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HILBERT SCHEME OF SMOOTH SPACE CURVES

LAWRENCE EIN

Denote by $H_{d,g,n}$ the open subscheme of the Hilbert Scheme parametrizing the smooth irreducible curves of degree d and genus g in \mathbb{P}^n . The purpose of this paper is to prove that $H_{d,g,3}$ is irreducible when $d \geq g+3$. We also prove that every irreducible reduced curve in \mathbb{P}^3 with $d \geq P_d+2$ is smoothable in \mathbb{P}^3 . These results answer two questions proposed by Hartshorne and Hirschowitz ([5], 1.4). I would also like to remark that these results were asserted by Severi with an incomplete proof ([8], p. 370).

Let $\mathcal{C} \rightarrow M_{g,m}$ be the universal family of smooth curves over the fine moduli space of genus g curves with level m structure. Suppose $\mathcal{P}ic \mathcal{C}$ is the relative Picard scheme. Set $\mathcal{W}_d^r = \{(\mathcal{L}, C) \in \mathcal{P}ic \mathcal{C} \mid \mathcal{L} \text{ is a degree } d \text{ line bundle on a curve } C \text{ and } h^0(\mathcal{L}) \geq r+1\}$. Now suppose that \mathcal{L} is a degree d very ample line bundle with

$$h^0(\mathcal{L}) = r+1 \quad \text{and} \quad h^1(\mathcal{L}) = \delta > 0.$$

We show that if Y is an irreducible component of \mathcal{W}_d^r containing the point corresponding to (\mathcal{L}, C) , then $\dim Y \leq 5g-1-4\delta-d$. We also show that the above inequality implies that $H_{d,g,3}$ is irreducible when $d \geq g+3$. More generally we prove that $H_{d,g,n}$ is irreducible when

$$d > \frac{(2n-3)g+n+3}{n}.$$

I should also point out that Joe Harris has found an example where $H_{d,g,n}$ is reducible when $d \geq g+n$. Throughout the paper we shall work over the complex numbers.

I would like to thank Mark Green and Rob Lazarsfeld for many helpful discussions.

LEMMA 1. — *Let E be a rank m locally free sheaf on a smooth irreducible curve C . Let $X = \mathbb{P}(E)$ and $\pi: X \rightarrow C$ be the projection map. We denote by U the tautological line bundle of $\mathbb{P}(E)$. Suppose $V \subseteq H^0(U)$ is a $r+1$ -dimensional subspace. Then,*

(a) *The natural map $V \otimes \mathcal{O}_X \rightarrow U$ is surjective, if and only if $V \otimes \mathcal{O}_C \rightarrow \pi_* U = E$ is surjective.*

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(b) Assume that $|V|$ gives a birational morphism

$$f: X \rightarrow f(X) = Y \subseteq \mathbb{P}^r. \quad \text{Set } F = \ker(V \otimes \mathcal{O}_C \rightarrow E).$$

Then there is an exact sequence,

$$0 \rightarrow (\Lambda^m E)^* \otimes \mathcal{O}_C \left(\sum_1^{r-m} p_j \right) \rightarrow F \rightarrow \sum_1^{r-m} \mathcal{O}_C(-p_j) \rightarrow 0$$

where p_j 's are general points on C .

Proof. — (a) Suppose that $V \otimes \mathcal{O}_X \rightarrow U$ is surjective. Let $M = \ker(V \otimes \mathcal{O}_X \rightarrow U)$. If $R = \pi^{-1}(x)$ then

$$M|_R \cong \Omega_{\mathbb{P}^{r-m-1}}^1(1) \oplus (r+1-m) \mathcal{O}_{\mathbb{P}^{r-m-1}}.$$

Hence, $R^1 \pi_* M = 0$. It follows that $V \otimes \mathcal{O}_C \rightarrow \pi_* U = E$ is surjective. Conversely, if $V \otimes \mathcal{O}_C \rightarrow E$ is surjective, then the composition $V \otimes \mathcal{O}_X \rightarrow \pi^* E \rightarrow U$ is also surjective.

(b) Set $Y = f(X)$. Choose $r-m$ general points y_1, y_2, \dots, y_{r-m} in Y . We may assume that $\{y_1, y_2, \dots, y_{r-m}\}$ spans a $(r-m-1)$ -plane L in \mathbb{P}^r .

By the uniform position lemma [2], we may assume that

$$L \cap Y = \{y_1, y_2, \dots, y_{r-m}\}.$$

Furthermore we shall assume that $f^{-1}(y_i) = q_i$ and f is an isomorphism in a neighborhood of q_i . Set

$$Q = \{q_1, q_2, \dots, q_{r-m}\}.$$

Consider the exact sequence

$$0 \rightarrow I_Q \otimes U \rightarrow U \rightarrow U|_Q \rightarrow 0,$$

where I_Q is the ideal sheaf of Q in X . Set $p_i = \pi(q_i)$ and $P = \pi(Q)$. Observe that the restriction map $V \rightarrow H^0(U|_Q)$ is surjective.

Let $W = \ker(V \rightarrow H^0(U|_Q))$. Observe that the natural map

$$\pi_* U = E \rightarrow \pi_*(U|_Q) = \sum_{i=1}^{r-m} \mathcal{O}_{p_i} = \mathcal{O}_P$$

is surjective. Set $E' = \pi_*(I_Q \otimes U)$. Observe that E' is a rank m locally free sheaf and $R^1 \pi_*(I_Q \otimes U) = 0$.

Consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & M' & \rightarrow & W \otimes \mathcal{O}_X & \xrightarrow{\alpha} & I_Q \otimes U \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & M & \rightarrow & V \otimes \mathcal{O}_X & \rightarrow & U \rightarrow 0 \\
 & & \downarrow \beta & & \downarrow & & \downarrow \\
 & & r-m & & & & \\
 0 & \rightarrow & \sum_{i=1}^{r-m} I_{q_i} & \rightarrow & H^0(U|_Q) \otimes \mathcal{O}_X & \rightarrow & U|_Q \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where

$$M = \ker(V \otimes \mathcal{O}_X \rightarrow U) \quad \text{and} \quad M' = \ker(W \otimes \mathcal{O}_X \rightarrow I_Q \otimes U).$$

Observe that α is surjective because $f^{-1}(L \cap Y) = Q$.

It follows from the snake lemma β is also surjective.

Let $f_i = \pi^{-1}(p_i) \cong \mathbb{P}^{m-1}$. Consider the exact sequences,

$$0 \rightarrow \text{Tor}_1(I_Q \otimes U, \mathcal{O}_{f_i}) \rightarrow M' \otimes \mathcal{O}_{f_i} \rightarrow W \otimes \mathcal{O}_{f_i} \rightarrow I_Q \otimes U \otimes \mathcal{O}_{f_i} \rightarrow 0,$$

and

$$0 \rightarrow k(q_i) \rightarrow I_Q \otimes U \otimes \mathcal{O}_{f_i} \rightarrow I_{q_i/f_i}(1) \rightarrow 0,$$

where $k(q_i)$ is the residue field of q_i in I_{q_i/f_i} is the ideal sheaf of q_i in f_i . It follows from a local computation that the map

$$H^0(W \otimes \mathcal{O}_{f_i}) \rightarrow H^0(I_Q \otimes U \otimes \mathcal{O}_{f_i})$$

is surjective.

Also observe that $\text{Supp}(\text{Tor}_1(I_Q \otimes U, \mathcal{O}_{f_i})) \subset q_i$.

Hence $H^1(M' \otimes \mathcal{O}_{f_i}) = 0$. M' is torsion free and it is flat over C . It follows from the theorem of base changes that $R^1 \pi_* M' = 0$.

There is the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \pi_* M' & \rightarrow & W \otimes \mathcal{O}_C & \rightarrow & E' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & F & \rightarrow & V \otimes \mathcal{O}_C & \rightarrow & E \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & r-m & & & & r-m \\
 0 & \rightarrow & \sum_{i=1}^{r-m} \mathcal{O}_C(-p_i) & \rightarrow & H^0(U|_Q) \otimes \mathcal{O}_C & \rightarrow & \sum_{i=1}^{r-m} \mathcal{O}_{p_i} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

This showed that $F \rightarrow \sum_{i=1}^{r-m} \mathcal{O}(-p_i)$ is surjective.

Now

$$\text{rank}(\pi_* M') = 1 \quad \text{and} \quad \pi_* M' \cong (\wedge^m E')^* = (\wedge^m E)^* \otimes \mathcal{O}_C(P).$$

Remark. — The above construction is inspired by the techniques of Gruson and *et. al.* [6].

The fine moduli space of smooth irreducible genus g curves with level m structure is denoted by $M_{g,m}$. Suppose that $\mathcal{C} \rightarrow M_{g,m}$ is the universal family of curves. Let $\mathcal{P}ic \mathcal{C}$ be the relative Picard scheme. Set,

$$\mathcal{W}_d^r = \{ (\mathcal{L}, C) \in \mathcal{P}ic \mathcal{C} \mid \text{deg } \mathcal{L} = d \quad \text{and} \quad h^0(\mathcal{L}) \geq r+1 \}.$$

For the rest of the paper we shall use the following notations. We shall denote by C , a smooth irreducible genus g curve. \mathcal{L} is a degree d line bundle on C . We shall assume $h^0(\mathcal{L}) = r+1$, $h^1(\mathcal{L}) = \delta > 0$, and $|\mathcal{L}|$ has no base points. We denote by f the natural map:

$$f: C \rightarrow f(C) = C' \underset{\text{def}}{\subseteq} \mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^r.$$

Suppose that $\mathcal{O}(1)$ is the tautological line bundle of $\mathbb{P}(H^0(\mathcal{L}))$. $P^1(\mathcal{O}(1))$, the first principal part of $\mathcal{O}(1)$, is isomorphic to $H^0(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}^r}$. Set $M = f^*(\Omega_{\mathbb{P}^r}^1(1))$ and $P^1(\mathcal{L}) = \text{first principal part of } \mathcal{L}$. There is the following diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & H^0(\mathcal{L}) \otimes \mathcal{O}_C & \rightarrow & \mathcal{L} \rightarrow 0 \\ & & \downarrow \text{df} & & \downarrow & & \parallel \\ 0 & \rightarrow & K \otimes \mathcal{L} & \rightarrow & P^1(\mathcal{L}) & \rightarrow & \mathcal{L} \rightarrow 0, \end{array}$$

where K is the canonical sheaf of C . Observe that $P^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1} \cong P^1(K)$. Hence there is the following diagram:

$$(1. A) \quad \begin{array}{ccccccc} 0 & \rightarrow & M \otimes K \otimes \mathcal{L}^{-1} & \rightarrow & H^0(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1} & \rightarrow & K \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & K^2 & \rightarrow & P^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1} & \rightarrow & K \rightarrow 0. \end{array}$$

Consider the map:

$$(1. B) \quad \mu: H^0(\mathcal{L}) \otimes H^0(K \otimes \mathcal{L}^{-1}) \rightarrow H^0(P^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1}).$$

$H^0(P^1(\mathcal{L}) \otimes K \otimes \mathcal{L}^{-1})$ is naturally isomorphic to the cotangent space of $\mathcal{P}ic \mathcal{C}$ at the point (\mathcal{L}, C) . The image of μ is the annihilator of the Zariski tangent space of \mathcal{W}_d^r at the point (\mathcal{L}, C) . See [1] for more details.

THEOREM 2. — *Suppose that \mathcal{L} is a very ample degree d line bundle on a smooth irreducible curve C such that $h^0(\mathcal{L}) = r+1$ and $h^1(\mathcal{L}) = \delta > 0$, where $r \geq 3$. Then,*

$$(a) \text{rank}(\mu) \geq 3\delta - 2 + r = 4\delta + d - g - 2. \quad (1. B).$$

(b) If Y is an irreducible component of \mathcal{W}_d^r containing the point (\mathcal{L}, C) , then $\dim Y \leq 5g - 4\delta - d - 1$.

(c) Let N be the normal sheaf of C in $\mathbb{P}(\mathbf{H}^0(\mathcal{L}))$. Then $h^1(N) \leq (r-2)(\delta-1)$.

Proof. – Consider the natural embedding,

Let N^* be the conormal sheaf of C in \mathbb{P}^r . There is the following exact sequence:

$$0 \rightarrow N^* \otimes \mathcal{L} \rightarrow H^0(\mathcal{L}) \otimes \mathcal{O}_C \rightarrow P^1(\mathcal{L}) \rightarrow 0.$$

Consider the natural map

$$F: \mathbb{P}(P^1(\mathcal{L})) \rightarrow T \subseteq \mathbb{P}^r.$$

T is the tangent surface of C , and F is a birational morphism. By Lemma 1,

$$h^1(N) = h^0(N^* \otimes K) \leq \sum_{i=1}^{r-2} h^0(K \otimes \mathcal{L}^{-1}(-p_i)) + h^0\left(\mathcal{L}^{-3} \left(\sum_{i=1}^{r-2} p_i \right)\right) = (r-2)(\delta-1).$$

But $H^0(N^* \otimes K) = \ker \mu$. Thus

$$\text{rank}(\mu) \geq (r+1)\delta - (r-2)(\delta-1) = 3\delta - 2 + r = 4\delta + d - g - 2.$$

Since the image of μ is the annihilator of the Zariski tangent space of \mathcal{W}_d^r at (\mathcal{L}, C) , it follows that,

$$\dim Y \leq (4g-3) - (4\delta + d - g - 2) = 5g - 4\delta - d - 1.$$

COROLLARY 3. – Assume that $r \geq 3$ and

$$f: C \rightarrow f(C) = C' \subseteq \mathbb{P}(H^0(\mathcal{L}))$$

is a birational map. Furthermore assume either f is unramified or $P_a(C') < g + 3d - (r-2)$. Then,

(a) $\text{rank}(\mu) \geq 4\delta + d - g - 2$.

(b) If Y is an irreducible component of \mathcal{W}_d^r containing the point (\mathcal{L}, C) , then $\dim Y \leq 5g - 1 - 4\delta - d$.

Proof. – Consider the natural map

$$\varphi: H^0(\mathcal{L}) \otimes \mathcal{O}_C \rightarrow P^1(\mathcal{L}).$$

Set

$$E = \text{Im}(\varphi), \quad N^* \otimes \mathcal{L} = \ker(\varphi) \quad \text{and} \quad D = \text{cok}(\varphi).$$

Observe that $\text{cok} \varphi$ is equal to $\text{cok}(df: f^* \Omega_{\mathbb{P}^r} \otimes \mathcal{L} \rightarrow \Omega_C^1 \otimes \mathcal{L})$.

It follows that $\text{cok} \varphi$ is isomorphic to $\Omega_C^1 \otimes \mathcal{L} \otimes \mathcal{O}_R$, where R is the ramification divisor.

Let $X = \mathbb{P}(E)$. Consider the natural map

$$F: X \rightarrow F(X) = T \subseteq \mathbb{P}(H^0(\mathcal{L})).$$

T is the closure of the tangent surface of the smooth part of C' . $F: X \rightarrow T$ is birational. Now

$$P_a(C') - g = \sum_{p \in C} \text{length}(\mathcal{O}_{p, C/\mathcal{O}_{f(p), C'}}).$$

Observe that

$$\text{deg } R = \sum_{p \in C} \text{length}(\mathcal{I}_{p, C/\mathcal{O}_{p, C} \cdot \mathcal{I}_f(p), C'}).$$

It follows that $\text{deg } R \leq P_a(C') - g$. By Lemma 1, we can

LEMMA 1. — *We can construct the following exact sequence:*

$$0 \rightarrow \mathcal{L}^{-3} \otimes_{\mathcal{O}_C} (R) \otimes_{\mathcal{O}_C} \left(\sum_{i=1}^{r-2} p_i \right) \rightarrow N^* \otimes K \rightarrow \sum_{i=1}^{r-2} K \otimes \mathcal{L}^{-1}(-p_i) \rightarrow 0.$$

Since $\text{deg}(R) \leq P_a - g$, it follows from our assumption

$$h^0 \left(\mathcal{L}^{-3} \left(R + \sum_{i=1}^{r-2} p_i \right) \right) = 0.$$

Thus $\dim \ker \mu = h^0(N^* \otimes K) \leq (r-2)(\delta-1)$. As in Theorem 2, we conclude that $\text{rank } \mu \geq 4\delta + d - g - 2$ and $\dim Y \leq 5g - 1 - 4\delta - d$.

The open set of the Hilbert scheme corresponding to smooth irreducible degree d genus g curves in \mathbb{P}^3 is denoted by $H_{d, g, 3}$. If $X \in H_{d, g, 3}$, then $\chi(N_{X/\mathbb{P}^3}) = h^0(N_{X/\mathbb{P}^3}) - h^1(N_{X/\mathbb{P}^3}) = 4d$.

As in [7], one can show that each irreducible component of $H_{d, g, 3}$ has dimension greater or equal to $4d$.

THEOREM 4. — *If $d \geq g + 3$, then $H_{d, g, 3}$ is irreducible.*

Proof. — There is an irreducible open set of $H_{d, g, 3}$ corresponding to nonspecial curves ($h^1(\mathcal{O}_C(1)) = 0$) ([5], 6.2). Suppose for contradiction that $H_{d, g, 3}$ is reducible. Then there is an irreducible component W of $H_{d, g, 3}$ such that the general curve C in the family W satisfies

$$h^0(\mathcal{O}_C(1)) = r + 1 \quad \text{and} \quad h^1(\mathcal{O}_C(1)) = \delta > 0.$$

We denote by $H_{d, g, 3}^m$ the Hilbert scheme of degree d genus g smooth irreducible curves in \mathbb{P}^3 with level m structure. Let W_m be an irreducible component of $H_{d, g, 3}^m$ which maps onto W . Then $\dim W = \dim W_m$.

There is a natural map from

$$h: W_m \rightarrow \mathcal{W}_d^r \subseteq \mathcal{P}ic \mathcal{C}.$$

Let Y be an irreducible component of \mathcal{W}_d^r containing $h(W_m)$. Let x be a general point of W_m , then

$$\dim h^{-1} h(x) \leq \dim G(4, d+1+\delta-g) + \dim \text{Aut } \mathbb{P}^3$$

where $G(4, d+1+\delta-g)$ is the Grassman variety of 4 dimensional subspaces in a $d+1+\delta-g$ -dimensional vector space. Then

$$\dim W = \dim W_m \leq \dim h^{-1} h(x) + \dim Y \leq 4d-1$$

by Theorem 2. This is a contradiction. Hence, $H_{d,g,3}$ is irreducible.

Remark. — In [4], Harris has proved that $H_{d,g,3}$ is irreducible while $d > 5/4g + 1$.

Suppose that C' is an irreducible reduced degree d curve in \mathbb{P}^3 . Let

$$N_{C'/\mathbb{P}^3} = \mathcal{H} \text{ om } \mathcal{O}_{\mathbb{P}^3}(I_{C'}, \mathcal{O}_{C'}) = \mathcal{H} \text{ om } \mathcal{O}_{C'}(I_{C'}|_{C'}^2, \mathcal{O}_{C'})$$

be the normal sheaf of C' .

LEMMA 5. — $\chi(N_{C'/\mathbb{P}^3}) = h^0(N_{C'/\mathbb{P}^3}) - h^1(N_{C'/\mathbb{P}^3}) = 4d$. Hence every irreducible component of the Hilbert scheme containing C' has dimension greater or equal to $4d$.

Proof. — C' is locally Cohen Macaulay. We can construct an exact sequence:

$$0 \rightarrow E_2 \rightarrow E_1 \rightarrow I_{C'} \rightarrow 0$$

where E_1 and E_2 are locally free sheaves on \mathbb{P}^3 .

Consider the following exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{H} \text{ om}(I_{C'}, \mathcal{O}_{\mathbb{P}^3}) \rightarrow E_1^* \xrightarrow{\varphi_1} E_2^* \rightarrow \omega_{C'}(4) \rightarrow 0, \\ 0 \rightarrow \mathcal{H} \text{ om}(I_{C'}, \mathcal{O}_{C'}) \rightarrow E_1^*|_{C'} \xrightarrow{\varphi_2} E_2^*|_{C'} \rightarrow \mathcal{E} \text{ xt}^1(I_{C'}, \mathcal{O}_{C'}). \end{aligned}$$

Observe that $\varphi_2 = \varphi_1 \otimes \mathcal{O}_{C'}$. Thus

$$\text{Cok } \varphi_2 = \text{Cok } \varphi_1 \otimes \mathcal{O}_{C'} = \omega_{C'}(4).$$

Observe that

$$c_1(E_1^*) = c_1(E_2^*) \quad \text{and} \quad \text{rank } E_1^* = 1 + \text{rank } E_2^*.$$

It follows from the Riemann-Roch theorem,

$$\chi(N_{C'/\mathbb{P}^3}) = \chi(E_1^*|_{C'}) + \chi(\omega_{C'}(4)) - \chi(E_2^*|_{C'}) = 1 - P_d + \chi(\omega_{C'}) + 4d = 4d.$$

C' is codimension two Cohen-Macaulay. It follows that there is no local obstructions to the deformations of C' ([3], 5.1). Hence the obstructions to the deformations of C' in \mathbb{P}^3 is given by $H^1(N_{C'/\mathbb{P}^3})$. As in [7], one can show that this implies the inequality of dimension as claimed.

THEOREM 6. — Suppose that X is an irreducible reduced degree d curve in \mathbb{P}^3 . If $d \geq P_d(X) + 2$, then X is smoothable.

Proof. — Let W be an irreducible component of the Hilbert scheme containing the point corresponding to X . If the general member of W is smooth, then X is smoothable. Assume for contradiction that a general curve C' in W is singular. Let $S \rightarrow W$ be the universal family of curves. Let $p: \tilde{S} \rightarrow S \rightarrow W$ be the normalization of S . Let $U \subseteq W$ be the open set where p is smooth. Suppose the normalization of C' is a smooth curve of genus g . We can construct a variety U_m étale over U such that there is a map $h: U_m \rightarrow \mathcal{P}ic \mathcal{C}$. We shall divide the proof into five cases. Consider the normalization map $\pi: C \rightarrow C'$. Set $\pi^* \mathcal{O}_{C'}(1) = \mathcal{O}_C(1)$.

Since $g < P_a(C')$, $\deg \mathcal{O}_C(1) \geq g + 3$.

Case 1. — Assume that $g = 0$.

Then $\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^1}(d)$. C' is obtained by projecting the d -uple embedding of \mathbb{P}^1 . The generic projection gives a smooth curve. Thus,

$$\dim W < \dim G(4, d+1) + \dim \text{Aut } \mathbb{P}^3 - \dim \text{Aut } \mathbb{P}^1 = 4d.$$

Case 2. — Assume that $g = 1$.

As in Case 1, we can prove that

$$\dim W < \dim G(4, d) + \dim \text{Aut } \mathbb{P}^3 - \dim \text{Aut } C + \dim \mathcal{P}ic \mathcal{C} = 4d.$$

Case 3. — Assume that $g \geq 2$, $\dim h(U_m) = \dim \mathcal{P}ic \mathcal{C} = 4g - 3$, and $h^1(\mathcal{O}_C(1)) = 0$.

The generic line bundle of degree $d \geq g + 3$ is very ample. Let x be a general point of U_m . Then $\dim h^{-1}h(x) < \dim G(4, d+1-g) + \dim \text{Aut } \mathbb{P}^3$.

Hence, $\dim W = 4g + 3 + \dim h^{-1}h(x) < 4d$.

Case 4. — Assume that $h^1(\mathcal{O}_C(1)) = 0$, $g \geq 2$, and $\dim h(U_m) < 4g - 3$, in this case

$$\begin{aligned} \dim W = \dim U_m = \dim h^{-1}h(x) + \dim h(U_m) &< \dim G(4, d+1-g) \\ &+ \dim \text{Aut } \mathbb{P}^3 + (4g-3) \leq 4d. \end{aligned}$$

Case 5. — Assume that $g \geq 2$, and $h^1(\mathcal{O}_C(1)) = \delta > 0$.

Using Corollary 3, we can show that

$$\dim W = \dim U_m \leq 4d - 1,$$

as in Theorem 2.

In each of the five cases, we show that $\dim W < 4d$.

This is impossible. Thus a general curve in W is smooth.

LEMMA 7. — Assume $f: C \rightarrow C' \subseteq \mathbb{P}(H^0(\mathcal{L}) = \mathbb{P}^r)$ is a birational map. Also assume that $d \geq g$.

(a) Consider the multiplication map:

$$\mu_0: H^0(\mathcal{L}) \otimes H^0(K \otimes \mathcal{L}^{-1}) \rightarrow H^0(K).$$

Then $\text{rank } (\mu_0) \geq 2\delta + r - 1 = 3\delta + d - g - 1$.

$$(b) \quad \delta \leq \frac{2g+1-d}{3}.$$

Proof. — Consider the exact sequence:

$$0 \rightarrow M \rightarrow H^0(\mathcal{L}) \otimes \mathcal{O}_C \rightarrow \mathcal{L} \rightarrow 0 \quad \text{when } M = f^* \Omega_{\mathbb{P}^r}^1(1).$$

By Lemma 1, we can construct an exact sequence:

$$0 \rightarrow \mathcal{L}^{-1} \otimes \mathcal{O} \left(\sum_{i=1}^{r-1} p_i \right) \rightarrow M \rightarrow \sum_{i=1}^{r-1} \mathcal{O}(-p_i) \rightarrow 0.$$

Observe that,

$$\begin{aligned} h^1 \left(\mathbb{K} \otimes \mathcal{L}^{-2} \otimes \mathcal{O} \left(\sum_{i=1}^{r-1} p_i \right) \right) &= h^0 \left(\mathcal{L}^2 \otimes \mathcal{O} \left(- \sum_{i=1}^{r-1} p_i \right) \right) \\ &= 2d+1-g-(r-1) = -\chi \left(\mathbb{K} \otimes \mathcal{L}^{-2} \otimes \mathcal{O} \left(- \sum_{i=1}^{r-1} p_i \right) \right). \end{aligned}$$

Thus

$$h^0 \left(\mathbb{K} \otimes \mathcal{L}^{-2} \otimes \mathcal{O} \left(\sum_{i=1}^{r-1} p_i \right) \right) = 0.$$

Hence,

$$h^0(M \otimes \mathbb{K} \otimes \mathcal{L}^{-1}) = \dim \ker \mu_0 \leq (r-1)(\delta-1).$$

Thus $\text{rank } \mu_0 \geq 3\delta + d - g - 1$. Since $g \geq \text{rank}(\mu_0)$, it follows that $\delta \leq (2g+1-d)/3$.

THEOREM 8. — *Let $H_{d,g,n}$ be the open set of the Hilbert scheme of smooth irreducible degree d genus g curves in \mathbb{P}^n ($n \geq 3$). If $d > ((2n-3)g+n+3)/n$, then $H_{d,g,n}$ is irreducible.*

Proof. — Let C be a smooth irreducible degree d genus g curve in \mathbb{P}^n . Then $\chi(N_{C/\mathbb{P}^n}) = (n+1)d + (n-3)(1-g)$.

It follows that the dimension of each irreducible component of $H_{d,g,n}$ is at least $(n+1)d + (n-3)(1-g)$. Assume that $H_{d,g,n}$ has an irreducible component W such that the general curve in the family satisfies the property $h^0(\mathcal{L}) = r+1$ and $h^1(\mathcal{L}) = \delta > 0$.

Then,

$$\begin{aligned} \dim W &\leq 5g - 1 - 4\delta - d + \dim G(n+1, r+1) \\ &\quad + \dim \text{Aut } \mathbb{P}^n = 5g - 2 - 4\delta - d + (n+1)(\delta + d - g + 1), \end{aligned}$$

Since

$$\delta \leq \frac{2g+1-d}{3} \quad \text{and} \quad d > \frac{(2n-3)g+n+3}{n},$$

it follows that $\dim W < (n+1)d + (n-3)(1-g)$ which is a contradiction.

Remark. — The above result is an improvement of a theorem of Joe Harris. In ([4], p. 72), Harris proved that $H_{d, g, n}$ is irreducible while $d > \frac{(2n-1)g}{n+1} + 1$.

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