) \leq F(\varphi'(x)) + (F'(\varphi'(y_t(x))), \varphi'(y_t(x)) - \varphi'(x))$$

which along with (2.17) yields

$$(2.18) \quad t^{-1}(\varphi(x) - (S(t)\varphi)(x)) - F(\varphi'(x)) \leq 0.$$

Similarly, by (2.8) and (2.16) it follows that

$$\begin{aligned} (S(t)\varphi)(x) - \varphi(x) &\leq (\varphi'(y_t(x)), y_t(x) - x) + tF^*(t^{-1}(x - y_t(x))) = \\ &= -t(\varphi'(y_t(x)), F'(\varphi'(y_t(x)))) + tF^*(F'(\varphi'(y_t(x)))) = -tF'(\varphi'(y_t(x))). \end{aligned}$$

Hence

$$(2.19) \quad t^{-1} \left(\varphi(x) - (S(t)\varphi)(x) \right) - F(\varphi'(y_t(x))) \geq 0.$$

By (2.18) and (2.19) we see that

$$(2.20) \quad 0 \leq F(\varphi'(x)) - t^{-1} \left(\varphi(x) - (S(t)\varphi)(x) \right) \leq F(\varphi'(x)) - F(\varphi'(y_t(x))) \leq \\ \leq (F'(\varphi'(x)), \varphi'(x) - \varphi'(y_t(x))) ; \quad t > 0, x \in H.$$

Since φ' and F' are bounded on every Σ_R it follows by (2.6) that

$$|x - y_t(x)| \leq C_R t \quad \forall x \in \Sigma_R, t > 0.$$

Inasmuch as φ' is uniformly continuous on every Σ_R , by (2.20) we deduce (2.12) as claimed.

REMARK. Let L_0 be the operator defined in $C(H)$ by

$$(2.21) \quad L_0 \varphi = F(\varphi_x) \quad \forall \varphi \in D(L_0)$$

where $D(L_0)$ consists of the set of all $\varphi \in C^1(H) \cap K$ such that φ' is uniformly continuous on every bounded subset of H .

By (2.11) and (2.12) it follows that L_0 is accretive in $C(H)$ i.e.,

$$(2.22) \quad |\varphi - \psi + \lambda(L_0 \varphi - L_0 \psi)|_R \geq |\varphi - \psi|_R, \quad \forall R > 0; \lambda > 0$$

for every pair $(\varphi, \psi) \in D(L_0) \times D(L_0)$.

On the other hand, (2.21) implies that $L_0 \subset L_1$ where L_1 is the infinitesimal generator of $S(t)$.

3. - Equation (1.1) with $A \equiv 0$.

We begin with the homogeneous Cauchy problem

$$(3.1) \quad \begin{cases} \varphi_t(t, x) + F(\varphi_x(t, x)) = 0 ; & t \geq 0, x \in H \\ \varphi(0, x) = \varphi_0(x), & x \in H \end{cases}$$

where F satisfies assumptions (b), (c) and $\varphi_0 \in \overset{\circ}{K}$.

Define the function $\varphi: R^+ \times H \rightarrow R^1$,

$$(3.2) \quad \varphi(t, x) = (S(t)\varphi_0)(x), \quad t \geq 0, x \in H$$

where $S(t)$ is the semigroup defined by formula (2.7).

By Lemma 1 it follows that $\varphi(t, \cdot) \in C^1(H)$ for every $t > 0$ and $\varphi(t, \cdot) \in K$ for every $t \geq 0$. Furthermore, it follows by (2.10) and (2.12) that for each $\varphi_0 \in K$, the right derivative $(d^+/dt)S(t)\varphi_0$ exists at every $t > 0$ and

$$(3.3) \quad \frac{d^+}{dt} S(t)\varphi_0 + F((S(t)\varphi_0)_x) = 0, \quad \forall t > 0.$$

We have also used the fact that for each $\varphi_0 \in K$, $(S(t)\varphi_0)_x$ is uniformly continuous on bounded subsets (see (2.14)). In (3.3) d^+/dt is taken in the sense of topology of $C(H)$. If $\varphi_0 \in C^1(H)$ then eq. (3.3) remains valid for $t = 0$. In particular, it follows by (3.3) that φ is a strong solution to eq. (3.1) in the sense that

$$(3.4) \quad \frac{d^+}{dt} \varphi(t, x) + F(\varphi_x(t, x)) = 0 \quad \text{for every } t > 0, x \in H.$$

It is worth noting also that by (2.14) and Lemma 1 it follows that $\varphi(t, \cdot) \in C^1(H)$ for every $t > 0$ and

$$(3.5) \quad \sup \{|\varphi(t, \cdot)|_{1,R}; t \in [\delta, T]\} \leq C_{\delta,R} \quad \text{for every } \delta \in]0, T[\text{ and } R > 0.$$

If $\varphi_0 \in C^1(H)$ then by (2.6) and (2.14) it follows that $\varphi \in C([0, T]; C^1(H))$, i.e.,

$$(3.6) \quad \sup \{|\varphi(t, \cdot)|_{1,R}; t \in [0, T]\} \leq C_R \quad \forall R > 0.$$

On the other hand the accretivity of the operator $L_0(\varphi) = F(\varphi_x)$ on $C(H)$ (see (2.22)) implies via a standard argument the uniqueness of the strong solution φ .

Summarising, we get

THEOREM 1. *Let F satisfy assumptions (b), (c). Then for each $\varphi_0 \in K$, the Cauchy problem (3.1) has a unique strong solution given by formula (3.2). More precisely, $\varphi(t, \cdot) \in C^1(H) \cap K$ for all $t > 0$, satisfies (3.5) and as a function of t from $]0, +\infty[$, $\varphi(t)$ is continuous, everywhere differentiable from the right and satisfies eq. (3.4). Moreover the map $\varphi_0 \rightarrow \varphi$ is a contraction from $C(H)$ to $C([0, T]; C(H))$.*

If in addition $\varphi_0 \in C^1(H)$ then $\varphi \in C([0, T]; C^1(H))$ and equation (3.4) is satisfied for all $t \geq 0$.

REMARK. We have incidentally shown that the semigroup $S(t)$ has smoothing effect on initial data (see [3], [9] for other classes of contraction semigroups having this property).

We shall consider now the nonhomogeneous Cauchy problem

$$(3.7) \quad \begin{cases} \varphi_t(t, x) + F(\varphi_x(t, x)) = g(t, x); & t \in [0, T], x \in H \\ \varphi(0, x) = \varphi_0(x) \end{cases}$$

where F satisfies assumptions (b) and (c).

One assumes in addition that

(e) K is a closed convex cone of $C(H)$.

Further we shall assume that

$$(3.8) \quad \varphi_0 \in C^1(H) \cap K, \quad g \in C([0, T]; C^1(H)) \cap \mathcal{K}$$

where $\mathcal{K}$ is the closed convex cone of $C([0, T]; C(H))$ defined by

$$(3.9) \quad \mathcal{K} = \{\varphi \in C([0, T]; C(H)); \varphi(t) \in K, \forall t \in [0, T]\}.$$

Consider the approximating equation

$$(3.10) \quad \begin{cases} \varphi_t(t, x) + \varepsilon^{-1}(\varphi(t, x) - (S(\varepsilon)\varphi)(t, x)) = g(t, x) \\ \varphi(0, x) = \varphi_0(x), \quad t \in [0, T], x \in H, \varepsilon > 0 \end{cases}$$

or equivalently

$$(3.11) \quad \varphi(t, x) = \exp(\varepsilon^{-1}t)\varphi_0(x) + \int_0^t \exp(\varepsilon^{-1}(t-s))g(s, x)ds + \\ + \varepsilon^{-1} \int_0^t \exp(\varepsilon^{-1}(t-s))(S(\varepsilon)\varphi)(s, x)ds, \quad t \in [0, T].$$

By assumptions (e), (3.8) and by Lemma 2 it follows that the operator defined by the right hand side of eq. (3.11) maps every $C([0, T]; C(\Sigma_R))$ into itself and is contractant. Thus for every $\varepsilon > 0$, eq. (3.11) ((3.10)) has a unique solution $\varphi_\varepsilon \in \mathcal{K} \cap C^1([0, T]; C(H))$. Since, as proved earlier, $S(\varepsilon)\varphi \in C^1(H)$ for every $\varphi \in K$ and $\varepsilon > 0$ it follows by (3.11) that $\varphi_\varepsilon \in C([0, T]; C^1(H))$. Furthermore, recalling that (see (2.6) and (2.14)),

$$(3.12) \quad (S(\varepsilon)\varphi)_x(x) = \varphi'(y_\varepsilon(x)), \quad x \in H, \varepsilon > 0$$

we see that $(\varphi_\varepsilon)_x = \varphi'_\varepsilon$ is the solution to

$$(3.13) \quad \frac{d}{dt} \varphi'_\varepsilon(t, x) + \varepsilon^{-1} (\varphi'_\varepsilon(t, x) - \varphi'_\varepsilon(t, y_\varepsilon(t, x))) = g_x(t, x)$$

where

$$(3.14) \quad y_\varepsilon(t, x) + F'(\varphi'_\varepsilon(t, y_\varepsilon(t, x))) = x; \quad x \in H, \quad t \in [0, T].$$

Then by an easy computation involving eq. (3.13) and the Gronwall lemma it follows that

$$(3.15) \quad |\varphi_\varepsilon(t)|_{1,R} \leq C_R (|\varphi_0|_{1,R} + \sup_{0 \leq t \leq T} |g(t)|_{1,R}), \quad t \in [0, T].$$

(By C_R we shall denote several positive constants independent of ε .) Parenthetically we notice that since by (3.14) and assumption (e) the mapping $x \rightarrow \varphi'_\varepsilon(x, y_\varepsilon(t, x))$ is Lipschitzian on every Σ_R , it follows by (3.13) that if $\varphi'_0 \in C^0_{\text{Lip}}(H, H)$ and $g_x \in C([0, T]; C^0_{\text{Lip}}(H, H))$ then

$$(3.16) \quad (\varphi_\varepsilon)_x \in C([0, T]; C^0_{\text{Lip}}(H, H)).$$

Next by (2.20), (3.12) and (3.13) one has

$$\begin{aligned} |\varepsilon^{-1} (\varphi_\varepsilon(t, x) - (S(\varepsilon)\varphi_\varepsilon)(t, x)) - F(\varphi'_\varepsilon(t, x))| &\leq \\ &\leq (F'(\varphi'_\varepsilon(t, x)), \varphi'_\varepsilon(t, x) - \varphi'_\varepsilon(t, y_\varepsilon(t, x))) = \\ &= -\varepsilon \left(F'(\varphi'_\varepsilon(t, x)), \frac{d}{dt} \varphi'_\varepsilon(t, x) \right) + \varepsilon (F'(\varphi'_\varepsilon(t, x)), g_x(t, x)) = \\ &= \varepsilon (F'(\varphi'_\varepsilon(t, x)), g_x(t, x)) - \varepsilon \frac{d}{dt} F(\varphi'_\varepsilon(t, x)). \end{aligned}$$

Integrating the latter over $]0, T[$, we get by (3.15) the estimate

$$\begin{aligned} (3.17) \quad &\int_0^T |\varepsilon^{-1} (\varphi_\varepsilon(t, x) - (S(\varepsilon)\varphi_\varepsilon)(t, x)) - F(\varphi'_\varepsilon(t, x))| dt \leq \\ &\leq \varepsilon \int_0^T |F'(\varphi'_\varepsilon(t, x))| |g_x(t, x)| dt + \varepsilon (F(\varphi'_0(x)) - F(\varphi'_\varepsilon(T, x))) \leq C_R \varepsilon \quad \forall x \in \Sigma_R \end{aligned}$$

and therefore by (3.10)

$$(3.18) \quad \begin{cases} (\varphi_\varepsilon)_t(t, x) + F((\varphi_\varepsilon)_x(t, x)) = g(t, x) + \eta_\varepsilon(t, x), & t \in [0, T] \\ \varphi_\varepsilon(0, x) = \varphi_0(x) & x \in H \end{cases}$$

where

$$\int_0^T |\eta_\varepsilon(t, x)| dt \leq C_R \varepsilon \quad \forall x \in \Sigma_R, \varepsilon > 0.$$

Now coming back to equation (3.11) it follows by Lemma 2 (part (2.11)) and the Gronwall lemma that the mapping $(\varphi_0, g) \xrightarrow{G_\varepsilon} \varphi$ is Lipschitzian from $C(H) \times C([0, T]; C(H))$ to $C([0, T]; C(H))$. More precisely, one has

$$(3.19) \quad |G_\varepsilon(\varphi_0, g)(t) - G_\varepsilon(\psi_0, h)|_R \leq \\ \leq |\varphi_0 - \psi_0|_R + \int_0^t |g(s) - h(s)|_R ds \quad \forall R > 0; t \in [0, T]$$

for all $\varphi_0, \psi_0 \in K$ and $g, h \in \mathcal{K}$. By (3.18) and (3.19) one concludes that

$$|\varphi_\varepsilon(t) - \varphi_\lambda(t)|_R \leq C_R T(\varepsilon + \lambda) \quad \forall t \in [0, T]; \varepsilon, \lambda > 0.$$

Hence $\lim_{\varepsilon \downarrow 0} \varphi_\varepsilon = \varphi$ exists in $C([0, T]; C(H))$. Clearly $\varphi \in \mathcal{K}$ and by (3.15) we see that for each $t \in [0, T]$, $\varphi(t, \cdot)$ is Lipschitzian on every bounded subset of H and

$$|\varphi(t)|_{L^{10}, R} \leq C_R (|\varphi_0|_{1, R} + \sup_{t \in [0, T]} |g(t)|_{1, R}), \quad t \in [0, T].$$

Summarising, we have shown that there exists a sequence $\{\varphi_\varepsilon\} \subset C([0, T]; C(H))$ satisfying

$$(3.20) \quad \varphi_\varepsilon \in C([0, T]; C^1(H)) \cap \mathcal{K} \quad \forall \varepsilon > 0$$

$$(3.21) \quad \varphi_\varepsilon \in C^1([0, T]; C(H)) \quad \forall \varepsilon > 0$$

$$(3.22) \quad \varphi_\varepsilon \rightarrow \varphi \text{ in } C([0, T]; C(H)) \quad \text{for } \varepsilon \rightarrow 0$$

$$(3.23) \quad \{\varphi_\varepsilon\} \text{ is bounded in } C([0, T]; C^1(H)) \text{ and } (\varphi_\varepsilon)_t \text{ in } L^1(0, T; C(H))$$

$$(3.24) \quad \begin{cases} (\varphi_\varepsilon)_t(t, x) + F((\varphi_\varepsilon)_x(t, x)) \rightarrow g(t, x) & \text{in } L^1(0, T; C(H)) \\ \varphi_\varepsilon(0, x) = \varphi_0(x). \end{cases}$$

Here the convergence in the space $L^1(0, T; C(H))$ is understood in the local convex topology given by the family of seminorms (1.13).

DEFINITION 1. A function φ satisfying conditions (3.20) up to (3.24) is called weak solution to the Cauchy problem (3.7). We notice that by (3.19),

(3.22), (3.23) and (3.24) the weak solution $\varphi = G(\varphi_0, g)$ is unique and

$$(3.25) \quad |G(\varphi_0, g) - G(\varphi_0, h)|_R \leq |\varphi_0 - \psi_0|_R + \int_0^T |g(s) - h(s)|_R ds$$

for all $\varphi_0, \psi_0 \in K$ and $g, h \in \mathcal{K}$ satisfying condition (3.8). We have therefore proved the following theorem

THEOREM 2. *Assume that hypotheses (a), (b), (c) and (e) are satisfied. Then for any pair of functions $(\varphi_0, g) \in K \times \mathcal{K}$ satisfying condition (3.8), the Cauchy problem (3.7) has a unique weak solution φ which satisfies*

$$(3.26) \quad \sup \{|\varphi(t)|_{L^p, R}; t \in [0, T]\} < \infty.$$

Furthermore, the map $(\varphi_0, g) \rightarrow \varphi$ is Lipschitzian from $C(H) \times C([0, T]; C(H))$ to $C([0, T]; C(H))$.

REMARKS. 1° It is worth noting that another way to prove Theorem 2 is to apply the Bénéilan existence result (see [2], [6]) to nonlinear evolution equation

$$(3.27) \quad \frac{d\varphi}{dt} + L\varphi = g, \quad t \in [0, T]; \varphi(0) = \varphi_0$$

in the space $C(H)$. Here L is the closure of L_0 (see (2.22)) in $C(H) \times C(H)$.

2° Assumptions (b), (c) and (e) are verified by a large class of functions F which includes functions of the form

$$(3.28) \quad F(x) = \zeta(|x|^2) \quad x \in H$$

where ζ is a real valued, convex and differentiable function on $[0, \infty[$ which satisfies the following conditions

$$(3.29) \quad \zeta'(0) > 0; \quad \lim_{r \rightarrow +\infty} \zeta(r)/r^{\frac{1}{2}} = +\infty.$$

4. - Existence and uniqueness for equation (1.2).

We shall study here the Cauchy problem

$$(4.1) \quad \begin{cases} \varphi_t(t, x) + \frac{1}{2}|\varphi_x(t, x)|^2 + (Ax, \varphi_x(t, x)) = g(t, x) \\ \varphi(0, x) = \varphi_0(x); \quad x \in D(A), t \in [0, T] \end{cases}$$

where

$$(4.2) \quad \varphi_0 \in C^1(H)$$

and

$$(4.3) \quad g \in C([0, T]; C^1(H)) .$$

Further we shall assume that

$$(4.4) \quad \varphi'_0 \in C^0_{\text{Lip}}(H, H) ; \quad g_x \in C([0, T]; C^0_{\text{Lip}}(H, H)) .$$

In this case K is the set of all convex functions $\varphi \in C(H)$ such that $\partial\varphi(0) \in 0$.

We start with the approximating equation

$$(4.5) \quad \begin{aligned} \varphi(t, x) = & \exp(-t/\varepsilon) \varphi_0(\exp[-tA]x) + \\ & + \int_0^t \exp(-(t-s)/\varepsilon) g(s, \exp(-(t-s)A)x) ds + \\ & + \varepsilon^{-1} \int_0^t \exp(-(t-s)/\varepsilon) (S(\varepsilon)\varphi)(s, \exp(-(t-s)A)x) ds \quad \text{for } t \in [0, T], x \in H \end{aligned}$$

where by formula (2.8), $S(\varepsilon)$ is given by

$$(4.6) \quad (S(\varepsilon)\varphi)(x) = \inf \left\{ \frac{|x-y|^2}{2\varepsilon} + \varphi(y); y \in H \right\} = \varphi(y_\varepsilon(x)) + \frac{|x-y_\varepsilon(x)|^2}{2\varepsilon}$$

and

$$(4.7) \quad y_\varepsilon(x) = (1 + \varepsilon\varphi')^{-1}x .$$

Applying the contraction principle on the closed convex cone of $C([0, T]; C(H))$

$$(4.8) \quad \mathcal{K} = \{\varphi \in C([0, T]; C(H)); \varphi(t) \in K \quad \forall t \in [0, T]\}$$

we see that eq. (4.5) has a unique solution $\varphi_\varepsilon \in \mathcal{K}$. Moreover, in as much as $(S(\varepsilon)\varphi)'(x) = \varphi'(y_\varepsilon(x))$, we see that $\varphi_\varepsilon \in C([0, T]; C^1(H))$ and

$$(4.9) \quad \begin{aligned} \varphi'_\varepsilon(t, x) = & \exp(-t/\varepsilon) \exp(-tA^*) \varphi'_0(\exp(-tA)x) + \\ & + \int_0^t \exp(-(t-s)/\varepsilon) \exp(-(t-s)A^*) g'(s, \exp(-(t-s)A)x) ds + \\ & + \varepsilon^{-1} \int_0^t \exp(-(t-s)/\varepsilon) \exp(-(t-s)A^*) \varphi'_\varepsilon(s, y_\varepsilon(s, \exp(-(t-s)A)x)) ds . \end{aligned}$$

By (4.5) it follows that for each $x \in D(A)$, $\varphi_\varepsilon(t, x)$ is differentiable on $[0, T]$ and satisfies the equation

$$(4.10) \quad (\varphi_\varepsilon)_t(t, x) + \varepsilon^{-1}(\varphi_\varepsilon(t, x) - S(\varepsilon)\varphi_\varepsilon(t, x)) + (Ax, (\varphi_\varepsilon)_x(t, x)) = \\ = g(t, x); \quad t \in [0, T], x \in D(A).$$

By an easy computation involving the Gronwall lemma and eq. (4.9) it follows

$$(4.11) \quad |\varphi_\varepsilon(t)|_{1,R} \leq C_R \left(|\varphi_0|_{1,R} + \sup_{0 \leq t \leq T} |g(t)|_{1,R} \right), \quad t \in [0, T].$$

Next, since the mapping $x \rightarrow y_\varepsilon(x)$ is nonexpansive and in virtue of (2.6),

$$\varphi'_\varepsilon(x, y_\varepsilon(t, x)) = \varepsilon^{-1}(x - y_\varepsilon(t, x))$$

it follows by (4.4) and (4.9) that $\varphi'_\varepsilon(t) \in C_{\text{Lip}}^0(H, H)$ for every $t \in [0, T]$. Moreover, using once again the Gronwall lemma one finds the estimate

$$(4.12) \quad \|\varphi'_\varepsilon(t)\|_{0,R} \leq C_R \left(\|\varphi'_0\|_{0,R} + \sup_{0 \leq t \leq T} \|g'(t)\|_{0,R} \right)$$

(we recall that $\varphi' = \varphi_x$ stands for the Fréchet derivative with respect to x .)

Next by inequality (2.20) and (4.11), (4.12)

$$(4.13) \quad \begin{cases} |\varepsilon^{-1}(\varphi_\varepsilon(t, x) - (S(\varepsilon)\varphi_\varepsilon)(t, x)) - F(\varphi'_\varepsilon(t, x))| \leq |F'(\varphi'_\varepsilon(t, x))| \\ \|\varphi'_\varepsilon(t)\|_{0,R} |x - y_\varepsilon(t, x)| \leq C_R \varepsilon \end{cases}$$

for all $x \in \Sigma_R$ and $t \in [0, T]$.

Since the mapping $(\varphi_0, g) \rightarrow \varphi_\varepsilon$ is Lipschitzian from $C(H) \times C([0, T]; C(H))$ to $C([0, T]; C(H))$ (see (3.19)) we may infer by (4.13) that

$$|\varphi_\varepsilon(t) - \varphi_\lambda(t)|_R \leq C_R T(\varepsilon + \lambda); \quad t \in [0, T]; \quad \varepsilon, \lambda > 0$$

and therefore $\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon = \varphi$ exists in $C([0, T]; C(H))$. Clearly $\varphi \in \mathcal{K}$ and

$$(4.14) \quad \sup_{t \in [0, T]} |\varphi(t)|_{\text{Lip}, R} < \infty, \quad \forall R > 0.$$

By (4.10)-(4.12) and (4.13) we see that the function φ is a weak solution to problem (4.1) in the sense of Definition 1, i.e. there exists a sequence

$\{\varphi_\varepsilon\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ such that for $\varepsilon \rightarrow 0$

$$(4.15) \quad \varphi_\varepsilon \rightarrow \varphi \quad \text{in } C([0, T]; C(H))$$

$$(4.16) \quad \varphi_\varepsilon \in C^1([0, T]; C(D(A))) ; \quad \varphi_\varepsilon(0, x) = \varphi_0(x)$$

$$(4.17) \quad (\varphi_\varepsilon)_t(t, x) + \frac{1}{2}|(\varphi_\varepsilon)_x(t, x)|^2 + (Ax, (\varphi_\varepsilon)_x(t, x)) \rightarrow g(t, x) \\ \text{in } C([0, T]; C(H))$$

$$(4.18) \quad \{(\varphi_\varepsilon)_x\} \text{ is bounded in } C([0, T]; C_{\text{Lip}}^0(H, H)).$$

Here the space $D(A)$ is endowed with the graph norm.

Summarising, we have proved the following theorem

THEOREM 3. *Suppose that assumptions (a), (e) are satisfied and φ_0, g satisfy conditions (4.2), (4.3) and (4.4). Then the Cauchy problem (4.1) has a unique weak solution $\varphi \in \mathcal{K}$ which satisfies (4.14). Moreover, the map $(\varphi_0, g) \rightarrow \varphi$ is Lipschitzian from $C(H) \times C([0, T]; C(H))$ to $C([0, T]; C(H))$ and for every $x \in D(A)$ the function $\varphi(t, x)$ is absolutely continuous on $[0, T]$.*

Our next concern is a regularity result for the solutions to equation (4.1). To this purpose we return to approximating sequence $\{\varphi_\varepsilon\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ and set

$$E^\varepsilon(t, x) = \varphi'_\varepsilon(t, x) \quad t \in [0, T], x \in H.$$

As seen above $E^\varepsilon \in C([0, T]; C^0(H, H))$ is the solution to (see (4.9))

$$(4.19) \quad E^\varepsilon(t, x) = \exp\left(-\frac{t}{\varepsilon}\right) \exp(-tA^*) \varphi'_0(\exp(-tA)x) + \\ + \varepsilon^{-1} \int_0^t \exp\left(-\frac{(t-s)}{\varepsilon}\right) \exp(-(t-s)A^*) E^\varepsilon(s, \exp(-(t-s)A)x) ds + \\ + \int_0^t \exp\left(-\frac{(t-s)}{\varepsilon}\right) \exp(-(t-s)A^*) g'(s, \exp(-(t-s)A)x) ds$$

where

$$(4.20) \quad E^\varepsilon(t, x) = (S(\varepsilon)\varphi_\varepsilon)'(t, x) = E^\varepsilon(1 + \varepsilon E^\varepsilon)^{-1}(t, x).$$

In addition to (4.2), (4.3) and (4.4) we shall assume that

$$(4.21) \quad \varphi'_0 \in C_{\text{Lip}}^1(H, H); \quad g'_x \in C([0, T]; C_{\text{Lip}}^1(H, H)).$$

We shall prove that under these conditions $E^s \in C([0, T]; C^1_{\text{Lip}}(H, H))$. To this end we introduce the following convex cone of $C^0(H, H)$

$$(4.22) \quad \Pi = \{E \in C^0(H, H); E \text{ monotone and } E(0) = 0\}.$$

For every $E \in \Pi$ we set $E_\varepsilon = E(1 + \varepsilon E)^{-1}$ (1 is the identity operator in H). We notice that for every $\varepsilon > 0$ the operator $(1 + \varepsilon E)^{-1}$ is well defined and nonexpansive on H . In the next lemma we gather for later use some elementary properties of E_ε .

LEMMA 3. For all $\varepsilon > 0$ and $R > 0$ one has

$$(4.23) \quad |E_\varepsilon|_{0,R} \leq |E|_{0,R} \quad \forall E \in \Pi$$

$$(4.24) \quad |E_\varepsilon|_{1,R} \leq |E|_{1,R} \quad \forall E \in \Pi \cap C^1(H, H)$$

$$(4.25) \quad \|E_\varepsilon\|_{1,R} \leq \|E\|_{1,R} + \varepsilon |E|_{1,R} (\|E\|_{1,R} \|E\|_{0,R} + \|E\|_{1,R}) \quad \forall E \in \Pi \cap C^1_{\text{Lip}}(H, H).$$

Moreover, if $\|E\|_{1,R}$ and $\|\tilde{E}\|_{1,R}$ are $\leq \alpha$ then there exists $\eta(\alpha) > 0$ such that

$$(4.26) \quad |E - \tilde{E}|_{1,R} \leq (1 + \varepsilon \eta(\alpha)) |E - \tilde{E}|_{1,R}.$$

The proof is standard and relies on the formula

$$E'_\varepsilon(x) = E'((1 + \varepsilon E)^{-1}x) (1 + \varepsilon E'((1 + \varepsilon E)^{-1}x)^{-1}).$$

(By E' we shall denote the Fréchet derivative of the operator E .)

In the space $C([0, T]; C^1(H, H))$ consider the approximating equation

$$(4.27) \quad \begin{aligned} E(t, x) = & \exp\left(-\frac{\varepsilon}{t}\right) \exp(-tA^*) E_0(\exp(-tA)x) + \\ & + \varepsilon^{-1} \int_0^t \exp\left(-\frac{(t-s)}{\varepsilon}\right) \exp(-(t-s)A^*) E(s, \exp(-(t-s)A)x) ds + \\ & + \int_0^t \exp\left(-\frac{(t-s)}{\varepsilon}\right) \exp(-(t-s)A^*) G(s, \exp(-(t-s)A)x) ds, \\ & x \in H; t \in [0, T] \end{aligned}$$

where $E_0 = \varphi'_0$ and $G = g'$. We consider the following closed convex cone of $C([0, T]; C^1(H, H))$

$$(4.28) \quad Q = \{E \in C([0, T]; C^1_{\text{Lip}}(H, H)); E(t) \in \Pi, \|E(t)\|_{1,R} \leq \alpha_R \text{ for every } t \in [0, T]\}$$

where $\alpha_R \rightarrow +\infty$ as $R \rightarrow +\infty$.

Let Γ be the operator defined by the right hand of equation (4.27). For the beginning we shall assume that $\|E_0\|_{1,R} \leq \alpha_R/2$ and $\|G(t)\|_{1,R} \leq \alpha_R/2$ for $t \in [0, T]$. Then for all sufficiently small T , Γ maps Q into itself and by (4.25) one has

$$\|(\Gamma E)(t) - (\Gamma \tilde{E})(t)\|_{1,R} \leq \varrho_R \sup \{ \|E(t) - \tilde{E}(t)\|_{1,R}; 0 \leq t \leq T \}$$

for all $E, \tilde{E} \in Q$. Here $0 < \varrho_R < 1$ for every $R > 0$. Hence eq. (4.27) (equivalently (4.19)) has a unique solution $E = E^\varepsilon \in C([0, T']; C_{\text{Lip}}^1(H, H))$ where $[0, T']$ is some subinterval of $[0, T]$. Next after some calculations involving equation (4.19), estimates (4.23), (4.25) and the Gronwall lemma it follows that

$$(4.29) \quad \|E^\varepsilon(t)\|_{1,R} \leq C_R \quad \forall t \in [0, T'], \varepsilon > 0$$

where C_R is independent of T' . This implies by a standard procedure that $E^\varepsilon \in C([0, T]; C_{\text{Lip}}^1(H, H))$ and inequality (4.29) extends on the whole interval $[0, T]$.

On the other hand, using once again estimate (4.26) we see that

$$(4.30) \quad \|\Phi_\varepsilon(t, E_0, G) - \Phi_\varepsilon(t, \tilde{E}_0, \tilde{G})\|_{k,R} \leq C_R \left(\|E_0 - \tilde{E}_0\|_{k,R} + \int_0^t \|G(s) - \tilde{G}(s)\|_{k,R} ds \right)$$

for $t \in [0, T]$ and $k = 0, 1$, where $\Phi_\varepsilon(t, E_0, G) = E^\varepsilon$ is the solution to (4.27).

On the other hand, we have

$$(4.31) \quad \varepsilon^{-1}(E(x) - E_\varepsilon(x)) - E'(x)E(x) = \\ = \int_0^t \left(E'(sx + (1-s)y_\varepsilon(x)) E(y_\varepsilon(x)) - E'(x)E(x) \right) ds, \quad x \in H$$

where $y_\varepsilon(x) = (1 + \varepsilon E)^{-1}x$. Along with estimates (4.29) and (4.30) the latter yields

$$(4.32) \quad \|\varepsilon^{-1}(E^\varepsilon(t) - E_\varepsilon^\varepsilon(t)) - (E^\varepsilon)'(t)E^\varepsilon(t)\|_{0,R} \leq C_R \varepsilon, \quad t \in [0, T].$$

Then by (4.32) it follows that

$$\|E^\varepsilon(t) - E^\lambda(t)\|_{0,R} \leq C_R(\varepsilon + \lambda), \quad t \in [0, T]; \varepsilon, \lambda > 0.$$

Hence there exists $E \in C([0, T]; C_0(H, H))$ such that for $\varepsilon \rightarrow 0$,

$$E_\varepsilon \rightarrow E \quad \text{in } C([0, T]; C^0(H, H)).$$

By the uniqueness of the limit we infer that $E(t, x) = \varphi_x(t, x)$ where φ is the weak solution to equation (4.1). We have therefore proved that

$$(4.33) \quad (\varphi_\varepsilon)_x \rightarrow \varphi_x \quad \text{in } C([0, T]; C_0(H, H)) .$$

Then by (4.17) we may infer that

$$\varphi \in C^1([0, T]; D(A)) \cap C([0, T]; C^1(H, H))$$

and

$$(4.34) \quad \begin{cases} \varphi_t(t, x) + \frac{1}{2} |\varphi_x(t, x)|^2 + (Ax, \varphi_x(t, x)) = g(t, x) \\ \varphi(0, x) = \varphi_0(x) \quad \text{for } x \in D(A), t \in [0, T] . \end{cases}$$

This amounts to saying that φ is a classical solution to equation (4.1). We have therefore proved

THEOREM 4. *In Theorem 3 suppose in addition that φ_0 and g satisfy conditions (4.21). Then φ is a classical solution to equation (4.1).*

Now we notice that by (4.10), $E^\varepsilon = \varphi'_\varepsilon$ satisfy

$$(4.35) \quad \begin{aligned} E^\varepsilon(t, x) = & \exp(-tA^*) \varphi'_0(\exp(-tA)x) + \\ & + \int_0^t \exp(-(t-s)A^*) (E^\varepsilon)_x E^\varepsilon(s, \exp(-(t-s)A)x) ds + \\ & + \int_0^t \exp(-(t-s)A^*) g'(s, \exp(-(t-s)A)x) ds + \delta_\varepsilon(t, x) \end{aligned}$$

where $\delta_\varepsilon \rightarrow 0$ in $C([0, T]; C^0(H, H))$ for $\varepsilon \rightarrow 0$, while by (4.33)

$$(4.36) \quad E^\varepsilon \rightarrow E = \varphi_x \quad \text{in } C([0, T]; C^0(H, H)) .$$

Keeping in mind that the equation

$$(4.37) \quad \begin{aligned} E(t, x) = & \exp(-tA^*) E_0(\exp(-tA)x) + \\ & + \int_0^t \exp(-(t-s)A^*) (E_x E)(s, \exp(-(t-s)A)x) ds + \\ & + \int_0^t \exp(-(t-s)A^*) G(s, \exp(-(t-s)A)x) ds ; \quad t \in [0, T], x \in H \end{aligned}$$

is the «mild» form of equation (1.3), we may say that $E = \varphi_x$ is a *weak solution* to this equation.

We have therefore the following existence result

THEOREM 5. *Under assumptions of Theorem 4, $E(t, x) = \varphi_x(t, x)$ is a weak solution to operator equation (1.3) where $E_0 = \varphi'_0$ and $G = g_x$.*

5. - An example in control theory.

The relevance of the Hamilton-Jacobi equations in control theory and the calculus of variations is well-known (see e.g. [3] and [12] for recent results concerning infinite dimensional problems). Here we shall study the connection between equation (1.1) and the following optimal control problem:

$$(5.1) \quad \text{Minimize} \quad \int_0^T (g(x(t)) + h(u(t))) dt + \varphi_0(x(T))$$

over all $u \in L^2(0, T; U)$ and $x \in C([0, T]; H)$ subject to state equation

$$(5.2) \quad \begin{cases} x' + Ax = Bu, & t \in [0, T] \\ x(0) = x_0. \end{cases}$$

Here B is a linear continuous operator from U to H , $g: H \rightarrow \mathbf{R}^1$, $h: U \rightarrow \mathbf{R}^1$, $\varphi_0: H \rightarrow \mathbf{R}^1$ are given lower semicontinuous convex functions and U is a real Hilbert space identified with its own dual and with inner product $\langle \cdot, \cdot \rangle$.

We shall denote by $W^{1,2}(0, T; H)$ the space

$$\{x \in L^2(0, T; H); x' \in L^2(0, T; H)\}$$

where x' is the derivative in the sense of distributions. We shall assume that $x_0 \in D(A)$ and $-A$ is the infinitesimal generator of an analytic semigroup of contractions on H . Then for each control $u \in L^1(0, T; U)$, system (5.2) has a unique solution $x_u \in W^{1,2}(0, T; H)$ with $Ax_u \in L^2(0, T; H)$.

We associate with problem (5.1) the equation

$$(5.3) \quad \begin{cases} \psi_t(t, x) - h^*(-B^* \psi_x(t, x)) - (Ax, \psi_x(t, x)) + g(x) = 0, \\ \psi(T, x) = \varphi_0(x) \end{cases} \quad \begin{matrix} x \in D(A), t \in [0, T]. \end{matrix}$$

where h^* is the conjugate of h and B^* is the dual operator of B .

We observe that by substitution $\varphi(t, x) = \psi(T - t, x)$, eq. (5.3) can be written in the form (1.1) where $F(y) = h^*(-B^*y)$ for all $y \in H$.

By analogy with Definition 1, we say that the function $\psi \in \mathcal{K}$ is a weak solution to equation (5.3) if there exists a sequence $\{\psi_\varepsilon\} \subset C([0, T]; C^1(H)) \cap \mathcal{K}$ such that for $\varepsilon \rightarrow 0$,

$$(5.4) \quad \psi_\varepsilon \in C^1([0, T]; C(D(A))) ; \quad \psi_\varepsilon(T) = \varphi_0$$

$$(5.5) \quad \psi_\varepsilon \rightarrow \psi \quad \text{in } C([0, T]; C(H))$$

$$(5.6) \quad (\psi_\varepsilon)_t - h^*(-B^*(\psi_\varepsilon)_x) - (Ax, (\psi_\varepsilon)_x) + g \rightarrow 0 \quad \text{in } C([0, T]; C(H)) .$$

$$(5.7) \quad \text{The sequence } \{(\psi_\varepsilon)_x\} \text{ belongs to } C([0, T]; C_{\text{Lip}}^0(H, H)) \text{ and it is bounded in } C([0, T]; C^1(H, H)).$$

Here $\mathcal{K}$ is defined by (3.9) and K is the set of all convex functions $\varphi \in C(H)$ such that $0 \in \partial\varphi(0)$ and

$$(5.8) \quad ((h^*)_x(-B^*y), B^*x) \leq 0 \quad \forall [x, y] \in \partial\varphi .$$

The results proved in sect. 3 and 4 give existence and uniqueness of the weak solution ψ to equation (5.3) in several situations. For instance if $A \equiv 0$ and K is a closed convex cone of $C(H)$, Theorem 2 gives existence and uniqueness of a weak solution under the following assumptions:

$$(5.9) \quad \varphi_0, \quad g \in K \cap C^1(H); \quad \varphi'_0, \quad g' \in C_{\text{Lip}}^0(H, H)$$

$$(5.10) \quad h^* \in C^1(H) \text{ and } F(y) = h^*(-B^*y) \text{ satisfies (b) and (c)} .$$

If $h(u) = |u|^2/2$ and the range of B is all of H we may apply Theorem 3 to obtain existence and uniqueness under conditions (4.2), (4.3) and (4.4).

PROPOSITION 1. *Let $\psi \in \mathcal{K}$ be a weak solution to equation (5.3) where $(h^*)_x \in C_{\text{Lip}}^0(H, H)$. Then for every $y \in D(A)$ and $t \in [0, T]$ one has*

$$(5.11) \quad \psi(t, y) = \\ = \inf \left\{ \int_t^T (g(x_u(s)) + h(u(s))) ds + \varphi_0(x_u(T)); u \in L^2(t, T; U), x_u(t) = y \right\} .$$

Moreover, if u^* is an optimal control in problem (5.1) then it is expressed as a

function of the optimal state x^* by the feedback formula

$$(5.12) \quad u^*(t) = (h^*)_x(-B^* \partial \psi(t, x^*(t))) \quad \text{a.e. } t \in [0, T].$$

Here $\partial \psi$ is the subdifferential of $\psi(t, \cdot)$.

Formula (5.12) gives the optimal feedback law of control problem (5.1). In particular if ψ happens to be a classical solution to (5.3) (in particular this is the case if the conditions of Theorem 4 are satisfied) then $\partial \psi = \psi_x$ and it follows by (5.12) that $u^* \in C^1([0, T]; U)$.

PROOF OF PROPOSITION 1. Let $t \in [0, T]$ and $u \in L^2(t, T; U)$ be fixed. Let x_u the solution to (5.2) on $[t, T]$ such that $x_u(t) = y$. The obvious equality

$$\frac{d}{ds} \psi_\varepsilon(s, x_u(s)) = (\psi_\varepsilon)_s(s, x_u(s)) + ((\psi_\varepsilon)_x(s, x_u(s)), x'_u(s)), \quad \text{a.e. } s \in]t, T[$$

along with (5.4) and (5.6) implies that $\psi_\varepsilon(s, x_u(s))$ is absolutely continuous on $[t, T]$ and

$$(5.13) \quad \begin{aligned} \frac{d}{ds} \psi_\varepsilon(s, x_u(s)) + g(x_u(s)) + h(u(s)) &= h^*(-B^*(\psi_\varepsilon)_x(s, x_u(s)) + \\ &+ h(u(s)) + \langle B^*(\psi_\varepsilon)_x(s, x_u(s)), u(s) \rangle + \delta_\varepsilon(s) \quad \text{a.e. } s \in]t, T[\end{aligned}$$

where $\delta_\varepsilon \rightarrow 0$ uniformly on $[t, T]$.

Recalling that

$$h(u) + h^*(\tilde{u}) \geq \langle u, \tilde{u} \rangle \quad \forall u, \tilde{u} \in U$$

we deduce by (5.13) that

$$\psi_\varepsilon(t, y) \leq \int_t^T (g(x_u(s)) + h(u(s))) ds + \varphi_0(x_u(T)) - \int_t^T \delta_\varepsilon(s) ds.$$

Therefore

$$(5.14) \quad \begin{aligned} \psi(t, y) &\leq \\ &\leq \inf \left\{ \int_t^T (g(x_u(s)) + h(u(s))) ds + \varphi_0(x_u(T)); u \in L^2(t, T; U), x_u(t) = y \right\}. \end{aligned}$$

Now consider the Cauchy problem

$$(5.15) \quad \begin{cases} x' + Ax = B(h^*)_x(-B^*(\psi_\varepsilon)_x(s, x)), & s \in]t, T[\\ x(t) = y. \end{cases}$$

For each $\varepsilon > 0$ and $y \in D(A)$, problem (5.15) has a unique solution $x_\varepsilon \in W^{1,2}(t, T; H)$. Here is the argument.

Since $(\psi_\varepsilon)_x \in C([0, T]; C_{\text{Lip}}^0(H, H))$ and $(h^*)_x \in C_{\text{Lip}}^0(H, H)$, we deduce by a standard argument that (5.15) has a unique continuous local solution x_ε . In as much as $\psi_\varepsilon(s, \cdot) \in K$ for all $s \in [0, T]$ it follows by (5.15) that

$$(5.16) \quad |x_\varepsilon(s)| \leq C \quad s \in [t, T']$$

where $[t, T']$ is the maximal interval of definition for x_ε .

Estimate (5.16) then implies by a standard device that x_ε can be extended as a solution (in the « mild » sense) to (5.15) on the whole interval $[t, T]$. Clearly $x_\varepsilon \in W^{1,2}(t, T; H)$ and equation (5.15) is satisfied a.e. on $]t, T[$.

Now in (5.13) we take $u = u_\varepsilon = (h^*)_x(-B^*(\psi_\varepsilon)_x(s, x_\varepsilon))$ and obtain

$$\frac{d}{ds} \psi_\varepsilon(s, x_\varepsilon(s)) + g(x_\varepsilon(s)) + h(u_\varepsilon(s)) = \delta_\varepsilon(s) \quad \text{a.e. } s \in]t, T[$$

and therefore

$$\psi_\varepsilon(t, y) - \int_t^T (g(x_\varepsilon(s)) + h(u_\varepsilon(s))) ds + \varphi_0(x_\varepsilon(T)) \rightarrow 0 \quad \text{for } \varepsilon \rightarrow 0.$$

Along with (5.14) the latter implies (5.11) as claimed.

Let $u^* \in L^2(0, T; U)$ be any optimal control of the problem and let $x^* \in W^{1,2}(0, T; H)$ be the corresponding optimal state. By (5.11) it follows that

$$(5.17) \quad \psi(t, x^*(t)) = \int_t^T (g(x^*(s)) + h(u^*(s))) ds + \varphi_0(x^*(T)), \quad t \in [0, T].$$

Next, in (5.13) we take $u = u^*$ and $x_u = x^*$. We get

$$\begin{aligned} \psi_\varepsilon(t, x^*(t)) &= \int_t^T (g(x^*(s)) + h(u^*(s))) ds + \varphi_0(x^*(T)) - \\ &- \int_t^T (h^*(-B^*(\psi_\varepsilon)_x(s, x^*(s))) + h(u^*(s)) + \langle B^*(\psi_\varepsilon)_x(s, x^*(s)), u^*(s) \rangle + \delta_\varepsilon(s)) ds \end{aligned}$$

and by (5.6), (5.17) we see that

$$\begin{aligned} \int_t^T (h(u^*(s)) + h^*(-B^*(\psi_\varepsilon)_x(s, x^*(s))) + \langle B^*(\psi_\varepsilon)_x(s, x^*(s)), u^*(s) \rangle) ds \rightarrow 0 \\ \text{uniformly on } [0, T] \end{aligned}$$

which implies in particular that

$$(5.18) \quad \int_t^T (h^*(-B^*(\psi_\varepsilon)_x(s, x^*(s)) - h^*(v))) ds \leq \\ \leq \int_t^T \langle -B^*(\psi_\varepsilon)_x(s, x^*(s)) - v(s), u^*(s) \rangle ds + \eta_\varepsilon, \quad \forall v \in L^2(0, T; U)$$

where $\eta_\varepsilon \rightarrow 0$ for $\varepsilon \rightarrow 0$.

On the other hand, it follows by (5.7) that $\{(\psi_\varepsilon)_x(t, x^*(t))\}$ is bounded in $L^\infty(0, T; H)$. Thus we may assume that

$$(5.19) \quad (\psi_\varepsilon)_x(t, x^*) \rightarrow q \quad \text{weak star in } L^\infty(0, T; H)$$

and letting ε tend to zero in (5.18) we get (because the convex integrand is weakly lower semicontinuous),

$$\int_0^T (h^*(-B^*q(t)) - h^*(v)) dt \leq \int_0^T \langle -B^*q(t) - v(t), u^*(t) \rangle dt, \quad \forall v \in L^2(0, T; U).$$

Equivalently,

$$(5.20) \quad u^*(t) \in (h^*)_x(-B^*q(t)) \quad \text{a.e. } t \in]0, T[.$$

Similarly by (5.19) it follows that

$$q(t) \in \partial\psi(t, x^*(t)) \quad \text{a.e. } t \in]0, T[$$

which along with (5.20) implies (5.12) thereby completing the proof.

6. - Additional regularity properties for the equation (1.3).

LEMMA 4. For all $\varepsilon > 0$ and $R > 0$ one has

$$(6.1) \quad \|E_\varepsilon\|_{2,R} \leq (1 + \alpha\|E\|_{0,R}) (\|E\|_{2,R} + 3\alpha\|E\|_{1,R}^2) \quad \forall E \in \Pi \cap C_{\text{Lip}}^2(H, H)$$

$$(6.2) \quad |E_\varepsilon - \tilde{E}_\varepsilon|_{2,R} \leq (1 + BR^2)\{1 + 2\varepsilon BR(1 + R) + \\ + \varepsilon^2 B^2 R^2 \cdot (1 + BR + 3R)\} |E - \tilde{E}|_{2,R} \quad \forall E, \tilde{E} \in \Pi \cap C^2(H, H)$$

where $B = \sup \{\|E\|_{2,R}, \|\tilde{E}\|_{2,R}\}$.

PROOF. The proof is standard and relies on the formula:

$$(6.3) \quad E''_{\varepsilon}(x) = E''(J_{\varepsilon}(x))(J'_{\varepsilon}(x), J'_{\varepsilon}(x)) + E'(J_{\varepsilon}(x))J''_{\varepsilon}(x)$$

where

$$(6.4) \quad \begin{cases} J_{\varepsilon}(x) = (1 + \varepsilon E)^{-1}(x) \\ J'_{\varepsilon}(x) = (1 + \varepsilon E'(J_{\varepsilon}(x)))^{-1} \\ J''_{\varepsilon}(x) = -\varepsilon J'_{\varepsilon}(x) E''(J_{\varepsilon}(x))(J'_{\varepsilon}(x), J'_{\varepsilon}(x)). \end{cases}$$

LEMMA 5. Assume that $E_0, \tilde{E}_0 \in \Pi \cap C^2_{\text{Lip}}(H, H)$ and $G, \tilde{G} \in C([0, T]; C^2_{\text{Lip}}(H, H))$ with $G(t) \in \Pi, \forall t \in [0, T]$; then equation (4.27) has unique solutions $E, \tilde{E} \in C([0, T]; C^2_{\text{Lip}}(H, H))$ and it is:

$$(6.5) \quad \|E(t, \cdot)\|_{2,R} \leq \exp(c_R t) \|E_0\|_{2,R} + \int_0^t \exp(c_R(t-s)) \|G(s, \cdot)\|_{2,R} ds$$

$$(6.6) \quad |E(t, \cdot) - \tilde{E}(t, \cdot)|_{1,R} \leq \exp(c_R t) |E_0 - \tilde{E}_0|_{1,R} + \int_0^t \exp(c_R(t-s)) |G(s, \cdot) - \tilde{G}(s, \cdot)|_{1,R} ds$$

where c_R is independent from α .

PROOF. The proof is quite similar to that of Theorem 4 (using estimates (6.1) and (6.2)).

THEOREM 6. Assume that $E_0 \in C^2_{\text{Lip}}(H, H) \cap \Pi$ and $G \in C([0, T]; C^2_{\text{Lip}}(H, H))$ with $G(t) \in \Pi, \forall t \in [0, T]$. Then there exists a unique solution $E \in C([0, T]; C^1(H, H))$ to equation (4.37).

PROOF. Consider the approximating equation

$$E^{\varepsilon}(t, x) = \exp(-tA^*) E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot [\gamma_{\varepsilon}(E^{\varepsilon})(s, \exp(-(t-s)A)x) + G(s, \exp(-(t-s)A)x)] ds$$

where

$$\gamma_{\varepsilon}(f) = (f - f_{\varepsilon})/\varepsilon \quad \forall f \in \Pi.$$

We write γ_{ε} in the following form:

$$(6.7) \quad \gamma_{\varepsilon}(f) = f_x f + R_{\varepsilon}(f)$$

where

$$(6.8) \quad R_{\varepsilon}(f)(x) = \int_0^t \{f_x(\xi x + (1-\xi)(1 + J_{\varepsilon}(x))f_{\varepsilon}(x) - f_x(x)f(x))\} d\xi.$$

If $f \in C_{\text{Lib}}^2(H, H)$ it is:

$$(6.9) \quad \lim_{\varepsilon \rightarrow 0} |R_\varepsilon(f)|_{1R} = 0.$$

Let now $\mu > 0$, it is:

$$E^\mu(t, x) = \exp(-tA^*)E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot [\gamma_\varepsilon(E^\mu) + R_\varepsilon(E^\mu) - R_\mu(E^\mu)] \cdot (s, \exp(-(t-s)A)x) ds.$$

Recalling (6.6), (6.9) and estimating $|E^\varepsilon(t, \cdot) - E^\mu(t, \cdot)|_{1R}$ via the Gromwall lemma we get:

$$\lim_{\varepsilon \rightarrow 0} E^\varepsilon = E \quad \text{in } C([0, T]; C^1(H, H))$$

and from (6.5) it follows that E is a solution to equation (4.37). To prove uniqueness consider two solutions E_1 and E_2 , then for every $\beta > 0$ it is:

$$E_i(t, x) = \exp(-tA^*)E_0(\exp(-tA)x) + \int_0^t \exp(-(t-s)A^*) \cdot [\gamma_\beta(E_i) + G - R_\beta(E_i)](s, \exp(-(t-s)A)x) ds \quad i = 1, 2.$$

Using again (6.6), (6.9) and the Gromwall lemma we get $E_1 = E_2$.

Equation (4.37) is a « mild » form of equation (1.3). To find classical solution we consider the semigroup in $C_0(H, H)$ defined by

$$(6.10) \quad G_t(f)(x) = \exp(-tA)f(\exp(-tA)x) \quad \forall f \in C^0(H, H).$$

We remark that G_t applies II in itself; we put

$$\left\{ \begin{array}{l} D(\mathcal{A}) = \left\{ f \in C^0(H, H); \exists \lim_{h \rightarrow 0^+} \frac{1}{h} (G_h(f)(x) - f(x)) \in C^0(H, H) \right\} \\ \mathcal{A}(f)(x) = \lim_{h \rightarrow 0^+} \frac{1}{h} (G_h(f)(x) - f(x)) \end{array} \right.$$

LEMMA 6. - Assume that $f \in D(\mathcal{A}) \cap C^1(H, H)$ and $x \in D(A)$; then it is $f(x) \in D(A^*)$ and moreover:

$$(6.11) \quad \mathcal{A}(f)(x) = A^*f(x) + f_x(x)Ax.$$

PROOF. Put $g = \mathcal{A}(f)$ and take $x, y \in D(A)$; it is:

$$\begin{aligned} \langle g(x), y \rangle &= \frac{d}{dh} \langle \exp(-hA^*) f(\exp(-hA)x), y \rangle|_{h=0} = \\ &= \frac{d}{dh} \langle f(\exp(-hA)x), \exp(-hA)y \rangle|_{h=0} = -\langle f_x(x)Ax, y \rangle - \langle f(x), Ay \rangle, \end{aligned}$$

therefore the linear mapping

$$y \rightarrow \langle f(x), Ay \rangle = \langle g(x), y \rangle - \langle f_x(x)Ax, y \rangle$$

is continuous; consequently it is $f(x) \in D(A^*)$ and (6.11) is fulfilled.

We write now equation (1.3) in the following form

$$(6.12) \quad \begin{cases} E_t + \mathcal{A}(E) + E_x E = G \\ E(0) = E_0. \end{cases}$$

THEOREM 7. Assume that $E_0 \in C_{\text{Lip}}^2(H, H) \cap \Pi \cap D(\mathcal{A})$, $\mathcal{A}(E_0) \in C^1(H, H)$, $G, G_t \in C([0, T]; C_{\text{Lip}}^2(H, H))$ with $G(t, \cdot) \in \Pi$.

Then the equation (1.3) has a unique classical solution.

PROOF. We can solve (in the « mild » form) the following linear problem:

$$(6.13) \quad \begin{cases} V_t + \mathcal{A}(V) + V_x E + E_x V = G_t \\ V(0) = G(0, \cdot) - \mathcal{A}(E_0) - E_{0x} E_0 \end{cases}$$

where $E \in C([0, T]; C^1(H, H))$ is the solution to (4.37).

Let us consider the approximating problems:

$$\begin{cases} E_t^n + \mathcal{A}_n(E^n) + E_x^n E^n = G \\ E^n(0) = E_0 \\ V_t^n + \mathcal{A}_n(V^n) + V_x^n V^n + E_x^n V^n = G_t \\ V^n(0) = G(0, \cdot) - \mathcal{A}_n(E_0) - E_{0x} E_0 \end{cases}$$

where $\mathcal{A}_n(E) = A_n^* E + E_x A_n$ and $A_n = n - n^2 (n + A)^{-1}$: it is clear that $V^n = E_t^n$ and it is easy to show that

$$V^n \rightarrow V, \quad E^n \rightarrow E \quad \text{in } C([0, T]; C^1(H, H))$$

this implies $V = E_t$ and the thesis follows.

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