ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 60, série *Mathématiques*, n° 13 (1976), p. 1-29

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Ann. Sci. Univ. Clermont, Sér. Math., fasc. 13, 1976, pp. 1-29

INTERPRETING SET THEORY IN THE ENDOMORPHISM SEMI-GROUP OF A FREE ALGEBRA OR IN A CATEGORY

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ABSTRACT. - We prove that in the endomorphism semi-group of a free algebra F_λ with λ-generators, we can interprete $<H(\lambda^+), \epsilon>$ and get similar results for categories. The result was announced in [S 3].

I would like to thank R. McKenzie, that seeing the proof, pointed out the holding of 3.3 (A) (by [X]). I thank also M. Rubin for stimulating discussions on the problem during its solution, and for reading the previous version, and detecting errors. The readers should thank him for urging me to rewrite the paper, and in particular to state explicitely that $\underline{B}$ is a Boolean algebra.

§ 0. INTRODUCTION.

This paper has two lines of thought as motivation : comparing category theory with set theory, and investigating the complexity of the theories of some natural structures.

Lawvere [L] proved that in the category of all maps between sets, we can interpret set theory (which is not surprising in view of Rabin [R] interpreting a general two-place relation by two one-place functions).

Eklof asked whether in Ab (the category of abelian group) we can define the free group of cardinality λ . He got a positive answer for $\lambda < \aleph_{(\omega\omega)}, \lambda = 2^{\aleph_0}$, and λ the first strongly compact cardinal. Feferman asked whether for some μ $Ab_\mu \prec Ab$ (Ab_μ - the category of abelian group of cardinality $< \mu$).

It is natural to replace the class of abelian group by any variety, and to concentrate on the free members. (In fact, our results hold for more general categories ; see § 5).

From another direction, there was interest in the first order theories of permutations groups. Mycielski [My] asked it.

This problem was dealt in Ershov [E],

McKenzie [M], Pinus [P], Shelah [S1] , [S2] where it was totally solved, in fact.

The semi-group of endomorphism of a free algebra is a natural generalization.

For simplicity we shall restrict ourselves to $\lambda > |L|$, L the language of V.

Let for regular λ , $H(\lambda)$ be the set of sets of hereditary power $< \lambda$, and for singular λ, $H(\lambda) = \bigcup_{\mu < \lambda} H(\mu^+)$. Our main result is that for a variety V, in Cat_λ (the category of members of V of cardinality $< \lambda$), we can interpret uniformly a model M_λ consisting of some copies of $(H(\lambda), \epsilon)$. For $\lambda = \infty$ this generalizes Lauvere theorem, and it also solves Feferman problem (i.e. reduce it to a problem of set theory). In categories in which the set of free members is definable, Eklof problem is answered too (e.g. for Ab).

There is an example showing that not always we can define the set of free members. However, we can characterize the algebras of cardinality $\leq \mu$ if e.g. $\mu^{|L|} = \mu$ (L-the language of V), and μ is definable in $H(\lambda)$ and we can characterize algebras which are free sums of subalgebras of cardinality $\leq 2^{|L|}$.

REMARKS : (1) By [S1], [S2], we can give a total analysis of the category of one-to-one maps ; by which we cannot interprete set theory in it.

(2) It is natural to ask what we can interpret in the automorphism group of a free algebra [or the category of monomorphism]. Clearly here the result depends on the variety. This converges with the question of M. Rubin who asked on the classificated of first-order theories, by biinterpretability of their saturated models automorphism groups.
In [Ru] he solved the problem for Boolean algebras. If we allow quantification over elements, we can essentially solved the problem.

(3) It will be interesting, to change somewhat our main theorem 5.5 to get biinterpretability. Of course, if the set of identities of variety is definable in $H(\lambda^*)$ and there are no non-trivial beautiful terms, this holds for M_{λ^*}.

(4) Sabbagh and the author note that if in Cat_λ , F_1 is definable then each automorphism of the skelton of Cat_λ (i.e. the full subcategory of a set of representatives from each isomorphism class of objects) is induced by an automorphism of the (multi-sorted) algebras of terms.

NOTATION :

Let V be a fixed variety, and Cat : the category of all algebras in V with all homomorphisms. Let K be a fixed subcategory of Cat, usually we assume K is a full subcategory. Let F_λ be the

free algebra (in V) generated by λ free generators $\{a_t : t \in I\}$. Let G_λ be the endomorphism semi-group of F_λ. Elements of I will be denoted by t, s and a_t, I, t, s will appear in no other context. Several times, we deal with $K = G_\lambda$.

Two endomorphisms (or elements, or subalgebras) of F_λ are called conjugate if some automorphism of F_λ takes one to the other. We denote elements of algebras by a, b, c, d (a-usually a generator) and also by x, y, z which serve as individual constants too.

For any f let Rnf be its range, and for a set $B \subseteq A \in V$, Cl B is its closure in A, $\tilde{b} = Cl\ \{b\}$. Let $\bar{x}, \bar{y}, \bar{z}$ denote finite sequences of variables, $\bar{x} = \langle x_0, ..., x_{n-1} \rangle$ usually. For a term τ we write $\tau = \tau(x_0, ..., x_{n-1}) = \tau(\bar{x})$, if every variable appearing in τ belong to $\{x_0, ..., x_{n-1}\}$. We can assume w.l.o.g. that if $\tau(\bar{x},\bar{y}) = \tau(\bar{x},\bar{z})$ is an identity (of V), $\bar{x}, \bar{y}, \bar{z}$ are pairwise disjoint, then for some term $\sigma(\bar{x})$, $\sigma(\bar{x}) = \tau(\bar{x},\bar{y})$ is an identity. A term $\tau(x_0, ..., x_{n-1})$ is called reduced if for no i and σ is

$$\sigma(x_0, ..., x_{i-1}, x_{i+1}, ..., x_{n-1}) = \tau(x_0, ..., x_{n-1})$$

an identity.

Clearly for every term $\tau(x_0, ...)$ there is a reduced term $\sigma(x_{i(0)}, ...)$ such that (i(0), i(1), ... are distinct and

$$\tau(x_0, x_1, ...) = \sigma(x_{i(0)}, x_{i(1)}, ...)$$

is an identity.

Clearly for every $b \in Cl\ B$ ($B \subseteq A \in V$) there is a reduced τ and distinct $b_i \in B$ such that $b = \tau(b_0, b_1, ...)$. Also if $\tau(\bar{x})$ is reduced, $t_i \in I$ are distinct, $\tau(a_{t_1}, ..., a_{t_n}) = \sigma(a_{s_1}, ..., a_{s_m})$ (all in some F_λ !) then $\{t_1, ..., t_n\} \subseteq \{s_1, ..., s_m\}$. We say $\tau(a_{t_0}, ...)$ is reduced if $\tau(x_0, ...)$ is reduced and the t_i s are distinct. Notice that any function $h : \{a_t : t \in I\} \to A \in V$, has a unique extension $\hat{h} \in Hom(F_\lambda, A)$. If $h : I \to I$, $\hat{h} \in G_\lambda$ is defined by $\hat{h}(a_t) = a_{h(t)}$.

We consider any K as a model : with two universes (the algebras and the morphisms, and the relations $f \in Hom(A,B)$, $g = f \circ h$. Naturally a first-order language L is associated with it. For convenience, we can consider this language for G_λ too.

§ 1. ON DEEPNESS.

DEFINITION 1.1. : Let h be a function from B into B. For every $x \in B$, we define its depth $Dp(x) = Dp(x,h)$ as an ordinal or ∞ by defining when it is $\geq \alpha$:

$Dp(x) \geq 0 \Leftrightarrow x \in B$

$Dp(x) \geq \delta \Leftrightarrow Dp(x) \geq \alpha$ for every $\alpha < \delta$ (where δ denote a limit ordinal)

$Dp(x) \geq \alpha + 1 \Leftrightarrow$ for some $y \in B$, $f(y) = x$ and $Dp(y) \geq \alpha$.

LEMMA 1.1. : Let $\{a_t : t \in I\}$ freely generate F_λ , and h is a function from I into I. Define $\hat{h}$ by $\hat{h}(\tau(a_{t(1)}, ..., a_{t(n)})) = \tau(a_{h(t(1))}, ..., a_{h(t(n))})$. Then

(A) $\hat{h} \in G$.

(B) $Dp[\tau(a_{t(1)}, ..., a_{t(n)}), \hat{h}] \geq \min_{e=1,n} Dp[t(e),h]$.

(C) If in (B), $\tau(x_1, ..., x_n)$ is reduced and the t(e) are distinct, then equality holds.

PROOF.

(A) Immediate.

(B) We prove by induction on α that

$$\min_{e=1,n} Dp[t(e),h] \geq \alpha \Rightarrow Dp[\tau(a_{t(1)}, ..., a_{t(n)}] \geq \alpha$$

and this suffices. For $\alpha = 0$ or α limit, it is trivial. For $\alpha = \beta + 1$, by the assumption and definition of depth, there are $s(e) \in I$, $h(s(e)) = t(e)$ and $Dp(s(e),h) \geq \beta$. Then $\min_e Dp[s(e), h] \geq \beta$, hence by the induction hypothesis $Dp[\tau(a_{s(1)}, ..., a_{s(n)}), \hat{h}] \geq \beta$ but

$$\hat{h}(\tau(a_{s(1)}, ..., a_{s(n)})) = \tau(a_{t(1)}, ..., a_{t(n)}), \text{ hence}$$

$$Dp[\tau(a_{t(1)}, ..., a_{t(n)}), \hat{h}] \geq \beta + 1 = \alpha .$$

(C) It suffices to prove by induction on α that

$$Dp[\tau(a_{t(1)}, ..., a_{t(n)}), \hat{h}] \geq \alpha \Rightarrow Dp(t(e), h) \geq \alpha$$

(for all e). For $\alpha = 0$ or α a limit ordinal, this is trivial. For $\alpha = \beta + 1$, by the definition of depth, there is a reduced term $\sigma(x_1, ..., x_m)$ and distinct s(e) (e = 1,m) such that

(i) $\hat{h}(\sigma(a_{s(1)}, ..., a_{s(m)}) = \tau(a_{t(1)}, ..., a_{t(n)})$ and

(ii) $Dp[\sigma(a_{s(1)}, ..., a_{s(m)}), \hat{h}] \geq \beta$.

By (ii) and the induction hypothesis $Dp[s(e),h] \geq \beta$ for e = 1,m.

By (i) and the definition of $\hat{h}$,

$$\sigma(a_{h(s(1))}, ..., a_{h(s(m))}) = \tau(a_{t(1)}, ..., a_{t(n)}).$$

As $\tau(x_1, ..., x_n)$ is reduced and the t(e) are distinct,

$\{t(1), ..., t(n)\} \subseteq \{h(s(1)), ..., h(s(m))\}$. So for each $e = 1,n$ there is k_e, $1 \leq k_e \leq m$ such that $t(e) = h(s(k_e))$ hence $Dp[t(e), h] \geq Dp[s(k_e), h] + 1 \geq \beta + 1 = \alpha$.

LEMMA 1.2. : Let h_1, h_2 be functions from B into B which commute i.e. $h_1 \circ h_2 = h_2 \circ h_1$. Then for any $x \in B$

$$Dp[x, h_1] \leq Dp[h_2(x), h_1] .$$

PROOF : We prove by induction on α that

$$Dp[x, h_1] \geq \alpha \Rightarrow Dp[h_2(x), h_1] \geq \alpha .$$

For $\alpha = 0$, or α a limit ordinal, it is immediate.

For $\alpha = \beta + 1$.

If $Dp[x, h_1] \geq \beta + 1$, then for some $y \in B$, $h_1(y) = x$ and $Dp[y, h_1] \geq \beta$.

So $h_1(h_2(y)) = h_1 \circ h_2(y) = h_2 \circ h_1(y) = h_2(h_1(y)) = h_2(x)$, and by the induction hypothesis $Dp[h_2(y), h_1] \geq \beta$ (as $Dp[y, h_1] \geq \beta$) hence

$Dp[h_2(x), h_1] \geq Dp[h_2(y), h_1] + 1 \geq \beta + 1 = \alpha$.

LEMMA 1.3. : Let $\{a_t : t \in I\}$ freely generates F_λ , $J \subseteq I, |I - J| = |I|$, $J = \bigcup_{\alpha < a(0)} J_\alpha$ and let $B = Cl\{a_t : t \in J\}$. Then we can find $f \in G_\lambda$, so that

(A) if $t \in J_\alpha$ then $Dp[a_t, f] = \alpha$

(B) if $g \in G_\lambda$, g and f commute, and g maps B into B, then for every $\alpha < a(0)$, $t \in J_\alpha$, $g(a_t) \in Cl\left\{a_s : s \in \bigcup_{\alpha \leq \beta < a(0)} J_\beta\right\}$

(C) every function g from $\{a_t : t \in J\}$ into B satisfying the condition from (B), can be extended to an endomorphism of G_λ , commuting with f and mapping B into B

(D) f is the form mentioned in 1.1.

PROOF : By renaming we can assume $I-J = I_0 \cup \{<0, t, \eta> : t \in J_\alpha , \alpha < a(0)$ $\ell(\eta) > 0$, η a decreasing sequence ordinals, $\eta(0) < \alpha\} \cup \{<1, t, n> : 0 < n < \omega\}$ and we identify $\langle 0, t, <> \rangle$ and $<1, t, 0>$ with t. Let us define a function h on I :

(i) for $s \in I_0$ $h(s) = s$

(ii) for $t \in J$, $\langle 0, t, \eta^\frown<\beta> \rangle \in I$

$$h(\langle 0, t, \eta^\frown<\beta> \rangle) = <0, t, \eta>$$

(iii) for $t \in J$, $h(<1, t, n>) = <1, t, n+1>$.

Let f be $\hat{h}$ as defined in lemma 1.1. It is easy to prove that if $t \in J$, $\ell(\eta) = n$, then $Dp[< 0, t, \eta > , h] = \eta(n-1)$. Hence (A) follows immediately by 1.1 (C), and $f \in G_\lambda$ by 1.1 (A). As for part (B), if $g \in G_\lambda$ commutes with f, g maps B into B, $t \in J_\alpha$, $\alpha < \alpha(0)$, let $g(a_t) = \sigma(a_{s(1)}, ..., a_{s(n)})$ be reduced. As g maps B into B, $g(a_t) \in B$, hence necessarily $s(e) \in J$ for $e = 1, n$; so let $s(e) \in J_{\alpha(e)}$. By 1.1 (C) and a remark above

$$Dp[g(a_t), f] = Dp[\sigma(a_{s(1)}, ..., a_{s(e)}), f] = \min_e Dp[s(e), h] = \min_e \alpha(e).$$

On the other hand by lemma 1.2, as g commutes with f,

$$\alpha = Dp[a_t, f] \leq Dp[g(a_t), f].$$

Combining both we get $\alpha \leq \alpha(e)$ for $e = 1,n$. Hence $g(a_t) \in Cl\{a_t : t \in J_\beta, \alpha \leq \beta < \alpha(0)\}$, so we proved (B).

As for (C), extend g to a function g_1 from $\{a_t : t \in I\}$ into F_λ, by :

if $g(t) = \tau(a_{t(1)}, ..., a_{t(n)})$

let $g_1(a_{<0,t,\eta>}) = \tau(a_{<0,t(1),\eta>}, a_{<0,t(2),\eta>}, ..., a_{<0,t(n),\eta>})$

$g_1(a_{<1,t,m>}) = \tau(a_{<1,t(1),m>}, a_{<1,t(2),m>}, ..., a_{<1,t(n),m>})$

and $g_1(a_s) = a_s$ for $s \in I_0$.

It is easy to check that g_1 is well defined (because $t \in J_\alpha$, $t(e) \in J_\beta \Rightarrow \alpha \leq \beta$) and it has a unique extension to $g_2 \in G_\lambda$. In order to check that g_2 and f commute, it suffices to prove that for every $s \in I$, $f \circ g_2(a_s) = g_2 \circ f(a_s)$ which is quite easy.

Now (D) holds by the definition of f.

§ 2. SIMPLE PROPERTIES EXPRESSIBLE IN FIRST-ORDER LOGIC.

LEMMA 2.1. : Each of the following properties (in a full subcategory K) is expressible by formulas (of L) :

(A) proj (f) - which means f is a projection

(B) f is an automorphism, aut(f) in short

(C) g is a projection, and $Rn(f) \subseteq Rn(g)$

(D) g is a projection, $Rn(f \restriction Rng) \subseteq Rn(g)$

(E) g is a projection, $Rn(f_e) \subseteq Rn(g)$ for $e = 1, 2, 3$ and

$$[f_3 \restriction Rn(g)] = [f_1 \restriction Rn(g)] \circ [f_2 \restriction Rn(g)]$$

(F) g, f are projections and $Rn(g) = Rn(f)$.

REMARK : Clearly for any $f \in Hom(A, B)$, Rn(f) is a subalgebra of B.

PROOF : We give the expression or an indication of it in each case

(A) $f \circ f = f$

(B) $(\exists g, A)(f \circ g = g \circ f = 1_A)$ [1_A - the identity of A define by

$$(\forall f,B)(f \in Hom(B,A) \to h \circ f = f)]$$

(C) proj (g) $\wedge$ $g \circ f = f$.

As g is a projection, $x \in Rn(g) \Leftrightarrow g(x) = x$. Hence when proj (g), and $f : B \to A$, $g : A \to A$.

$$g \circ f = f \Leftrightarrow (\forall x \in B)\ (g(f(x)) = f(x)) \Leftrightarrow (\forall x)\ [f(x) \in Rn(g)] \Leftrightarrow$$
$$\Leftrightarrow Rn(f) \subseteq Rn(g)$$

(D) proj (g) $\wedge$ $g \circ f \circ g = f \circ g$

(the proof similar to the previous one)

(E) proj (g) $\wedge \overset{3}{\underset{i=1}{\wedge}} g \circ f_i = f_i \wedge f_1 \circ f_2 \circ g = f_3 \circ g$

(F) Immediate by (C).

CLAIM 2.2.

(A) If B is the range of some projection $f \in G_\lambda$, then any homomorphism $h : B \to B$ (B considered as a subalgebra) can be extended to an $h' \in G_\lambda$

(B) If $B = Cl\{a_t : t \in J\}$ where $J \subseteq I$ and $\{a_t : t \in I\}$ freely generates F_λ, then a homomorphism $h : B \to B$ is onto B iff for some homomorphism $g : B \to B$ $h \circ g$ is the identity.

PROOF.

(A) Clearly $h \circ f \in G_\lambda$ extend h and its domain is F_λ.

(B) Clearly if $h \circ g$ is the identity (on B), then for any $y \in B$, $g(y) \in B$ and $h(g(y)) = y$, hence h is into B.

Let h be onto B, so for every $t \in J$, $a_t = h(\tau^t(a_{s(1,t)}, \ldots))$ for some term τ^t and $s(e,t) \in J$. There is a unique homomorphism $g : B \to B$, $g(a_t) = \tau^t(a_{s(1,t)}, \ldots)$. Clearly for any $t \in J$, $h \circ g(a_t) = a_t$ hence $h \circ g$ is the identity.

REMARK. Sabbagh had proved that «f is one-to-one» and «f is onto» are (first-order) definable in Cat_λ.

§ 3. BEAUTIFUL TERMS.

DEFINITION 3.1. : The term $\tau(x_1, ..., x_n)$ is called beautiful if

(A) For any term $\sigma(x_1, ..., x_m)$

$$\tau(\sigma(x_1^1, x_2^1, ..., x_m^1), \sigma(x_1^2, ..., x_m^2), ..., \sigma(x_1^n, ..., x_m^n)) =$$

$$\sigma(\tau(x_1^1, x_1^2, ..., x_1^n), \tau(x_2^1, ..., x_2^n), ..., \tau(x_m^1, ..., x_m^n))$$

is an identity (of F_λ)

(B) $\tau(\tau(x_1^1, x_2^1, ..., x_n^1), \tau(x_1^2, ..., x_n^2), ..., \tau(x_1^n, ..., x_n^n)) = \tau(x_1^1, x_2^2, ..., x_n^n)$

is an identity

(C) $\tau(x, ..., x) = x$ is an identity.

REMARK : The beautiful terms for $n > 1$ cause us much trouble. For the free abelian group, only x is beautiful and reduced. However, if F is a free algebra of the identities $T(T_1(x), T_2(x)) = x$, $T_1(T(x,y)) = x$ and $T_2(T(x,y)) = y$ and $T_3(x,y) = T(T_1(x), T_2(y))$, then the identities which hold in $(|F_\lambda| ; T_3)$ define a variety for which $T_3(x,y)$ is a beautiful term.

LEMMA 3.1.

(A) The set of beautiful terms is closed under substitution, i.e. if

$\sigma(x_1, ..., x_m)$, $\tau_i(x_1, ..., x_{n(i)})$ $(i = 1, ..., n)$ are beautiful terms, then so is

$$\sigma^*(y_1, ..., y_k) = \sigma(\tau_1(y_{j(1,1)}, y_{j(1,2)}, ..., y_{j(1,n(1))}), \tau_2(y_{j(2,1)}, ...), ...,$$

$$\tau_k(y_{j(k,1)}, y_{j(k,2)}, ..., y_{j(k,n(k))}))$$

(B) x is a beautiful term, and there is no other beautiful term $\tau(x)$;
and $\tau(x_1, ..., x_n) = x_i$ is beautiful.

(C) The two-place beautiful terms ($\tau(x,y)$) generate by substitution all the beautiful terms.

PROOF.

(A) The checking has no problems.

(B) By the third demand in Def. 3.1 ; the second phrase - by checking.

(C) If $\tau = \tau(x_1, ..., x_n)$, $(n > 2)$ let $\tau_1(x,y) = \tau(x, ..., x, y)$

$\tau_2(x_1, ..., x_{n-1}) = \tau(x_1, ..., x_{n-1}, x_{n-1})$. So $\tau(x_1, ..., x_n) =$

$\tau_2(x_1, ..., x_{n-2}, \tau_1(x_{n-1}, x_n))$ is an identity as

$\tau_2(x_1, ..., x_{n-2}, \tau_1(x_{n-1}, x_n)) =$

$\tau(x_1, ..., x_{n-2}, \tau_1(x_{n-1}, x_n), \tau_1(x_{n-1}, x_n)) =$

$\tau(\tau(x_1, ..., x_1), ..., \tau(x_{n-2}, ..., x_{n-2}), \tau(x_{n-1}, ..., x_{n-1}, x_n), \tau(x_{n-1}, ..., x_{n-1}, x_n)) =$

$\tau(x_1, ..., x_{n-2}, x_{n-1}, x_n)$.

So we can prove our assertion by induction on n.

DEFINITION 3.2. : The beautiful Boolean algebra of a variety is the Boolean algebra $\underline{B}$ such that

(1) Its elements are beautiful terms of the form $\tau(x,y)$ (x,y here are fixed).

(2) Its zero is y, its unit is x.

(3) The intersection is defined by

$$\tau_1(x,y) \cap \tau_2(x,y) = \tau_1(\tau_2(x,y), y) = \tau_2(\tau_1(x,y), y).$$

(4) The complement is define by $\tau(x,y)^c = \tau'(x,y)$ where $\tau(x,y) = \tau'(y,x)$.

DEFINITION 3.3. : For any filter T of $\underline{B}$, let $\approx_T$ be the relation (defined on elements of members of Cat)

$$a \approx_T b \text{ iff } a = \tau(b,a) \text{ iff } b = \tau(a,b)$$

and it is similarly defined on each Hom(A,B), A, B $\in$ Cat (see Def. 3.4 (A)).

DEFINITION 3.4. :

(A) If $f_i : A \to B$ (in Cat), τ a beautiful term, $\tau(f_1, ..., f_n) : A \to B$ is defined by

$$\tau(f_1, ..., f_n)(a) = \tau(f_1(a), ..., f_n(a))$$

(B) If A_i (i = 1, ..., n) belong to Cat, τ a beautiful term, the algebra $\tau(A_1, ..., A_n)$ is defined as follows :

its elements are $< a_1, ..., a_n >$ where $a_i \in A_i$ and $<a_1, ..., a_n> = < b_1, ..., b_n>$ iff for each i, $a_i \approx_{T_i} b_i$ where T_i is the filter generated by $\tau(y, ..., y, x, y, ..., y)$ (the x-in the ith place) ; for a term σ , $\sigma(< a^1_1, ..., a^1_n >, < a^2_1, ..., a^2_n >, ...) =$

$$< \sigma(a^1_1, a^2_1, ...), \sigma(a^1_2, a^2_2, ...), ... >$$

(C) If $f_i : A_i \to B_i$ (in Cat) τ a beautiful term, then

$$\tau(f_1, ..., f_n) : \tau(A_1, ..., A_n) \to \tau(B_1, ..., B_n)$$

is defined naturally.

REMARK : There is an ambiguity in the definition of $\tau(f_1, ..., f_n)$; so when both definitions are meaningful we prefer (A) ; and even better : in (B) if $\bigwedge_{i=1}^{n} A_i = A$ or $\tau(x_1, ..., x_n) = x_i$ is an identity, we make $\tau(A_1, ..., A_n) = A_i$. In fact $\tau(A_1, ...)$ is essentially defined up to isomorphism.

NOTATION : If $\bar{x}^e = < x^e_0, ..., x^e_{m-1} >$ then

$$\tau(\bar{x}^0, ..., \bar{x}^{n-1}) = < \tau(x^0_0, x^1_0, ...), \tau(x^0_1, x^1_1, ...), ... >$$

LEMMA 3.2. :

(A) The beautiful Boolean algebra is really a Boolean algebra.

(B) For any filter T of $\underline{B}$ and $A \in$ Cat, $\approx_T$ is an equivalence relation over A, and even a congruence relation, so $A/\approx_T$ is defined and $\in$ Cat and $\tau(x,y) \in T$ implies the identity $\tau(x,y) = x$ holds in $A/\approx_T$.

(C) For any $f_i : A \to B$ (in Cat) and any beautiful τ, $\tau(f_1, ..., f_n)$ is a homomorphism from A into B. If $A = B$, f_i a projection then $\tau(f_1, ..., f_n)$ is a projection.

(D) If $A_i \in$ Cat, τ is beautiful then $\tau(A_1, ..., A_n) \in$ Cat.

(E) If $f_i : A_i \to B_i$ in Cat, τ beautiful then

$\tau(f_1, ..., f_n) : \tau(A_1, ..., A_n) \to \tau(B_1, ..., B_n)$ is in Cat.

(F) If $T \subsetneq \underline{B}$ is an ultrafilter, for the category $\{A/\approx_T : A \in \text{Cat}\}$, there are no beautiful terms. It is the category of the variety, whose identities are those of V and $\{\tau(x,y) = x : \tau \in T\}$. This holds for filters T too.

PROOF : Easy.

LEMMA 3.3. :

(A) For every $\tau_1(x,y) = \tau_2(y,x)$, let T_e be the filter generated by $\tau_e(x,y)$. Then any $A \in$ Cat is the direct product of $A/\approx_{T_1}$ and $A/\approx_{T_e}$. In $A/\approx_{T_e}$, $\tau_e(x,y) = x$ is an identity.

(B) If our variety V is the modules over a ring R, then the beautiful two-place terms are $ax + (1-a)y$ where a is in the center of R and is idempotent (i.e. $a^2 = a$).

PROOF : Easy.

LEMMA 3.4. : Let τ be a beautiful term.

(A) Let $K \subseteq$ Cat be a subcategory, such that for $f_i : A \to B$ in K, $\tau(f_1, ..., f_n)$ is in K. In the proper language L^* (i.e. with variables for objects, and for morphisms, and a partial operation of composition, and the relation $f \in \text{Hom}(A,B)$ plus our τ) for every first order formula $\phi_0(f_1, ... ; A_1, ...)$, there is a finite set Φ of formulas of L^* with the same free variables such that :

(i) $\phi_0 \in \Phi$

(ii) if $g_i^j : B_{i(1)} \to B_{i(2)}$, $g_i = \tau(g_i^1, ..., g_i^n)$, then the sets

$$\Phi^j = \{\phi(f_1, ... ; A_1, ...) \in \Phi : K \models \phi[g_1^j, g_2^j, ... ; B_1, ...]\}$$

totally determine $\Phi^* = \{\phi(f_1, \ldots ; A_1, \ldots) \in \Phi : K \models \phi [g_1, g_2, \ldots ; B_1, \ldots]\}$

hence in particular the truth of $\phi_0(g_1, \ldots ; B_1, \ldots)$

(iii) if in (ii) the Φ^j's are equal then $\Phi^* = \Phi^j$.

(B) We can expand L^* by letting in variables for elements, and more terms of our variety ; and also let in (ii) $g^j_i : B^j_{i(1)} \to B^j_{i(2)}$; and the same conclusion holds, provided that we are careful for the question when

$$\tau(B_{i(1,1)}, B_{i(1,2)}, \ldots) = \tau(B_{i(2,1)}, B_{i(2,2)}, \ldots).$$

REMARK : This is just a varient of Feferman Vaught [FV] , as refined by Weinstein [W] and Galvin [G] .

PROOF :

(A) We define $\Phi = \Phi_{\phi_0}$ by induction on the structure of ϕ_0.

Let $\tau^j(x,y) = \tau(y, \ldots, y, x, y, \ldots, y)$ (x - in the jth place) for j = 1,n.

If ϕ_0 is $f_i \in \mathrm{Hom}(A_{i(1)}, A_{i(2)})$ or $A_i = A_j$ let $\Phi_{\phi_0} = \{\phi_0\}$

If ϕ_0 is $f_{k(1)} = f_{k(2)} \circ f_{k(3)}$ let

$$\Phi_{\phi_0} = \{f_{k(1)} = \tau^j(f_{k(2)} \circ f_{k(3)}, f_{k(1)}) : j = 1,n\} \cup \{\phi_0\}$$

(if $\tau(f,g)$ is not defined, $h = \tau(f,g)$ is false)

If $\phi_0 = f_k = \tau(f_{k(1)}, \ldots, f_{k(n)})$ let

$$\Phi_{\phi_0} = \{f_k = \tau^j(f_{k(1)}, \ldots, f_{k(n)}) : j = 1,n\} \cup \{\phi_0\}$$

If $\phi_0 = \neg\psi$ or $\phi_0 = \psi_1 \wedge \psi_2$ then $\Phi_{\phi_0} = \Phi_\psi \cup \{\phi_0\}$, $\Phi_{\phi_0} = \Phi_{\psi_1} \cup \Phi_{\psi_2} \cup \{\phi_0\}$ resp.

If $\phi_0 = (\exists g)\psi$ then $\Phi_{\phi_0} = \{(\exists g)\psi_1 : \psi_1$ a boolean combination of $\Phi_\psi\}$.

It is now easy to prove (ii) by induction and (iii) follows trivially by choosing $g^j_i = g^1_i$ for all i, j.

(B) Left to the reader.

§ 4. DEFINING ARBITRARY SETS WITH PARAMETERS.

THEOREM 4.1. : There is a formula ϕ^b_m such that the following holds. Let K be a full subcategory of Cat, and $F_\mu \in K$. Suppose A, B $\in$ K, $|A| + |B| \leq \mu$, and $\bar{f}_i$ ($i < i(0) < \mu^+$) is an m-tuple of members of Hom (A,B). Then we can find $\bar{g}$ (from K) such that :

$$K \models \phi^b_m [\bar{f},\bar{g}] \quad \underline{\text{iff}} \quad \bar{f} = \tau (\bar{f}_{i(1)}, ..., \bar{f}_{i(n)})$$

for some beautiful term τ and $i(1) < ... < i(n)$.

REMARK : (1) The f^*_i are added as individual constant during the proof, in order not to mention them explicitely during the proof. In the end, we include them in $\bar{g}$.

(2) Let $\phi^b_1 = \phi^b$.

PROOF : Let $\{a_t : t \in I^*\}$ freely generate F_μ, and let $J \subseteq I^*$, $\mu = |J| = |I^* - J|$. For notational simplicity let $J = \{<\alpha,\beta> : \alpha,\beta < \mu\}$, $a_{<\alpha,\beta>} = a^\beta_\alpha$, and $i(0) = \mu$ (as we can allow repeations).

MAIN LEMMA 4.2. : There is a formula $\phi(f)$, such that $G_\mu \models \phi[f]$ iff there is a beautiful $\tau = \tau(x_1, ..., x_n)$ and ordinals $\alpha(1), ..., \alpha(n)$, so that for every $\beta < \mu$, $f(a^\beta_0) = \tau(a^\beta_{\alpha(1)}, ..., a^\beta_{\alpha(n)})$.

PROOF OF 4.1 FROM 4.2 :

Let $f^*_0 : F_\mu \mapsto A$, be such that it maps $\{a^\alpha_0 : \alpha < \mu\}$ onto A, and $f^*_0(a^\beta_\alpha) = f^*_0(a^\beta_0)$. Let $\bar{f}_i = <f^0_i, ..., f^{m-1}_i>$, and let $f^*_{0,e} : F_\mu \to B$, be such that $f^*_{0,e}(a^\beta_\alpha) = f^e_\alpha \circ f^*_0(a^\beta_\alpha)$, for $e < m$.

Let $f^*_1 : F_\mu \to F_\mu$ maps $\{a_t : t \in I^*\}$ onto $\{a^\beta_0 : \beta < \mu\}$

Let $\phi^b_m(f_0, ..., f_{m-1})$ says that there is $f : F_\mu \to F_\mu$, such that $\phi(f)$ and

$$\bigwedge_{e<m} f_e \circ f^*_0 \circ f^*_1 = f^*_{0,e} \circ f \circ f^*_1 .$$

PROOF OF 4.2 :

We partition our proof to three cases [it suffice to prove each one separately, and then by comining the formulas and choosing the parameters for each case, we can easily find a unique formula] .

We shall use § 2 freely : and restrict ourselves to F_μ and G_μ .

Case I : $\mu = \aleph_0$

Let $f_2^* \in G_\mu$ be such that $f_2^*(a_t) = a_0^0$ for each $t \in I^*$ and $f_i^* \in G_\mu$ $(i = 3, 7)$ be such that :

$t \in I^* - J$ implies $f_i^*(a_t) = a_0^0$, $f_3^*(a_n^m) = a_{n+1}^m$, $f_4^*(a_n^m) = a_n^{m+1}$,

$f_5^*(a_n^m) = a_m^n$, and $f_7^*(a_n^m) = a_n^0$; $f_6^*(a_n^m) = a_n^n$. Let $B_1 = Cl\{a_n^0 : n\}$,

$B_2 = Cl\{a_n^m : n,m\}$, $B_0 = Cl\{a_0^0\}$.

Clearly there are first order formulas $\phi_e(X)$ such that :

(1) $\phi_1(f)$ says that $f \restriction B_1$ is into B_1 and it commutes with $f_3^* \restriction B_1$
(by 2.1 D, as f_7^* is a projection, and $Rn\, f_7^* = B_1$)

(2) $\phi_2(f)$ says that $\phi_1(f)$ and $f \restriction B_2$ is into B_2 and it commutes with $f_4^* \restriction B_2$

(3) $\phi_3(f)$ says that $\phi_2(f)$ and for any g, $\phi_2(g)$ implies that $f \restriction B_2$ and $(f_5^* \circ g \circ f_5^*) \restriction B_2$ commute.

(4) $\phi_4(f)$ says that $\phi_3(f)$ and $(f_5 \circ f \circ f_5^* \circ f) \restriction B_0 = (f_6^* \circ f) \restriction B_0$

(5) $\phi_5(f)$ says that $\phi_4(f)$ and $(f_2^* \circ f) \restriction B_0 = f_2^* \restriction B_0$

(6) $\phi_6(f)$ says that for some f', $f' \restriction B_3 = f \restriction B_3$ and $\phi_5(f')$ where $B_3 = cl\{a_0^m : m < \omega\}$

Now we notice

(1*) $G_\mu \models \phi_1[f]$ iff $f(a_n^0) = \tau(a^0_{n+\ell(1)}, ..., a^0_{n+\ell(k)})$ (for some $\tau, \ell(i)$, for every n)

[Clearly the «if» part holds ; for the other direction, as $f(a_0^0) \in B_1$,

$f(a_0^0) = \tau(a^0_{\ell(1)}, ..., a^0_{\ell(k)})$ for some τ, $\ell(i)$; apply f_3^* n times and we get our result].

(2*) $G_\mu \models \phi_2[f]$ iff $f(a_n^m) = \tau(a^m_{n+\ell(1)}, ..., a^m_{n+\ell(k)})$

the «if» part is immediate, and for the «only if» use (1*), and apply f_4^* m times on the right side

(3*) $G_\mu \models \phi_3[f]$ iff $f(a_n^m) = \tau(a^m_{n+\ell(1)}, ..., a^m_{n+\ell(k)})$, (the $\ell(i)$ distinct) and $\tau(x_1, ..., x_k)$ satisfies condition (A) for being beautiful.

[Suppose $G_\mu \models \phi_3[f]$, then clearly $G_\mu \models \phi_2[f]$ hence by (2*) for some τ, $\ell(i)$,

for every n,m $f(a^m_n) = \tau(a^m_{n+\ell(1)}, ..., a^m_{n+\ell(k)})$.

Now for every term $\sigma(x_1,..., x_p)$ let $g \in G_\lambda$ be such that $g(a^m_n) = \sigma(a^m_{n+1}, ..., a^m_{n+p})$.

So clearly $f^*_5 \circ g \circ f^*_5(a^m_n) = \sigma(a^{m+1}_n, ..., a^{m+p}_n)$. By (2^*), $G_\lambda \models \phi_2[g]$, hence

by ϕ_3's definition, $(f^*_5 \circ g \circ f^*_5) \circ f[a^0_0] = f \circ (f^*_5 \circ g \circ f^*_5)[a^0_0]$, computing we get

$$\tau(\sigma(a^1_{\ell(1)}, ..., a^p_{\ell(1)}), \sigma(a^1_{\ell(2)}, ..., a^p_{\ell(2)}), ..., \sigma(a^1_{\ell(k)}, ..., a^p_{\ell(k)}))$$

$$= \sigma(\tau(a^1_{\ell(1)}, ..., a^1_{\ell(k)}), \tau(a^2_{\ell(1)}, ..., a^2_{\ell(k)}), ..., \tau(a^p_{\ell(1)}, ..., a^p_{\ell(k)})).$$

As this holds for any σ, we prove that τ satisfies condition (A) from Definition 3.1. The other direction in (3^*) should be easy now].

(4^*) : $G_\mu \models \phi_4(f)$ iff $G_\lambda \models \phi_3[f]$, and the τ from (3^*) satisfies condition (B) from Def. 3.1.

[Assuming $G_\mu \models \phi_3[f]$, τ as in (4^*), $f^*_5 \circ f \circ f^*_5 \circ f \restriction B_2 = f^*_6 \circ f \restriction B_2$ is equivalent to the equality of

$$f^*_6 \circ f(a^0_0) = f^*_6(\tau(a^0_{\ell(1)}, ..., a^0_{\ell(k)})) = \tau(a^{\ell(1)}_{\ell(1)}, ..., a^{\ell(k)}_{\ell(k)})$$

and $f^*_5 \circ f \circ f^*_5 \circ f(a^0_0) = f^*_5 \circ f \circ f^*_5(\tau(a^0_{\ell(1)}, ..., a^0_{\ell(k)}))$

$$= f^*_5 \circ f(\tau(a^{\ell(1)}_0, ..., a^{\ell(k)}_0))$$

$$= f^*_5(\tau(\tau(a^{\ell(1)}_{\ell(1)}, a^{\ell(1)}_{\ell(2)}, ...), \tau(a^{\ell(2)}_{\ell(1)}, a^{\ell(2)}_{\ell(2)}, ...), ..., \tau(a^{\ell(k)}_{\ell(1)}, a^{\ell(k)}_{\ell(2)}, ...))$$

$$= \tau(\tau(a^{\ell(1)}_{\ell(1)}, a^{\ell(2)}_{\ell(1)}, ...), \tau(a^{\ell(1)}_{\ell(2)}, a^{\ell(2)}_{\ell(2)}, ...,), \tau(a^{\ell(1)}_{\ell(k)}, a^{\ell(2)}_{\ell(k)}, ...)$$

This clearly is equivalent to condition (B) from Def. 3.1 on τ. The other direction is easy.]

(5^*) $G_\mu \models \phi_5[f]$ iff $f(a^m_n) = \tau(a^m_{n+\ell(1)}, ..., a^m_{n+\ell(k)})$ (the $\ell(i)$ are distinct) where $\tau(x_1, ..., x_k)$ is beautiful.

[Because, assuming $G_\lambda \models \phi_4(f)$, $f^*_2 \circ f \restriction B_0 = f^*_2 \restriction B_0$ is equivalent to $a^0_0 = \tau(a^0_0, ..., a^0_0)$]

(6^*) $G_\mu \models \phi_6[f]$ iff $f(a^n_0) = \tau(a^n_{\ell(1)}, ..., a^n_{\ell(k)})$

for some beautiful τ, and $\ell(i) < \omega$, for every $n < \omega$.

[Immediate, by (5), noticing $f^*_5 \circ f^*_7 \circ f^*_5$ is a projection onto B_3].

So we clearly finish case I.

Case II : $\mu = |J|$ is regular $> \aleph_0$.

Let $I^* - J = I_0 \cup \{ < \alpha, \delta, n >: \alpha < \mu, \delta < \mu_1, \text{cf } \delta = \aleph_0, n < \omega \}$.

$|I_0| = \mu$, and let us denote $a^{\beta,n}_\alpha = a_{<\alpha,\beta,n>}$; where $\mu_1 = \mu$. But for using in case III, we from now up to the end of the proof of claim 4.3, assume only $\mu_1 \leq \mu$, μ_1 is regular. For each limit $\delta < \mu_1$, cf $\delta = \aleph_0$ choose an increasing sequence $\delta(n)$ $(n < \omega)$ of ordinals $< \delta$, whose limit is δ; such that for each $\beta < \mu_1$, $n < \omega$, $\{\delta < \mu_1 : \beta = \delta(n)\}$ is a stationary subset of μ_1 (see e.g. Solovay [So]). Let us define some f^*_e's, by defining $f^*_e(a_t)$ $(t \in I^*)$ understanding that when $f^*_e(a_t)$ is not explicitely defined, it is a^0_0. So let,

for $\alpha < \mu$, $\beta < \mu_1$, $f^*_2(a^\beta_\alpha) = a^0_0$, $f^*_3(a^\beta_\alpha) = a^0_\alpha$, $f^*_4(a^\beta_\alpha) = a^\beta_0$, $f^*_5(a^\beta_\alpha) = a^\alpha_\beta$,

$f^*_6(a^\beta_\alpha) = a^\alpha_\alpha$, and when $\delta < \mu_1$, cf $\delta = \aleph_0$, $f^*_7(a^\delta_\alpha) = a^{\delta,0}_\alpha$;

cf $\beta \neq \aleph_0 \rightarrow f^*_7(a^\beta_\alpha) = a^\beta_\alpha$; $f^*_8(a^\beta_\alpha) = a^\beta_\alpha$; $f^*_8(a^{\delta,n}_\alpha) = a^{\delta,n+1}_\alpha$,

$f^*_9(a^\beta_\alpha) = a^\beta_\alpha$, $f^*_9(a^{\delta,n}_\alpha) = a^{\delta(n)}_\alpha$. Let $B_0 = \text{Cl}\{a^0_0\}$, $B_1 = \text{Cl}\{a^0_\alpha : \alpha < \mu\}$,

$B_2 = \text{Cl}\{a^\beta_0 : \beta < \mu_1\}$, $B_3 = \text{Cl}\{a^\beta_\alpha : \alpha < \mu, \beta < \mu_1\}$,

$B_4 = \text{Cl}\{a^\beta_\alpha, a^{\beta,n}_\alpha : \alpha < \mu, \beta < \mu_1, n < \omega\}$ and $B_5 = \text{Cl}\{a^\beta_0, a^{\beta,n}_0 : \beta < \mu_1, n < \omega\}$,

$B_6 = \text{Cl}\{a^\beta_0 : \beta < \mu_1, \text{cf}\beta = \aleph_0\}$, $B_7 = \text{Cl}\{a^\beta_\alpha : \alpha < \mu, \beta < \mu_1, \text{cf}\beta = \aleph_0\}$

Clearly f^*_3, f^*_4, f^*_9 are projections onto B_1, B_2, B_3 resp. and let $f^*_{10}, f^*_{11}, f^*_{12}$ be projections onto B_4, B_5, B_6 resp.

Now we apply lemma 1.3, with $\{ <\alpha, \beta> : \alpha < \mu, \beta < \mu_1\}$ for J, and $\{ <\alpha, \beta> : \alpha < \mu \}$ for J_β, and I^* for I, and get $f^*_{13} \in G_\lambda$ as mentioned there.

Let the first order formula $\phi_1(f,g)$ says that f,g are conjugate to f^*_2, Rnf, Rng $\subseteq B_3$ and there is $h \in G_\mu$ commuting with f^*_{13}, and mapping B_3 into itself, such that $h \circ f = g$. We shall write $\phi_1(f,g)$ also in the form $f \trianglelefteq g$. So by 1.3, if f,g are conjugate to f^*_2 and

$f(a^0_0) = a^{\beta}_{\alpha}$, $g(a^0_0) = \tau\,(a^{\beta(1)}_{\alpha(1)}, ..., a^{\beta(k)}_{\alpha(k)})$ <u>then</u> $f \trianglelefteq g$ iff $\beta \trianglelefteq \beta(1) \wedge ... \wedge \beta \trianglelefteq \beta(k)$.

Let the first order formula $\phi_2(f)$ says that $f \restriction B_2$, $f \restriction B_5$, $f \restriction B_0$, $f \restriction B_6$ are into B_3, B_4, B_1, B_7 resp. and for any g conjugate to f^*_2, if Rng $\subseteq B_2$, then $g \trianglelefteq f \circ g$; and $f \restriction B_5$ commute with f^*_7, f^*_8, f^*_9 and $f \restriction B_3$ commute with f^*_3.

CLAIM 4.3 : $G_\lambda \models \phi_2[f]$ iff for each $\beta < \mu_1$

$f(a^0_0) = \tau(a^{\beta}_{\alpha(1)}, ..., a^{\beta}_{\alpha(k)})$, $f(a^{\beta,n}_0) = \tau\,(a^{\beta,n}_{\alpha(1)}, ..., a^{\beta,n}_{\alpha(k)})$

for some τ , α (e) (which do not depend on β !).

PROOF : Assume $G_\mu \models \phi_2$ [f] , and let

$f(a^{\beta}_0) = \tau_{\beta}\,(a^{\gamma(\beta,1)}_{\alpha(\beta,1)}, a^{\gamma(\beta,2)}_{\alpha(\beta,2)}, ..., a^{\gamma(\beta,k(\beta))}_{\alpha(\beta,k(\beta))})$.

$\alpha(\beta,e)$ increase with e,w.l.o.g. By part of ϕ_2 saying : g conjugate to f^*_2 , Rng $\subseteq B_2$ implies $g \trianglelefteq f \circ g$; it follows that $\beta \trianglelefteq \gamma(\beta,e)$ (choose g such that $g(a_t) = a^{\beta}_0$)).

As μ_1 is regular, $\mu_1 > \aleph_0$, for any $\beta_0 < \mu_1$

$\sup \{ \gamma(\beta,e) : e = 1, ..., k(\beta), \beta < \beta_0 \} < \mu_1$, hence

$S = \{ \beta_0 : \beta_0 < \mu_1$; and $\beta < \beta_0$, $1 \trianglelefteq e \trianglelefteq k(\beta)$ implies $\gamma(\beta,e) < \beta_0 \}$

is an unbounded subset of μ_1 ; and by its definition it is closed.

Now we shall prove that for $\delta \in S$, cf $\delta = \aleph_0$ implies $\gamma(\delta,e) = \delta$. For suppose $\gamma = \gamma(\delta,e_0) \neq \delta$ then as said above $\delta < \gamma(\delta,e_0)$. As $\delta_1(n)$ is increasing (as a function of n) and its limit is δ_1 , (or $\delta_1(n)$ is not defined) for some $n < \omega$ big enough, $\delta < \gamma(\delta,e_0)(n)$, and $\gamma(\delta,e_1) \neq \gamma(\delta,e_2) \Rightarrow \gamma(\delta,e_1)(n) \neq \gamma(\delta,e_2)(n)$ (when they are defined).

As $f \restriction B_6$ is into B_7 necessarily $\gamma(\delta,e)$ has cofinality $\aleph_0$, and as $f \restriction B_5$ commutes with f^*_7

$$f(a^{\delta,0}_0) = \tau_\delta\,(a^{\gamma(\delta,1),0}_{\alpha(\delta,1)}, a^{\gamma(\delta,2),0}_{\alpha(\delta,2)}, ...)$$

As $f \restriction B_5$ commutes with f^*_8

$$f(a^{\delta,n}_0) = \tau_\delta\,(a^{\gamma(\delta,1),n}_{\alpha(\delta,1)}, a^{\gamma(\delta,2),n}_{\alpha(\delta,2)}, ...)$$

As $f \restriction B_5$ commutes with f^*_9

$$f(a^{\delta(n)}_0) = \tau_\delta\ (a^{\gamma(\delta,1)(n)}_{a(\delta,1)}, a^{\gamma(\delta,2)(n)}_{a(\delta,1)}, \ldots)$$

and $\tau_\delta\ (a^{\gamma(\delta,1)(n)}_{a(\delta,1)}, \ldots)$ is reduced (by the choice of τ_δ and of n).

But $f(a^{\delta(n)}_0)$ is equal also to $\tau_{\delta(n)}\ (a^{\gamma(\delta(n),1)}_{a(\delta(n),1)}, \ldots)$ which is reduced too.

Hence $\gamma(\delta,e_0)(n) \in \{\gamma(\delta(n),e) : 1 \leq e \leq k(\delta(n))\}$

but on the one hand $\delta < \gamma(\delta,e_0)(n)$ by the choice of e_0 and n,

and on the other hand $\delta(n) < \delta \Rightarrow \gamma(\delta(n),e) < \delta$ as $\delta \in S$, contradiction.

Hence $\gamma(\delta,e) = \delta$ for every $\delta \in S$.

Now for every $\beta < \mu_1$, n we know $\{\delta < \mu : \mathrm{cf}\,\delta = \aleph_0, \delta(n) = \beta\}$

is stationary, so there is $\delta \in S$, $\mathrm{cf}\,\delta = \aleph_0$ such that $\delta(n) = \beta$.

As before we can show that $f(a^{\delta(n)}_0) = \tau_\delta\ (a^{\delta(n)}_{a(\delta,1)}, \ldots, a^{\delta(n)}_{a(\delta,k(\delta))})$

$= \tau_{\delta(n)}(a^{\gamma(\delta(n),1)}_{a(\delta(n),1)}, \ldots, a^{\gamma(\delta(n),k(\delta(n)))}_{a(\delta(n),k(\delta(n)))})$ and the last is reduced.

Hence $\gamma(\delta(n),e) \in \{\delta(n)\}$ for each e, hence $\gamma(\delta(n),e) = \delta(n)$, that is $\gamma(\beta,e)$

for each β and e. Hence, as $\tau_\beta\ (a^{\gamma(\beta,1)}_{a(\beta,1)}, \ldots)$ is reduced, the ordinals $a(\beta,e)$,

$1 \leq e \leq k(\beta)$ are distinct.

As $f \restriction B_3$ commutes with f^*_3, for every β

$$\tau_\beta\ (a^0_{a(\beta,1)}, \ldots, a^0_{a(\beta,k(\beta))}) = \tau_0(a^0_{a(0,1)}, \ldots, a^0_{a(0,k(0))})$$

As the $a(\beta,e)$ are distinct, necessarily $\{a(\beta,e) : 1 \leq e \leq k(e)\} =$

$\{a(0,e) : 1 \leq e \leq k(0)\}$, but as $a(\beta,e)$ is increasing with e (for each β, by the choice of

$\tau_\beta(\ ,\ldots,))$ necessarily $a(\beta,e) = a(0,e)$, $k(\beta) = k(0)$ and we can assume that $\tau_\beta = \tau_0$.

Now clearly as before

$$f(a^{\beta,n}_0) = \tau_0\ (a^{\beta,n}_{a(0,1)}, \ldots)$$

So we finish one direction of claim 3.3 where as the other is immediate.

* * *

Let $\phi_3(f)$ say that f maps B_2 into B_3 and if there is $f_1, f_1 \restriction B_2 \quad f \restriction B_2$ and $\phi_2(f_1)$.

It is easy to check that for every β ; for some τ , γ (e) :

$G_\mu \models \phi_3[f] \Leftrightarrow f(a^\beta_0) = \tau(a^\beta_{\gamma(1)}, \ldots a^\beta_{\gamma(k)})$.

Let $\phi_4(f)$ say that f maps B_2 into B_3 , $\phi_3(f)$, and if $\phi_3(g)$, then

$g \circ f_5 \circ f \restriction \tilde{a}^0_0 = f_5 \circ f \circ f_5 \circ g \restriction \tilde{a}^0_0$ and $f^*_5 \circ f \circ f^*_5 \circ f \restriction \tilde{a}^0_0 = f^*_6 \circ f \restriction \tilde{a}^0_0$ and

$f^*_2 \circ f \restriction \tilde{a}^0_0 = f^*_2 \restriction \tilde{a}^0_0$. (note that $\tilde{a}^0_0 = \mathrm{Rn}\, f^*_2$). As in case I we can check that

$G_\mu \models \phi_4[f] \Leftrightarrow f(a^\beta_0) = \tau_0(a^\beta_{\gamma(1)}, \ldots, a^\beta_{\gamma(k)})$ (w.l.o.g. reduced)

and τ_0 is beautiful.

Case III : μ a singular cardinal.

We give this case with less details.

We let $\mu_1 < \mu$ and B_1, B_2, B_3 be as in case II, μ_1 regular , $\mu_1 > \aleph_0$. By case II we can define f^*_e's properly, so that, for some ϕ^0 $G_\lambda \models \phi^0(f)$ iff there are τ and distinct $a_1, \ldots, a_n$ so that for each $\beta < \mu_1$, $f(a^\beta_0) = \tau(a^\beta_{a_1}, \ldots, a^\beta_{a_n})$.

Let $\phi^1(f)$ say that $f(\tilde{a}^0_0) \subseteq B_2$ and for every g, $\phi^0(g)$ implies that

$(f \circ g) \restriction \tilde{a}^0_0 = (g \circ f) \restriction \tilde{a}^0_0$. It is easy to check that $G_\lambda \models \phi^1[f]$ iff there are σ and distinct $\beta(1), \ldots, \beta(m)$ such that for every α , $f(a^0_\alpha) = \sigma(a^{\beta(1)}_\alpha, \ldots, a^{\beta(m)}_\alpha)$,

σ satisfying (A) of Def. 3.1.

As μ_1 is regular, we can use case II. So there is ϕ^2 such that $G_\lambda \models \phi^2[f]$ iff there are a beautiful term σ and distinct $\beta(i)$ such that for every α , $f(a^0_\alpha) = \sigma(a^{\beta(1)}_\alpha, \ldots, a^{\beta(m)}_\alpha)$.

Let $\mu = \sum_{i < \mathrm{cf}\mu} \mu(i)$, $\mu(i) < \mu$, $\mu(i)$ increasing.

We just prove that for every $\gamma < \mathrm{cf}\mu$, there is $\bar{f}^*_\gamma$ as constructed in case II and above, such that

(i) $G_\lambda \models \phi^2[f, \bar{f}^*_\gamma]$ iff there are a beautiful term σ and distinct $\beta(i) < \mu(\gamma)^+$, such that for every $\alpha < \mu$, $f(a^0_\alpha) = \sigma(a^{\beta(1)}_\alpha, \ldots, a^{\beta(m)}_\alpha)$

(ii) $\bar{f}^*_\gamma(0)$ is a projection onto $\mathrm{Cl}\{a^\beta_\alpha : \alpha < \mu, \beta < \mu(\gamma)^+\}$

Checking the construction carefully, we see that :

if τ, σ are beautiful terms, $\beta(i) < \mu$ $(\gamma_\ell)^+$ $(i = 1, ..., m,\ \ell = 1, ..., n)$

and for every $a < \mu$, $f(a^0_a) = \sigma(a^{\beta(1)}_a, ..., a^{\beta(m)}_a)$, then $G_\lambda \models \phi^2(f, \tau(\bar{f}^*_{\gamma_1}, ..., \bar{f}^*_{\gamma_n}))$.

Now checking the proof of 4.1 from 4.2, the existence of the $\bar{f}^*_\gamma$ mentioned above, is sufficient to prove 4.1 when $i(0) < \mu$.

Hence there are ϕ^3 and g^* such that

$G_\mu \models \phi^3[\bar{f}, g^*]$ iff $\bar{f} = \tau(\bar{f}^*_{\gamma(1)}, ..., \bar{f}^*_{\gamma(n)})$ for some beautiful τ. So let the formula $\phi^4[f, g^*]$ say «there is $\bar{f}_1$, such that $\phi^3[\bar{f}_1, \bar{g}^*]$ and for every $\bar{f}_2$ which satisfies $\phi^3[\bar{f}_2, \bar{g}^*] \wedge Rn\ \bar{f}_1(0) \subseteq Rn\ \bar{f}_2(0)$ satisfies also $\phi^2[f, \bar{f}_2]$ ».

Clearly $G_\mu \models \phi^4[f, \bar{g}^*]$ iff for some beautiful τ and $\beta(i) < \mu$, for every $a < \mu$, $f(a^0_a) = \tau(a^{\beta(1)}_a, ...)$ and $\phi^4(f^*_5 \circ f \circ f^*_5, g^*)$ is our desired formula.

THEOREM 4.4 : We can find formulas ϕ^b, ϕ^{eq}, ϕ^r, ϕ^x (in the language of G_λ) such that for any ultrafilter $T \subseteq \underline{B}$, λ, and free set of generators $\{a_t : t \in I^*\} \subseteq F_\lambda$ and $f_t \in F_\lambda$ defined by $f_t(a_s) = a_t$; there is $\bar{f}^* \in G_\lambda$ such that $f^*_0 = f_t$ for some t and

(A) $G_\lambda \models \phi^b[f, \bar{f}^*]$ iff $f = \tau(f_{t_1}, ...)$ for some beautiful τ .

(B) For any f,g, (see Def. 3.3)

$$G_\lambda \models \phi^{eq}[f, g ; \bar{f}^*] \text{ iff } \phi^b[f, \bar{f}^*] \wedge \phi^b[g, \bar{f}^*] \text{ and } f \approx_T g$$

(C) For any $h : I^* \to I^*$ and $f, g \in G_\lambda$,

$\phi^{eq}[f, g : \bar{f}^*]$ implies $\phi^{eq}[\hat{h} \circ f, \hat{h} \circ g ; \bar{f}^*]$

(D) For any two-place relation R on the class of ϕ^{eq}-equivalence classes, there is $\bar{g} \in G$ such that for any $f_1, f_2 \in G_\lambda$

$$G_\lambda \models \phi^r[f_1, f_2 ; \bar{g}] \text{ iff } \phi^b[f_1 ; \bar{f}^*] \wedge \phi^b[f_2 ; \bar{f}^*] < f_1/\phi^{eq}, f_2/\phi^{eq} > \in R$$

(E) $G_\lambda \models \phi^x[f]$ iff $f \in G_\lambda$ is a projection onto some $Cl\{\tau_e(a_{t(e,1)}, ...) : e < n\}$ (for some $n < \omega$, beautiful τ_e, and $t(e,1), ... \in I^*$.

PROOF : The set

$$R = \{<f,g> : f, g \in \{\tau(f_{t(1)}, ...) : \tau \text{ beautiful}, t(e) \in I^*\} \text{ and for some } \tau\ (x,y) \in T, f = \tau(g,f)\}$$

is closed under beautiful terms, hence by 4.1 is definable by some ϕ.
This observation suffice for (A), (B). Now (C) is a statement on $\approx_T$, in fact, which is easy to verify.
Now (D) follows by (C), as by (C) we can define arbitrary one-place functions from the set of $\phi^{eq}(X,Y\ ;\ \bar{f}^*)$-equivalence classes into itself, so by Rabin [Ra] , we can define arbitrary relations. Lastly, (E) follows by 4.1.

Now any such subalgebra is the range of a projection, because by Def. 3.1 and 3.1 (A) we can assume the subalgebra is

$$B = \mathrm{cl}\{\tau(a_{t(i,0)}, a_{t(i,1)}, \ldots, a_{t(i,m-1)}) : i < n\}\ .$$

Choose $b \in B$ and let

$$h(a_t) = \tau(c_{s(t,0)}, \ldots, c_{s(t,m-1)})$$

where $c_{s(t,j)} = a_t$ if for some i, $t(i,j) = t$ and $c_{s(t,j)} = b$ otherwise.

§ 5. INTERPRATING SET THEORY IN A CATEGORY.

Here K will be a full subcategory of Cat,

$A \in K \to |A| < \lambda^*$, and $\lambda < \lambda^* \to F_\lambda \in K$, and $|L(V)| < \lambda^*$ (in fact $(\forall \lambda < \lambda^*)(\exists \mu \geq \lambda)(F_\mu \in K)$ suffice).

In the formulas from 4.4 we now put F_μ explicitely as a parameter.

DEFINITION 5.1 : Let $\bar{f}$ be of the length of $\bar{f}^*$ (of 4.4) and let $\psi^a(\bar{f},A)$ be the formula saying (in K) :

(0) $f_e \in$ End A (= Hom(A,A))

(1) f_0 is a projection

(2) $\phi^b(g;\bar{f},A)$ implies g is conjugate to f_0 (so $g \in$ End A)

and $\phi^b(g_1,\bar{f},A) \wedge \phi^b(g_2,\bar{f},A) \to g_2 = g_2 \circ f_1$

(3) $\phi^{eq}(g_1,g_2;\bar{f},A)$ is an equivalence relation $E_{\bar{f}}$ over

$W^{\bar{f}}_A = \{g : \phi^b(g;\bar{f},A)\}$ with >1 equivalence classes

(4) $\phi^b(g;\bar{f},A)$ implies $\phi^x(g;\bar{f},A)$ which implies $g \in$ End $A \wedge$ proj g

(5) Suppose $\phi^x(g_1;\bar{f},A) \wedge \phi^x(g_2;\bar{f},A)$, then there is a g_3 such that :

(i) $\phi^x(g_3;\bar{f},A) \wedge Rn\, g_1 \subseteq Rn\, g_3 \wedge Rn\, g_2 \subseteq Rn\, g_3$

(ii) if $\phi^b(g_4;\bar{f},A) \wedge Rn\, g_4 \subseteq Rn\, g_3$

then $\bigvee_{e=1}^{2} (\exists g)[\phi^b(g;\bar{f},A) \wedge Rn\, g \subseteq Rn\, g_e \wedge \phi^{eq}(g_4, g;\bar{f},A)]$

(6) $\phi^P(g_1,g_2,g_3;\bar{f},A)$ represent a pairing function on the set of $E_{\bar{f}}$-equivalence classes.

(7) If (i) $\phi^b(g_e;\bar{f},A)$ $(e = 1,6)$, $h \in$ End A,

(ii) $(\forall g)[\phi^x(g;\bar{f},A) \wedge Rn\, g_1 \subseteq Rn\, g \wedge Rn\, g_2 \subseteq Rn\, g \to Rn\, g_3 \subseteq Rn\, g]$

(iii) $h \circ g_e = g_{e+3}$ for $e = 1,2,3$

then for $e = 1,2$ $\phi^{eq}(g_3,g_e;\bar{f},A) \equiv \phi^{eq}(g_6,g_{e+3};\bar{f},A)$

(at least one of them holds by (5))

(8) If $\phi^x(g^*,\bar{f},A)$ then there is g^*_2, such that $\phi^b(g^*_2,\bar{f},A)$ and for each g^*_1 satisfying $\phi^b(g^*_1,\bar{f},A) \wedge Rn\, g^*_1 \subseteq Rn\, g^*$ the following holds : $\neg\phi^{eq}(g^*_1,g^*_2;\bar{f},A)$ and

for every g_3, g_4 satisfying $\bigwedge_{e=3}^{4} \phi^b(g_e, \bar{f}, A)$ there is $h \in$ End A such that

$h \circ g_1^* = g_3$, $h \circ g_2^* = g_4$ and $(\forall g)\ [\phi^b(g,\bar{f},A) \to \phi^b(h \circ g, \bar{f}, A)]$

(9) If $\phi^b(g_e, \bar{f}, A)$ $(e = 1,2)$ then for some $h \in$ End A, $h \circ g_e = g_{1-e}$

and $(\forall g)(\phi^b(g,\bar{f},A) \equiv \phi^b(h \circ g, \bar{f}, A))$

CLAIM 5.1 :

(A) For each $\lambda < \lambda^*$, let $\bar{f}^*_\lambda \in$ End F_λ be as constructed in 4.4.

Then $K \models \psi^a[\bar{f}^*_\lambda, F_\lambda]$

(B) Suppose $K \models \psi^a[\bar{f}, A]$. Then

(1) If $\bigwedge_{e=1}^{n+1} \phi^b(g_e, \bar{f}, A)$ and $Rn\, g_{n+1} \subseteq Cl \bigcup_{e=1}^{n} Rn\, g_e$

then $\bigvee_{e=1}^{n} \phi^{eq}(g_{n+1}, g_e; \bar{f}, A)$

(2) $\{g : \phi^b(g, \bar{f}, A)\}$ is closed under beautiful terms

(3) There is an ultrafilter $T = T(\bar{f}, A)$ such that

$\bigwedge_{e=1}^{2} \phi^b(g_e, \bar{f}, A)$ implies $\phi^{eq}(g_e, \tau(g_1, g_2), \bar{f}, A)$ where $e = 1 \Leftrightarrow \tau(x,y) \in T \Leftrightarrow e \neq 2$.

(4) $E_{\bar{f}}$ has infinitely many equivalence classes

PROOF :

(A) Immediate.

(B) (1) Easy, by part (5) (and (4)) of Def. 5.1 (for $n = 1$, use (2)).

(B) (2), (3). By part (8) of Def. 5.1 we can define inductively g_i^* $(i < \omega)$ such that

$\phi^b(g_i^*, \bar{f}, A)$ and if $i < j$, $\phi^b(g^1, \bar{f}, A) \wedge \phi^b(g^2, \bar{f}, A)$ then for some $h \in$ End A

$h \circ g_i^* = g^1$, $h \circ g_j^* = g^2$ and $(\forall g)(\phi^b(g, \bar{f}, A) \to \phi^b(h \circ g, \bar{f}, A))$.

Let $\tau(x,y)$ be a beautiful term .

Let us apply 3.4 to Φ^b, and get $\Phi = \{\psi_e : e < k < \omega\}$

So there are $i < j \leq 2^k$ such that $\bigwedge_{e<k} \psi_e(g_i, \bar{f}, A) \equiv \phi_e(g_j, \bar{f}, A)$ hence

$\phi^b(\tau(g_i, g_j), \bar{f}, A)$. Now for any g^1, g^2 satisfying $\phi^b(g^1, \bar{f}, A) \wedge \phi^b(g^2, \bar{f}, A)$,

there is h such that $h \circ g_i = g^1$, $h \circ g_j = g^2$ and

$(\forall g)(\phi^b(g, \bar{f}, A) \to \phi^b(h \circ g, \bar{f}, A))$. Hence $\tau(g^1, g^2) = h(\tau(g_i, g_j))$ satisfy $\phi^b(x, \bar{f}, A)$; hence (2) is proved. For (3) note that by (B) $\phi^{eq}(\tau(g_i, g_j), g_p, f, A)$, hence by (7) for any g^1, g^2 as above $\phi^{eq}(\tau(g^1, g^2), g^q, \bar{f}, A)$ where $p = i \leftrightarrow q = 1$.

(B) (4). By Def. 5.1, (7) and (8).

LEMMA 5.2 :

(A) There is a formula ψ^s such that

(1) $K \models \psi^s[\bar{f}_\lambda, F_\lambda] \wedge (\forall \bar{f}, A)[\psi^s(\bar{f}, A) \to \psi^a(\bar{f}, A)]$

(2) If $K \models \psi^s[\bar{f}, A]$ then for any subset R of $W^{\bar{f}}_A/E_{\bar{f}}$, $|R| < \lambda^*$, there is $g^* \in K$ such that $g/E_{\bar{f}} \in R \Leftrightarrow K \models \phi^r[g, g^*, \bar{f}, A]$

(3) Like (2), for a two-place relation R.

(B) There is a formula ψ^t such that

if $K \models \psi^s[\bar{f}^e, A^e]$ $(e = 1,2)$ then :

$K \models \psi^t[\bar{f}^1, A^1, \bar{f}^2, A^2]$ iff $T(\bar{f}^1, A^1) = T(\bar{f}^2, A^2)$

(C) There are formulas ψ^e, θ such that

(1) $K \models \psi^e[\bar{f}_\lambda, F_\lambda] \wedge (\forall f, A)[\psi^e(\bar{f}, A) \to \psi^s[\bar{f}, A)]$

(2) $K \models \psi^e[\bar{f}^1, A^1] \wedge \psi^e[\bar{f}^2, A^2] \wedge \psi^t[\bar{f}^1, A^1, \bar{f}^2, A^2]$

implies : for every $g^1_i \in W^{\bar{f}^1}_{A^1}$

pairwise non $E_{\bar{f}^1}$-equivalent, $(i < \lambda < \lambda^*)$ and $g^2_i \in W^{\bar{f}^2}_{A^2}$

there is $\bar{h}$ such that

$\theta[g^1, g^2, \bar{h}] \Leftrightarrow (\exists i)[g^1 E_{\bar{f}^1} g^1_i \wedge g^2 E_{\bar{f}^2} g^2_i]$

(D) There is a formula ψ^m such that

(1) $K \models \psi^m[\bar{f}_\lambda, F_\lambda] \wedge (\forall \bar{f}, A)[\psi^m(\bar{f}, A) \to \psi^e(\bar{f}, A)]$

(2) $K \models \psi^m[f, A]$ implies $E_{\bar{f}}$ has $< \lambda^*$ equivalence classes.

PROOF OF 5.2 :

(A) Immediate by 4.1 and claim 5.1 (B) (2), (3) (for (3), use (2) and ϕ^{P}).

(B) Suppose $K \models \psi^{s}\ [\bar{f}^{e}, A^{e}]$ $(e = 1,2)$ and $T = T(\bar{f}^{1}, A^{1}) = T(\bar{f}^{2}, A^{2})$.

For $e = 1,2$ let $g^{e}_{n} \in W^{\bar{f}^{e}}_{A^{e}}$ $(n < \omega)$ be pairwise non-$E_{\bar{f}^{e}}$-equivalent.

Let $|A^{1}| + |A^{2}| \leq \lambda < \lambda^{*}$, let $\{a_i : i < \lambda\ \omega\}$ freely generate F_{λ}, and $f_n \in \text{End } F_{\lambda}$ be a projection onto $\{a_i : \lambda\, n \leq i < \lambda\ (n+1)\}$.

Let $\bar{f}^{*}$ be such that

$\underline{\phi}^{b}(g, \bar{f}^{*}, F_{\lambda}) \Leftrightarrow g = \tau(f_{\beta(1)}, \ldots), \tau$ beautiful

$\phi^{eq}(g_1, g_2, \bar{f}^{*}, F_{\lambda}) \Leftrightarrow \phi^{b}(g_1, \bar{f}^{*}, F_{\lambda}) \wedge \phi^{b}(g_2, \bar{f}^{*}, F_{\lambda})$

and for some $\tau(x,y) \in T$

$g_1 = \tau(g_2, g_1)$

Let h_e maps $\{a_i : \lambda\, n \leq i < \lambda\ (n+1)\}$ onto $\text{Rn}\, g^{e}_{n}$.

Let $\theta^{2}(g^{1}, g^{2}) = \theta^{2}(g^{1}, g^{2}; h_1, h_2, f, \bar{f}^{1}, \bar{f}^{2}, A^{1}, A^{2}, \bar{f}^{*}, F_{\lambda})$ says :

$\phi^{b}(g^{e}, \bar{f}^{e}, A^{e})$ $(e = 1,2)$ and for some g^{e}_{1} $(e = 1,2)$

$$g^{e}\, E_{\bar{f}^{e}}\, g^{e}_{1}$$

and there is f, $\phi^{b}(f, \bar{f}^{*}, F_{\lambda})$ such that

$\text{Rn}\, h_e \circ f \subseteq \text{Rn}\, g^{e}_{1}$ and for every g^{e}_{0} $(e = 1,2)$

$g^{e}_{0} \in \text{End } A^{e} \wedge \text{proj}\, g^{e}_{0} \wedge \text{Rn}\, h_e \circ f \subseteq g^{e}_{0} \to \text{Rn}\, g^{e}_{0} \subseteq \text{Rn}\, g^{e}_{0}$

Then $\theta^{2}(g^{1}, g^{2})$ <u>iff</u> there are beautiful τ, and

$n(1), \ldots$ such that

$g^{e}\, E_{\bar{f}^{e}}\, \tau(g^{e}_{n(1)}, \ldots)$ for $e = 1, 2$

Let $\psi^{t}[\bar{f}^{1}, A^{1}, \bar{f}^{2}, A^{2}]$ say that $\psi^{s}[f^{e}, A^{e}]$, $e = 1,2$ and for some h_1, h_2, f^{*}, B, $\theta^{2}(g^{1}, g^{2}) =$

$= \theta^{2}(g^{1}, g^{2}, h_1, h_2, \bar{f}^{1}, \bar{f}^{2}, A^{1}, A^{2}, \bar{f}^{*}, B)$.

define a one-to-one map from a subset of $W^{\bar{f}^{1}}_{A^{1}}$ into $W^{\bar{f}^{2}}_{A^{2}}$; and this set is infinite (i.e. is

ordered in an order of type ω, see (A)).

We have just proved that

$\psi^s [\bar{f}^e, A^e]$ (e = 1,2) and $T(\bar{f}^1, A^1) = T(\bar{f}^2, A^2)$ implies $\psi^t [\bar{f}^1, A^1; \bar{f}^2, A^2]$.

Suppose the conclusion holds, then clearly $\psi^s [\bar{f}^e, A^e]$; if $\tau(x,y) \in T(\bar{f}^1, A^1)$, $\tau(x,y) \notin T(\bar{f}^2, A^2)$ we get contradiction by 3.4.

(C) The same proof as (B), essentially.

(D) Let $\psi^m(\bar{f}, A)$ says that $\psi^e(\bar{f}, A)$ and there is A^1 such that for every $\bar{f}^1$, if $\psi^t [\bar{f}, A; \bar{f}^1, A^1]$ (and there is at least one such $\bar{f}^1$) then for some $\bar{f}^2$ and automorphism h of A^1 :

(0) $\psi^e(\bar{f}^2, A^1)$

(1) for some $\bar{h}$, $\theta(g^1, g^2, \bar{h})$ define a map from the $E_{\bar{f}^2}$-equivalence classes onto the $E_{\bar{f}}$-equivalence classes

(2) if $\phi^b(g^1, \bar{f}^2, A^1) \wedge \phi^b(g^2, \bar{f}^2, A^1) \wedge \neg\phi^{eq}(g^1, g^2, \bar{f}^2, A^1)$ then $g^1 \circ h \circ f^1_0 \neq g^2 \circ h \circ f^1_0$.

For proving D(1) choose $A^1 = F_\lambda$, and $\bar{f}^2$ is chosen like $\bar{f}_\lambda$, but the range of the projections, is freely generated by $\aleph_0$ elements. For D(2), if $E_{\bar{f}}$ has $\geq \lambda^*$ equivalence classes, choose $\bar{f}^1$ such that $Rn\, f^1_0$ is finitely generated.

DEFINITION 5.2 : We say $< g, \bar{f}, A >$ represent the model (N,R) if :

$g \in$ End A, and let $\{ g_i : i < a^* \}$ be representatives of the $E_{\bar{f}}$-equivalence classes, and let

$J = \{ i < a^* : (\exists f)[\phi^r(f,g,\bar{f},A) \wedge (\exists f_2)\ \phi^p(f, f_2, g_i, \bar{f}, A)] \}$

and there is H, a one-to one function from J onto R, and :

$<H(i), H(j)> \in R$ iff $(\exists f\ f_2\ g)\ (\phi^r(f,g,\bar{f},A) \wedge \phi^p(f,f_2,g,\bar{f},A) \wedge \phi^p(f_2, g_i, g_j, \bar{f}, \bar{A}))$

LEMMA 5.4 : There are formulas ψ^f, ψ^h such that

(A) $\psi^f(\bar{g}, \bar{f}, A)$ iff $\psi^m(\bar{f}, A)$ and $< g, \bar{f}, A >$ represent a well founded model satisfying extensionality.

(B) $\psi^h(g^1, \bar{f}^1, A^1; g^2, \bar{f}^2, A^2)$ iff $\psi^f(g^e, \bar{f}^e, A^e)$ (e = 1, 2) and $T(\bar{f}^1, A^1) = T(\bar{f}^2, A^2)$ and $< g^e, \bar{f}^e, A^e >$ (e = 1,2) represent isomorphic models.

PROOF : Easy by 5.3.

THEOREM 5.5 : In K we can interpret a model M_{λ^*} consisting of μ (V) disjoint copies of $H(\lambda^*)$ (μ(V) - the number of ultrafilters of $\underline{B}$).

PROOF : Let the elements of the model be triples $< \bar{g}, \bar{f}, A >$ satisfying ψ^h. Equality to define by ψ^h, and ϵ is defined naturally.

DEFINITION 5.3 :

(A) The model $M = (A_1, ..., A_m ; R_1, ..., R_k)$ (A_i-universes, R_i-relations) is an explicitely interpratable expansion of the model $N = (A_1, ..., A_n ; R_1, ..., R_\ell)$ if there are formulas $\phi^0_i(\bar{x})$, $\phi^1_i(\bar{x},\bar{y})$, ψ_j in L(N) ($n < i \leq m, 1 < j \leq k$) and function F_i ($n < i \leq m$) such that

(1) F_i is a function from $\{ \bar{a} : N \models \phi^0_i [\bar{a}] \}$ onto A_i

(2) $F_i(\bar{a}_1) = F_i(\bar{a}_2)$ iff $N \models \phi^1_i [\bar{a}_1, a_2]$

(3) $M \models R_j [F_{i(1)}(\bar{a}_1), ...]$ iff $N \models \psi_j [\bar{a}_1, ...]$.

(B) The scheme of the explicite interpratable expansion is all the syntactical information involved.

DEFINITION 5.4 :

If M^e is an explicite interpratable expansion of N^e (e = 1,2) by the same scheme, and by the function F^e_j, and N^1 is a submodel of N^2, then the function G, $G \restriction A^1_i$ = the identity ($1 \leq i \leq n$), $G(F^1_j(\bar{a})) = F^2_j(\bar{a})$ when $N^1 \models \phi^0_j [\bar{a}]$, is called the natural embedding of M^1 into M^2.

CLAIM 5.6 : If in Def. 5.4, N^1 is an elementary submodel of N_2 then the natural embedding of M_1 into M_2 is an elementary embedding.

PROOF : Trivial.

DEFINITION 5.5. : Let M(K) be the model with the universes and relations listed below : (functions and partial functions will be encode by relations).

(A) Those of K.

(B) $A_1 = \{ < a, T > : a \in H(\lambda^*), T \in S(\underline{B}) \}$ where $S(\underline{B})$ is the set of ultrafilters of $\underline{B}$. E will be an equivalence relation on A_1 defined by : $< a_1, T_1 > E < a_2, T_2 >$ iff $T_1 = T_2$. R_1 is defined by : $< a_1 , T_1 > R_1 < a_2 , T_2 >$ iff $a_1 \in a_2$ and $T_1 = T_2$.

(C) $A_2 = \{\langle <a_T, T> : T \in S(\underline{B}) \rangle : a_T \in H(\lambda^*)\}$ and

$<a, T_0> R_2 \langle <a_T, T> : T \in S(\underline{B}) \rangle$ iff $a = a_{T_0}$

(D) $A_3 = S(B)$

$F_1(<a, T>) = T$

$F_2(B, T) = B \approx_T$ for an algebra $B \in K$.

THEOREM 5.7 : M(K) is an explicitely interpratable expansion of K, assuming $2^{|L(V)|} < \lambda^*$.

REMARK : All the information we know on Eklof and Feferman problems mentioned in the introduction can be easily extracted from this theorem. We assume $2^{|L(V)|} < \lambda^*$ in order to simplify the definition of M(K).

PROOF : We go through Def. 5.5.

(A) No problem.

(B) Essentially, this was proved in 5.5.

(C) (D) Left to the reader.

REFERENCES

[E] J.L. ERSHOV, Undecidability of theories of symmetric and simple finite groups, Dokl. Akad. Nauk SSSR. N 4, 158, 777-779.

[FV] FEFERMAN and VAUGHT, The first order properties of algebraic systems, Fund. Math. 47 (1959) 57-103.

[G] G. GALVIN, Horn sentences, Annals of Math. Logic 1 (1970) 389-422.

[L] LAWVERE F.W., The category of categories as a foundation of math., conference on categorical algebra, La-Jolla 65, ed. Eilenberg, Harrison, Maclane, Rohre, Springer-Verlag, Berlin 65.

[MK] R. McKENZIE, On elementary types of symmetric groups, Algebra Universalis 1 (1971) N 1, 13-20.

[My] J. MYCIELSKI, Problem 324, colloq. Math. 8 (1961) 279.

[P] A.G. PINUS, Elementary definability of symmetric groups. Algebra Universalis. 3 (1973), 59-66.

[Ra] M.O. RABIN, A simple method for undecidability proofs, Proc. of the 1964 international Congress for Logic, ed. Bar-Hillel, North Holland 1965, 58-68.

[Ru] M. RUBIN, The automorphism group of homogeneous and saturated Boolean algebras, Algebra Universalis, submitted.

[S 1] S. SHELAH, First-order theory of permutation groups, Israel J. of Math. 14 (1973) 149-162.

[S 2] ———— Errata to : first order theory of permutation groups, Israel J. of Math. 15 (1973) 437-441.

[S 3] ———— Various results in mathematical logic. Notices of A.M.S. 21 (1974, Aug.) A-502.

[So] R.M. SOLOVAY, Real-valued mesurable cardinals, Proceedings of Symposia in Pure Math. XIII part I, ed. D. Scott, A.M.S. Providence R.I. 1971.

[W] J.M. WEINSTEIN, First order formulas preserved by direct product, Ph. D. thesis, Univ. of Wisconsin, Madison, Wisc. 1965.