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On the Approximation of Elliptic Operators with Discontinuous Coefficients.

WILLIAM H. MC CONNELL (*)

Summary. – *Implicit in the application of many physical theories is the replacement of differential equations with discontinuous coefficients by ones with constant coefficients. When the coefficients depend upon only one independent variable, sufficient conditions are proven which determine if such a procedure truly approximates the actual solution. The new system is constructed and applications to boundary value problems, spectra and physical examples are discussed. It is interesting that, in general, the constant coefficents are neither close in a pointwise sense nor in an average sense to the discontinuous coefficients.*

The elastic deformation of most structural metals and composite media is properly modeled by partial differential equations with discontinuous coefficients. Since representations of such solutions are difficult to obtain, more tractable approximate theories intending to characterize the mollification of the actual solution are often used. The approximating theory is to reflect the gross response of the body while masking the fine structure of the material. In particular, *the replacement of discontinuous coefficients by constant ones is common in applications to structural metals* (Timoshenko and Goodier [12]) *and composite media* (Pagano [8]). In Theorem 2 we prove sufficient conditions to determine when such an approximation is valid in the case of *laminated* composite media, that is, when the coefficients depend upon one and only one coordinate. In fact, Theorem 2 contains the explicit criterion for selection of suitable constant coefficients. The proof of Theorem 2 is based on Theorem 1, which considers sequences of elliptic systems whose coefficients need not converge in L_1 but are still shown to affect a limiting system. Similar results for spectra are given in Theorem 4.

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Spagnolo [11] has developed compactness arguments for sequences of single uniformly elliptic operators demonstrating that, like Theorem 1, there always exists a limiting operator. Marino and Spagnolo [7] have determined the form of this new operator for the special case when the coefficients of the sequence are diagonal and products of functions of one variable. Khoroshun [5] and Pagano [9] have considered special (piecewise quadratic) solutions to derive theories similar to Theorem 2, but their methods do not generalize to deal with *every* H^1 solution.

1. – Interior estimates and results.

Let $\Omega \subset E^n$ be a bounded domain with a C^1 boundary.

$L_p(\Omega)$ will denote the classical Banach space of equivalent real-valued, measurable, pth-power summable functions under the norm

$$\|f\|_{L_p(\Omega)} \equiv \Big[\int_\Omega |f|^p\,dx\Big]^{1/p}.$$

A vector function $\boldsymbol{f}(\boldsymbol{x}) = (f_i(x_j))$, $i, j = 1, \ldots, n$, is said to belong to $L_p(\Omega)$ iff each component $f_i \in L_p(\Omega)$, and

$$\|\boldsymbol{f}\|_{L_p(\Omega)} \equiv \||\boldsymbol{f}|\|_{L_p(\Omega)}.$$

The *Sobolev Space* $H^1(\Omega)$ is the completion of $C^\infty(\bar{\Omega})$ under the norm

$$\|u\|_{H^1(\Omega)} \equiv \Big\{\int_\Omega [u^2 + |\nabla u|^2]\,dx\Big\}^{\frac{1}{2}}. \tag{1}$$

A vector valued function $\boldsymbol{u}(\boldsymbol{x})$ is said to belong to $H^1(\Omega)$ iff each component does and

$$\|\boldsymbol{u}\|_{H^1(\Omega)} = \Big\{\int_\Omega [u_i u_i + u_{i,j} u_{i,j}]\,dx\Big\}.$$

Repeated subscripts are to be summed 1 to n. The *strong derivatives* of u are denoted $u_{,j}$. The *Sobolev Space* $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm (1).

The bulk of the paper deals with systems with n dependent variables in E^n, but the general ideas apply to single equations with only superficial alterations. A function $\boldsymbol{u} \in H^1(\Omega)$ is said to be a *weak solution* to

$$(C_{ijkl} u_{k,l})_{,j} = 0, \quad i = 1, \ldots, n,$$

iff for all $\boldsymbol{\eta} \in H_0^1(\Omega)$

$$\int_\Omega C_{ijkl} u_{k,l} \eta_{i,j} \, dx = 0 . \tag{2}$$

It will be assumed that $\boldsymbol{C}$ possess the ellipticity property that there exist Λ, $\lambda > 0$ such that for all *symmetric* $\boldsymbol{e}$ (1)

$$\lambda e_{ij} e_{ij} \leqslant C_{ijkl} e_{ij} e_{kl} \tag{3}$$

and

$$\Lambda \geqslant \max_{i,j,k,l} \operatorname*{ess\,sup}_{x\in\Omega} |C_{ijkl}(x)| .$$

Also $\boldsymbol{C}$ will be assumed to satisfy the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} . \tag{3'}$$

Gurtin [5] discuss the physical basis of these assumptions for elasticity.

For the most part, the analysis deals with coefficients $\boldsymbol{C}$ that depend only upon the n-th independent coordinate, $x = (x_1, \dots, x_n)$

$$\boldsymbol{C} \equiv \boldsymbol{C}(x_n) . \tag{4}$$

Lemma 1, Theorems 1, 3 and 4 deal with families $\{\boldsymbol{C}^\sigma\}_{\sigma\in(0,1)}$ such that (3), (3′) and (4) hold uniformly. For solutions of (2) the stress τ^σ is defined

$$\tau^\sigma_{ij} \equiv \tfrac{1}{2} C^\sigma_{ijkl}(u^\sigma_{k,l} + u^\sigma_{l,k}) = C^\sigma_{ijkl} u^\sigma_{k,l} .$$

LEMMA 1. *Let $\boldsymbol{u}^\sigma$ be a weak solution of* (2) *associated with $\boldsymbol{C}^\sigma$. Suppose* (3), (3′) *and* (4) *hold with $\boldsymbol{C} = \boldsymbol{C}^\sigma$. If there exists $c < \infty$ where $\|\boldsymbol{u}^\sigma\|_{H^1(\Omega)} < c$, then there exists $\boldsymbol{u} \in H^1(\Omega)$, $\boldsymbol{\tau} \in L_2(\Omega)$ and a sequence $\{\boldsymbol{u}^{\sigma_r}, \boldsymbol{\tau}^{\sigma_r}\}_r$ such that as $r \to \infty$,*

$$\sigma_r \to 0 , \quad \boldsymbol{u}^{\sigma_r} \xrightarrow[H^1(\Omega)]{} \boldsymbol{u} , \quad \boldsymbol{\tau}^{\sigma_r} \xrightarrow[L_2(\Omega)]{} \boldsymbol{\tau}$$
$$\boldsymbol{u}^{\sigma_r}_{\alpha} \xrightarrow[H^1_{\mathrm{loc}}(\Omega)]{} \boldsymbol{u}_{,\alpha} , \quad \alpha = 1, \dots, n-1 ,$$

and

$$\tau^{\sigma_r}_{in} \xrightarrow[H^1_{\mathrm{loc}}(\Omega)]{} \tau_{in} , \quad i = 1, \dots, n . \text{ (2)}$$

(1) This is more restrictive than *strong ellipticity* but is advantageous in that no continuity assumptions are required to bound $\|\nabla u\|_{L_2(\Omega)}$.

(2) $\overrightarrow{Y}$ means weak convergence in the space Y.

REMARK. The bound $\|\boldsymbol{u}^\sigma\|_{H^1(\Omega)} < c$ may be obtained from suitable boundary values. These well-known arguments will be sketched in the next section, but Lemma 1 and Theorem 1 are *not* subordinate to any particular boundary condition: they are interior results.

Cauchy's inequality will be used frequently. It states that for any $e_{ij} = e_{ji}$, $d_{ij} = d_{ji}$ and $\varepsilon > 0$,

$$e_{ij} C^\sigma_{ijkl} d_{kl} \leqslant \frac{\varepsilon}{2} C^\sigma_{ijkl} e_{ij} e_{kl} + \frac{1}{2\varepsilon} C^\sigma_{ijkl} d_{ij} d_{kl} ,$$

since

$$0 \leqslant \left(\sqrt{\varepsilon}\, e_{ij} - \frac{1}{\sqrt{\varepsilon}} d_{ij}\right) C^\sigma_{ijkl} \left(\sqrt{\varepsilon}\, e_{kl} - \frac{1}{\sqrt{\varepsilon}} d_{kl}\right) .$$

PROOF OF LEMMA 1. The existence of $\boldsymbol{u}$ and $\boldsymbol{\tau}$ follow from the hypothesis that $\|\boldsymbol{u}^\sigma\|_{H^1(\Omega)} < c$ and the weak compactness of closed balls in $H^1(\Omega)$ and $L_2(\Omega)$. For $h \neq 0$

$$\Delta^h_\alpha u_i(x) \equiv \frac{1}{h}\left(u_i(x + \boldsymbol{e}_\alpha h) - u_i(x)\right) , \qquad \alpha = 1, \ldots, n-1$$

and

$$\Omega_\varrho \equiv \{x | x \in \Omega \text{ and } \operatorname{dist}(x, \partial\Omega) > \varrho\} .$$

The following difference-quotient technique is similar to that discussed in Ladyzhenskaya and Ural'tseva [6]. If $\boldsymbol{\eta} \in H^1_0(\Omega_{\varrho/2})$ and $|h| < \varrho/2$,

$$\int_\Omega C^\sigma_{ijkl} (\Delta^h_\alpha u^\sigma_h)_{,l} \eta_{i,j} \, dx = 0 .$$

Define $\zeta \in C^\infty_0(\Omega_{\varrho/2})$ such that $\zeta = 1$ on Ω_ϱ. Then choosing

$$\eta_i = \Delta^h_\alpha u^\sigma_i \zeta^2$$

$$v_i \equiv \Delta^h_\alpha u^\sigma_i$$

$$\int_\Omega C^\sigma_{ijkl} v_{i,j} v_{k,l} \zeta^2 \, dx = -2 \int_\Omega C^\sigma_{ijkl} \zeta \zeta_{,j} v_i v_{k,l} \, dx \equiv J .$$

From (3) and (3′)

$$\frac{\lambda}{2} \int_\Omega (v_{i,j} v_{i,j} + v_{i,j} v_{j,i}) \zeta^2 \, dx \leqslant J .$$

Using the integration by parts formulae,

$$\int_\Omega v_{i,j}v_{j,i}\zeta^2\,dx = \int_\Omega (v_{i,i})^2\zeta^2\,dx + 2\int_\Omega v_{i,i}v_j\zeta\zeta_{,j}\,dx - 2\int_\Omega v_{i,i}v_j\zeta_{,i}\zeta\,dx\,.$$

Using Cauchy's inequality and the above identity,

$$\frac{\lambda}{2}\int_\Omega v_{i,j}v_{i,j}\zeta^2\,dx \leqslant J + \int_\Omega\left[2\,\varepsilon\,v_{i,j}v_{i,j}\zeta^2 + \frac{2\lambda^2}{\varepsilon}|\boldsymbol{v}|^2|\nabla\zeta|^2\right]dx\,.$$

Decomposing J similarly and absorbing the gradient terms on the left, one has

$$\Big\|\sum_{i,j=1}^{n}(\Delta_\alpha^h u_{i,j}^\sigma)\Big\|_{L_1(\Omega_\varrho)} \leqslant c_0(\lambda, \Lambda, n, \Omega, \varrho)\|\Delta_\alpha^h \boldsymbol{u}^\sigma\|_{L_2(\Omega_{\varrho/2})}$$

where c_0 depends upon the parameters listed. Its explicit form is peripheral; such constants will often be lumped together. This implies

$$\|\boldsymbol{u}_{,\alpha}^\sigma\|_{H^1(\Omega_\varrho)} \leqslant c_1(\lambda, \Lambda, n, \Omega, \varrho, c)$$

therefore,

$$\|\tau_{ij,\alpha}^\sigma\|_{L_2(\Omega_\varrho)} \leqslant c_2(\lambda, \Lambda, n, \Omega, \varrho, c)\,.$$

For $\boldsymbol{\eta}\in H_0^1(\Omega)$,

$$\sum_{i=1}^{n}\int_\Omega \tau_{in}^\sigma\eta_{i,n}\,dx = -\sum_{i=1}^{n}\sum_{\alpha=1}^{n-1}\int_\Omega \tau_{i\alpha}^\sigma\eta_{i,\alpha}\,dx = \sum_{i=1}^{n}\sum_{\alpha=1}^{n-1}\int_\Omega \tau_{i\alpha,\alpha}^\sigma\eta_i\,dx\,,$$

therefore

$$\tau_{in,n}^\sigma = -\sum_{\alpha=1}^{n-1}\tau_{i\alpha,\alpha}^\sigma$$

$$\|\tau_{ni}^\sigma\|_{H^1(\Omega_\varrho)} \leqslant c_3(\lambda, \Lambda, n, \Omega, \varrho, c)\,,\qquad \text{for } i = 1, \dots, n\,.$$

The conclusion follows from the weak compactness of closed balls in H^1. □

Let a^σ and b^σ be measurable functions such that $a^\sigma \rightharpoonup a^0$ and $b^\sigma \rightharpoonup b^0$ is L_2. Suppose also that $a^\sigma b^\sigma$ converges weakly in L_1. The limit will be denoted $a^\sigma b^\sigma \rightharpoonup (ab)^0$. Generally $(ab)^0 \neq a^0 b^0$ unless a^σ or b^σ converge strongly in L_2.

The submatrix $[C_{injn}^\sigma]$, $i, j = 1, \dots, n$ is nonsingular (with smallest eigenvalue not less than $\lambda/2$) by (3), so call the inverse S_{ij}^σ.

THEOREM 1. *Let $\boldsymbol{u}^\sigma$ be a solution of* (2) *associated with* $\boldsymbol{C}^\sigma$. *Suppose* $\{\boldsymbol{C}^\sigma\}_{\sigma\in(0,1)}$ *satisfy* (3), $\boldsymbol{C}_{,\alpha}=0$ *for* $\alpha=1,\ldots,n-1$, *and* $\|\boldsymbol{u}^\sigma\|_{H^1(\Omega)}\leqslant c$. *Let* σ_r *be any subsequence such that* $\boldsymbol{u}^{\sigma_r}$, $\boldsymbol{\tau}^{\sigma_r}$, $\boldsymbol{S}^{\sigma_r}$, $\boldsymbol{C}^{\sigma_r}$, $C^{\sigma_r}_{ijnk}S^{\sigma_r}_{ks}C^{\sigma_r}_{snmq}$ *and* $C^{\sigma_r}_{ijnk}S^{\sigma_r}_{kt}$ *converge weakly to* $\boldsymbol{u}$ *in* $H^1(\Omega)$, *to* $\boldsymbol{\tau}$, $\boldsymbol{S}^0$, $\boldsymbol{C}^0$, $(C_{ijnk}S_{ks}C_{snmq})^0$ *and* $(C_{ijnk}S_{kt})^0$ *respectively in* $L_2(\Omega)$. (*These exist since closed balls in* $L_2(\Omega)$ *are weakly compact.*) *Define*

$$A_{ijkl}\equiv C^0_{ijkl}-(C_{ijnq}S_{qs}C_{snkl})^0+(C_{ijnq}S_{qs})^0(S^0_{sl})^{-1}(S_{lt}C_{ntmq})^0 .$$

Then

$$\tau_{ij}=A_{ijkl}u_{k,l}$$

and

$$(A_{ijkl}u_{k,l})_{,j}=0$$

in the weak sense.

PROOF.

$$\tau^\sigma_{in}=\sum_{j=1}^{n}\sum_{\alpha=1}^{n-1}C^\sigma_{inj\alpha}u^\sigma_{j,\alpha}+\sum_{j=1}^{n}C^\sigma_{injn}u^\sigma .$$

So (using explicit summation)

$$u^\sigma_{j,n}=\sum_{i=1}^{n}S^\sigma_{ij}\tau^\sigma_{in}-\sum_{i,m=1}^{n}\sum_{\alpha=1}^{n-1}S^\sigma_{ji}C^\sigma_{inm\alpha}u^\sigma_{m,\alpha} . \tag{5}$$

By Lemma 1 and the compactness theorem of Rellich [10], $\{\tau^{\sigma_r}_{in}\}_r$ and $\{\boldsymbol{u}^{\sigma_r}_{,\alpha}\}_r$ converge strongly in $L_2^{\mathrm{loc}}(\Omega)$. Let $\varrho>0$. Then

$$u_{j,n}=\sum_{i=1}^{n}S^0_{ji}\tau_{in}-\sum_{m,i=1}^{n}\sum_{\alpha=1}^{n-1}(S_{ij}C_{inm\alpha})^0u_{m,\alpha}\qquad\text{a.e. in }\Omega_\varrho ,$$

so

$$\tau_{in}=\sum_{j,t,,m,q=1}^{n}(S^0_{ij})^{-1}(S_{jq}C_{qnmt})^0u_{m,t}\qquad\text{a.e. in }\Omega_\varrho . \tag{6}$$

In general,

$$\tau^\sigma_{ij}=\sum_{\alpha=1}^{n-1}\sum_{k=1}^{n}C^\sigma_{ijk\alpha}u^\sigma_{k,\alpha}+\sum_{k=1}^{n}C^\sigma_{ijkn}u^\sigma_{k,n} .$$

Using (5),

$$\tau^\sigma_{ij}=\sum_{\alpha=1}^{n-1}\Big[\sum_{m=1}^{n}C^\sigma_{ijm\alpha}-\sum_{t,k=1}^{n}C^\sigma_{ijkn}S^\sigma_{kt}C^\sigma_{tnm\alpha}\Big]u^\sigma_{m,\alpha}+\sum_{,k=1}^{n}C^\sigma_{ijkn}S^\sigma_{kt}\tau^\sigma_{tn} .$$

Again $\boldsymbol{u}^{\sigma_r}_{,\alpha}$ and $\tau^{\sigma_r}_{tn}$ converge strongly in $L_2(\Omega_\varrho)$ so

$$\tau_{ij}=\sum_{\alpha=1}^{n-1}\Big[\sum_{m=1}^{n}C^0_{ijm\alpha}-\sum_{k,t=1}^{n}(C_{ijkn}S_{kt}C_{tnm\alpha})^0\Big]u_{m,\alpha}+\sum_{k,t=1}^{n}(C_{ijkn}S_{kt})^0\tau_{tn} ,\qquad\text{a.e. in }\Omega_\varrho .$$

Using (6) and summing to n all repeated indices,

$$\tau_{ij} = [C^0_{ijmq} - (C_{ijkn}S_{ks}C_{snmq})^0 + (C_{ijkn}S_{ks})^0(S^0_{sr})^{-1}(S^{rt}C_{tnmq})^0]u_{m,q} \quad \text{a.e. in } \Omega_\varrho .$$

Since ϱ is arbitrary, this must hold a.e. in Ω. □

One would expect that $\boldsymbol{A}$ satisfies ellipticity conditions like (3). This is indeed the case.

LEMMA 2. *There exists $\lambda' > 0$ and $\Lambda' < \infty$ such that*

$$\lambda' e_{ij}e_{ij} \leqslant A_{ijkl}e_{ij}e_{kl} \quad \textit{and} \quad \max_{i,j,k,l} \operatorname*{ess\,sup}_{x\in\Omega} |A_{ijkl}(x)| \leqslant \Lambda' .$$

PROOF. The existence of Λ' follows from the boundedness of each term in $\boldsymbol{A}$. The search for λ' seems to require some machinery. Let $e_{ij}e_{ij} = 1$. Consider the function

$$(7) \qquad f^\sigma(\boldsymbol{e}) \equiv (C^\sigma_{ijkl} - C^\sigma_{ijnq}S^\sigma_{qr}C^\sigma_{nrkl} + (C_{ijnq}S_{qk})^0(S^0_{kr})^{-1}(S^{rt}C_{ntkl})^0)\, e_{ij}e_{kl}$$

which has a Lipschitz bound independent of x and σ. Then in $L_2(\Omega)$,

$$f^\sigma(e) \rightharpoonup e_{ij}A_{ijkl}e_{kl} .$$

$f^\sigma(\boldsymbol{e})$ will be shown to be bounded below by a positive constant λ'.

Define

$$d_{kl} = 2^{-1}(\delta_{nl}S^\sigma_{qk} + \delta_{nk}S^\sigma_{ql})\,C^\sigma_{qnmp}e_{mp} .$$

If $\boldsymbol{d} = 0$, then the second term in (7) is zero and trivially $f^\sigma(e) \geqslant \lambda$. Henceforth assume $\boldsymbol{d} \neq 0$. Now from (3),

$$(e_{ij} + \alpha d_{ij})\,C^\sigma_{ijkl}(e_{kl} + \alpha d_{kl}) \geqslant \lambda(e_{ii} + \alpha d_{ij})(e_{ij} + \alpha d_{ij}) .$$

Minimizing with respect to α yields

$$(8) \qquad (C^\sigma_{ijkl} - C^\sigma_{ijnq}S^\sigma_{qr}C^\sigma_{nrkl})\,e_{ij}e_{kl} \geqslant \lambda\big(1 - (e_{ij}d_{ij})(d_{kl}d_{kl})^{-1}\big) .$$

Finally, define

$$M = \{\varepsilon_{ij} | \varepsilon_{ij} = y_i\delta_{jn} + y_j\delta_{in} \text{ for } y \in E^n \text{ and } \varepsilon_{ij}\varepsilon_{ij} = 1\} .$$

The existence of λ' follows from considering three regions on the unit ball for $\boldsymbol{e}$. If $\boldsymbol{e}\in \boldsymbol{M}$

$$f^{\sigma}(\boldsymbol{e}) = e_{ij}(C_{ijnq}S_{qk})^0(S^0_{kr})^{-1}(S_{rt}C_{ntkl})^0 e_{kl} \geqslant \frac{\lambda}{2} .$$

Then by the uniform Lipschitz bound for f^σ, there exists a neighborhood (independent of σ and x) where $f^\sigma(\boldsymbol{e})\geqslant \lambda/4$. Since $\boldsymbol{d}(d_{ij}d_{ij})^{-1}\in \boldsymbol{M}$, if $\boldsymbol{e}$ is outside this neighborhood, (8) implies that there exists $\theta<1$ so that $f^\sigma(\boldsymbol{e})\geqslant\lambda(1-\theta)$. Define $\lambda'\equiv\min\big(\lambda(1-\theta), \lambda/4\big)$. □

The central problem this paper addresses is to determine sufficient conditions for a constant coefficient system to approximate, in the average, a system with discontinuous coefficients. Theorem 2 addresses this problem directly. To that end, define

$$C(\lambda, \Lambda) \equiv \{\boldsymbol{C}|\boldsymbol{C} \text{ satisfies (3), (3') and (4)}\}$$

$$\mathcal{K} \equiv K(x)|K(x)\subset\Omega,\ K(x) = [x_1, x_1 + b_1]\times\ldots\times[x_n, x_n + b_n],$$

$$\frac{R}{2} < (b_i b_i)^{\frac{1}{2}} < R, \text{ and } \frac{1}{2} < \max_{i,j}(b_i/b_j) < 2 .$$

If $f\in L_1(\Omega)$ and $K(x)\in\mathcal{K}_R$, then

$$f_{(K)}(x) \equiv \frac{1}{m(K)}\int\limits_{K(x)} f(\xi)\,d\xi .$$

THEOREM 2. *For every* $\varepsilon>0$ *and* $R>0$, *there exists* $\delta>0$ *such that if* $\boldsymbol{C}^\sigma\in C(\lambda,\Lambda)$, $\boldsymbol{A}$ *is constant and*

$$\operatorname*{ess\,sup}_{x\in\Omega_R} |C_{ijkl_{(K_1)}}(x) - (C_{ijsn}S_{st}C_{ntkl})_{(K_2)}(x) + \\ + (C_{ijsn}S_{st})_{(K_3)}(S_{tp_{(K_4)}})^{-1}(S_{pm}C_{nmkl})_{(K_5)}(x) - A_{ijkl}| < \delta$$

for all $K_i\in\mathcal{K}_R$, $i=1,\ldots,5$; *then to every solution* $\boldsymbol{u}$ *of* $(C_{ijkl}u_{k,l})_{,j}=0$ (*in the weak sense*) *where* $\|\boldsymbol{u}\|_{H^1(\Omega)}\leqslant 1$, *there exists a solution* $\boldsymbol{v}$ *of* $(A_{ijkl}v_{k,l})_{,j}=0$ *where* $\|\boldsymbol{v}\|_{H^1(\Omega)}\leqslant 1$,

$$\|\boldsymbol{u}-\boldsymbol{v}\|_{L^2(\Omega_R)} < \varepsilon ,$$

$$\|(\nabla\boldsymbol{u})_{(K)} - (\nabla\boldsymbol{v})_{(K)}\|_{L_2(\Omega_{R2})} < \varepsilon ,$$

and

$$\|(C_{ijkl}u_{k,l})_{(K)} - (A_{ijkl}v_{k,l})_{(K)}\| < \varepsilon$$

for all $K = K(x)\in\mathcal{K}_R$ *and* $x\in\Omega_{2R}$.

PROOF. Define the solution space

$$\mathcal{S}(A) \equiv \{u \mid \|u\|_{H^1(\Omega)} \leqslant 1 \text{ and } (A_{ijkl}u_{k,l})_{,j} = 0\} .$$

To gain a contradiction, suppose there exist $\varepsilon > 0$, $R > 0$ and sequences $\{C^r\}_r \subset C(\lambda, \Lambda)$, $\{A^r\}_r$ and $\{u^r\}_r$ such that

$$|C^r_{ijkl(K_1)}(x) - (C^r_{ijsn}S^r_{st}C^n_{ntkl})_{(K_2)}(x) + (C^r_{ijsn}S^r_{st})_{(K_3)} \cdot (S^r_{tp(K_4)})^{-1} \cdot$$
$$\cdot (S^r_{pm}C^r_{mnkl})_{(K_5)}(x) - A^r_{ijkl}| < \delta = \frac{1}{r}$$

for all $K_i \in \mathcal{K}_R$, $\|u^r\|_{H^1(\Omega)} \leqslant 1$ and, say

$$\inf_{v \in \mathcal{S}(A^r)} \|u^r - v\|_{L_2(\Omega_R)} \geqslant \varepsilon .$$

Since $\{A^r\}_r$ is bounded, there exists $A^{r_m} \to A^0$ as $m \to \infty$; and since $C(\lambda, \Lambda)$ is weakly compact, there are weak limits C^0_{ijkl}, $(C_{ijsn}S_{st}C_{ntkl})^0$, S^0_{tp}, and $(C_{ijsn}S_{st})^0$ of appropriate subsequences such that

$$C^0_{ijkl_{(K_1)}}(x) - (C_{ijsn}S_{st}C_{ntkl})^0_{(K_2)}(x) + (C_{ijsn}S_{st})^0_{(K_3)}(S^0_{tp_{(K_4)}})^{-1}(S_{pm}C_{nmkl})^0_{(K_5)}(x) = A^0_{ijkl}$$

for all $K_i \in \mathcal{K}_R$. The K_i are independent, so treating each integral separately, and differentiating with respect to the length of each edge yields

$$C^0_{ijkl} - (C_{ijsn}S_{st}C_{ntkl})^0 + (C_{ijsn}S_{st})^0(S^0_{tp})^{-1}(S_{pm}C_{nmkl})^0 = A^0_{ijkl} \qquad \text{a.e. in } \Omega_R .$$

Notice that by Lemma 2, $A^{r_m} \in C(\lambda'/2, 2\Lambda')$ for large m. Now from Theorem 1, there exists a weak limit u such that $u^{r_m} \to u$ in $L_2(\Omega_R)$ and $u \in \mathcal{S}(A^0)$. Clearly there exists $v^{r_m} \in \mathcal{S}(A^{r_m})$ such that $v^{r_m} \to u$ in $H^1(\Omega_R)$ (e.g., choose $v^{r_m} - u \in H^1_0(\Omega)$) contradicting the assumption that the distance from u^r to $\mathcal{S}(A^r)$ is at least ε. The other cases are handled similarly. $\square$

If such C and A exist, the v associated with u of Theorem 2 is not unique, but a natural identification would be to choose v to attain the same boundary data as u .

2. – Global results for boundary value problems.

The usual boundary value problems are one of three types: $(C_{ijkl}u_{k,l})_{,j} = 0$ and

(A) $u = \varphi$ on $\partial\Omega$,

(B) $\tau_{ij}n_j = \psi_i$ on $\partial\Omega$ where $\boldsymbol{n}$ is the outer unit normal,

or

(C) $\boldsymbol{u} = \boldsymbol{\varphi}$ on S_1, and $\tau_{ij}n_j = \psi_i$ on S_2, $S_1 \cup S_2 = \partial\Omega$.

To use Theorem 1, one must establish that (A), (B) and (C) can be interpreted in the setting of $H^1(\Omega)$ and that these solutions have uniformly bounded $H^1(\Omega)$ norms. The following are standard Sobolev Space interpretations by Ladyzhenskaya and Ural'tseva [6]; they can be motivated by formally using the divergence theorem.

(A) Suppose $\boldsymbol{\varphi} \in H^1(\Omega)$, then a weak solution $\boldsymbol{u}$ to (2), where $\boldsymbol{u} - \boldsymbol{\varphi} \in H^1_0(\Omega)$, is said to be a *weak solution to* (A).

(B) Suppose $\boldsymbol{\psi} \in L_2(\partial\Omega)$, and $\boldsymbol{\eta} \in H^1(\Omega)$, then $\boldsymbol{u}$ is a *weak solution to* (B) iff

$$\int_\Omega C_{ijkl} u_{k,l} \eta_{i,j}\, dx = \int_{\partial\Omega} \psi_i \eta_i\, dS\,,$$

$$\int_\Omega u_i\, dx = 0\,,$$

and

$$\int_\Omega [u_{i,j} - u_{j,i}]\, dx = 0\,.$$

Here the values of $\boldsymbol{\eta}$ on $\partial\Omega$ are interpreted as the trace of $\boldsymbol{\eta}$. For example, $\boldsymbol{u} \in H^1(\Omega)$ is said to *vanish on* $S_1 \subset \partial\Omega$ iff there exists a $C^\infty(\Omega)$ sequence approximating $\boldsymbol{u}$ in $H^1(\Omega)$ where each function vanishes in a strip adjacent to S_1.

(C) Suppose $\boldsymbol{\psi} \in L_2(S_2)$, $\boldsymbol{\eta}, \boldsymbol{\varphi} \in H^1(\Omega)$, and $\boldsymbol{\eta}$ vanishes on S_1. Then $\boldsymbol{u}$ is a *weak solution to* (C) iff

$$\int_\Omega C_{ijkl} u_{k,l} \eta_{i,j}\, dx = \int_{S_2} \psi_i \eta_i\, dS$$

and $\boldsymbol{u} - \boldsymbol{\varphi}$ vanishes on S_1.

The existential questions for (A), (B), and (C) are discussed for single equations by Ladyzhenskaya and Ural'tseva [6] and their methods readily generalize to the systems considered here. The following well-known results are summarized for later discussion.

LEMMA 3. *If* $\boldsymbol{u} \in H^1(\Omega)$ *is a weak solution to* (A), (B) *or* (C) *with* $\boldsymbol{C}$ *satisfying* (3), (3′) *but not necessarily* (4), *then there exists a number* c *independent* *of* $\boldsymbol{C}$ *such that*

$$\|\boldsymbol{u}^\sigma\|_{H^1(\Omega)} \leqslant c .$$

Here c *depends on* Ω, λ, Λ *and the boundary values.*

REMARK. The technique of proof is similar to that used for a single equation by Ladyzenskaya and Ural'tseva [6] except in place of the usual uniform-ellipticity inequality to bound the L_2 norm of the derivatives, one uses (3) and Korn's inequality as discussed by Gurtin [4] or Fichera [3]. It states that if $\boldsymbol{v} \in H^1(\Omega)$ and there exists $S_1 \subset \partial\Omega$ where surface measure of $S_1 > 0$ and $\boldsymbol{v}$ vanishes on S_1, or

$$\int_\Omega (v_{i,j} - v_{j,i})\, dx = 0 , \qquad i, j = 1, \ldots, n ,$$

then there exists a k such that

$$k\int_\Omega (v_{i,j} + v_{i,j})(v_{i,j} + v_{i,j})\, dx \geqslant \int_\Omega v_{i,j} v_{i,j}\, dx .$$

The global version of Theorem 1 is

THEOREM 3. *Suppose* $\{\boldsymbol{C}^\sigma\}$ *satisfies* (3), (3') *and* (4). *Consider fixed boundary values and let* $\boldsymbol{u}^\sigma$ *satisfy either* (A), (B) *or* (C). *Then there exist weak limits* $\boldsymbol{u}$ *and* $\boldsymbol{\tau}$ *of Lemma* 1 *such that*

i) $\tau_{ij,j} = 0$ *in the weak sense,*

ii) $\boldsymbol{u}$ *and* $\boldsymbol{\tau}$ *satisfy the same boundary values as* $\boldsymbol{u}^\sigma$ *and* $\boldsymbol{\tau}^\sigma$,

iii) $\tau_{ij} = A_{ijmq} u_{m,q}$ a.e. *in* Ω, *as in Theorem* 1, *and*

iv) *Upon restriction to the subsequence* $\{\boldsymbol{C}^{\sigma_r}\}_{r=1}^\infty$ *that generates* $\boldsymbol{A}$, *there is a unique limit* $\boldsymbol{u}$.

PROOF. The existence of $\boldsymbol{u}$ and $\boldsymbol{\tau}$ follow from Lemmas 1 and 3. $\boldsymbol{\tau}^{\sigma_r} \xrightarrow[L_2(\Omega)]{} \boldsymbol{\tau}$ implies that as $r \to \infty$ and $\forall \boldsymbol{\eta} \in H^1(\Omega)$,

$$\int_\Omega \tau_{ij}^{\sigma_r} \eta_{i,j}\, dx \to \int_\Omega \tau_{ij} \eta_{i,j}\, dx$$

so it satisfies the same equation; hence (i). If $\boldsymbol{u}^\sigma - \boldsymbol{\varphi} = 0$ on $S_1 \subset \partial\Omega$, then since $\boldsymbol{u}^{\sigma_r} \xrightarrow[H^1(\Omega)]{} \boldsymbol{u}$, $\boldsymbol{u} - \boldsymbol{\varphi} = 0$ on S_1 too; hence (ii). Theorem 1 implies (iii). The solution $\boldsymbol{u}\in H^1(\Omega)$ to

$$\int_\Omega A_{ijm\alpha}u_{m,\alpha}\eta_{i,j}\,dx = 0$$

$\forall\boldsymbol{\eta}\in H_0^1(\Omega)$ is unique by Lemma 2 for boundary conditions (A), (B) or (C). (If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two solutions, choose $\boldsymbol{\eta}=\boldsymbol{u}-\boldsymbol{v}$. Subtracting the equation $\boldsymbol{v}$ satisfies from the one $\boldsymbol{u}$ satisfies and applying Korn's inequality implies $\boldsymbol{\eta}=0$.) Hence the sequence $\{\boldsymbol{u}^{\sigma_r}\}_{r=1}^\infty$ has a unique accumulation point. This is the limit point by the compactness theorems. □

REMARK. If $\boldsymbol{C}^\sigma\to\boldsymbol{C}^0$ in $L_2(\Omega)$, then $\boldsymbol{C}^0=\boldsymbol{A}$, as is well known, for example see Ladyzhenskaya and Ural'tseva [6], or Spagnolo [11].

3. – Spectra.

Results analogous to Lemma 1 and Theorem 1 hold for eigenvalues and eigenfunctions. Define the *eigenvalue* μ^σ *and the eigenfunction* $\boldsymbol{u}^\sigma$ for weight ϱ^σ of $\boldsymbol{L}^\sigma(\boldsymbol{u})\equiv(C^\sigma_{ijkl}u_{k,l})_{,j}$ to be a number and a function such that $\forall\boldsymbol{\eta}\in H_0^1(\Omega)$,

$$\int_\Omega C^\sigma_{ijkl}u^\sigma_{k,l}\eta_{i,j}\,dx = \mu^\sigma\int_\Omega u_i^\sigma\eta_i\varrho^\sigma\,dx \qquad \text{and} \int_\Omega |\boldsymbol{u}^\sigma|^2\varrho^\sigma\,dx = 1$$

One can consider boundary value problems (A), (B), or (C) with homogeneous boundary values $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$. A result parallel to Lemma 1 and Theorem 3 is

THEOREM 4. *Suppose* $\boldsymbol{C}^\sigma$ *satisfies* (3), (3′) *and* (4), $\varrho^\sigma\equiv\varrho^\sigma(x_n)$, *and* $\lambda\leqslant\varrho^\sigma\leqslant\Lambda$. *Consider boundary conditions* (A), (B) *or* (C) *and let* $\boldsymbol{u}^\sigma_{(j)}$, *and* μ^σ_j *be the* j-th *eigenfunction and eigenvalue* ($\mu^\sigma_j\leqslant\mu^\sigma_{j+1}$) *of* $\boldsymbol{L}^\sigma$ *with weight* ϱ^σ. *If there exist numbers* c_j *such that* $\mu^\sigma_j<c_j$, *then there exists a subsequence* $\sigma_r\to0$ *as* $r\to\infty$ *such that for all* $j=1,2,\ldots$, *and* $1<p<\infty$,

$$\varrho^{\sigma_r}\xrightarrow[L_p(\Omega)]{}\varrho\,,\quad \mu\to\mu_j^{\sigma_r},\quad \boldsymbol{u}^{\sigma_r}_{(j)}\xrightarrow[H^1(\Omega)]{}\boldsymbol{u}_{(j)}\,,\quad \boldsymbol{\tau}^{\sigma_r}_{(j)}\xrightarrow[L_2(\Omega)]{}\boldsymbol{\tau}_{(j)}\,,$$
$$\boldsymbol{u}^{\sigma_r}_{(j),\alpha}\xrightarrow[H^1_{\mathrm{loc}}(\Omega)]{}\boldsymbol{u}_{(j),\alpha}\,,\qquad \alpha=1,\ldots,n-1$$

and

$$\tau^{\sigma_r}_{in(j)}\xrightarrow[H^1_{\mathrm{loc}}(\Omega)]{}\tau_{in(j)}\,,\qquad i=1,\ldots,n\,.$$

$$\tau_{lm(j)}=A_{lmst}u_{s(t),t}$$

where $\boldsymbol{A}$ *is generated from* $\{\boldsymbol{C}^{\sigma_r}\}_{r=1}^{\infty}$ *as in Theorem* 1. *Finally*

$$\tau_{lm(j),m} = \varrho\mu_j u_{l(j)}$$

in the weak sense and $\boldsymbol{u}$ *and* $\boldsymbol{\tau}$ *satisfy the same boundary values as* $\boldsymbol{u}^\sigma$ *and* $\boldsymbol{\tau}^\sigma$.

REMARK. There are standard arguments to find the c_j's. They will be outlined in Lemma 4.

PROOF. By Korn's inequality and the normalization

$$\|\boldsymbol{u}^\sigma_{(j)}\|_{H^1(\Omega)} < c(c_j, \Omega, n, \lambda, k) .$$

Since $|\mu^\sigma_j| < c_j$, by the compactness of the real line, there exists a converging subsequence $\mu_j^{\sigma_r} \to \mu_j$. If $\varrho^\sigma \rightharpoonup \varrho$ in $L_\infty(\Omega)$, the conclusion follows as in Theorem 1. Otherwise, from the stronger imbedding theorem [6] that the unit ball of $H^1(\Omega)$ is compact in $L_q(\Omega)$, $q < 2n/(n-2)$, $\varrho^{\sigma_r}u^{\sigma_r} \rightharpoonup \varrho u$ in $L_2(\Omega)$. □

In fact, μ_j is the j-th eigenvalue of $\boldsymbol{A}(\boldsymbol{u}) \equiv (A_{ijmq}u_{m,q})_{,j}$. This will be discussed in

LEMMA 4. *Suppose* $\boldsymbol{C}^\sigma$ *satisfies* (3) *and* (3′) *but not necessarily* (4), *and* $\lambda \leqslant \varrho^\sigma \leqslant \Lambda$. *Let* $\boldsymbol{u}^\sigma_{(j)}$ *and* μ^σ_j *be the eigenfunctions and eigenvalue of* $\boldsymbol{L}^\sigma$ *with weight* ϱ^σ *for homogeneous problems* (A), (B) *or* (C). *Then there exist numbers* c_j *such that* $\mu^\sigma_j < c_j$ *and a subsequence* $\{\sigma_r\}_{r=1}^\infty$ *such that as* $r \to \infty$

$$\sigma_r \to 0\,, \quad \boldsymbol{u}^{\sigma_r}_{(j)} \xrightarrow[H^1(\Omega)]{} \boldsymbol{u}_{(j)}\,, \quad \boldsymbol{\tau}^{\sigma_r}_{(j)} \xrightarrow[L_2(\Omega)]{} \boldsymbol{\tau}_{(j)}$$

$$\mu_j^{\sigma_r} \to \mu_j \quad \text{and} \quad \varrho^{\sigma_r} \xrightarrow[L_p(\Omega)]{} \varrho \qquad \text{for } 1 < p < \infty\,.$$

Furthermore

$$\tau_{ml(j),l} = \mu_j\,\varrho u_{m(j)}$$

in the weak sense and $\boldsymbol{u}_{(j)}$ *and* $\boldsymbol{\tau}_{(j)}$ *satisfy the same boundary values as* $\boldsymbol{u}^\sigma_{(j)}$ *and* $\boldsymbol{\tau}^\sigma_{(j)}$ *do. Lastly, suppose* $\boldsymbol{w} \in H^1(\Omega)$ *and* $(\boldsymbol{w}, \varrho\boldsymbol{u}_{(j)}) = 0$ *for all* j. *In addition if for problem* (A), $\boldsymbol{w} \in H^1_0(\Omega)$, *if for* (B),

$$\int_\Omega (w_{j,i} - w_{i,j})\, dx = 0\,,$$

or if for problem (C) $\boldsymbol{w}$ *vanishes on* S_1; *then* $\boldsymbol{w} = 0$.

REMARK. In this general setting, no $\boldsymbol{\tau} \sim \nabla \boldsymbol{u}$ relationship is determined.

PROOF. If suffices to bound the eigenvalues of (A) since they are the largest. Let l_j be the j-th eigenvalue of the Laplacian operator with weight $\varrho = 1$ for the homogeneous Dirichlet boundary data. From Korn's inequality and (3), if $\boldsymbol{v} \in H_0^1(\Omega)$ and $\boldsymbol{v} \neq 0$,

$$\frac{\lambda}{4K\Lambda} \frac{\int_\Omega v_{i,j} v_{i,j}\, dx}{(\boldsymbol{v}, \boldsymbol{v})} \leqslant \frac{\int_\Omega C^\sigma_{ijkl} v_{i,j} v_{k,l}\, dx}{(\boldsymbol{v}, \varrho^\sigma \boldsymbol{v})} \leqslant \frac{n^4 \Lambda}{\lambda} \frac{\int_\Omega v_{i,j} v_{i,j}\, dx}{(\boldsymbol{v}, \boldsymbol{v})}$$

where

$$(\boldsymbol{u}, \boldsymbol{w}) \equiv \int_\Omega u_i w_i \, dx \,.$$

Applying the obvious generalization of the maximum-minimum characterization of the eigenvalues by Courant and Hilbert [1],

$$\frac{\lambda}{4k\Lambda} l_j \leqslant \mu_j^\sigma \leqslant \frac{n^4 \Lambda}{\lambda} l_j \equiv c_j \,. \tag{9}$$

The converging subsequences exist due to compactness.

The completeness for problem (A) of $\{\boldsymbol{u}_{(j)}\}_{j=1}^\infty$ in $H_0^1(\Omega)$ follows easily from the growth of l_j. Assume, to gain a contradiction, that there exists a $\boldsymbol{w} \in H_0^1(\Omega)$ such that $\boldsymbol{w} \neq 0$ and $(\boldsymbol{w}, \varrho \boldsymbol{u}_{(j)}) = 0$ for all j.

$$\mu_j^\sigma = \inf_{\substack{\boldsymbol{v} \in H_0^1(\Omega) \\ \boldsymbol{v} \neq 0 \\ (\boldsymbol{v}, \varrho^\sigma \boldsymbol{u}^\sigma_{(s)}) = 0 \\ s = 1, 2, \ldots, j-1}} \frac{\int_\Omega C^\sigma_{ijkl} v_{i,j} v_{k,l}\, dx}{(\boldsymbol{v}, \varrho \boldsymbol{v})} \,,$$

But the test function

$$\boldsymbol{v} = \boldsymbol{w} - \sum_{i=1}^{j-1} (\boldsymbol{u}^\sigma_{(i)}, \varrho^\sigma \boldsymbol{w}) \boldsymbol{u}^\sigma_{(i)}$$

implies that

$$\mu_j \leqslant \frac{n^4 \Lambda}{\lambda(\boldsymbol{w}, \boldsymbol{w})} \int_\Omega w_{i,k} w_{i,k} \, dx \,,$$

or from (9), $\{l_j\}_{j=1}^\infty$ is bounded. This is a clear contradiction, so $w = 0$. The argument is similar for boundary conditions (B) and (C). □

REMARK. The set of eigenvalues of the C^σ system map onto the set of eigenvalues of the limit system.

4. – Generalizations.

The arguments of Theorems 1 and 3 generalize to an equation of the form

$$(C^\sigma_{ijkl}u_{k,l})_{,j} = f^\sigma_i$$

provided C^σ satisfies (3), (3′) and there exists a number M such that

$$\|C^\sigma_{,\alpha}\|_{L_2(\Omega)}, \qquad \|f^\sigma\|_{L_2(\Omega)}, \qquad \|f^\sigma_{,\alpha}\|_{L_2(\Omega)} < M$$

for $\alpha = 1, \ldots, n-1$. The proofs also serve to outline a similar development for a single uniformly elliptic equation with the limiting matrix A,

$$A_{ij} = c^0_{ij} - (c_{in}sc_{nj})^0 + (c_{in}s)^0(s_0)^{-1}(sc_{nj})^0$$

where

$$s^\sigma \equiv (c^\sigma_{nn})^{-1} \xrightarrow[L_p(\Omega)]{} s^0 ,$$
$$(c^\sigma_{ij}u^\sigma_{,j})_{,i} = 0 .$$

The results would extend to any strongly elliptic system provided a Gärding-type [2] inequality holds. This has only been proven for continuous coefficients.

Of course, any domain and $\{C^\sigma\}_{\sigma\in(0,1)}$ for which there exists a local coordinate transformation mapping them into the setting of Lemma 1, can be analyzed. For example, concentric cylinders with discontinuities in the r-coordinate direction can be handled.

5. – Examples.

The following two examples illustrate how A is computed and are themselves interesting.

Suppose $c^\sigma_{ij} = f^\sigma(x_3)\,\delta_{ij}$ for a single equation in E^3,

$$f^\sigma(x_3) = f^\sigma(x_3 + \sigma) ,$$
$$f(x_3) = \begin{cases} \alpha_1 & \text{if } x \in [0, \theta\sigma) , \; 0 < \theta < 1 , \\ \alpha_2 & \text{if } x \in [\theta\sigma, \sigma) , \end{cases}$$

$$c^0_{ij} = \{\theta\alpha_1 + (1-\theta)\alpha_2\}\delta_{ij}\,,$$

$$s^\sigma = \frac{1}{f^\sigma(x_3)}\,,$$

$$s^0 = \frac{\theta}{\alpha_1} + \frac{1-\theta}{\alpha_2}\,,$$

$$A = \begin{bmatrix} \theta\alpha_1 + (1-\theta)\alpha_2 & 0 & 0 \\ 0 & \theta\alpha_1 + (1-\theta)\alpha_2 & 0 \\ 0 & 0 & \left(\dfrac{\theta}{\alpha_1} + \dfrac{1-\theta}{\alpha_2}\right)^{-1} \end{bmatrix}.$$

For a system of equally spaced isotropic laminates with Lamé con stants λ_β and μ_β, $\beta = 1, 2$,

$$C^\sigma_{ijkl} = \mu_\beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda_\beta\delta_{ij}\delta_{kl}$$

$$\boldsymbol{C}^\sigma(x_3) = \boldsymbol{C}^\sigma(x_3 + \sigma)$$

$$\lambda_\beta, \mu_\beta = \begin{cases} \lambda_1, \mu_1 & \text{if } x \in [0, \sigma/2) \\ \lambda_2, \mu_2 & \text{if } x \in [\sigma/2, \sigma)\,. \end{cases}$$

Then

$$A_{11\,11} = A_{22\,22} = \mu_1 + \mu_2 + \frac{1}{2}\left\{\lambda_1 + \lambda_2 - \frac{\lambda_1^2}{\lambda_1 + 2\mu_1} - \frac{\lambda_2^2}{\lambda_2 + 2\mu_2} + \right.$$
$$\left. + \left(\frac{\lambda_1}{\lambda_1 + 2\mu_1} + \frac{\lambda_2}{\lambda_2 + 2\mu_2}\right)^2 \left(\frac{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}{\lambda_1 + \lambda_2 + 2(\mu_1 + \mu_2)}\right)\right\},$$

$$A_{11\,22} = A_{11\,11} - \mu_1 - \mu_2\,,$$

$$A_{11\,33} = A_{11\,22} + \left(\frac{\lambda_1}{\lambda_1 + 2\mu_1} + \frac{\lambda_2}{\lambda_2 + 2\mu_2}\right)\left(\frac{[\lambda_1 + 2\mu_1][\lambda_2 + 2\mu_2]}{\lambda_1 + \lambda_2 + 2\mu_1 + 2\mu_2}\right)\cdot$$
$$\cdot\left(\frac{\mu_1}{\lambda_1 + 2\mu_1} + \frac{\mu_2}{\lambda_2 + 2\mu_2}\right) - \left(\frac{\mu_1\lambda_1}{\lambda_1 + 2\mu_1} + \frac{\mu_2\lambda_2}{\lambda_2 + 2\mu_2}\right),$$

$$A_{22\,33} = A_{11\,33}\,,$$

$$A_{33\,33} = A_{11\,11} + 2\left(\frac{\lambda_1}{\lambda_1 + 2\mu_1} + \frac{\lambda_2}{\lambda_2 + 2\mu_2}\right)\left(\frac{[\lambda_1 + 2\mu_1][\lambda_2 + 2\mu_2]}{\lambda_1 + \lambda_2 + 2\mu_1 + 2\mu_2}\right)\cdot$$
$$\cdot\left(\frac{\mu_1}{\lambda_1 + 2\mu_1} + \frac{\mu_2}{\lambda_2 + 2\mu_2}\right) - 2\left(\frac{\mu_1\lambda_1}{\lambda_1 + 2\mu_1} + \frac{\mu_2\lambda_2}{\lambda_2 + 2\mu_2}\right)$$
$$- 2\left(\frac{\mu_1 + \mu_2}{4\mu_1\mu_2} - \mu_1 - \mu_2\right) +$$
$$+ 4\left(\frac{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}{\lambda_1 + \lambda_2 + 2\mu_1 + 2\mu_2}\right)\left(\frac{\mu_1}{\lambda_1 + 2\mu_1} + \frac{\mu_2}{\lambda_2 + 2\mu_2}\right)^2$$
$$- 4\left(\frac{\mu_1^2}{\lambda_1 + 2\mu_1} + \frac{\mu_2^2}{\lambda_2 + 2\mu_2}\right),$$

$$A_{12\,12} = \frac{\mu_1 + \mu_2}{2},$$

$$A_{23\,23} = A_{13\,13} = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2}.$$

All other $A_{ijmq} = 0$. A consequence is $A_{1111} = A_{1122} + 2A_{1212}$, so this coefficient matrix is transversely isotropic (hexagonal) about the x_3 axis: the representation of A is invariant under rotations about the x_3 axis. This is physically reasonable.

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