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On some connections between Boolean algebras and Heyting algebras

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On some connections between Boolean algebras and Heyting algebras.

par

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Abstract.

We present a finitely axiomatizable class of Heyting algebras (with identity $(-x) + (--x) = 1$) such that this class and the class of Boolean algebras with n distinguished ideals are mutually first order definable.

As a corollary some results on countably categoricity, finitely axiomatizibility, decidability, prime and countably saturated models for Heyting algebras are obtained.

I - Constructions of Boolean algebras with particular subsets from Heyting algebras, and converse problems of representation.

For each element x of an Heyting algebra $H = (H, +, ., \rightarrow, 0, 1)$, the element $x \rightarrow 0$ is denoted by $-x$; x is called dense if $-x = 0$, and regular if $--x = x$.

It is well-known (see for example [14]) that one obtains a Boolean algebra $A(H)$ by endowing the set of all regular elements of H with the constants 0 and 1, and with the operations $+^*$, $.$, $-$, where $+^*$ is defined by $x +^* y = --(x+y)$. Similarly the set $\nabla(H)$ of all dense elements of H is an implicative lattice and a filter in H.

Notation.

Let H *be an Heyting algebra. For any* $a \in A(H)$ *and* $b \in \nabla(H)$ *put* :

$$\nabla_a = \{c \in \nabla(H) : a \leq c\} \quad \textit{and}$$

$$P_b = \{c \in A(H) : c \leq b\}.$$

It is shown in [7] that each ∇_a is a filter in H, that each P_b is decreasing (i.e. $x \leq y$ and $y \in P_b$ imply $x \in P_b$), and that H satisfies the identity $(-x) + (--x) = 1$ if an only if P_b is an ideal of A (H) for each $b \in \nabla(H)$ (it is well - known that this identity is equivalent to the identity $(-x) + (-y) = --((-x) + (-y)))$.

Now let A be a Boolean algebra and $\{\nabla_a\}_{a \in A}$ be a set of filters in an implicative lattice ∇. Put $L = (\{(a,b) : a \in A,\ b \in \nabla\} / \sim\ ; \leq / \sim)$ where $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \rightarrow b_2 \in \nabla a_1$; $(a_1, b_1) \sim (a_2, b_2)$ if $(a_1, b_1) \leq (a_2, b_2)$ and $(a_2, b_2) \leq (a_1, b_1)$.
It is proved in [7] that L is a lower semilattice.

The following question may then be interesting : when L is an Heyting algebra ?

A necessary condition for A and $\{\nabla_a\}_{a \in A}$ to generate an Heyting algebra L was presented in the theorem 1 of [7].

However, S.I. Mardaev have given an example showing that this condition is not sufficient.

So, the theorem 1 of [7] saying that the presented condition is necessary and sufficient, an its corollary 2, turned out to be wrong.

Call P-algebra each Boolean algebra with distinguished decreasing subsets. As a consequence of the previous situation the following lemma, based on this corollary 2, has lost a proof :

<u>Lemma</u> [7].

Let us consider a finite implicative lattice $\nabla = (\{b_1, \ldots, b_n\}, \leq)$ *and a* P*-algebra* $(A, P_{b_1}, \ldots, P_{b_n})$. *There exists an Heyting algebra* H *such that* $\nabla = \nabla(H)$ *and* $(A, P_{b_1}, \ldots, P_{b_n}) = (A(H), P_{b_1}, \ldots, P_{b_n})$ *if and only if* :

(*) $\quad P_c \cap P_d = P_{c.d}$ *and* $1 \in P_e \Leftrightarrow e = 1$ *for any* $c, d, e \in \nabla$.

It is obvious that the condition (*) is necessary for existing such an Heyting algebra H. So it is natural to consider the following problems :

Problem 1. Do every implicative lattice ∇ and P-algebra $(A, P_b)_{b \in \nabla}$ verifying (*) can be represented as $\nabla = \nabla(H)$ and $(A, P_b)_{b \in \nabla} = (A(H), P_b)_{b \in \nabla}$ for some Heyting algebra H ?

A P-algebra $(A, P_j)_{j \in J}$ will be called I-algebra if every P_j is an ideal.

Problem 1_i is the problem 1 for I-algebra $(A, P_b)_{b \in \nabla}$; problem 1_f is the problem 1 for P-algebra $(A, P_b)_{b \in \nabla}$ with finite ∇, and problem 1_{if} is the problem 1 for I-algebra $(A, P_b)_{b \in \nabla}$ with finite ∇.

Problem 2. Does every I-algebra $(A, I_1 \ldots, I_n)$ can be represented by $A = A(H)$ and $I_j = P_{b_j}$ for some Heyting algebra H with finite set $\nabla(H)$ and $b_1, \ldots, b_n \in \nabla(H)$?

Problem 3. Does every I-algebra $(A, I_1, \ldots I_n)$ can be represented as $A = A(H)$ and $I_j = P_{b_j}$ for some Heyting algebra H and $b_1, \ldots, b_n \in \nabla(H)$?

Problem 4. Does every I-algebra $(A, I_1, \ldots, I_n)$ can be represented as $A = A(H)$ and $I_j = P_{b_j}$ for some Heyting algebra H and some definable $b_1, \ldots, b_n \in \nabla(H)$ (it means that there exist formulas $\phi_1(x), \ldots, \phi_n(x)$ of the Heyting algebra first order language such that $\{c \in H : H \models \phi_i(c)\} = \{b_i\}$) ?

Problems 2', 3' and 4' are problems 2, 3 and 4 for Heyting algebra H satisfying the identity $(-x) + (--x) = 1$.

Interest to these problems is connected with the following result :

Theorem [7].

Let H *and* H' *be Heyting algebras.*

Then $H \simeq H'$ *if and only if there exists an isomorphism* $\phi : \nabla(H) \to \nabla(H')$ *such that* $(A(H), P_b)_{b \in \nabla(H)} \simeq (A(H'), P_{\phi(b)})_{b \in \nabla(H)}$

A special interest to problems 1 ,.1 f and 2 is connected with the following facts :

Theorem [7] .

Let H *and* H' *be Heyting algebras and* $\nabla(H)$ *be finite. Then* $H \equiv H'$ *if and only if there exists an isomorphism*

$$\phi : \nabla(H) \rightarrow \nabla(H')$$

such that $(A(H), P_b)_{b \in \nabla(H)} \equiv (A(H'), P_{\phi(b)})_{b \in \nabla(H)}$.

Corollary [7].

Let H *be an Heyting algebra with* $\nabla(H)$ *finite.*

The theory of H *is countably categorical (finitely axiomatizable, decidable) if and only if the theory of* $(A(H), P_b)_{b \in \nabla(H)}$ *is the same* .

Interest to problems 1 i , 1 f i , 2' , 3' and 4' is also connected with results on different model theoretical properties of Boolean algebras with distinguished ideals (see all bibliographic references except [14] .

II - Some negative answers

Proposition 1 .

Let ∇ *be an implicative semilattice,* $(A, P_b)_{b \in \nabla}$ *be a* P*-algebra verifying the condition (*) and such that there exist* b_0 *with* P_{b_0} *maximal for inclusion in* $\{P_b\}_{b \neq 1}$ *, and* $a \in A$ *with* $a, -a \notin P_{b_0}$ *. Then there does not exist an Heyting algebra* H *with* $\nabla = \nabla(H)$ *and* $(A, P_b)_{b \in \nabla} \simeq (A(H), P_{b(H)})_{b \in \nabla}$.

<u>Proof</u>. Suppose that such an H exists.

Then $a \rightarrow b_0 \geq b_0 \in \nabla(H)$ gives $P_{b_0} \subsetneq P_{a \rightarrow b_0}$ and $a \notin P_{b_0}$ gives $P_{a \rightarrow b_0} \neq P_1$; as moreover $-a = a \rightarrow 0 \leq a \rightarrow b_0$ implies $-a \in P_{a \rightarrow b_0}$, we obtain a contradiction with the maximality of P_{b_0} .

Corollary 1 .

Problems 1 , 1 i , 1 f *and* 1 i f *have a negative solution.*

For each element a of a Boolean algebra, let (a) *be the principal ideal generated by* a.

Proposition 2 .

Let H *be an Heyting algebra with* A (H) *atomless. If there exists* $d \in A(H)$ *with* $d \neq 0$ *and* $(d) \cap P_f = (0)$ *for some* $f \in \nabla(H)$, *then* $\nabla(H)$ *is infinite.*

Proof. We can take an infinite sequence of regular elements $d_1 < d_2 < \ldots < d$. Suppose $f + d_i = f + d_{i+1}$ for one i : then $d_{i+1} \, . \, (-d_i) \leq (f + d_i) \, . \, (-d_i) \leq f$ gives $d_{i+i} \, . \, (-d_i) \in (d) \cap P_f = (0)$, so that $d_{i+1} \leq --d_i = d_i$ which is contradictory.
Thus the $f + d_i$'s form a strictly increasing sequence of elements of $\nabla(H)$.

Corollary 2 .

Problems 2 *and* 2' *have a negative solution.*

III - A finitely axiomatizable class of Heyting algebras.

If an Heyting algebra H contains a least dense element a_1, $\nabla(H)$ is an Heyting algebra for the operations $+, ., \rightarrow$, and the constants a_1 and 1. We then put $\nabla^1(H) = \nabla(H)$, $\nabla^2(H) = \nabla(\nabla^1(H))$, and continue the process by putting $\nabla^{i+1}(H) = \nabla(\nabla^i(H))$ as long as $\nabla^i(H)$ contains a least dense element a_{i+1}.
Notice that $a_{n+1} = 1$ for a number n if and only if $\nabla^n(H)$ is a Boolean algebra.

Definition .

For a number n *different from* 0 , *let* C_n *be the class of all Heyting algebras verifying* :

1) $(-x) + (--x) = 1$,
2) $\nabla^i(H)$ *has a least element* a_i *for each* $i \in \{1, \ldots, n\}$, *and* $\nabla^n(H)$ *is a Boolean algebra* ,

3) *For each* $x \in [\alpha_i, \alpha_{i+1}]$ *there exists* $y \in A(H)$ *satisfying* $x = y \cdot \alpha_{i+1} + \alpha_i$ (*with* $i \in \{0, \dots, n\}$, *putting* $\alpha_0 = 0$ *and* $\alpha_{n+1} = 1$).

Notice that if α_1 exists in an Heyting algebra, the condition 3) is satisfied for $i = 0$ by taking $y = --x$. It is easy to see that :

Remark.

C_n *is finitely axiomatizable.*

Definition.

Let $\mathcal{A} = (A, I_1, \dots, I_n)$ *be a Boolean algebra* A *with distinguished ideals* $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$.

We put (identifying $(A/I_i)/(I_{i+1}/I_i)$ *with* A/I_{i+1}) :

$H(\mathcal{A}) = \{(a_0, a_1, \dots, a_n) \in A \times (A/I_1) \times \dots \times (A/I_n) : a_0/I_1 \geq a_1$ and $a_i/(I_{i+1}/I_i) \geq a_{i+1} \ \forall i \in \{1, \dots, n-1\}\}$,

and we endow this set with the pointwise order (i.e. $(a_0, \dots, a_n) \leq (b_0, \dots, b_n)$ *if* $a_i \leq b_i$ *for* $i \in \{0, \dots, n\}$) .

Proposition 3.

$H(\mathcal{A})$ *is an Heyting algebra of the class* C_n, *with*

$A(H(\mathcal{A})) = \{(a_0, \dots, a_n) \in H(\mathcal{A}) : a_i = a_0/I_i \ \ \forall i \in \{1, \dots, n\}\}$,

and for $i \in \{1, \dots, n\}$:

$\nabla^i(H(\mathcal{A})) = \{(a_0, \dots, a_n) \in H(\mathcal{A}) : a_0 = 1, a_1 = 1/I_1, \dots, a_{i-1} = 1/I_{i-1}\}$

(*so that* $\alpha_i = (1, 1/I_1, \dots, 1/I_{i-1}, 0/I_i, \dots 0/I_n)$).

Proof. Obviously $H(\mathcal{A})$ is a bounded lattice with $+$ and $.$ computed pointwise, such that for each elements $(a_0, \dots a_n)$ and $(b_0, \dots, b_n)$ there exists $(c_0, \dots c_n) = (a_0, \dots, a_n) \rightarrow (b_0, \dots b_n)$, defined by

$c_0 = b_0 + (-a_0)$ and

$c_i = ((b_0 + (-a_0))/I_i)$ ou au moins $((b_1 + (-a_1))/(I_i/I_1)) \dots$

$(b_{i-1} + (a_{i-1}))/(I_i/I_{i-1})) . (b_i + (-a_i))$.

So $H(\mathcal{A})$ is an Heyting algebra, and for each $(a_0, \dots, a_n) \in H(\mathcal{A})$ we have

$$-(a_0, \dots, a_n) = \left(-a_0, -a_0/I_1, -a_0/I_2, \dots, -a_0/I_n \right),$$ which shows that $H(A)$

verifies the identity $(-x) + (--x) = 1$ and gives the expected description of $A(H(\mathcal{A}))$ and $\nabla^1(H(\mathcal{A}))$.

Supposing now that α_i exists for $i \geq 1$ and takes the value given in the proposition, we have

$(a_0, \dots, a_n) \to \alpha_i = (1, 1/I_1, \dots, 1/I_{i-1}, -a_i, -a_i/(I_{i+1}/I_i), \dots, -a_i/(I_n/I_i))$,

which gives the existence and value of α_{i+1} if $i < n$, and shows that

$\nabla^n(H(\mathcal{A}))$ is a Boolean algebra isomorphic to A/I_n if $i = n$.

Finally each element x of $[\alpha_i, \alpha_{i+1}]$ has the form $x = (1, 1/I_1, \dots, 1/I_{i-1}, a_i, 0/I_{i+1}, \dots, 0/I_n)$; taking $a_0 \in A$ with $a_i = a_0/I_i$ and putting $y = (a_0, a_0/I_1, \dots, a_0/I_n)$ we obtain $y \in A(H(\mathcal{A}))$ with $x = y \cdot \alpha_{i+1} + \alpha_i$.

Definition.

For each Heyting algebra H *of the class* C_n, *put* $I_i(H) = A(H) \cap (\alpha_i]$ *for* $i \in \{1, \dots, n\}$ *(where* $(\alpha_i] = \{x \in H : x \leq \alpha_i\}$*), and put* $\mathcal{A}(H) = (A(H), I_1(H), \dots, I_n(H))$.

Theorem 1.

a) *Let* $\mathbf{A} = (A, I_1, \dots, I_n)$ *be a Boolean algebra* A *with distinguished ideals* $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n$. *Then* $\mathcal{A} \simeq \mathcal{A}(H(\mathcal{A}))$.

b) *Let* H *be an Heyting algebra of the class* C_n. *Then* $H \simeq H(\mathcal{A}(H))$.

<u>Proof</u>.

a) From proposition 3 we see that we define an isomorphism $f : A \to A(H(A))$ by putting $f(a) = (a, a/I_1, \dots, a/I_n)$ for each $a \in A$. Moreover for each $i \in \{1, \dots, n\}$:

$$f(a) \in I_i(H(A)) \iff f(a) \leq (1, 1/I_1, \dots, 1/I_{i-1}, 0/I_i, \dots, 0/I_n)$$
$$\iff a \in I_i$$

b) If $a \in H$ then for each $i \in \{0, \dots, n\}$ there exists $b_i \in A(H)$ with $a \cdot \alpha_{i+1} + \alpha_i = b_i \cdot \alpha_{i+1} + \alpha_i$. Notice that if b'_i is suitable too, then $b_i/I_i(H) = b'_i/I_i(H)$ (do not write $I_i(H)$ when $i = 0$) : indeed

$b_i . a_{i+1} \leq b'_i . a_{i+1} + a_i$ gives $b_i . (-b'_i) . a_{i+1} \leq a_i$, so that $b_i . (-b'_i) \leq a_{i+1} \rightarrow a_i = a_i$ and thus $b_i . (-b'_i) \in I_i(H)$; as $1 = b'_i + (-b'_i)$ we see that $b_i / I_i(H) = (b_i . b'_i) / I_i(H)$, and obtain $b'_i / I_i(H) = (b_i . b'_i) / I_i(H)$ by a similar argument. This allows us to define $\phi : H \rightarrow H(A(H))$ by putting $\phi(a) = (b_0, b_1 / I_1(H), \dots, b_n / I_n(H))$.

Conversely we define $\psi : H(A(H)) \rightarrow H$ by putting $\psi(x) = x_0 . (x_1 + a_1) \dots (x_n + a_n)$ for each element $x = (x_0, x_1 / I_1(H), \dots, x_n / I_n(H))$ of $H(A(H))$.

Then by some simple verifications we see that ϕ and ψ preserve the orders and that each of them is inverse of the other, which gives the result.

IV. Positive answers

First notice that from any ideals $I_1, \dots, I_n$ of a Boolean algebra, an increasing sequence (for inclusion) of $2^n - 1$ ideals can be computed by using Heyting algebras operations, such that $I_1, \dots, I_n$ are computable from them in a similar way. We can thus apply the results of the previous section to any Boolean algebra with distinguished ideals $I_1, \dots, I_n$:

Corollary 3.

Problems 3 , 3', 4 *and* 4' *have a positive solution.*

Corollary 4.

For any Heyting algebras H, H' *of the class* C_n :

a) $H \simeq H'$ *if and only if* $A(H) \simeq A(H')$,

b) $H \equiv H'$ *if and only if* $A(H) \equiv A(H')$,

c) *The theory of* H *is countably categorical (finitely axiomatizable, decidable) if and only if the theory of* $A(H)$ *is the same* ,

d) *The theory of* H *has a prime (countably saturatel) model if and only if the theory of* $A(H)$ *has a same model* ,

e) H is prime (countably saturated) if and only if $A(H)$ is the same.

The decidability of the theory of Boolean algebras with a sequence of distinguished ideals is due to Rabin [13] ; a description of countably categorical Boolean algebras with a finite number of distinguished ideals is due to Macintyre and Rosenstein [2] (see also [6] and [17]) ; descriptions of finitely axiomatizable such algebras are given in [8] and [17] ; numbers of theories of (A, I)'s according to fixed A's, or decidability or undecidability of theories of all (A, I)'s for fixed A's, are given in [1] and [4] , descriptions of elementary equivalence of Boolean algebras with dintinguished ideals are presented in [3], [6], [8], [15] and [16] ; a complete description of decidability of elementary theories of Boolean algebras with distinguished ideals is obtained in [8] ; a classification of complete theories using an axiomatization of structures of definable ideals is given in [17] ; prime models were studied in [5] , [9], [12] and [17], countably saturated in [5] and [12].

In particular these results imply :

Corollary 5.

1) *The theory of the class* C_n *is decidable.*

2) *For any non-superatomic Boolean algebra* A *there exists a continuum of Heyting algebras* H *with* $A \simeq A(H)$, *satisfying* $(-x) + (--x) = 1$, *and having different theories which* :

   a) *have note a prime model*,

   b) *have a prime model but no countably saturated model*,

   c) *have a countably saturated model*.

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REFERENCES

[1] P.F. JURIE and A. TOURAILLE :
Idéaux élémentairement équivalents dans une algèbre booléienne,
Comptes-Rendus des Séances de l'Académie des Sciences. Série I : Mathématique, vol 299 (1984) pp. 415-418.

[2] A. MACINTYRE and J.G. ROSENSTEIN,
$\aleph_0$-categoricity for rings without nilpotent elements and for Boolean structures ,
Journal of Algebra, vol. 43 (1976) pp. 129-154.

[3] B. MOLZAN,
On the theory of Boolean algebras with Ramsey quantifiers,
Proceedings of the third Easter conference on model theory (Gros-Köris, 1985) ,
Seminarberichte N_0 70,
Sektion Mathematik, Humboldt-Universität, Berlin, 1985, pp. 186-192.

[4] D.E. PAL'CHUNOV,
On undecidability of theories of Boolean algebras with distinguished ideals,
Algebra and Logic, vol. 25 (1986) , n°3 , pp. 326-346.

[5] D.E. PAL'CHUNOV,
On prime and countably saturated Boolean algebras with distinguished ideals,
Proceedings of the 8th Soviet Union Mathematical Logic Conference, Moskow, 1986, p. 147.

[6] D.E. PAL'CHUNOV,
Countably categorical Boolean algebras with distinguished ideals,
Studia Logica, vol. XLVI (1987) , n° 2, pp. 121-135.

[7] D.E. PAL'CHUNOV,
On Heyting algebras with finite number of dense elements,
Computable invariants in the algebraic system theory, Novosibirsk, 1987, pp. 35-45.

[8] D.E. PAL'CHUNOV,

Finitely axiomatizable Boolean algebras with distinguished ideals,

Algebra and Logic, vol. 26 (1987) , n° 4 , pp. 435-455.

[9] D.E. PAL'CHUNOV,

On the prime models of the theory of Boolean algebras with distinguished ideals,

Proceedings of the Summer School and Conference "Kleene-90 ", Sofia, 1990 , p. 59.

[10] D.E. PAL'CHUNOV,

Unlocal Boolean algebras with distinguished ideals,

Proceedings of the Soviet and France Colloquium on Model Theory , Karaganda, 1990, pp. 35-36.

[11] D.E. PAL'CHUNOV,

Direct Summands of Boolean algebras with distinguished ideals,

Proceedings of the Logic Colloquium' 90 , Helsinki, 1990, p. 60.

[12] D.E. PAL'CHUNOV,

Prime and countably saturated Boolean algebras,

Proceedings of the Institute of Mathematics, Nauka, Novosibirsk, to appear.

[13] M.O. RABIN,

Decidability of second-order theories and automata on infinite trees,

Transactions of the American Mathematical Society, vol. 141 (1969) , pp. 1 - 35.

[14] H. RASIOWA and R. SIKORSKI,

The Mathematics of Metamathematics,

Polska Akademia Nauk Monographi Mathematyczne 41, Warsawa, 1963.

[15] A. TOURAILLE,

Elimination des quantificateurs dans la théorie élémentaire des algèbres de Boole munies d'une famille d'idéaux distingués,

Comptes-Rendus des Séances de l'Académie des Sciences , Série I : Mathématique, vol. 300 (1985) , pp. 125-128.

[16] A. TOURAILLE,

Théories d'algèbres de Boole munies d'idéaux distingués I,

Journal of Symbolic Logic, vol. 52 (1987), pp. 1027-1043.

[17] A. TOURAILLE,

Théories d'algèbres de Boole munies d'idéaux distingués II,

Journal of Symbolic Logic, vol. 55 (1990), pp. 1192-1212.

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