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Convergence of nonlinear integrodifferential reaction-diffusion equations via Mosco$\times\Gamma$-convergence

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Abstract

We study the convergence of sequences of nonlinear integrodifferential reaction-diffusion equations when the Fickian terms belong to a class of convex functionals defined on a Hilbert space, equipped with the Mosco-convergence, and the non Fickian terms belong to a class of convex functionals, whose restrictions to a compactly embedded subspace are equipped with the Γ-convergence. As a consequence we prove a homogenization theorem for this class under a stochastic homogenization framework.

1. Introduction

In the spirit of [2, 4], we investigate the compactness or the stability in terms of convergence, of integrodifferential reaction-diffusion problems defined in $L^2(0,T,X)$ by

$$(\mathcal{P})\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}(t)+\partial\Phi(u(t))+\displaystyle\int_0^t K(t-s)\partial\Psi(u(s))\,\mathrm{d}s\ni F(t,u(t)) \text{ for a.e. } t\in(0,T)\\ u(0)=u^0,\ u^0\in\operatorname{dom}(\partial\Phi),\end{cases}$$

under suitable variational convergences on the classes of functionals Φ and Ψ (see Theorem 5.1). As usually the integral in the first member is taken in the sense of Bochner. The domain of the lower semicontinuous convex functionals $\Phi,\Psi : X\to\mathbb{R}\cup\{+\infty\}$ is a subspace V compactly embedded in the Hilbert space X. More precisely, the class of functionals Φ is equipped with the Mosco-convergence; the class of the restrictions to V of functionals Ψ, is equipped with the sequential Γ-convergence associated with the weak topology of V (for definitions see Appendix D, and for more details on Γ-convergence we refer to [5, 13]). When dealing with concrete functionals Φ and Ψ of the calculus of variations, problems of type $(\mathcal{P})$ arise from the conservation of mass when the flux is splitted into two terms: the Fickian flux whose divergence is $\partial\Phi(u(t,\cdot))$ and the non Fickian flux which takes time memory effects into account, and whose divergence is $\int_0^t K(t-s)\partial\Psi(u(s))\,\mathrm{d}s$; we refer to Section 6 for various examples in the framework of

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stochastic homogenization. For integrodifferential equations and non-local effects induced by homogenization see [3] and references therein.

We say that u is a solution of $(\mathcal{P})$ if $u \in L^2([0,T],X)$ is absolutely continuous in time and satisfies $(\mathcal{P})$. Well posedness in the sense of existence of solutions has been extensively studied using maximal monotone operator techniques under an extrinsic condition involving the scalar product of $\partial\Phi$ with the Yosida approximation of $\partial\Psi$, not easy to handle, even for elementary examples; see [12, Example 2]. We refer to the pioneers [7, 8, 12, 18, 24] and references therein when F is a source without reaction term. In the paper we treat the problem under a coercivity condition on $\partial\Phi \circ (\partial\Psi)^{-1}$, which is more flexible and stable for the product of the Mosco-convergence with the Γ-convergence. For existence of solutions of nonlinear Volterra integrodifferental equations under a weak formulation but with nonsmooth kernels, refer to [15]. For existence results related to integrodifferential equations of non convolution type or to integrodifferential equations whose source includes a delay term see [10, 11, 24]. For recent developments in non-Fickian diffusion and its applications to viscoelastic materials, we refer to [14, 16] and references therein.

Plan of the paper

In Section 2, we make precise the structure of the first member of $(\mathcal{P})$ and discuss on the coercivity condition on $\partial\Phi \circ (\partial\Psi)^{-1}$ and its stability with respect to the variational convergences of functionals Φ and Ψ. Section 3 and 4 are devoted to local and global existence respectively. Precisely in Theorems 4.1, 4.4 we establish existence of a strong solution of $(\mathcal{P})$ with a right derivative at each $t \in [0,T[$. The main result of the paper is stated in Section 5, Theorem 5.1, where we establish the convergence of sequences in the class of problems $(\mathcal{P})$ when the classes of functionals Φ and Ψ are equipped with the two variational convergences mentioned above. This theorem can be seen as a compactness result for the class of problems $(\mathcal{P})$, and, under some additional conditions, as a stability result (see Remark 5.2). In the concrete case when $X = L^2(\Omega)$, in Theorem 5.3, we extend the convergence to reaction diffusion problems. In Section 6 we apply Theorem 5.3 to the stochastic homogenization analysis of problems of the type $(\mathcal{P})$ when Φ and Ψ are concrete random functionals of the calculus of variations. More precisely, in Section 6.1 we address the stochastic homogenization of a random problems modeled from a Fick's law with delay: the non Fickian flux is superimposed on the first flow at each time t. According to the models, it may account for maturation period, resource regeneration time, mating processes, or incubation period. In Section 6.2, we treat the stochastic homogenization of general nonlinear integrodifferential reaction-diffusion equations in one dimension space in the setting of a Poisson point process. The general problem considered is a randomization of [12, Example 2] with a reaction source. For recent developments in periodic homogenization of parabolic problems in $L^2(0,T,L^2(\mathbb{R}^d))$ with a convolution type operator, refer to [22, 23]. By contrast, our results fall within the scope of stochastic homogenization of parabolic problems with non-local operators in the context of boundary value problems.

2. Preliminaries

In the following X denotes a Hilbert space endowed with a scalar product denoted by $\langle\cdot,\cdot\rangle$ and its associated norm $\|\cdot\|_X$. In Section 5.2 and Section 6, $X = L^2(\Omega)$ where Ω is a C^1-regular domain in $\mathbb{R}^N$. We denote by V a reflexive Banach space compactly embedded in X, by $\|\cdot\|_V$ its norm, and by V' its dual. Therefore, for $T > 0$ fixed all along the paper, the following continuous and compact embeddings hold:

$$V \overset{c}{\hookrightarrow} X \overset{c}{\hookrightarrow} V'$$

$$L^2(0,T,V) \overset{c}{\hookrightarrow} L^2(0,T,X) \overset{c}{\hookrightarrow} L^2(0,T,V')$$

where X and $L^2(0,T,X)$ are identified with their duals. For a proof and general results concerning compact embeddings for vector valued spaces, refer to [1] and references therein. We assume that $\langle u,v\rangle_{V',V} = \langle u,v\rangle$ whenever $u \in X$ and $v \in V$.

2.1. Structure of the first member of $(\mathcal{P})$

We are given two lower-semicontinuous (lsc in short) convex proper functionals $\Phi,\Psi : X \to]-\infty,+\infty]$ with domain V, satisfying $\inf_X \Phi > -\infty$, $\inf_X \Psi > -\infty$, and such that $\mathrm{dom}(\partial\Phi) \subset \mathrm{dom}(\partial\Psi)$. We denote by $\Psi_{\lfloor V}$ the restriction of Ψ to V. Clearly, since $\mathrm{dom}(\Psi) = V$, the subdifferentials of $\Psi_{\lfloor V}$ and Ψ are connected through the relation

$$\partial\Psi = \partial(\Psi_{\lfloor V}) \cap X.$$

When there is no ambiguity, to simplify the notation, for $u \in \mathrm{dom}(\partial\Phi)$ we write $\partial\Phi(u)$ and $\partial\Psi(u)$ to denote any element of the sets $\partial\Phi(u)$ and $\partial\Psi(u)$ respectively. However, generally speaking, the element considered in each set remains fixed in the equation at hand. We assume that $0 \in \mathrm{dom}(\partial\Psi)$ with $\partial\Psi(0) = \{0\}$. The functional $\Psi_{\lfloor V}$ is assumed to be strongly convex in the following sense:

$$\left\langle \partial(\Psi_{\lfloor V})(u) - \partial(\Psi_{\lfloor V})(v), u - v \right\rangle_{V',V} \geq \alpha_\Psi \|u - v\|_V^2 \quad \text{for all } (u,v) \in \mathrm{dom}(\partial(\Psi_{\lfloor V}))^2. \tag{2.1}$$

Note that when $(u,v) \in \mathrm{dom}(\partial\Psi)^2$, then (2.1) yields

$$\langle \partial\Psi(u) - \partial\Psi(v), u - v\rangle \geq \alpha_\Psi \|u - v\|_V^2,$$

hence, since $\partial\Psi(0) = \{0\}$, for all $u \in \mathrm{dom}(\partial\Psi)$, we infer that

$$\langle \partial\Psi(u), u\rangle \geq \alpha_\Psi \|u\|_V^2.$$

We assume that the subdifferentials $\partial\Phi$ and $\partial\Psi$ are connected via the following coercivity condition on $\partial\Phi \circ \partial\Psi^{-1}$: there exist two constants $\alpha_{\Phi,\Psi} > 0$ and $\beta_{\Phi,\Psi} \geq 0$ such that for all $u^* \in R_{\partial\Phi}(\partial\Psi)$,

$$\left\langle \partial\Phi((\partial\Psi)^{-1}(u^*)), u^* \right\rangle \geq \alpha_{\Phi,\Psi} \|u^*\|_X^2 - \beta_{\Phi,\Psi}. \tag{2.2}$$

For the definition of $(\partial\Psi)^{-1}$ and the relative range of A with respect to B, refer to Appendix C. Here $R_{\partial\Phi}(\partial\Psi) = \partial\Psi(\mathrm{dom}(\partial\Phi))$, since $\mathrm{dom}(\partial\Phi) \subset \mathrm{dom}(\partial\Psi)$. This condition must be understood in the sense of a set relation, i.e. for any u^* in $\mathrm{dom}(\partial\Phi \circ (\partial\Psi)^{-1})$, for any corresponding u such that $u^* \in \partial\Psi(u)$ (i.e. $u \in (\partial\Psi)^{-1}(u^*)$) and also $u \in \mathrm{dom}(\partial\Phi)$, and for any $\xi^* \in \partial\Phi(u)$ we have

$$\langle \xi^*, u^*\rangle \geq \alpha_{\Phi,\Psi} \|u^*\|_X^2 - \beta_{\Phi,\Psi}.$$

In short, 2.2 is equivalent to

$$\langle \partial\Phi(u), \partial\Psi(u)\rangle \geq \alpha_{\Phi,\Psi} \|\partial\Psi(u)\|_X^2 - \beta_{\Phi,\Psi}$$

for all $u \in \mathrm{dom}(\partial\Phi)$, with the following notation convention: let A and B be two sets of Hilbert space $(X, \|\cdot\|_X)$, then

- $\langle A, B\rangle := \{\langle a, b\rangle : a \in A,\ B \in B\}$;
- $\langle A, B\rangle \geq \|B\|_X^2$ stands for $\langle a, b\rangle \geq \|b\|_X^2$ for all $\langle a, b\rangle \in \langle A, B\rangle$.

Condition (2.2) replaces condition

$$\langle \partial\Phi(u), \partial\Psi_\lambda(u)\rangle \geq -\beta(\|\partial\Phi(u)\|_X^2 + \|u\|_X^2 + 1) \text{ for all } u \in \mathrm{dom}(\partial\Phi) \tag{2.3}$$

in [8, 12, 24], or [7, (d) p. 253], which links $\partial\Phi$ and $\partial\Psi_\lambda$, the subdifferential of the Moreau–Yosida envelope at $\lambda > 0$ of Ψ. We say that (2.1) and (2.2) hold uniformly if the constants α_Ψ, $\alpha_{\Phi,\Psi}$ and $\beta_{\Phi,\Psi}$ do not depend on the functionals Φ and Ψ.

Example 2.1. Consider $\Psi : X \to]-\infty, +\infty]$ lsc convex proper, and let $G : X \to]-\infty, +\infty[$ be a lsc convex functional, continuous at a point of V. Assume that there exists $\beta_{\Psi,G} \geq 0$ such that

$$\inf_{u \in \mathrm{dom}(\partial\Psi) \cap \mathrm{dom}(\partial G)} \langle \partial\Psi(u), \partial G(u)\rangle \geq -\beta_{\Psi,G} \tag{2.4}$$

in the sense that $\inf_{u\in \mathrm{dom}(\partial\Psi)\cap\mathrm{dom}(\partial G)} \inf_{\xi\in\partial\Psi(u),\zeta\in\partial G(u)} \langle \xi,\zeta\rangle > -\infty$. Assume furthermore that $\partial\Psi$ is single valued, then the functional Ψ and its perturbation $\Phi = \Psi + G$ by G, satisfy (2.2).

Indeed, $\mathrm{dom}(\partial\Phi) = \mathrm{dom}(\partial\Psi)\cap\mathrm{dom}(\partial G) \subset \mathrm{dom}(\partial\Psi)$, and from [6, Theorem 9.5.4], $\partial\Phi = \partial\Psi + \partial G$. Hence for all $u \in \mathrm{dom}(\partial\Phi)$

$$\begin{aligned}\langle \partial\Phi(u), \partial\Psi(u)\rangle &= \|\partial\Psi(u)\|_X^2 + \langle \partial\Psi(u), \partial G(u)\rangle \\ &\geq \|\partial\Psi(u)\|_X^2 + \inf_{u\in\mathrm{dom}(\partial\Psi)\cap\mathrm{dom}(\partial G)} \langle \partial\Psi(u), \partial G(u)\rangle \\ &\geq \|\partial\Psi(u)\|_X^2 - \beta_{\Psi,G}.\end{aligned}$$

As a particular case, take Ψ satisfying (2.1), and $G = b\|\cdot\|_X^2$ where $b \geq 0$. For all $u \in \mathrm{dom}(\partial\Psi)$ we have from (2.1)

$$\langle \partial\Psi(u), \partial G(u)\rangle = 2b\,\langle \partial\Psi(u), u\rangle \geq 2b\|u\|_V^2.$$

Consequently Ψ and G satisfy (2.4) since

$$\inf_{u\in\mathrm{dom}(\partial\Psi)\cap\mathrm{dom}(\partial G)} \langle \partial\Psi(u), \partial G(u)\rangle \geq 0.$$

Therefore the functionals Ψ and $\Phi = \Psi + b\|\cdot\|_X^2$ satisfy (2.2). Existence in the case when $b = 0$, i.e. $\Phi = \Psi$ has been established in [19]. Other examples are provided in Section 6.2.

Remark 2.2. Without being able to prove it, it seems that (2.2) and (2.3) are not comparable. Nevertheless if we assume that $\partial\Psi$ is univalent and we let $\lambda \to 0$ in (2.3), we obtain (see [6, Proposition 17.2.2])

$$\langle \partial\Phi(u), \partial\Psi(u)\rangle \geq -\beta(\|\partial\Phi(u)\|_X^2 + \|u\|_X^2 + 1)$$

which is less restrictive than (2.2). This is why we can only say that at the limit, i.e. when $\lambda \to 0$, (2.3) is less restrictive than (2.2). However from Remark 2.4 below, in the absence of equi-coercivity of the class of functionals $\Psi_{\lfloor V}$, we can suspect that they are not equivalent.

Define the class $\mathcal{F}$ of pairs of functionals (Φ,Ψ) by

$$(\Phi,\Psi)\in\mathcal{F} \iff \begin{cases}\Phi,\Psi : X \to]-\infty,+\infty] \text{ are lsc convex proper,}\\ \mathrm{dom}(\Phi) = \mathrm{dom}(\Psi) = V,\\ \mathrm{dom}(\partial\Phi) \subset \mathrm{dom}(\partial\Psi),\\ 0 \in \mathrm{dom}(\partial\Psi) \text{ and } \partial\Psi(0) = \{0\},\end{cases}$$

and let $(\Phi_n, \Psi_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$. Then we write

$$\Phi_n \xrightarrow{\mathrm{M}} \Phi,$$
$$\Psi_{n\lfloor V} \xrightarrow{\Gamma_{w\text{-}V}} \Psi_{\lfloor V},$$

to denote the Mosco-convergence of the sequence $(\Phi_n)_{n\in\mathbb{N}}$ to some lsc convex proper functional $\Phi : X \to]-\infty, +\infty]$ and the $\Gamma_{w\text{-}V}$-convergence of the sequence $(\Psi_{n\lfloor V})_{n\in\mathbb{N}}$ to the restriction to V of some lsc convex proper functional $\Psi : X \to]-\infty, +\infty]$ with $\mathrm{dom}(\Psi) = V$. Let us equip $X \times X$ with the strong topology product, $V \times V'$ with the product of the weak topology of V with the strong topology of V', and denote by $G_{s,s}$ and $G_{w,s}$ the associated graph convergence (see Appendix C). Then the following implications hold (see Theorem C.5):

$$\begin{aligned} \Phi_n \xrightarrow{\mathrm{M}} \Phi &\Longrightarrow \partial\Phi_n \xrightarrow{G_{s,s}} \partial\Phi, \\ \Psi_{n\lfloor V} \xrightarrow{\Gamma_{w\text{-}V}} \Psi_{\lfloor V} &\Longrightarrow \partial(\Psi_{n\lfloor V}) \xrightarrow{G_{w,s}} \partial(\Psi_{\lfloor V}). \end{aligned} \tag{2.5}$$

Note that converse implications hold true, up to normalization.

Acccording to above considerations, we endow $\mathcal{F}$ with the product $\mathrm{M} \times \Gamma_{w\text{-}V}$-convergence. The class $\mathcal{F}$ is not closed for this convergence, however the proposition below shows that conditions (2.1) and (2.2), which are essential in establishing existence in Sections 3 and 4, are in some sense stable in $\mathcal{F}$, then well suited to the convergence analysis of Section 5 (see Remark 5.2), and Section 6.

Proposition 2.3. *Assume that* (2.1) *and* (2.2) *are satisfied uniformly with respect to all elements of* $\mathcal{F}$. *For every sequence* $(\Phi_n, \Psi_n)_{n\in\mathbb{N}}$ *of* $\mathcal{F}$ *and every lsc convex proper functionals* $\Phi, \Psi : X \to]-\infty, +\infty]$, *if* $\mathrm{dom}(\Psi) = V$, $\Phi_n \xrightarrow{M} \Phi$ *and* $\Psi_{n\lfloor V} \xrightarrow{\Gamma_{w\text{-}V}} \Psi_{\lfloor V}$, *then* (Φ, Ψ) *satisfies* (2.1), (2.2) *and* $\mathrm{dom}(\partial\Phi) \subset \mathrm{dom}(\partial\Psi)$.

Proof. We denote by α_Ψ and $\alpha_{\Phi,\Psi}$ the two uniform constants appearing in (2.1) and (2.2).

Stability of (2.1). According to (2.5), $\partial(\Psi_{n\lfloor V}) \xrightarrow{G_{w,s}} \partial(\Psi_{\lfloor V})$. Hence, from Proposition C.4, for $(u, v) \in \mathrm{dom}(\partial(\Psi_{\lfloor V}))^2$ there exists $(u_n, v_n) \in \mathrm{dom}(\partial(\Psi_{n\lfloor V}))^2$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } V, \\ \partial(\Psi_{n\lfloor V})(u_n) \longrightarrow \partial(\Psi_{\lfloor V})(u) \text{ strongly in } V', \\ v_n \rightharpoonup v \text{ weakly in } V, \\ \partial(\Psi_{n\lfloor V})(v_n) \longrightarrow \partial(\Psi_{\lfloor V})(v) \text{ strongly in } V' \end{cases}$$

where, by notation convention, $\partial(\Psi_{n\lfloor V})(u_n) \to \partial(\Psi_{\lfloor V})(u)$ and $\partial(\Psi_{n\lfloor V})(v_n) \to \partial(\Psi_{\lfloor V})(v)$ stand for $u_n^* \to u^*$ and $v_n^* \to v^*$ with $u_n^* \in \partial(\Psi_{n\lfloor V})(u_n)$, $v_n^* \in \partial(\Psi_{n\lfloor V})(v_n)$, and $u^* \in \partial(\Psi_{\lfloor V})(u_n)$, $v^* \in \partial(\Psi_{\lfloor V})(v_n)$. From now on, we use a similar notation convention for all convergences below. The claim then follows from the convergences above by passing to the limit on

$$\langle \partial(\Psi_{n\lfloor V})(u_n) - \partial(\Psi_{n\lfloor V})(v_n), u_n - v_n \rangle_{V',V} \geq \alpha_\Psi \|u_n - v_n\|_V^2.$$

Stability of (2.2) *associated with* (2.1). Let $u \in \mathrm{dom}(\partial\Phi)$. From $\partial\Phi_n \overset{G_{s,s}}{\to} \partial\Phi$ and Proposition C.4, there exists $u_n \in \mathrm{dom}(\partial\Phi_n)$ such that

$$\begin{cases} u_n \longrightarrow u \text{ strongly in } X, \\ \partial\Phi_n(u_n) \longrightarrow \partial\Phi(u) \text{ strongly in } X. \end{cases} \tag{2.6}$$

For all $n \in \mathbb{N}$ we have

$$\langle \partial\Phi_n(u_n), \partial\Psi_n(u_n) \rangle \geq \alpha_{\Phi,\Psi} \|\partial\Psi_n(u_n)\|_X^2 - \beta_{\Phi,\Psi}. \tag{2.7}$$

From (2.6) and (2.7) we deduce that

$$\sup_{n\in\mathbb{N}} \|\partial\Psi_n(u_n)\|_X < +\infty,$$

which combined with (2.1) gives

$$\sup_{n\in\mathbb{N}} \|u_n\|_V < +\infty.$$

Hence, there exist a (non relabeled) subsequence $(u_n)_{n\in\mathbb{N}}$ and $\xi \in X$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } V \quad \text{(and strongly in } X), \\ \partial\Psi_n(u_n) \rightharpoonup \xi \text{ weakly in } X \text{ thus } \partial\Psi_n(u_n) \longrightarrow \xi \text{ strongly in } V'. \end{cases} \tag{2.8}$$

Since $\partial(\Psi_{n\lfloor V}) \overset{G_{w,s}}{\to} \partial(\Psi_{\lfloor V})$, we conclude from above that $\xi \in \partial(\Psi_{\lfloor V})(u) \cap X = \partial\Psi(u)$. From (2.6) and (2.8), and by passing to the limit $n \to +\infty$ on (2.7), we obtain

$$\langle \partial\Phi(u), \partial\Psi(u) \rangle \geq \alpha_{\Phi,\Psi} \liminf_{n\to+\infty} \|\partial\Psi_n(u_n)\|_X^2 - \beta_{\Phi,\Psi} \geq \alpha_{\Phi,\Psi} \|\partial\Psi(u)\|_X^2 - \beta_{\Phi,\Psi}$$

where ξ is denoted by $\partial\Psi(u)$. This completes the proof. □

Remark 2.4. It is not clear that (2.3) is stable in the following sense: let $((\Phi_n, \Psi_n)_{n\in\mathbb{N}}, (\Phi, \Psi))$ be a sequence of lsc convex proper functionals from X into $]-\infty, +\infty]$ with $\mathrm{dom}(\Psi_n) = \mathrm{dom}(\Psi) = V$ such that (Φ_n, Ψ_n) satisfies (2.3) and converges to (Φ, Ψ) for the product $\mathrm{M} \times \Gamma_{w\text{-}V}$-convergence, then (Φ, Ψ) satisfies (2.3). However under an additional equi-coerciveness condition, one can establish this stability. Indeed, let $(\Phi_n, \Psi_n)_{n\in\mathbb{N}}$ be a sequence of lsc convex proper functionals $\Phi_n, \Psi_n : X \to]-\infty, +\infty]$ such that $\mathrm{dom}(\Psi_n) = V$, which satisfy (2.3), and let $\Phi, \Psi : X \to]-\infty, +\infty]$ be lsc convex proper functionals such that $\mathrm{dom}(\Psi) = V$ and $(\Phi_n, \Psi_n) \to (\Phi, \Psi)$ for the product $\mathrm{M} \times \Gamma_{w\text{-}V}$ convergence. Assume furthermore that $(\Psi_n)_{n\in\mathbb{N}}$ fulfills the following equi-coerciveness condition: for all $r \in \mathbb{R}$, there exists a weakly compact subset K_r of V such that for all $n \in \mathbb{N}$

$$[\Psi_{n\lfloor V} \leq r] \subset K_r.$$

We claim that (Φ, Ψ) satisfies (2.3). Indeed take $u \in \mathrm{dom}(\partial\Phi)$. From $\partial\Phi_n \overset{G_{s,s}}{\to} \partial\Phi$ and Proposition C.4 we infer that there exists $u_n \in \mathrm{dom}(\partial\Phi_n)$ satisfying (2.6). From (2.3) we have

$$\langle \partial\Phi_n(u_n), \partial\Psi_{n,\lambda}(u_n)\rangle \geq -\beta(\|\partial\Phi_n(u_n)\|_X^2 + \|u_n\|_X^2 + 1). \tag{2.9}$$

Since $\partial\Psi_{n,\lambda}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$ (see [6, Proposition 17.2.1]), we have

$$\|\partial\Psi_{n,\lambda}(u_n) - \partial\Psi_{n,\lambda}(u)\|_X \leq \frac{1}{\lambda}\|u_n - u\|_X. \tag{2.10}$$

By combining (2.9) and (2.10), we infer that

$$\begin{aligned}&\langle \partial\Phi_n(u_n), \partial\Psi_{n,\lambda}(u)\rangle\\ &\qquad\geq -\beta(\|\partial\Phi_n(u_n)\|_X^2 + \|u_n\|^2 + 1) - \frac{1}{\lambda}\|\partial\Phi_n(u_n)\|_X\|u_n - u\|_X. \end{aligned}\tag{2.11}$$

On the other hand, by using the equi-coerciveness hypothesis and the compact embedding $V \overset{c}{\hookrightarrow} X$, it is easy to show that

$$\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\longrightarrow} \Psi_{\lfloor V} \Longrightarrow \Psi_n \overset{\mathrm{M}}{\longrightarrow} \Psi,$$

for a proof see Proposition D.2. Consequently (refer to [5, Proposition 3.29])

$$\partial\Psi_{n,\lambda}(u) \longrightarrow \partial\Psi_\lambda(u)$$

strongly in X for all $\lambda > 0$. Therefore, by passing to the limit $n \to +\infty$ in (2.10), we obtain

$$\langle \partial\Phi(u), \partial\Psi_\lambda(u)\rangle \geq -\beta(\|\partial\Phi\|_X^2 + \|u\|^2 + 1),$$

which completes the claim.

Regarding the kernel of the Bochner integral, we assume that $K : [0,T] \to \mathbb{R}_+$ belongs to $C^1([0,T])$. For every $v \in L^2(0,T,X)$ we adopt the notation

$$K \star v(t) := \int_0^t K(t-s)v(s)\,\mathrm{d}s.$$

In case $\partial\Psi$ is linear, to obtain the uniqueness of the solution we assume that K satisfies the additional conditions

$$K \in C^2(]0,T]),\; K(0) > 0,\; \text{and } (-1)^k K^{(k)}(t) \geq 0 \text{ for } t \in]0,T[\text{ and } k = 0,1,2, \tag{2.12}$$

which imply that for every $v \in L^2(0,T,X)$ such that $v(t) \in \mathrm{dom}(\partial\Psi)$ for a.e. $t \geq 0$,

$$\int_0^t \langle v(s), K \star \partial\Psi(v(s))\rangle \,\mathrm{d}s \geq \frac{K(t)}{2}\left\langle \partial\Psi\left(\int_0^t v(s)\,\mathrm{d}s\right), \int_0^t v(s)\,\mathrm{d}s\right\rangle$$
$$\geq 0 \text{ for all } t \geq 0. \tag{2.13}$$

For a proof see [8, 17].

2.2. Structure of the reaction functional

The reaction functional $F : [0,T] \times X \to X$ is a Borel measurable map satisfying:

(C_1) there exists $L \in L^2(0,T)$ such that $\|F(t,u) - F(t,v)\|_X \leq L(t)\|u - v\|_X$ for all $(u,v) \in X^2$ and all $t \in [0,T]$;

(C_2) the map $t \mapsto \|F(t,0)\|_X$ belongs to $L^2(0,T)$;

(C_3) L belongs to $L^2(0,T) \cap W^{1,1}(0,T)$ and there exists a nonnegative $\Theta \in L^1(0,T)$ such that $\|F(t,u) - F(s,u)\|_X \leq \int_s^t \Theta(\sigma)\,\mathrm{d}\sigma$ for all $s < t$ and all $u \in X$.

When Ω is a bounded domain of $\mathbb{R}^N$ and $X = L^2(\Omega)$, we specify F as follows: let $l \in \mathbb{N}^*$, then for all $u \in L^2(\Omega)$ and for a.e. $x \in \Omega$, $F(t,u)(x) = r(t,x) \cdot g(u(x)) + q(t,x)$ where

- $r \in L^\infty((0,T) \times \mathbb{R}^N, \mathbb{R}^l) \cap W^{1,1}(0,T, L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^l))$,
- $q \in L^2(0,T, L^2_{\mathrm{loc}}(\mathbb{R}^N)) \cap W^{1,1}(0,T, L^2_{\mathrm{loc}}(\mathbb{R}^N))$,
- $g : \mathbb{R} \to \mathbb{R}^l$ is bounded and L_g-lipschitz continuous.

It is easy to check that F fulfills the conditions (C_1), (C_2) and (C_3) with $L = \|r\|_{L^\infty((0,T)\times\mathbb{R}^N,\mathbb{R}^l)} L_g$ and $\Theta(\tau) = M_g\left\|\frac{\mathrm{d}r}{\mathrm{d}t}(\tau,\cdot)\right\|_{L^2(\Omega,\mathbb{R}^l)} + \left\|\frac{\mathrm{d}q}{\mathrm{d}t}(\tau,\cdot)\right\|_{L^2(\Omega)}$ for all $\tau \in (0,T)$ where $M_g = \sup_{r\in\mathbb{R}} |g(r)|$. See Section 5.2 for sequences of functionals of this type, and Section 6 for more details when F is randomized.

3. Existence of a local solution

From now on, to simplify the notation, we assume that Φ and Ψ are Gâteaux-differentiable, i.e. $\partial\Phi$ and $\partial\Psi$ are univalent. We follow the standard strategy of [7, 8, 12, 24] consisting in regularizing the non Fickian term $\int_0^t K(t-s)\partial\Psi(u(s))\,\mathrm{d}s$ by means of the Yosida approximation of $\partial\Psi$. The novelty is the presence of a reaction term and the fact that we assume condition (2.2) in place of (2.3).

3.1. **The regularized problem** $(\mathcal{P}_\lambda)$

Consider the Cauchy problem

$$(\mathcal{P})\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}(t)+\partial\Phi(u(t))+K\star\partial\Psi(u)(t)=F(t,u(t)) \text{ for a.e. } t\in(0,T)\\ u(0)=u^0,\ u^0\in\mathrm{dom}(\partial\Phi),\end{cases}$$

and denote by Ψ_λ the Moreau–Yosida approximation of index $\lambda>0$ of Ψ. We begin by establishing the global existence and uniqueness of a strong solution for the approximate problem expressed in $L^2(0,T,X)$

$$(\mathcal{P}_\lambda)\begin{cases}\dfrac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)+\partial\Phi(u_\lambda(t))+K\star\partial\Psi_\lambda(u_\lambda)(t)=F(t,u_\lambda(t)) \text{ for a.e. } t\in(0,T)\\ u_\lambda(0)=u^0,\ u^0\in\mathrm{dom}(\partial\Phi).\end{cases}$$

Set $G_\lambda(t,u_\lambda):=F(t,u_\lambda(t))-K\star\partial\Psi_\lambda(u_\lambda)(t)$ for all $\lambda>0$ and all $t\in[0,T]$. We rewrite the approximate equation in $(\mathcal{P}_\lambda)$ as

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)+\partial\Phi(u_\lambda(t))=G_\lambda(t,u_\lambda). \tag{3.1}$$

Lemma 3.1. *Assume that* (C$_1$)*,* (C$_2$)*, and* (C$_3$) *hold. Then, there exists a unique solution* $u_\lambda\in C([0,T],X)$ *of* $(\mathcal{P}_\lambda)$*. Furthermore* $\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\in L^2(0,T,X)$*,* $\partial\Phi(u_\lambda)\in L^2(0,T,X)$*, and*

(S$_\lambda$) $u_\lambda(t)\in\mathrm{dom}(\partial\Phi)$ *for all* $t\in[0,T]$*, and admits a right derivative* $\frac{\mathrm{d}^+u_\lambda}{\mathrm{d}t}(t)$ *which satisfies for every* $t\in[0,T[$

$$\frac{\mathrm{d}^+u_\lambda}{\mathrm{d}t}(t)+\partial\Phi(u_\lambda(t))=G_\lambda(t,u_\lambda).$$

Proof. Since $\partial\Psi_\lambda$ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$, it is easy to show that for all $(u,v)\in C([0,T],X)\times C([0,T],X)$,

$$\|K\star\partial\Psi_\lambda(u)-K\star\partial\Psi_\lambda(v)\|_{C([0,T],X)}\le C_{\lambda,T}\|u-v\|_{C([0,T],X)} \tag{3.2}$$

where $C_{\lambda,T}:=\frac{1}{\lambda}\big(\int_0^T K(s)\,\mathrm{d}s\big)$. For each $u\in C([0,T],X)$, denote by Λu the unique solution in $C([0,T],X)$ with $\frac{\mathrm{d}\Lambda u}{\mathrm{d}t}\in L^2(0,T,X)$ of the Cauchy problem

$$(\mathcal{P}_u)\begin{cases}\dfrac{\mathrm{d}\Lambda u}{\mathrm{d}t}(t)+\partial\Phi(\Lambda u(t))=G_\lambda(t,u) \text{ for a.e. } t\in(0,T)\\ \Lambda u(0)=u_0,\ u_0\in\mathrm{dom}(\partial\Phi).\end{cases}$$

For existence and uniqueness of Λu, we only have to check that $G_\lambda\in L^2(0,T,X)$ (refer to [6, Theorem 17.2.5], or [9, Theorem 3.7]). The claim follows straightforwardly from (C$_1$), (C$_2$) and (3.2).

The method is to show that the iterated map Λ^n is a strict contraction for n large enough. Indeed, from existence of a unique fixed point u_λ for Λ^n we will deduce that Λu_λ is a fixed point too. Thus, from uniqueness $\Lambda u_\lambda = u_\lambda$, so that u_λ is a fixed point for Λ which clearly solves $(\mathcal{P}_\lambda)$ and satisfies $\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} \in L^2(0,T,X)$ and $\partial\Phi(u_\lambda) \in L^2(0,T,X)$.

Let $(u,v) \in C([0,T],X) \times C([0,T],X)$ satisfying for a.e. $t \in (0,T)$

$$\frac{\mathrm{d}\Lambda u}{\mathrm{d}t}(t) + \partial\Phi(\Lambda u(t)) = G_\lambda(t,u),$$
$$\frac{\mathrm{d}\Lambda v}{\mathrm{d}t}(t) + \partial\Phi(\Lambda v(t)) = G_\lambda(t,v).$$

From the monotonicity of $\partial\Phi$, we infer that for a.e. $\sigma \in (0,T)$

$$\left\langle \frac{\mathrm{d}\Lambda v}{\mathrm{d}t}(\sigma) - \frac{\mathrm{d}\Lambda u}{\mathrm{d}t}(\sigma), \Lambda v(\sigma) - \Lambda u(\sigma) \right\rangle \leq \langle G_\lambda(\sigma,u) - G_\lambda(\sigma,v), \Lambda v(\sigma) - \Lambda u(\sigma)\rangle ,$$

hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda v(\sigma) - \Lambda u(\sigma)\|_X^2 \leq \langle G_\lambda(\sigma,u) - G_\lambda(\sigma,v), \Lambda v(\sigma) - \Lambda u(\sigma)\rangle .$$

By integration, we have for all $t \in [0,T]$

$$\begin{aligned}\frac{1}{2}\|\Lambda v(t) - \Lambda u(t)\|_X^2 &\leq \int_0^t \langle G_\lambda(\sigma,u) - G_\lambda(\sigma,v), \Lambda v(\sigma) - \Lambda u(\sigma)\rangle \,\mathrm{d}\sigma \\ &\leq \int_0^t \|G_\lambda(\sigma,u) - G_\lambda(\sigma,v)\|_X \|\Lambda v(\sigma) - \Lambda u(\sigma)\|_X \mathrm{d}\sigma.\end{aligned}$$

Thus, according to Lemma B.1 with $p = 2$, it follows that for all $t \in [0,T]$

$$\|\Lambda v(t) - \Lambda u(t)\|_X \leq \int_0^t \|G_\lambda(\sigma,u) - G_\lambda(\sigma,v)\|_X \mathrm{d}\sigma.$$

From (3.2) and (C_1) we infer that for all $t \in [0,T]$

$$\|\Lambda v - \Lambda u\|_{C([0,t],X)} \leq \int_0^t L_{\lambda,T}(\sigma)\|u - v\|_{C([0,\sigma],X)}\mathrm{d}\sigma \tag{3.3}$$

where $L_{\lambda,T}(\sigma) := C_{\lambda,T} + L(\sigma)$. By iterating (3.3), and according to the formula

$$\int_0^t L_{\lambda,T}(\sigma_1)\int_0^{\sigma_1} L_{\lambda,T}(\sigma_2)\dots\int_0^{\sigma_{n-1}} L_{\lambda,T}(\sigma_n)\mathrm{d}\sigma_n\dots\mathrm{d}\sigma_1 = \frac{\left(\int_0^t L_{\lambda,T}(\sigma)\mathrm{d}\sigma\right)^n}{n!}$$

obtained by a standard calculus for multiple integrals, we obtain

$$\|\Lambda^n v - \Lambda^n u\|_{C([0,T],X)} \leq \frac{\left(\int_0^T L_{\lambda,T}(\sigma)\mathrm{d}\sigma\right)^n}{n!}\|u - v\|_{C([0,T],X)}.$$

The claim follows for n sufficiently large.

To prove that u_λ satisfies S_λ, we have to establish that $G_\lambda \in W^{1,1}(0,T,X)$ (see [6, Theorem 17.2.6], or [9, Theorem 3.7]). We first claim that $K \star \partial\Psi_\lambda(u_\lambda)$ belongs to $W^{1,2}(0,T,X)$. This follows from

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}K \star \partial\Psi_\lambda(u_\lambda)\right\|_{L^2(0,T,X)} \leq (K(0) + T^{\frac{1}{2}}\|K'\|_{L^2(0,T)})\|\partial\Psi_\lambda(u_\lambda)\|_{L^2(0,T,X)}$$

which is obtained from the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}K \star \partial\Psi_\lambda(u_\lambda)(t) = K(0)\partial\Psi_\lambda(u_\lambda)(t) + K' \star \partial\Psi_\lambda(u_\lambda)(t) \text{ for a.e. } t \in (0,T). \quad (3.4)$$

It remains to establish that $F(\cdot, u_\lambda) \in W^{1,1}(0,T,X)$. This follows from (C_3), and the following calculation: for all $(s,t) \in [0,T]^2$ with $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}
&\|F(t,u_\lambda(t)) - F(s,u_\lambda(s))\|_X \\
&\qquad \leq \|F(t,u_\lambda(t)) - F(s,u_\lambda(t))\|_X + |L(s)|\|u_\lambda(t) - u_\lambda(s)\|_X \\
&\qquad \leq \int_s^t \Theta(\sigma)\mathrm{d}\sigma + \left(L(0) + \int_0^T \left|\frac{\mathrm{d}L}{\mathrm{d}\sigma}(\sigma)\right|\mathrm{d}\sigma\right)\int_s^t \left\|\frac{\mathrm{d}u_\lambda}{\mathrm{d}\sigma}(\sigma)\right\|_X \mathrm{d}\sigma, \quad (3.5)
\end{aligned}$$

which proves that $F(\cdot, u_\lambda)$ is absolutely continuous. The proof is complete. □

3.2. Convergence of $(\mathcal{P}_\lambda)$ to $(\mathcal{P})$: existence of a local solution of $(\mathcal{P})$

The following lemma furnishes local estimates for the solution of $(\mathcal{P}_\lambda)$, needful for establishing the convergence of $(\mathcal{P}_\lambda)$ to $(\mathcal{P})$. Its proof is postponed to Appendix A

Lemma 3.2. *Assume that* (2.1)*,* (2.2) *and* (C_1)*,* (C_2)*,* (C_3) *hold. Then for every* $0 < \widetilde{T} \leq T$ *satisfying* $\widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)} + \alpha_\Psi\|L\|_{L^2(0,\tilde{T})} < \alpha_{\Phi,\Psi}$*, the following estimates hold:*

$$\sup_{\lambda>0} \|\partial\Psi(u_\lambda)\|_{L^2(0,\tilde{T},X)} < +\infty, \tag{3.6}$$

$$\sup_{\lambda>0} \|u_\lambda\|_{L^2(0,\tilde{T},V)} < +\infty, \tag{3.7}$$

$$\sup_{\lambda>0} \left\|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right\|_{L^2(0,\tilde{T},X)} < +\infty, \tag{3.8}$$

$$\sup_{\lambda>0} \|\partial\Phi(u_\lambda)\|_{L^2(0,\tilde{T},X)} < +\infty, \tag{3.9}$$

$$\sup_{\lambda>0} \|u_\lambda\|_{C(0,\tilde{T},X)} < +\infty, \tag{3.10}$$

$$\sup_{\lambda>0} \left\|\frac{\mathrm{d}^+u_\lambda}{\mathrm{d}t}(t)\right\|_X < +\infty \qquad \textit{for each } t \in]0,\widetilde{T}], \tag{3.11}$$

$$\sup_{\lambda>0} \|\partial\Psi(u_\lambda(t))\|_X < +\infty \qquad \textit{for each } t \in [0,\widetilde{T}], \tag{3.12}$$

$$\sup_{\lambda>0} \|\partial\Phi(u_\lambda(t))\|_X < +\infty \qquad \textit{for each } t \in [0,\widetilde{T}]. \tag{3.13}$$

Theorem 3.3 (Local solution)**.** *Assume that* (2.1)*,* (2.2)*,* (C_1)*,* (C_2)*,* (C_3) *hold, and let* $\widetilde{T} > 0$ *be a positive number satisfying* $\widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)} + \alpha_\Psi\|L\|_{L^2(0,\tilde{T})} < \alpha_{\Phi,\Psi}$*. Then* ($\mathcal{P}$) *admits a solution* $u_{\tilde{T}}$ *in* $C([0,\widetilde{T}],X)$ *which satisfies* $u_{\tilde{T}}(t) \in \mathrm{dom}(\partial\Phi)$ *for each* $t \in [0,\widetilde{T}]$.

Proof. To shorten the notation we write u for $u_{\tilde{T}}$. The proof falls into four steps.

Step 1: Compactness in $C([0,\widetilde{T}],X)$. We establish existence of $u \in C([0,\widetilde{T}],X)$ and a subsequence of $(u_\lambda)_{\lambda>0}$ (not relabeled), such that

$$u_\lambda \longrightarrow u \text{ in } C([0,\widetilde{T}],X), \tag{3.14}$$

$$J_\lambda u_\lambda(t) \longrightarrow u(t) \text{ in } X, \quad \text{for all } t \in [0,\widetilde{T}], \tag{3.15}$$

where $J_\lambda := (I+\lambda\partial\Psi)^{-1} : X \to X$ is the resolvent of index λ of $\partial\Psi$ (for the properties of J_λ see [6, Proposition 17.2.1]).

To prove (3.14), the method consists in applying Ascoli's theorem. From (3.8) and (3.10) we have

$$\sup_{\lambda>0} \|u_\lambda\|_{C([0,\tilde{T}],X)} < +\infty \text{ (equiboundedness)},$$

$$\|u_\lambda(t) - u_\lambda(s)\|_X \leq (t-s)^{\frac{1}{2}} \sup_{\lambda>0} \left\|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right\|_{L^2(0,\tilde{T},X)} \text{ for all } 0 \leq s \leq t \leq \widetilde{T} \text{ (equicontinuity)}.$$

It remains to show that for each $t \in]0, \widetilde{T}]$, the set $E_t := \{u_\lambda(t) : \lambda > 0\}$ is relatively compact in X (for $t = 0$, E_t is reduced to $\{u_0\}$). Let $t \in]0, \widetilde{T}]$. From the compact embedding $V \hookrightarrow X$, it suffices to establish that $\sup_{\lambda>0} \|u_\lambda(t)\|_V < +\infty$. The claim follows from (2.1) and (3.12) which yields

$$\begin{aligned}\|u_\lambda(t)\|_V^2 \leq \frac{1}{\alpha_\Psi} \langle \partial\Psi(u_\lambda(t)), u_\lambda(t)\rangle &\leq \frac{1}{\alpha_\Psi} \|\partial\Psi(u_\lambda(t))\|_X \|u_\lambda(t)\|_X \\ &\leq \frac{1}{\alpha_\Psi} \|\partial\Psi(u_\lambda(t))\|_X \|u_\lambda(t)\|_V,\end{aligned}$$

hence

$$\|u_\lambda(t)\|_V \leq \frac{1}{\alpha_\Psi} \sup_{\lambda>0} \|\partial\Psi(u_\lambda(t))\|_X.$$

Estimate (3.15) is established as follows: from the definition of J_λ, we have $J_\lambda u_\lambda(t) - u_\lambda(t) = \lambda\partial\Psi_\lambda(u_\lambda(t))$ so that, from

$$\|\partial\Psi_\lambda(u_\lambda(t))\|_X \leq \|\partial\Psi(u_\lambda(t))\|_X \tag{3.16}$$

(see [6, Proposition 17.2.2]), we infer that

$$\|J_\lambda u_\lambda(t) - u_\lambda(t)\|_X \leq \lambda\|\partial\Psi_\lambda(u_\lambda(t))\|_X \leq \lambda\|\partial\Psi(u_\lambda(t))\|_X.$$

Hence, from (3.12), $J_\lambda u_\lambda(t) - u_\lambda(t) \to 0$ in X for $t \in]0, \widetilde{T}]$ as $\lambda \to 0$.

Step 2. We prove that $u(t) \in \mathrm{dom}(\partial\Phi)$ for all $t \in [0, \widetilde{T}]$, and that (3.14), (3.15) hold in V equipped with its norm $\|\cdot\|_V$. More precisely

$$u_\lambda \longrightarrow u \text{ in } C([0, \widetilde{T}], V), \tag{3.17}$$

$$J_\lambda u_\lambda(t) \longrightarrow u(t) \text{ in } V \text{ for all } t \in [0, T]. \tag{3.18}$$

Fix $t \in [0, \widetilde{T}]$. From (3.13), there exist $B(t) \in X$ and a subsequence such that

$$\partial\Phi(u_\lambda(t)) \rightharpoonup B(t) \text{weakly in } X.$$

From (3.14) $u_\lambda(t) \to u(t)$ strongly in X, and since the maximal monotone operator $\partial\Phi$ is demi-closed (see [6, Proposition 17.2.4]), we deduce that $u(t) \in \mathrm{dom}(\partial\Phi)$ and $B(t) = \partial\Phi(u(t))$.

Observe that $u_\lambda(t)$ and $u(t)$ belong to $\mathrm{dom}(\partial\Phi) \subset \mathrm{dom}(\partial\Psi)$ for all $t \in [0, \widetilde{T}]$. Hence from (2.1), we deduce

$$\begin{aligned}\frac{1}{\alpha_\Psi}\|u_\lambda(t) - u(t)\|_V^2 &\leq \langle \partial\Psi(u_\lambda(t)) - \partial\Psi(u(t)), u_\lambda(t) - u(t)\rangle \\ &\leq \left(\sup_{\lambda>0} \|\partial\Psi(u_\lambda(t))\|_X + \|\partial\Psi(u(t))\|_X\right) \|u_\lambda(t) - u(t)\|_X.\end{aligned}$$

Hence (3.17) follows from (3.12) and (3.14). The proof of (3.18) is similar. More precisely, for all $t \in [0, \widetilde{T}]$

$$\begin{aligned}\|J_\lambda u_\lambda(t) - u_\lambda(t)\|_V^2 &\leq \langle \partial\Psi(J_\lambda u_\lambda(t)) - \partial\Psi(u_\lambda(t)), J_\lambda u_\lambda(t) - u_\lambda(t)\rangle \\ &\leq \sup_{\lambda>0}(\|\partial\Psi_\lambda(u_\lambda(t))\|_X + \|\partial\Psi(u_\lambda(t))\|_X)\|J_\lambda u_\lambda(t) - u_\lambda(t)\|_X.\end{aligned}$$

(recall that $\partial\Psi_\lambda(u_\lambda(t)) = \partial\Psi(J_\lambda u_\lambda(t))$, see [6, Proposition 17.2.1]). From (3.16) and (3.12)

$$\sup_{\lambda>0}\|\partial\Psi_\lambda(u_\lambda(t))\|_X \leq \sup_{\lambda>0}\|\partial\Psi(u_\lambda(t))\|_X < +\infty,$$

hence

$$\|J_\lambda u_\lambda(t) - u_\lambda(t)\|_V^2 \leq 2\sup_{\lambda>0}\|\partial\Psi(u_\lambda(t))\|_X\|J_\lambda u_\lambda(t) - u_\lambda(t)\|_X$$

so that (3.18) follows from (3.12) and (3.15).

Step 3. We establish that $G_\lambda(\,\cdot\,, u_\lambda) \rightharpoonup G(\,\cdot\,, u)$ in $L^2(0, \widetilde{T}, X)$ where the function $G(\,\cdot\,, u)$ is defined by $G(t, u) := F(t, u(t)) - K \star \partial\Psi(u)(t)$.

From (C_1), (C_2) and (3.14), $F(\,\cdot\,, u_\lambda)$ strongly converges to $F(\,\cdot\,, u)$ in $L^2(0, \widetilde{T}, X)$. We claim that $\partial\Psi_\lambda(u_\lambda) \rightharpoonup \partial\Psi(u)$ in $L^2(0, \widetilde{T}, X)$, from which we easily deduce that $K \star \partial\Psi_\lambda(u_\lambda) \rightharpoonup K \star \partial\Psi(u)$ in $L^2(0, \widetilde{T}, X)$. From (3.6) we have

$$\sup_{\lambda>0}\|\partial\Psi_\lambda(u_\lambda(t))\|_{L^2(0,\tilde{T},X)} \leq \sup_{\lambda>0}\|\partial\Psi(u_\lambda(t))\|_{L^2(0,\tilde{T},X)} < +\infty.$$

Thus, using the compact embedding $L^2(0, \widetilde{T}, X) \hookrightarrow L^2(0, \widetilde{T}, V')$, we infer that there exist a subsequence (not relabeled) and $C \in L^2(0, \widetilde{T}, X)$ such that successively,

$$\begin{aligned}\partial\Psi_\lambda(u_\lambda) &\rightharpoonup C \text{ weakly in } L^2(0, \widetilde{T}, X),\\ \partial\Psi_\lambda(u_\lambda) &\longrightarrow C \text{ strongly in } L^2(0, \widetilde{T}, V'),\\ \partial\Psi_\lambda(u_\lambda(t)) &\longrightarrow C(t) \text{ in } V' \text{ for a.e. } t \in (0, \widetilde{T}).\end{aligned}$$

Since $\partial\Psi_\lambda(u_\lambda) = \partial\Psi(J_\lambda u_\lambda)$, we deduce from above that $\partial\Psi(J_\lambda u_\lambda(t)) \to C(t)$ in V' for a.e. $t \in (0, T)$. As from (3.18), $J_\lambda u_\lambda(t) \to u(t)$ in V, from the maximality of $\partial\Psi$ we infer that $C(t) = \partial\Psi(u(t))$ for a.e. $t \in (0, T)$. This proves the claim.

Step 4: u solves ($\mathcal{P}$). To shorten the notation, we write $G_\lambda(t)$ for $G_\lambda(t, u_\lambda)$. Denote by Φ^* the Legendre–Fenchel conjugate of Φ. According to the Fenchel extremality condition (see [6, Proposition 9.5.1]), equation (3.1) is equivalent to

$$\Phi(u_\lambda(t)) + \Phi^*\left(G_\lambda(t) - \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\right) + \left\langle \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) - G_\lambda(t), u_\lambda(t)\right\rangle = 0$$

for a.e. $t \in (0,\widetilde{T})$, which, from the Legendre–Fenchel inequality, is in turn equivalent to

$$\int_0^{\tilde{T}} \left[\Phi(u_\lambda(t)) + \Phi^*\Big(G_\lambda(t) - \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\Big) + \Big\langle \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t) - G_\lambda(t), u_\lambda(t)\Big\rangle\right] \mathrm{d}t = 0.$$

Therefore, (3.1) is equivalent to

$$\int_0^{\tilde{T}} \left[\Phi(u_\lambda(t)) + \Phi^*\Big(G_\lambda(t) - \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\Big) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u_\lambda(t)\|^2 - \langle G_\lambda(t), u_\lambda(t)\rangle\right] \mathrm{d}t = 0,$$

hence to

$$\int_0^{\tilde{T}} \left[\Phi(u_\lambda(t)) + \Phi^*\Big(G_\lambda(t) - \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\Big)\right] \mathrm{d}t + \frac{1}{2}(\|u_\lambda(\widetilde{T})\|^2 - \|u_0\|^2) - \int_0^{\tilde{T}} \langle G_\lambda(t), u_\lambda(t)\rangle \; \mathrm{d}t = 0.$$

Equivalently

$$I_\Phi(u_\lambda) + I_{\Phi^*}\left(G_\lambda - \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right) + \frac{1}{2}(\|u_\lambda(\widetilde{T})\|^2 - \|u_0\|^2) - \int_0^{\tilde{T}} \langle G_\lambda(t), u_\lambda(t)\rangle \, \mathrm{d}t = 0 \quad (3.19)$$

where the integral functionals I_Φ and I_{Φ^*} are respectively defined in $L^2(0,\widetilde{T},X)$ by

$$I_\Phi(v) = \int_0^{\tilde{T}} \Phi(v(t))\,\mathrm{d}t \quad \text{and} \quad I_{\Phi^*}(v) = \int_0^{\tilde{T}} \Phi^*(v(t))\,\mathrm{d}t.$$

Combining $u_\lambda(\widetilde{T}) = u_0 + \int_0^{\tilde{T}} \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)\mathrm{d}t$ with $\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t}$ in $L^2(0,T,X)$ which is obtained from (3.8), we infer that

$$\liminf_{\lambda\to+\infty} \left\|u_\lambda(\widetilde{T})\right\|^2 \geq \left\|u(\widetilde{T})\right\|^2. \quad (3.20)$$

By passing to the lower limit in (3.19), from (3.20), (3.14), Step 3, and noticing that I_Φ and I_{Φ^*} are lower semicontinuous for the weak topology of $L^2(0,\widetilde{T},X)$, we obtain

$$\int_0^T \left[\Phi(u(t)) + \Phi^*\left(G(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t)\right)\right] \mathrm{d}t + \frac{1}{2}\big(\|u(T)\|^2 - \|u^0\|^2\big) - \int_0^T \langle G(t), u(t)\rangle \,\mathrm{d}t \leq 0$$

or equivalently,

$$\int_0^T \left[\Phi(u(t)) + \Phi^*\left(G(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t)\right) + \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}(t) - G(t), u(t)\right\rangle\right] \mathrm{d}t \leq 0. \quad (3.21)$$

From the Legendre–Fenchel inequality, we have

$$\Phi(u(t)) + \Phi^*\left(G(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t)\right) + \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}(t) - G(t), u(t)\right\rangle \geq 0,$$

so that (3.21) yields that for a.e. $t \in (0,T)$,

$$\Phi(u(t)) + \Phi^* \left(G(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t) \right) + \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}(t) - G(t), u(t) \right\rangle = 0$$

which is equivalent to

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial\Phi(u(t)) = G(t) \quad \text{for a.e. } t \in (0,T).$$

The proof is complete. □

4. Existence of solutions in $C([0,T],X)$

4.1. Existence of a global solution in $C([0,T],X)$: translation-induction method

Any local solution obtained in Theorem 3.3 can be continued on $[0,T]$ as follows: cover $[0,T]$ by the translated segments of $[0,\widetilde{T}]$, and stick together the $\widetilde{T}$-translated local solutions in $C([0,\widetilde{T}],X)$ of each suitably modified problem $(\mathcal{P})$. This process is a generalization of a standard method; see for instance [7, p. 243].

Theorem 4.1. *Given $\widetilde{T} > 0$ satisfying $\widetilde{T}^{\frac{1}{2}} \|K\|_{L^2(0,T)} + \alpha_\Psi \|L\|_{L^2(0,\tilde{T})} < \alpha_{\Phi,\Psi}$, any local solution $u_{\tilde{T}}$ of $(\mathcal{P})$ in $C([0,\widetilde{T}],X)$ obtained in Theorem 3.3 can be extended to a solution of $(\mathcal{P})$ in $C([0,T],X)$.*

Proof. For $i = 1,\dots,\ell$ where $\ell := \max\{k \in \mathbb{N} : k\widetilde{T} \le T\}$, set $T_i := i\widetilde{T}$, $T_{\ell+1} = T$, and denote by u_0 the solution $u_{\tilde{T}}$ of $(\mathcal{P})$ on $(0,\widetilde{T})$ whose existence has been established in Theorem 3.3. For $i = 1,\dots,\ell$ consider the Cauchy problem defined by induction:

$$(\mathcal{P}_i) \begin{cases} \dfrac{\mathrm{d}u_i}{\mathrm{d}t}(t) + \partial\Phi(u_i(t)) + K \star \partial\Psi(u_i)(t) = F(t+T_i, u_i(t)) - R_i(t) \text{ for a.e. } t \in (0,\widetilde{T}) \\ u_i(0) = u_{i-1}(\widetilde{T}) \end{cases}$$

where

$$R_i(t) := \sum_{k=1}^{i} \int_{T_{k-1}}^{T_k} K(t+T_i-s)\partial\Psi(u_{k-1}(s-T_{k-1}))\,\mathrm{d}s.$$

Existence of u_i can be obtained as in the proof of Theorem 3.3: substitute $F_i(t,u_i(t)) := F(t+T_i,u_i(t)) - R_i(t)$ for $F(t,u(t))$, and observe that $R_i \in W^{1,1}(0,\widetilde{T},X)$ so that F_i satisfies (C_1), (C_2), (C_3). Note that $u_{i-1}(\widetilde{T}) \in \mathrm{dom}(\partial\Phi)$ (repeat the first part of ***Step 2*** in the proof of Theorem 3.3 and reason by induction).

Finally we show that the function u defined by $u(t) := u_i(t-T_i)$ whenever $t \in [T_i, T_{i+1}]$, solves $(\mathcal{P})$. Indeed, for $t \in [T_i, T_{i+1}]$ the following calculation holds:

$$\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial\Phi(u(t)) + \int_0^t K(t-s)\partial\Psi(u(s))\,\mathrm{d}s - F(t,u(t)) \\
&\quad = \frac{\mathrm{d}u_i}{\mathrm{d}t}(t-T_i) + \partial\Phi(u_i(t-T_i)) + \sum_{k=1}^{i}\int_{T_{k-1}}^{T_k} K(t-s)\partial\Psi(u_{k-1}(s-T_{k-1}))\,\mathrm{d}s \\
&\qquad + \int_{T_i}^{t} K(t-s)\partial\Psi(u_i(s-T_i))\,\mathrm{d}s - F(t,u_i(t-T_i)).
\end{aligned}$$

Since $\sigma := t - T_i \in [0, T_{i+1} - T_i] = [0, \widetilde{T}]$, the second member is equal to

$$\begin{aligned}
&\frac{\mathrm{d}u_i}{\mathrm{d}t}(\sigma) + \partial\Phi(u_i(\sigma)) + \sum_{k=1}^{i}\int_{T_{k-1}}^{T_k} K(\sigma+T_i-s)\partial\Psi(u_{k-1}(s-T_{k-1}))\,\mathrm{d}s \\
&\qquad + \int_{T_i}^{\sigma+T_i} K(\sigma+T_i-s)\partial\Psi(u_i(s-T_i))\,\mathrm{d}s - F(\sigma+T_i,u_i(\sigma)) \\
&= \frac{\mathrm{d}u_i}{\mathrm{d}t}(\sigma) + \partial\Phi(u_i(\sigma)) + R_i(\sigma) + \int_0^{\sigma} K(\sigma-s)\partial\Psi(u_i(s))\,\mathrm{d}s - F(\sigma+T_i,u_i(\sigma)) \\
&= \frac{\mathrm{d}u_i}{\mathrm{d}t}(\sigma) + \partial\Phi(u_i(\sigma)) + K\star\partial\Psi(u_i)(\sigma) + R_i(\sigma) - F(\sigma+T_i,u_i(\sigma))
\end{aligned}$$

which, from $(\mathcal{P}_i)$, is equal to 0. Moreover $u(T_i^-) = u_{i-1}(\widetilde{T})$ and $u(T_i^+) = u_i(0) = u_{i-1}(\widetilde{T})$ so that $u \in C([0,T], X)$. □

4.2. Existence and uniqueness when Ψ is a quadratic functional

Proposition 4.2. *Under the conditions of Theorem 3.3, assume further that Ψ is a quadratic form in V. Then $(\mathcal{P})$ admits a unique solution.*

Proof. Let u_1 and u_2 be two solutions of $(\mathcal{P})$. This yields for a.e. $s \in (0,T)$,

$$\begin{aligned}
\frac{\mathrm{d}(u_1-u_2)}{\mathrm{d}t}(s) + (\partial\Phi(u_1(s)) - \partial\Phi(u_2(s))) + K\star\partial\Psi(u_1-u_2)(s)& \\
= F(s,u_1(s)) - F(s,u_2(s)).& \quad (4.1)
\end{aligned}$$

Form the scalar product of (4.1) with $u_1(s) - u_2(s)$ and integrate over $(0,t)$, taking into account the monotonicity of $\partial\Phi$ and (2.13), we obtain

$$\frac{1}{2}\|u_1(t)-u_2(t)\|^2 \le \int_0^t L(s)\,\|u_1(s)-u_2(s)\|^2\,\mathrm{d}s.$$

We conclude by applying the standard Grönwall's lemma. □

From Proposition 4.2 and Theorem 4.1, we have

Corollary 4.3. *When Ψ is a quadratic functional, then $(\mathcal{P})$ admits a unique solution in $C([0,T],X)$.*

4.3. **Existence of a right derivative of the solutions at each $t \in [0,T[$.**

Theorem 4.4 below is crucial to establish the convergence in Section 5. For its proof, condition (2.2) does not play any role.

Theorem 4.4. *Every solution u of $(\mathcal{P})$ admits a right derivative at every $t \in [0,T[$ which satisfies the equation:*

$$\frac{\mathrm{d}^+ u}{\mathrm{d}t}(t) + \partial\Phi(u(t)) + K \star \partial\Psi(u)(t) = F(t,u(t)),\ t \in [0,T[. \tag{4.2}$$

Proof.

Step 1. Fix t_0 in $[0,T[$ and write h to denote a sequence $(h_n)_{n\in\mathbb{N}}$ of positive numbers decreasing to 0. This step is devoted to the following estimate:

$$\limsup_{h\to 0} \left\| \frac{1}{h}(u(t_0+h) - u(t_0)) \right\|_X \leq \|-\partial\Phi(u(t_0)) - K \star \partial\Psi(u)(t_0) + F(t_0,u(t_0))\|_X. \tag{4.3}$$

Observe that the constant function $v := u(t_0)$ satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) + \partial\Phi(v(t)) = \partial\Phi(u(t_0)) \tag{4.4}$$

for each $t \in (0,T)$. Subtract (4.4) from

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial\Phi(u(t)) = -K \star \partial\Psi(u)(t) + F(t,u(t)),$$

form the scalar product with $u(t) - v(t)$ and integrate over (t_0, t_0+h). This yields

$$\frac{1}{2}\|u(t_0+h) - u(t_0)\|_X^2 \leq \int_{t_0}^{t_0+h} \langle -\partial\Phi(u(t_0)) - K \star \partial\Psi(u)(s) + F(s,u(s)), u(s) - u(t_0)\rangle \,\mathrm{d}s.$$

According to the Grönwall type lemma, Lemma B.1 with $p = 2$, it follows that

$$\left\| \frac{1}{h}(u(t_0+h) - u(t_0)) \right\|_X \leq \frac{1}{h}\int_{t_0}^{t_0+h} \|-\partial\Phi(u(t_0)) - K \star \partial\Psi(u)(s) + F(s,u(s))\|_X \mathrm{d}s.$$

The conclusion follows by passing to the upper limit when $h \to 0^+$.

Step 2. We prove that

$$\frac{1}{h}(u(t_0+h)-u(t_0)) \rightharpoonup -\partial\Phi(u(t_0))-K\star\partial\Psi(u)(t_0)+F(t_0,u(t_0)) \text{ weakly in } X. \quad (4.5)$$

From Step 1 a subsequence not relabeled of $\frac{1}{h}(u(t_0+h)-u(t_0))$ weakly converges to some w in X. For identifying w, take $(\xi,\xi^*) \in \partial\Phi$, i.e. $\xi^* = \partial\Phi(\xi)$. Then the constant function $v = \xi$ satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) + \partial\Phi(v(t)) = \xi^* \quad (4.6)$$

for all $t \in (0,T)$. Subtract (4.6) from

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial\Phi(u(t)) = -K\star\partial\Psi(u)(t) + F(t,u(t)),$$

and form the scalar product with $u(t)-\xi$ and integrate over (t_0,t_0+h). This yields

$$\frac{1}{2}\|u(t_0+h)-\xi\|_X^2 - \frac{1}{2}\|u(t_0)-\xi\|_X^2 \leq \int_{t_0}^{t_0+h} \langle -K\star\partial\Psi(u)(s) + F(s,u(s)) - \xi^*, u(s)-\xi\rangle \,\mathrm{d}s.$$

From the elementary inequality $2\langle a-b,b\rangle \leq \|a\|_X^2 - \|b\|_X^2$ we infer that

$$\left\langle \frac{1}{h}(u(t_0+h)-u(t_0)), u(t_0)-\xi \right\rangle \leq \frac{1}{h}\int_{t_0}^{t_0+h} \langle -K\star\partial\Psi(u)(s) + F(s,u(s)) - \xi^*, u(s)-\xi\rangle \,\mathrm{d}s.$$

Passing to the limit $h \to 0$ we find

$$\langle w, u(t_0)-\xi\rangle \leq \langle -K\star\partial\Psi(u)(t_0) + F(t_0,u(t_0)) - \xi^*, u(t_0)-\xi\rangle,$$
$$\langle -K\star\partial\Psi(u)(t_0) + F(t_0,u(t_0)) - w - \xi^*, u(t_0)-\xi\rangle \geq 0$$

for all $(\xi,\xi^*) \in \partial\Phi$, i.e.

$$(u(t_0), -K\star\partial\Psi(u)(t_0) + F(t_0,u(t_0)) - w)$$

is monotonically related to $\partial\Phi$ (see Definition C.1). Since $\partial\Phi$ is maximal monotone, from Proposition C.2 we deduce that $-K\star\partial\Psi(u)(t_0) + F(t_0,u(t_0)) - w = \partial\Phi(u(t_0))$.

Step 3: end of the proof. Combining (4.5), the lower semicontinuity of the norm, and (4.3), we deduce that

$$\lim_{h\to 0}\left\|\frac{1}{h}(u(t_0+h)-u(t_0))\right\|_X = \|-\partial\Phi(u(t_0)) - K\star\partial\Psi(u)(t_0) + F(t_0,u(t_0))\|_X.$$

Hence $\frac{1}{h}(u(t_0+h)-u(t_0)) \to -\partial\Phi(u(t_0))-K\star\partial\Psi(u)(t_0)+F(t_0,u(t_0))$ strongly in X, i.e.

$$\frac{\mathrm{d}^+u}{\mathrm{d}t}(t_0)+\partial\Phi(u(t_0))+K\star\partial\Psi(u)(t_0)=F(t_0,u(t_0))$$

which completes the proof. □

5. Convergence under Mosco×Γ-convergence

5.1. The abstract case

This section is placed within the framework defined in Section 2. Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of integrodifferential diffusion problems in $L^2(0,T,X)$ defined by

$$(\mathcal{P}_n)\begin{cases}\dfrac{\mathrm{d}u_n}{\mathrm{d}t}(t)+\partial\Phi_n(u_n(t))+\displaystyle\int_0^t K(t-s)\partial\Psi_n(u_n(s))\,\mathrm{d}s=F_n(t) \text{ for a.e. } t\in(0,T)\\ u_n(0)=u_n^0,\ u_n^0\in\operatorname{dom}(\partial\Phi_n),\end{cases}$$

where $F_n\in L^2(0,T,X)\cap W^{1,1}(0,T,X)$ satisfies (C_1), (C_2), and (C_3) with $F_n(t,u)=F_n(t)$. In next Sections 5.2 and 6, $X=L^2(\Omega)$ and the source F_n is structured as a reaction functional $F_n(t,u_n(t))$ as defined in Section 2.2. Recall that $\Phi_n,\Psi_n : X\to\,]-\infty,+\infty]$ are lsc convex proper functionals with domain V. Without loss of generality we assume that $\inf_X\Phi_n\geq 0$ and $\inf_X\Psi_n\geq 0$. The subdifferentials $\partial\Phi_n$ and $\partial\Psi_n$ are assumed to be univalent. Observe that this hypothesis is not closed under the Mosco and the $\Gamma_{w\text{-}V}$ convergence of $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_{n\lfloor V})_{n\in\mathbb{N}}$ respectively. We assume that $\operatorname{dom}(\partial\Phi_n)\subset\operatorname{dom}(\partial\Psi_n)$ and that conditions (2.1) and (2.2) hold uniformly in the sense that α_{ψ_n} and α_{Φ_n,Ψ_n} do not depend on n. We denote it by α_Ψ and $\alpha_{\Phi,\Psi}$ respectively.

Let $\widetilde{T}>0$ satisfy $\widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)}<\alpha_{\Phi,\Psi}$. By a particular solution of $(\mathcal{P}_n)$, we mean any solution in $C([0,T],X)$ obtained by translation-induction of a local solution $u_{n,\tilde{T}}\in C([0,\widetilde{T}],X)$, whose existence is established in Theorem 4.1. Note that when Ψ_n is quadratic, according to Proposition 4.2, a particular solution is nothing but the unique solution of $(\mathcal{P}_n)$.

Theorem 5.1. *Under the general conditions above, assume furthermore that*

(STAB$_1$) $F_n\rightharpoonup F$ *in* $L^2(0,T,X)$, $\sup_{n\in\mathbb{N}}\|F_n(t)\|_X<+\infty$ *for all* $t\in[0,T]$, *and* $\sup_{n\in\mathbb{N}}\left\|\frac{\mathrm{d}F_n}{\mathrm{d}t}\right\|_{L^1(0,T,X)}<+\infty$;

(STAB$_2$) $\sup_{n\in\mathbb{N}}\Phi_n(u_n^0)<+\infty$;

(STAB$_3$) $u_n^0\to u^0$ *strongly in* X;

(STAB$_4$) *there exists* $\Phi : X \to]-\infty, +\infty]$ *such that* $\Phi_n \overset{M}{\to} \Phi$*;*

(STAB$_5$) *there exists* $\Psi : V \to]-\infty, +\infty]$ *lsc convex proper, such that* $\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\to} \Psi$.

Then any particular sequence of solutions u_n *of* $(\mathcal{P}_n)$ *admits a subsequence which converges to a solution* $u \in C([0,T], X)$ *of the differential inclusion*

$$(\mathcal{P}) \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial\Phi(u(t)) + \displaystyle\int_0^t K(t-s)\big[\partial\Psi(u(s)) \cap X\big]\,\mathrm{d}s \ni F(t) \textit{ for a.e. } t \in (0,T) \\ u(0) = u^0,\ u^0 \in \mathrm{dom}(\partial\Phi). \end{cases}$$

Proof. We use the notation of the proof of Theorem 4.1, and do not relabel the various subsequences. Take $\widetilde{T} > 0$ satisfying $\widetilde{T}^{\frac{1}{2}} \|K\|_{L^2(0,T)} < \alpha_{\Phi,\Psi}$. Set $T_i := i\widetilde{T}$ for $0 = 1, \dots, \ell$ where $\ell := \max\{k \in \mathbb{N} : k\widetilde{T} \le T\}$, and $T_{\ell+1} := \widetilde{T}$. According to Theorem 4.1, for $i = 0, \dots, \ell$, the restriction of u_n to $[T_i, T_{i+1}]$ is given by $u_n(t) = u_{i,n}(t - T_i)$ where $u_{i,n}$ is a solution in $C([0,\widetilde{T}], X)$ of

$$(\mathcal{P}_{i,n}) \begin{cases} \dfrac{\mathrm{d}u_{i,n}}{\mathrm{d}t}(t) + \partial\Phi_n(u_{i,n}(t)) + K \star \partial\Psi_n(u_{i,n})(t) = F_{i,n}(t) \text{ for a.e. } t \in (0,\widetilde{T}) \\ u_{i,n}(0) = u_{i-1,n}(\widetilde{T}) \in \mathrm{dom}(\partial\Phi_n), \end{cases}$$

with

$$F_{i,n}(t) := F_n(t + T_i) - \sum_{k=0}^{i-1} \int_{T_k}^{T_{k+1}} K(t + T_i - s)\partial\Psi_n(u_{k,n}(s - T_k))\,\mathrm{d}s,$$

and by convention, $u_{-1,n}(\widetilde{T}) = u_n^0$ and $\sum_{k=0}^{-1} = 0$. In particular $F_{0,n}(t) = F_n(t)$. We set

$$G_{i,n}(t) := F_{i,n}(t) - K \star \partial\Psi_n(u_{i,n})(t),$$

then $(\mathcal{P}_{i,n})$ may be written as

$$(\mathcal{P}_{i,n}) \begin{cases} \dfrac{\mathrm{d}u_{i,n}}{\mathrm{d}t}(t) + \partial\Phi_n(u_{i,n}(t)) = G_{i,n}(t) \text{ for a.e. } t \in (0,\widetilde{T}) \\ u_{i,n}(0) = u_{i-1,n}(\widetilde{T}) \in \mathrm{dom}(\partial\Phi_n). \end{cases}$$

Our strategy is the following: for each $i = 0, \dots, \ell$, we show that $u_{i,n}$ converges to some u_i in $C([0,\widetilde{T}], X)$; next we claim that u defined by $u(t) = u_i(t - T_i)$ for $t \in [T_i, T_{i+1}]$ solves $(\mathcal{P})$ and that $u_n \to u$ in $C([0,T], X)$. We proceed in this way to check the uniform estimates similar to those of Lemma 3.2 which require $\widetilde{T}$ small enough, i.e. satisfying $\widetilde{T}^{\frac{1}{2}} \|K\|_{L^2(0,T)} < \alpha_{\Phi,\Psi}$.

Step 1. Reasoning by induction for $i = 0, \dots, \ell$, we prove the following three assertions:

(a)
$$\sup_{n\in\mathbb{N}} \|\partial\Psi_n(u_{i,n})\|_{L^2(0,\tilde{T},X)} < +\infty, \tag{5.1}$$
$$\sup_{n\in\mathbb{N}} \left\|\frac{\mathrm{d}u_{i,n}}{\mathrm{d}t}\right\|_{L^2(0,\tilde{T},X)} < +\infty, \tag{5.2}$$
$$\sup_{n\in\mathbb{N}} \|u_{i,n}\|_{C(0,\tilde{T},X)} < +\infty, \tag{5.3}$$
$$\sup_{n\in\mathbb{N}} \|\partial\Psi_n(u_{i,n}(t))\|_X < +\infty \text{ for each } t \in]0,\widetilde{T}], \tag{5.4}$$
$$\sup_{n\in\mathbb{N}} \|\partial\Phi_n(u_{i,n}(t))\|_X < +\infty \text{ for each } t \in]0,\widetilde{T}]. \tag{5.5}$$

(b) There exists a subsequence of $(u_{i,n})_{n\in\mathbb{N}}$ which uniformly converges to some u_i in $C([0,\widetilde{T}],X)$.

(c) For $k = 0,\dots,i$, there exists $\xi_k \in L^2(0,\widetilde{T},X)$ with $\xi_k(t) \in \partial\Psi(u_k(t)) \cap X$ such that
$$\partial\Psi_n(u_{k,n}) \rightharpoonup \xi_k \text{ weakly in } L^2(0,\widetilde{T},X).$$

Step $i = 0$.

Proof of (a). According to the uniform bounds (2.1), (2.2), (STAB$_1$), (STAB$_2$), $\inf_X \Phi_n \geq 0$, $\inf_X \Psi_n \geq 0$, and finally to the existence of a right derivative of $u_{i,n}$ at each $t \in]0,\widetilde{T}]$ (cf. Theorem 4.4), assertion a) is obtained by reproducing the proof of (3.6)–(3.13) with F_n substitute for F, Φ_n for Φ, and Ψ_n for Ψ_λ (unlike (3.12) and (3.13), we cannot claim that (5.4) and (5.5) hold for $t = 0$ because of the dependance on n of $u_{n,0}(0) = u^0_n$). We only establish (5.1), (5.2), (5.4) to highlight the importance of condition (2.2) and to emphasize the need for hypothesis (STAB$_1$).

For a.e. $t \in (0,\widetilde{T})$, form the scalar product in X of $\partial\Psi_n(u_{0,n}(t))$ with the equation of the first formulation of $(\mathcal{P}_{i,n})$ and integrate over $(0,\widetilde{T})$. This yields

$$\int_0^{\tilde{T}} \frac{\mathrm{d}}{\mathrm{d}t}\Psi_n(u_{0,n}(t))\mathrm{d}t + \int_0^{\tilde{T}} \left\langle \partial\Phi_n(u_{0,n}(t)), \partial\Psi_n(u_{0,n}(t))\right\rangle \mathrm{d}t$$
$$+ \int_0^{\tilde{T}} \left\langle K \star \partial\Psi_n(u_{0,n})(t), \partial\Psi(u_{0,n}(t))\right\rangle \mathrm{d}t = \int_0^{\tilde{T}} \left\langle F_n(t), \partial\Psi_n(u_{0,n}(t))\right\rangle \mathrm{d}t. \tag{5.6}$$

An easy calculation gives

$$\|K \star \partial\Psi_n(u_{0,n})\|_{L^2(0,\tilde{T},X)} \leq \widetilde{T}^{\frac{1}{2}} \|K\|_{L^2(0,T)} \|\partial\Psi_n(u_{0,n})\|_{L^2(0,\tilde{T},X)}. \tag{5.7}$$

Combining (5.6), (5.7) and (2.2) we conclude that

$$\left[\alpha_{\Phi,\Psi} - \widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)}\right]\|\partial\Psi_n(u_{0,n})\|^2_{L^2(0,\tilde{T},X)} \leq \widetilde{T}\beta_{\Phi,\Psi} + \sup_{n\in\mathbb{N}}\Psi_n(u_{0,n}) + \sup_{n\in\mathbb{N}}\|F_n\|_{L^2(0,\tilde{T},X)}\|\partial\Psi_n(u_{0,n})\|_{L^2(0,\tilde{T},X)}.$$

We deduce (5.1) from (STAB_1), (STAB_2), provided that $\widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)} < \alpha_{\Phi,\Psi}$.

We move on to the proof of (5.2). For a.e. $t \in (0,\widetilde{T})$, form the scalar product in X of $\frac{\mathrm{d}u_{0,n}}{\mathrm{d}t}(t)$ with the equation of $(\mathcal{P}_{0,n})$ and integrate over $(0,\widetilde{T})$. This yields

$$\left\|\frac{\mathrm{d}u_{0,n}}{\mathrm{d}t}\right\|^2_{L^2(0,\tilde{T},X)} \leq \Phi(u_n^0) - \inf_X \Phi_n + \left(\|K\star\partial\Psi_n(u_{0,n})\|_{L^2(0,\tilde{T},X)} + \|F_n\|_{L^2(0,\tilde{T},X)}\right)\left\|\frac{\mathrm{d}u_{0,n}}{\mathrm{d}t}\right\|_{L^2(0,\tilde{T},X)}$$

and (5.2) follows from (5.7), (5.1), and (STAB_1).

For establishing (5.4) first observe that by reproducing the proof of (A.7) in Appendix A, we have

$$\sup_{n\in\mathbb{N}}\|K\star\partial\Psi_n(u_{0,n})\|_{W^{1,2}(0,\tilde{T},X)} < +\infty. \tag{5.8}$$

Next, from (5.8) and (STAB_1) we deduce that

$$\sup_{n\in\mathbb{N}}\left\|\frac{\mathrm{d}G_{0,n}}{\mathrm{d}t}\right\|_{L^1(0,\tilde{T},X)} < +\infty. \tag{5.9}$$

Hence, combining the inequalities

$$\left\|\frac{\mathrm{d}^+u_{0,n}}{\mathrm{d}t}(t)\right\|_X \leq \frac{1}{t}\int_0^t\left\|\frac{\mathrm{d}u_{0,n}}{\mathrm{d}t}(s)\right\|_X \mathrm{d}s + \int_0^t\left\|\frac{\mathrm{d}G_{0,n}}{\mathrm{d}t}(s)\right\|_X \mathrm{d}s$$

for all $t \in \,]0,\widetilde{T}]$ (see [2, Lemma 3.3]), with (5.2) and (5.9), we obtain that for each $t \in \,]0,\widetilde{T}]$,

$$\sup_{n\in\mathbb{N}}\left\|\frac{\mathrm{d}^+u_{0,n}}{\mathrm{d}t}(t)\right\|_X < +\infty. \tag{5.10}$$

Take the scalar product of the equation

$$\frac{\mathrm{d}^+u_{0,n}}{\mathrm{d}t}(t) + \partial\Phi_n(u_{0,n}(t)) = G_{0,n}(t)$$

with $\partial\Psi_n(u_{0,n}(t))$ for each $t \in \,]0,\widetilde{T}]$. This yields

$$\left\langle\frac{\mathrm{d}^+u_{0,n}}{\mathrm{d}t}(t), \partial\Psi(u_{0,n}(t))\right\rangle + \alpha_{\Phi,\Psi}\|\partial\Psi(u_{0,n}(t))\|^2_X \leq \left\langle G_{0,n}(t), \partial\Psi_n(u_{0,n}(t))\right\rangle + \beta_{\Phi,\Psi}$$

from which we deduce

$$\alpha_{\Phi,\Psi}\|\partial\Psi_n(u_{0,n}(t))\|_X^2 \le \left(\left\|\frac{\mathrm{d}^+u_{0,n}}{\mathrm{d}t}(t)\right\|_X + \|G_{0,n}(t)\|_X\right)\|\partial\Psi_n(u_{0,n}(t))\|_X + \beta_{\Phi,\Psi}$$

for all $t \in]0,\widetilde{T}]$. The claim follows from (5.10), and $\sup_{n\in\mathbb{N}} \|G_{0,n}(t)\|_X < +\infty$ which is obtained according to

$$\|G_{0,n}(t)\|_X \le \|K\|_{L^2(0,T)}\|\partial\Psi_n(u_{0,n})\|_{L^2(0,\tilde{T},X)} + \|F_n(t)\|_X$$

and (5.1) and (STAB$_1$).

Proof of (b). From (5.2) and (5.3) we infer that the sequence $(u_{0,n})_{n\in\mathbb{N}}$ is bounded and uniformly equicontinuous in $C([0,\widetilde{T}],X)$. Assertion (b) then follows from the Ascoli compactness theorem provided that for each fixed $t \in [0,T]$, we establish that the set $E_0(t) := \{u_{0,n}(t) : n \in \mathbb{N}\}$ is relatively compact in X. For $t = 0$ we have $E_0(0) = \{u_n^0 : n \in \mathbb{N}\}$ so that the claim follows directly from (STAB$_3$). For $t \in]0,\widetilde{T}]$, (2.1) yields

$$\begin{aligned}\|u_{0,n}(t)\|_V^2 &\le \frac{1}{\alpha_\Psi}\left\langle \partial\Psi_n(u_{0,n}(t)), u_{0,n}(t)\right\rangle \le \frac{1}{\alpha_\Psi}\|\partial\Psi_n(u_{0,n}(t))\|_X\|u_{0,n}(t)\|_X \\ &\le \frac{1}{\alpha_\Psi}\|\partial\Psi_n(u_{0,n}(t))\|_X\|u_{0,n}(t)\|_V\end{aligned}$$

and the claim follows from (5.4) and the compact embedding $V \hookrightarrow X$.

Proof of (c). We have to establish the existence of $\xi_0 \in L^2(0,\widetilde{T},X)$ with $\xi_0 \in \partial\Psi(u_0(t))$ such that $\partial\Psi_n(u_{0,n}) \rightharpoonup \xi_0$ in $L^2(0,\widetilde{T},X)$. From (5.1) and the compact embedding $L^2(0,\widetilde{T},X) \hookrightarrow L^2(0,\widetilde{T},V')$, we infer that there exist a subsequence of $(\partial\Psi_n(u_{0,n}))_{n\in\mathbb{N}}$ and $\xi_0 \in L^2(0,\widetilde{T},X)$ such that successively

$$\begin{aligned}\partial\Psi_n(u_{0,n}) &\rightharpoonup \xi_0 && \text{weakly in } L^2(0,\widetilde{T},X),\\ \partial\Psi_n(u_{0,n}) &\longrightarrow \xi_0 && \text{strongly in } L^2(0,\widetilde{T},V'),\\ \partial\Psi_n(u_{0,n}(t)) &\longrightarrow \xi_0(t) && \text{strongly in } V' \text{ for a.e. } t \in (0,\widetilde{T}),\\ u_{0,n}(t) &\rightharpoonup u_0(t) && \text{weakly in } V \text{ for each } t \in]0,\widetilde{T}]\end{aligned}$$

(the last convergence follows from (2.1), (5.4), and (b) to identify the weak limit). According to (STAB$_5$) and the implication

$$\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\longrightarrow} \Psi \Longrightarrow \partial\Psi_{n\lfloor V} \overset{G_{w,s}}{\longrightarrow} \partial\Psi$$

(cf. Theorem C.5), the two last convergences above yield that for a.e. $t \in (0,\widetilde{T})$, $u_0(t) \in \mathrm{dom}(\partial\Psi)$ and $\xi_0(t) \in \partial\Psi(u_0(t)) \cap X$.

Step $i > 1$ from steps $0, \dots, i-1$. Fix $i > 1$. We assume that assertions (a), (b) and (c) hold for all $k = 0, \dots i-1$ and we establish that they hold for i.

Proof of (a). We first claim that $F_{i,n}$ satisfies (STAB$_1$) for $t \in [0, \widetilde{T}]$. From (c) for $k = 0, \dots, i-1$, we easily deduce that

$$\sum_{k=0}^{i-1} \int_{T_k}^{T_{k+1}} K(\cdot + T_i - s)\partial\Psi_n(u_{k,n}(s - T_k))\,\mathrm{d}s \rightharpoonup \sum_{k=0}^{i-1} \int_{T_k}^{T_{k+1}} K(\cdot + T_i - s)\xi_k(s - T_k)\,\mathrm{d}s. \quad (5.11)$$

in $L^2(0, \widetilde{T}, X)$. Hence from (STAB$_1$) and (5.11), $F_{i,n} \rightharpoonup F_i$ in $L^2(0, \widetilde{T}, X)$. On the other hand, from (STAB$_1$) and (5.4) for $k = 0, \dots, i-1$, we infer that $\sup_{n\in\mathbb{N}} \|F_{i,n}(t)\|_X$ for all $t \in [0, \widetilde{T}]$. Finally from (5.1), (STAB$_1$) and

$$\frac{\mathrm{d}F_{i,n}}{\mathrm{d}t}(t) := \frac{\mathrm{d}F_n}{\mathrm{d}t}(t + T_i) - \sum_{k=0}^{i-1} \int_{T_k}^{T_{k+1}} K^{'}(t + T_i - s)\partial\Psi_n(u_{k,n}(s - T_k))\,\mathrm{d}s,$$

we deduce that

$$\sup_{n\in\mathbb{N}} \left\| \frac{\mathrm{d}F_{i,n}}{\mathrm{d}t} \right\|_{L^1(0,\tilde{T},X)} < +\infty,$$

which proves the claim. By repeating the arguments of the proof of (a) at $i = 0$ where $F_{i,n}$ is substitute for F_n, we obtain the estimates of (a) provided that $\sup_{n\in\mathbb{N}} \Phi_n(u_{i,n}(0)) < +\infty$, that is to say $\sup_{n\in\mathbb{N}} \Phi_n(u_{i-1,n}(\widetilde{T})) < +\infty$ (this condition replace (STAB$_2$)). For that, first note that from (b) at index $i - 1$

$$\sup_{n\in\mathbb{N}} \left\| u_{i-1,n}(\widetilde{T}) \right\|_X < +\infty. \quad (5.12)$$

Next, fix $v \in \mathrm{dom}(\Phi)$. From (STAB$_3$) there exists a sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n \to v$ strongly in X and $\Phi_n(v_n) \to \Phi(v)$. The thesis then follows, from the convexity inequality

$$\Phi_n(u_{i-1,n}(\widetilde{T})) \leq \Phi_n(v_n) + \left\langle \partial\Phi_n(u_{i-1,n}(\widetilde{T})), u_{i-1,n}(\widetilde{T}) - v_n \right\rangle$$

and (5.5), (5.12).

Proof of (b). The proof of (b) is exactly the one of (b) at $i = 0$, by establishing that $E_i(t) := \{u_{i,n}(t) : n \in \mathbb{N}\}$ is relatively compact in X. Observe that for $t = 0$, $E_i(0) = \left\{u_{i-1,n}(\widetilde{T}) : n \in \mathbb{N}\right\}$ so that the claim follows directly from (b) at index $i - 1$.

Proof of (c). The proof is exactly the one of Step $i = 0$ by using estimates obtained in (a).

Step 2. By using a method similar to that of the proof of Theorem 3.3, and from the convergences obtained in Step 1, we are going to prove that u_i defined in (b), Step 1, solves the Cauchy problem

$$(\mathcal{P}_i)\begin{cases}\dfrac{\mathrm{d}u_i}{\mathrm{d}t}(t)+\partial\Phi(u_i(t))+K\star\partial\Psi(u_i)(t)\ni F_i(t) \text{ for a.e. } t\in(0,\widetilde{T})\\ u_i(0)=u_{i-1}(\widetilde{T})\in\mathrm{dom}(\partial\Phi).\end{cases}$$

We will infer that the function u defined by $u(t)=u_i(t-T_i)$ for $t\in[T_i,T_{i+1}]$ converges toward u in $C([0,T],X)$ and, according to Theorem 4.1, solves $(\mathcal{P})$.

By using the Fenchel extremality condition, the equation of $(\mathcal{P}_{i,n})$ written with $G_{i,n}$ as second member, is equivalent to

$$\int_0^{\tilde{T}}\left[\Phi_n(u_{i,n}(t))+\Phi_n^*\left(G_{i,n}(t)-\frac{\mathrm{d}u_{i,n}}{\mathrm{d}t}(t)\right)+\left\langle\frac{\mathrm{d}u_{i,n}}{\mathrm{d}t}(t)-G_{i,n}(t),u_{i,n}(t)\right\rangle\right]\mathrm{d}t=0,$$

where we have denoted by Φ_n^* the Legendre–Fenchel conjugate of Φ_n. Equivalently

$$I_{\Phi_n}(u_{i,n})+I_{\Phi_n^*}\left(G_{i,n}-\frac{\mathrm{d}u_{i,n}}{\mathrm{d}t}\right)+\frac{1}{2}\left(\|u_{i,n}(\widetilde{T})\|^2-\|u_{i,n}(0)\|^2\right)$$
$$-\int_0^{\tilde{T}}\left\langle G_{i,n}(t),u_{i,n}(t)\right\rangle\mathrm{d}t=0 \quad (5.13)$$

where the integral functionals I_{Φ_n} and $I_{\Phi_n^*}$ are defined in $L^2(0,\widetilde{T},X)$ by

$$I_{\Phi_n}(v)=\int_0^{\tilde{T}}\Phi_n(v(t))\mathrm{d}t \quad\text{and}\quad I_{\Phi_n^*}(v)=\int_0^{\tilde{T}}\Phi_n^*(v(t))\mathrm{d}t.$$

From (STAB_4) and [2, Lemma 4.1] we have

$$I_{\Phi_n}\xrightarrow{M}I_{\Phi}. \quad (5.14)$$

On the other hand, combining $u_{i,n}(\widetilde{T})=u_{i,n}^0+\int_0^{\tilde{T}}\frac{\mathrm{d}u_{i,n}}{\mathrm{d}t}(t)\mathrm{d}t$ with $\frac{\mathrm{d}u_{i,n}}{\mathrm{d}t}\rightharpoonup\frac{\mathrm{d}u_i}{\mathrm{d}t}$ in $L^2(0,\widetilde{T},X)$ which is obtained from (5.2), we infer that

$$\liminf_{n\to+\infty}\|u_{i,n}(\widetilde{T})\|^2\geq\|u_i(\widetilde{T})\|^2. \quad (5.15)$$

Finally, from (STAB_1) and assertion (c) of Step 1

$$G_{i,n}\rightharpoonup G_i:=F_i-K\star\xi_i \text{ weakly in } L^2(0,\widetilde{T},X) \quad (5.16)$$

where

$$F_i(t)=F(t+T_i)-\sum_{k=0}^{i-1}\int_{T_k}^{T_{k+1}}K(\,\cdot+T_i-s)\xi_k(s-T_k)\,\mathrm{d}s.$$

Hence, by passing to the liminf in (5.13), from (5.14), (5.15), Step 1 (b) and (5.16), we infer that

$$\int_0^T \left[\Phi(u_i(t)) + \Phi^*(G_i(t) - \frac{\mathrm{d}u_i}{\mathrm{d}t}(t))\right]\mathrm{d}t + \frac{1}{2}(\|u_i(T)\|^2 - \|u_i^0\|^2) - \int_0^T \langle G_i(t), u_i(t)\rangle\,\mathrm{d}t \leq 0$$

or equivalently,

$$\int_0^T \left[\Phi(u_i(t)) + \Phi^*(G_i(t) - \frac{\mathrm{d}u_i}{\mathrm{d}t}(t)) + \left\langle \frac{\mathrm{d}u_i}{\mathrm{d}t}(t) - G_i(t), u_i(t)\right\rangle\right]\mathrm{d}t \leq 0, \tag{5.17}$$

from which we conclude that

$$\frac{\mathrm{d}u_i}{\mathrm{d}t}(t) + \partial\Phi(u_i(t)) \ni G_i(t) \quad \text{for a.e. } t \in (0, T).$$

The initial condition $u_i(0) = u_{i-1}(\widetilde{T})$ is obtained from

$$u_i(0) = \lim_{n\to+\infty} u_{i,n}(0) = \lim_{n\to+\infty} u_{i-1,n}(\widetilde{T}) = u_{i-1}(\widetilde{T}).$$

Finally we claim that $u_{i-1}(\widetilde{T}) \in \mathrm{dom}(\partial\Phi)$. It comes from

$$u_{i-1,n}(\widetilde{T}) \in \mathrm{dom}(\partial\Phi_n), \quad \lim_{n\to+\infty} u_{i-1,n}(\widetilde{T}) = u_{i-1}(\widetilde{T}) \text{ strongly in } X,$$

(STAB_4) and Theorem C.5. This completes the proof. □

Remark 5.2. Let us strengthen (STAB_5) by:

(STAB_5') there exists $\Psi : X \to]-\infty, +\infty]$ lsc convex proper, such that $\mathrm{dom}(\Psi) = V$ and $\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\to} \Psi_{\lfloor V}$.

Then, with the notation above, we can assert that $\partial\Psi_{\lfloor V}(u(s)) \cap X = \partial\Psi$. The limit problem then becomes

$$(\mathcal{P})\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial\Phi(u(t)) + \displaystyle\int_0^t K(t-s)\partial\Psi(u(s))\,\mathrm{d}s \ni F(t) \text{ for a.e. } t \in (0,T)\\ u(0) = u^0,\ u^0 \in \mathrm{dom}(\partial\Phi).\end{cases}$$

Moreover, from Proposition 2.3, $\mathrm{dom}(\partial\Phi) \subset \mathrm{dom}(\partial\Psi)$ and Φ and Ψ fulfill conditions (2.1), (2.2). Therefore, under condition (STAB_5'), Theorem 5.1 may be considered as a stability result, although $\partial\Phi$ and $\partial\Psi$ are not univalent in general.

5.2. The case $X = L^2(\Omega)$

In this section, Ω is a bounded domain of $\mathbb{R}^N$, $X = L^2(\Omega)$, and $V = H_0^1(\Omega)$. We keep the same conditions on Φ_n, Ψ_n and u_n^0 but we further specify the structure of the source F_n. Given a positive integer l, $r_n \in L^\infty((0,T) \times \mathbb{R}^N, \mathbb{R}^l) \cap W^{1,1}(0,T, L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^l))$, $q_n \in L^2(0,T, L^2_{\mathrm{loc}}(\mathbb{R}^N)) \cap W^{1,1}(0,T, L^2_{\mathrm{loc}}(\mathbb{R}^N))$, and $g_n : \mathbb{R} \to \mathbb{R}^l$ a uniformly bounded and Lipschitz continuous function, we consider the reaction functional F_n defined for all $v \in L^2(\Omega)$ and all $x \in \Omega$ by

$$F_n(t,u)(x) = r_n(t,x) \cdot g_n(u(x)) + q_n(t,x).$$

We assume that

$$\begin{cases} \sup\limits_{n\in\mathbb{N}} \displaystyle\int_0^T \left\| \frac{\mathrm{d}r_n}{\mathrm{d}t} \right\|_{L^2(\Omega,\mathbb{R}^l)} \mathrm{d}t < +\infty, \\ \sup\limits_{n\in\mathbb{N}} \displaystyle\int_0^T \left\| \frac{\mathrm{d}q_n}{\mathrm{d}t} \right\|_{L^2(\Omega)} \mathrm{d}t < +\infty. \end{cases} \tag{5.18}$$

Denote by L_g the uniform Lipschitz constant of the functions g_n and by $M_g = \sup_{r\in\mathbb{R}} |g_n(r)|$ their uniform norm. Then, as noticed in Section 2.2, F_n fulfills (C_1), (C_2), (C_3) with $L_n = \|r_n\|_{L^\infty((0,T)\times\mathbb{R}^N,\mathbb{R}^l)} L_g$ and $\Theta_n(\tau) = M_g \|\frac{\mathrm{d}r_n}{\mathrm{d}t}(\tau,\cdot)\|_{L^2(\Omega,\mathbb{R}^l)} + \|\frac{\mathrm{d}q_n}{\mathrm{d}t}(\tau,\cdot)\|_{L^2(\Omega)}$. Theorem 5.3 below is a concrete version of Theorem 5.1 where, in addition, $F_n(t,u_n(t))$ is substituted for $F_n(t)$.

Theorem 5.3. *In addition to* (STAB_2)–(STAB_5)*, assume that*

(i) $\sup_{n\in\mathbb{N}} \|r_n\|_{L^\infty([0,T]\times\mathbb{R}^N,\mathbb{R}^l)} < +\infty$, *and* $r_n \rightharpoonup r$ *for the* $\sigma(L^\infty(0,T,L^2(\Omega,\mathbb{R}^l)), L^1(0,T,L^2(\Omega,\mathbb{R}^l)))$ *topology;*

(ii) $g_n \to g$ *pointwise in* $\mathbb{R}^l$*;*

(iii) *for all* $t \in [0,T]$, $\sup_{n\in\mathbb{N}} \|q_n(t,\cdot)\|_{L^2(\Omega)} < +\infty$, *and* $q_n \rightharpoonup q$ *weakly in* $L^2(0,T,L^2(\Omega))$.

Then any particular sequence of solutions u_n *of* ($\mathcal{P}_n$) *admits a subsequence which converges to* u *in* $C([0,T],X)$*, solution of*

$$(\mathcal{P}) \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) + \partial\Phi u(t) + \displaystyle\int_0^t K(t-s)\big[\partial(\Psi)(u(s)) \cap L^2(\Omega)\big]\,\mathrm{d}s \ni F(t,u(t)) \\ \hfill \text{for a.e. } t \in (0,T) \\ u(0) = u^0, \ u^0 \in \mathrm{dom}(\partial\Phi), \end{cases}$$

where $F(t,v)(x) = r(t,x)\cdot g(v(x)) + q(t,x)$ *for all* $(t,x) \in [0,T]\times\Omega$ *and all* $v \in L^2(\Omega)$.

Proof. We use the notation of the proof of Theorem 5.1. Clearly

$$\sup_{n\in\mathbb{N}} \|F_n(t, u_n(t))\|_{L^2(\Omega)} < +\infty$$

for all $t \in [0, T]$. On the other hand from (C_3) and the uniform bounds (5.18) we easily deduce that

$$\sup_{n\in\mathbb{N}} \left\| \frac{\mathrm{d}F_n}{\mathrm{d}t} \right\|_{L^1(0,T,L^2(\Omega))} < +\infty.$$

Therefore Step 1 of the proof of Theorem 5.1 is still valid. The rest of the proof mimics the one of Step 2. We only have to establish that $F_n(\cdot, u_{i,n}(\cdot)) \rightharpoonup F(\cdot, u_i(\cdot))$ in $L^2(0, \widetilde{T}, L^2(\Omega))$. This convergence is a straightforward consequence of the weak convergences $r_n \rightharpoonup r$, $q_n \rightharpoonup q$ and the pointwise convergence $g_n \longrightarrow g$ together with the uniform bound of g_n (for a complete proof refer to [2]). □

6. Stochastic homogenization of integrodifferential Cauchy problems

In this section Ω is a C^1-domain of $\mathbb{R}^N$, $X = L^2(\Omega)$ and $V = H^1_0(\Omega)$. For any Borel measurable function $W : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that for a.e. $x \in \mathbb{R}^N$, $\xi \mapsto W(x, \xi)$ is convex, when $\xi \mapsto W(x, \xi)$ is not Gâteaux differentiable we adopt the following convention: for any $v \in H^1(\Omega)$, we write indifferently $\mathrm{div}\partial_\xi W(\cdot, \nabla v)$ for the set

$$\left\{ \mathrm{div}\xi^* : \xi^* \in \partial_\xi W(\cdot, \nabla v) \right\}$$

or any element of this set.

6.1. Stochastic homogenization of a random problem modeled from a Fick's law with delay

To model the spatial environment, we consider a general probability space $(\Sigma, \mathcal{A}, \mathbb{P})$ equipped with a group $(T_z)_{z\in\mathbb{Z}^N}$ of $\mathbb{P}$-preserving transformations on Σ. We denote by $\mathcal{I}$ the σ-algebra of invariant sets of $\mathcal{A}$ by the group $(T_z)_{z\in\mathbb{Z}^N}$ and, for every $\mathbf{h}$ in the space $L^1_{\mathbb{P}}(\Sigma)$ of $\mathbb{P}$-integrable functions, by $\mathbb{E}^{\mathcal{I}}\mathbf{h}$ the conditional expectation of $\mathbf{h}$ with respect to $\mathcal{I}$. If $\mathcal{I}$ is made up of sets with probability 0 or 1, the discrete dynamical system $(\Sigma, \mathcal{A}, \mathbb{P}, (T_z)_{z\in\mathbb{Z}^N})$ is said to be ergodic. Under this condition, we have $\mathbb{E}^{\mathcal{I}}\mathbf{h} = \mathbb{E}\mathbf{h}$ where $\mathbb{E}\mathbf{h} = \int_\Sigma \mathbf{h}(\omega)\mathrm{d}\mathbb{P}(\omega)$ is the mathematical expectation of $\mathbf{h}$.

Let Ω be C^1-regular domain of $\mathbb{R}^N$ and denote by $u(\omega, t, x)$ a scalar state variable of a physical, biological or ecological model at position x and time t, subjected to an alea $\omega \in \Sigma$; according to the cases $u(\omega, \cdot, \cdot)$ is a concentration, or a density. We assume that for the model considered, the diffusion flux in Ω related to $u(\omega, t, x)$ has two contributions:

- the Fickian flux which locally has at each time t the direction of the negative spatial gradient of the state variable, given by $J_F(\omega,t,x) = -D(\omega,x)\nabla u(\omega,t,x)$,
- the non Fickian flux which locally has the direction of the negative spatial gradient of the state variable at some past time $\tau > 0$, given by $J_{NF}(\omega,t,x) = -D(\omega,x)\nabla u(\omega,t-\tau,x)$.

For example, in population dynamics, the non Fickian flux may account for maturation period, resource regeneration time, mating processes, or incubation period, which is superimposed on the first flow at each time t. The coefficient D accounts for the rate of movement in the heterogeneous spatial environment modeled according to $(\Sigma,\mathcal{A},\mathbb{P},(T_z)_{z\in\mathbb{Z}^N})$ in $\mathbb{R}^N$. From the mass conservation principle, for a given source $F(\omega,t,x)$, the variable u satisfies the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(\omega,t) + \mathrm{div}(J_F(\omega,t,x)) + \mathrm{div}(J_{NF}(\omega,t,x)) = F(\omega,t,x). \tag{6.1}$$

Assume τ small. Then we can express $\mathrm{div}(J_{NF}(\omega,t,x))$ as a divergence of the gradient field distributed following a suitable time kernel. Indeed, from $J_{NF}(\omega,t+\tau,x) = -D(\omega,x)\nabla u(t)$, using the first order time approximation, we have

$$J_{NF}(\omega,t+\tau,x) \sim J_{NF}(\omega,t,x) + \tau\frac{\partial J_{NF}}{\partial t}(\omega,t,x)$$

so that J_{NF} satisfies the first order differential equation

$$\tau\frac{\partial J_{NF}}{\partial t}(\omega,t,x) + J_{NF}(\omega,t,x) = -D(\omega,x)\nabla u(\omega,t,x).$$

By an elementary computation using the method of variation of constants, and assuming that $J_{NF}(\omega,0,x) = 0$, we see that J_{NF} is given by

$$J_{NF}(\omega,t,x) = -\frac{1}{\tau}\int_0^t \exp\Big(\frac{s-t}{\tau}\Big) D(\omega,x)\nabla u(\omega,s,x)\,\mathrm{d}s.$$

Therefore, (6.1) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t}(\omega,t,\cdot) - \mathrm{div}\big(D(\omega,\cdot)\nabla u(\omega,t,\cdot)\big) - \mathrm{div}\bigg(\frac{1}{\tau}\int_0^t \exp\Big(\frac{s-t}{\tau}\Big) D(\omega,\cdot)\nabla u(\omega,s,\cdot)\,\mathrm{d}s\bigg) = F(\omega,t,\cdot)$$

and can be written as

$$\frac{\mathrm{d}u}{\mathrm{d}t}(\omega,t,\cdot) - \mathrm{div}\big(D(\omega,\cdot)\nabla u(\omega,t,\cdot)\big) - \frac{1}{\tau}\int_0^t \exp\Big(-\frac{t-s}{\tau}\Big)\mathrm{div}\big(D(\omega,\cdot)\nabla u(\omega,s,\cdot)\big)\,\mathrm{d}s = F(\omega,t,\cdot),$$

which is an integrodifferential diffusion equation as treated in Section 3, with the kernel K defined by $K(t) = \frac{1}{\tau}\exp(-\frac{t}{\tau})$. To take into account the size of order ε of the spatial heterogeneities, the last integrodifferential diffusion equation becomes

$$\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}(\omega,t,\cdot) - \operatorname{div}\left(D\left(\omega,\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon(\omega,t,\cdot)\right) - \operatorname{div}\left(\frac{1}{\tau}\int_0^t \exp\left(\frac{s-t}{\tau}\right) D\left(\omega,\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon(\omega,s,\cdot)\mathrm{d}s\right) = F_\varepsilon(\omega,t,\cdot)$$

We are therefore led to consider the following more general problem, written in a mathematically rigorous formulation as follows: for $\mathbb{P}$-a.e. $\omega \in \Sigma$, the function $u_\varepsilon(\omega,\cdot) \in L^2(0,T,L^2(\Omega))$ solves

$$(\mathcal{P}_\varepsilon(\omega))\begin{cases}\dfrac{\mathrm{d}u_\varepsilon(\omega)}{\mathrm{d}t}(t) + \partial\Phi_\varepsilon(\omega,u_\varepsilon(\omega,t)) + \displaystyle\int_0^t K(t-s)\partial\Psi_\varepsilon(\omega,u_\varepsilon(\omega,s))\,\mathrm{d}s \\ \qquad\qquad = F_\varepsilon(\omega,t,u_\varepsilon(\omega,t)) \text{ for a.e. } t \in (0,T) \\ u_\varepsilon(\omega,0) = u_\varepsilon^0(\omega),\ u_\varepsilon^0(\omega) \in \operatorname{dom}(\partial\Phi_\varepsilon(\omega)).\end{cases}$$

The kernel K is given as in Section 2. For given $a > 0$ and $b \geq 0$, the functionals $\Phi_\varepsilon, \Psi_\varepsilon : L^2(\Omega) \to]-\infty,+\infty]$ are defined by

$$\Phi_\varepsilon(\omega,u) = \begin{cases} a\displaystyle\int_\Omega D\Big(\omega,\frac{x}{\varepsilon}\Big)\nabla u\cdot\nabla u\,\mathrm{d}x + \frac{b}{2}\int_\Omega u^2\mathrm{d}x & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise.}\end{cases}$$

$$\Psi_\varepsilon(\omega,u) = \begin{cases} \displaystyle\int_\Omega D\Big(\omega,\frac{x}{\varepsilon}\Big)\nabla u\cdot\nabla u\,\mathrm{d}x & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise,}\end{cases}$$

where the random matrix valued map

$$D = (d_{ij})_{i,j=1\dots N} : \Sigma\times\mathbb{R}^N \to \mathbb{M}^N$$

is $(\mathcal{A}\otimes\mathcal{B}(\mathbb{R}^N),\mathcal{B}(\mathbb{M}^N))$-measurable and covariant with respect to the group $(T_z)_{z\in\mathbb{Z}^N}$, that is

$$D(T_z\omega,x) = D(\omega,x+z)$$

for all $\omega \in \Sigma$ and all $x \in \mathbb{R}^N$. We also assume that there exist $\alpha > 0$ and $\beta > 0$ such that $\alpha|\xi|^2 \leq \sum_{i,j=1}^N d_{ij}(\omega,x)\xi_i\xi_j \leq \beta|\xi|^2$ for all $\omega \in \Sigma$, all $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$.

The random reaction functional F_ε is structured as follows: for all $u \in L^2(\Omega)$, all $t \in [0,T]$, and all $x \in \Omega$,

$$F_\varepsilon(\omega,t,u)(x) = r\left(\omega,t,\frac{x}{\varepsilon}\right)\cdot g(u(x)) + q\left(\omega,t,\frac{x}{\varepsilon}\right)$$

where

- $g : \mathbb{R} \to \mathbb{R}^l$ is a bounded L_g-Lipschitz continuous function;
- $r : \Sigma \times [0,T] \times \mathbb{R}^N \to \mathbb{R}^l$ is $(\mathcal{A} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^N), \mathcal{B}(\mathbb{R}^l))$-measurable; r is covariant with respect to the group $(T_z)_{z\in\mathbb{Z}^N}$, i.e. $r(T_z\omega,\cdot,\cdot) = r(\omega,\cdot,\cdot+z)$ for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and for all $z \in \mathbb{Z}^N$;
  for every $\omega \in \Sigma$, $r(\omega,\cdot,\cdot) \in L^\infty([0,T]\times\mathbb{R}^N,\mathbb{R}^l) \cap W^{1,1}(0,T,L^2_{\text{loc}}(\mathbb{R}^N,\mathbb{R}^l))$;
  for all bounded Borel sets B of $\mathbb{R}^N$, the real valued functions
  $$\omega \longmapsto \|r(\omega,t,\cdot)\|_{L^2(B,\mathbb{R}^l)} \text{ for all } t \in [0,T],$$
  $$\omega \longmapsto \int_0^T \left\| \frac{\mathrm{d}r}{\mathrm{d}t}(\omega,\tau,\cdot) \right\|_{L^2(B,\mathbb{R}^l)} \mathrm{d}\tau$$
  belong to $L_{\mathbb{P}}(\Sigma)$;
- $q : \Sigma \times [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is $(\mathcal{A} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^N), \mathcal{B}(\mathbb{R}))$-measurable; q is covariant with respect to the group $(T_z)_{z\in\mathbb{Z}^N}$, i.e. $q(T_z\omega,\cdot,\cdot) = q(\omega,\cdot,\cdot+z)$ for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and all $z \in \mathbb{Z}^N$;
  for all $\omega \in \Sigma$, $t \mapsto q(\omega,t,\cdot) \in W^{1,2}(0,T,L^2_{\text{loc}}(\mathbb{R}^N))$;
  for all bounded Borel sets B of $\mathbb{R}^N$, the real valued functions
  $$\omega \longmapsto \|q(\omega,t,\cdot)\|^2_{L^2(B)} \text{ for all } t \in [0,T], \tag{6.2}$$
  $$\omega \longmapsto \int_0^T \left\| \frac{\mathrm{d}q}{\mathrm{d}t}(\omega,\tau,\cdot) \right\|^2_{L^2(B)} \mathrm{d}\tau \tag{6.3}$$
  belong to $L_{\mathbb{P}}(\Sigma)$.

Taking the expression of each two subdifferentials $\partial\Phi_\varepsilon(\omega,\cdot)$ and $\partial\Psi_\varepsilon(\omega,\cdot)$ into account, the problem $(\mathcal{P}_\varepsilon(\omega))$ can be written as

$$(\mathcal{P}_\varepsilon(\omega)) \begin{cases} \dfrac{\mathrm{d}u_\varepsilon(\omega,\cdot)}{\mathrm{d}t}(t) - a\,\mathrm{div}\Big(D\Big(\omega,\dfrac{\cdot}{\varepsilon}\Big)\nabla u_\varepsilon(\omega,t)\Big) + b\,u_\varepsilon(\omega,t) \\ \quad - \displaystyle\int_0^t K(t-s)\mathrm{div}\Big(D\Big(\omega,\dfrac{\cdot}{\varepsilon}\Big)\nabla u_\varepsilon(\omega,s)\Big)\,\mathrm{d}s = F_\varepsilon(\omega,t,u_\varepsilon(\omega,t)) \\ \hfill \text{for a.e. } t \in (0,T) \\ u_\varepsilon(\omega,0) = u^0_\varepsilon(\omega),\ u^0_\varepsilon(\omega) \in \mathrm{dom}(\partial\Phi_\varepsilon(\omega)). \end{cases}$$

where

$$\mathrm{dom}(\partial\Phi_\varepsilon(\omega,\cdot)) = \mathrm{dom}(\partial\Psi_\varepsilon(\omega,\cdot)) = \left\{v \in H^1_0(\Omega) : \mathrm{div}(D(\omega,\cdot)\nabla v) \in L^2(\Omega)\right\}.$$

Condition (2.1) is clearly uniformly satisfied: take $\alpha_{\Psi_\varepsilon} = \alpha$. According to Examples 2.1, condition (2.2) is uniformly satisfied. Moreover, since $\Psi_\varepsilon(\omega,\cdot)$ is quadratic, from Proposition 4.2 $(\mathcal{P}_\varepsilon(\omega))$ admits a unique solution $u_\varepsilon(\omega,\cdot)$.

For all $\omega \in \Sigma$ and all $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$ set

$$W(\omega, x, \xi) := D(\omega, x)\xi \cdot \xi$$

and define $W_{\rm hom}$ for $\mathbb{P}$-a.e. $\omega \in \Sigma$ by

$$\begin{aligned} W_{\rm hom}(\omega, \xi) &= \lim_{n\to+\infty} \inf \left\{ \frac{1}{n^N} \int_{nY} W(\omega, y, \xi + \nabla u(y))\, {\rm d}y : u \in H_0^1(nY) \right\} \\ &= \inf_{n\in\mathbb{N}^*} \mathbb{E}^{\mathcal{I}} \inf \left\{ \frac{1}{n^N} \int_{nY} W(\omega, y, \xi + \nabla u(y))\, {\rm d}y : u \in H_0^1(nY) \right\}. \end{aligned}$$

It is well known that this limit exists for $\mathbb{P}$-a.e. $\omega \in \Sigma$ and is given by the formula above; for a proof, refer to [6, Proposition 12.4.3] and references therein. Note that if $(\Sigma, \mathcal{A}, \mathbb{P}, (T_z)_{z\in\mathbb{Z}^N})$ is ergodic, then $W_{\rm hom}$ is deterministic and given for $\mathbb{P}$-a.e. $\omega \in \Sigma$ by

$$\begin{aligned} W_{\rm hom}(\xi) &= \lim_{n\to+\infty} \inf \left\{ \frac{1}{n^N} \int_{nY} W(\omega, y, \xi + \nabla u(y))\, {\rm d}y : u \in H_0^1(nY) \right\} \\ &= \inf_{n\in\mathbb{N}^*} \mathbb{E} \inf \left\{ \frac{1}{n^N} \int_{nY} W(\cdot, y, \xi + \nabla u(y))\, {\rm d}y : u \in H_0^1(nY) \right\}. \end{aligned}$$

As a consequence of Theorem 5.3 we obtain

Corollary 6.1. *Assume that for* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$

(HOM$_1$) $\sup_{\varepsilon>0} \Phi_\varepsilon(u_\varepsilon^0(\omega)) < +\infty$*;*

(HOM$_2$) $u_\varepsilon^0(\omega) \to u^0(\omega)$ *strongly in* $L^2(\Omega)$*.*

Then for $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, the solution* $u_\varepsilon(\omega, \cdot)$ *of* $(\mathcal{P}_\varepsilon(\omega))$ *converges to* $u(\omega, \cdot)$ *in* $C([0, T], L^2(\Omega))$*, solution of the homogenized problem*

$$(\mathcal{P}(\omega)) \begin{cases} \dfrac{{\rm d}u(\omega)}{{\rm d}t}(t) - a\, {\rm div}(D_{\rm hom}(\omega)\nabla u(\omega, t)) + b\, u(\omega, t) \\ \quad - \displaystyle\int_0^t K(t-s)\, {\rm div}(D_{\rm hom}(\omega)\nabla u(\omega, s))\, {\rm d}s = F(\omega, t, u(\omega, t)) \\ \hfill \text{for a.e. } t \in (0, T) \\ u(\omega, 0) = u^0(\omega),\ u^0(\omega) \in {\rm dom}(\partial\Phi_{\rm hom}(\omega)) \end{cases}$$

with $D_{\rm hom}(\omega) = ((d_{\rm hom})_{ij}(\omega))_{i,j=1,\dots,N}$*,*

$$\begin{cases} (d_{\rm hom})_{ij}(\omega) = \dfrac{1}{2}(W_{\rm hom}(\omega, e_i + e_j) + W_{\rm hom}(\omega, e_i - e_j)), \\ {\rm dom}(\partial\Phi_{\rm hom}(\omega)) = H_0^1(\Omega) \cap H^2(\Omega), \end{cases}$$

where $(e_i)_{i=1,\dots,N}$ *is the canonical basis of* $\mathbb{R}^N$*. The homogenized reaction functional is given for every* $u \in L^2(\Omega)$*,* $\mathbb{P}$*-a.e.* $\omega \in \Sigma$*, and all* $(t,x) \in [0,T] \times \mathbb{R}^N$ *by*

$$\begin{cases} F_{\text{hom}}(\omega,t,u)(x) = r_{\text{hom}}(\omega,t) \cdot g(u(x)) + q_{\text{hom}}(\omega,t), \\ r_{\text{hom}}(\omega,t) = \mathbb{E}^{\mathcal{I}}\left(\int_{(0,1)^N} r(\omega,t,y)\,\mathrm{d}y\right), \\ q_{\text{hom}}(\omega,t) = \mathbb{E}^{\mathcal{I}}\left(\int_{(0,1)^N} q(\omega,t,y)\,\mathrm{d}y\right). \end{cases}$$

Proof. Firstly, by using arguments from ergodic theory of additive processes, we obtain that for $\mathbb{P}$-a.e. $\omega \in \Sigma$,

$$r_\varepsilon(\omega,\cdot,\cdot) \rightharpoonup r_{\text{hom}}(\omega,\cdot)$$

for the $\sigma(L^\infty(0,T,L^2(\Omega,\mathbb{R}^l)), L^1(0,T,L^2(\Omega,\mathbb{R}^l)))$ topology,

$$q_\varepsilon(\omega,\cdot,\cdot) \rightharpoonup q_{\text{hom}}(\omega,\cdot)$$

weakly in $L^2(0,T,L^2(\Omega))$, and

$$\sup_{\varepsilon>0}\left\| q\left(\omega,t,\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} < +\infty$$

for all $t \in [0,T]$. For a proof refer to [2, Lemma 5.2] and [2, proof of Theorem 5.1].

It remains to establish (STAB_4) and (STAB_5') of Remark 5.2, i.e. that for $\mathbb{P}$-a.e. $\omega \in \Sigma$: $\Phi_\varepsilon(\omega) \xrightarrow{M} \Phi(\omega,\cdot)$ and $\Psi_\varepsilon(\omega,\cdot)_{\lfloor H_0^1(\Omega)} \xrightarrow{\Gamma_{w-H_0^1}} \Psi(\omega,\cdot)_{\lfloor H_0^1(\Omega)}$ where

$$\Phi(\omega,u) = \begin{cases} a\displaystyle\int_\Omega D_{\text{hom}}(\omega)\nabla u \cdot \nabla u\,\mathrm{d}x + \frac{b}{2}\int_\Omega u^2\mathrm{d}x & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

$$\Psi(\omega,u) = \begin{cases} \displaystyle\int_\Omega D_{\text{hom}}(\omega)\nabla u \cdot \nabla u\,\mathrm{d}x & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Observe that the Γ-convergence of $\Psi_\varepsilon(\omega,\cdot)$ to $\Psi(\omega,\cdot)$ when $L^2(\Omega)$ is equipped with its strong topology yields the Γ-convergence of $\Psi_\varepsilon(\omega,\cdot)_{\lfloor H_0^1(\Omega)}$ to $\Psi(\omega,\cdot)_{\lfloor H_0^1(\Omega)}$ when $H_0^1(\Omega)$ is equipped with its weak topology. This property is a direct consequence of the uniform coercivity (see Proposition D.2):

$$\Psi_\varepsilon(\omega,u) \geq \alpha a \int_\Omega |u(x)|^2\mathrm{d}x, \quad \text{for all } u \in H_0^1(\Omega).$$

Noticing that $\Phi_\varepsilon(\omega,\cdot)$ is a continuous perturbation of $a\Psi_\varepsilon(\omega,\cdot)$ by $\frac{b}{2}\|\cdot\|^2_{L^2(\Omega)}$, these two convergences are straightforward consequences of [6, Theorem 12.1.1 (ii)] and [6, Theorem 12.4.7]. Finally, it is easily seen that the matrix $D_{\text{hom}}(\omega,\cdot)$ satisfies the bounds

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} (d_{\text{hom}})_{ij}(\omega)\xi_i\xi_j \leq \beta|\xi|^2$$

for $\mathbb{P}$-a.e. $\omega \in \Sigma$, and all $\xi \in \mathbb{R}^N$. Hence $\operatorname{dom}(\partial\Phi(\omega)) = H_0^1(\Omega)\cap H^2(\Omega)$. This completes the proof. □

6.2. Stochastic homogenization of nonlinear integrodifferential reaction-diffusion equations in one dimension space in the setting of a Poisson point process

Denote by $\mathcal{M}$ the set of all countable and locally finite sums of Dirac measures in $\mathbb{R}$, equipped with the σ-algebra generated by all the evaluation maps $\mathcal{E}_B : m \mapsto m(B)$ from $\mathcal{M}$ into $\mathbb{N} \cup \{+\infty\}$ when B belongs to $\mathcal{B}(\mathbb{R})$. Then, given $\theta > 0$, there exist a subset Σ of locally finite sequences $(\omega_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$, a probability space $(\Sigma, \mathcal{A}, \mathbb{P}_\theta)$ and a point process, called Poisson point process, $\mathcal{N} : \omega \mapsto \mathcal{N}(\omega,\cdot)$ from Σ into $\mathcal{M}$ satisfying

(i) $\mathcal{N}(\omega,\cdot) = \sum_{i\in\mathbb{N}} \delta_{\omega_i}$;

(ii) for every finite and pairwise disjoint family $(B_i)_{i\in I}$ of $\mathcal{B}(\mathbb{R})$, the random variables $(\mathcal{N}(\cdot, B_i))_{i\in I}$ are independent ;

(iii) for every bounded Borel set B and every $k \in \mathbb{N}$

$$\mathbb{P}_\theta([\mathcal{N}(\cdot, B) = k]) = \theta^k \mathcal{L}(B)^k \,\frac{\exp(-\theta\mathcal{L}(B))}{k!}.$$

We denote by $\mathbb{E}_\theta$ the expectation operator with respect to the probability $\mathbb{P}_\theta$. Note that for every bounded Borel set B in $\mathbb{R}$, we have $\mathcal{N}(\omega, B) = \#(\Sigma \cap B)$, and that $\mathbb{E}_\theta(\mathcal{N}(\cdot, B)) = \theta\mathcal{L}(B)$. We define the group $(T_z)_{z\in\mathbb{Z}^N}$ of $\mathbb{P}_\theta$-preserving transformation on $(\Sigma, \mathcal{A}, \mathbb{P}_\theta)$, by $T_z\omega = \omega - z$. From (ii), we can easily show that $(\Sigma, \mathcal{A}, \mathbb{P}_\theta, (T_z)_{z\in\mathbb{Z}})$ is ergodic, i.e. the σ-algebra of invariant sets of $\mathcal{A}$ is made up of sets with $\mathbb{P}_\theta$-measure 0 or 1. In the problem below, we use the dynamical system $(\Sigma, \mathcal{A}, \mathbb{P}_\theta, (T_z)_{z\in\mathbb{Z}^N})$ to describe the heterogeneous spatial environment.

Let Ω be an open bounded interval of $\mathbb{R}$. Let $\sigma^\pm \in C^1(\mathbb{R})$ be two scalar functions, and $a^\pm$ two positive real numbers satisfying

$$a^\pm \leq (\sigma^\pm)' \tag{6.4}$$

and set for all $\xi \in \mathbb{R}$

$$W^\pm(\xi) = \int_0^\xi \sigma^\pm(s)\,\mathrm{d}s.$$

We assume that there exists $(\alpha, \beta) \in \mathbb{R}_+^*$ such that $\alpha\xi^2 \leq W^{\pm}(\xi) \leq \beta(1+\xi^2)$. Such a condition is fulfilled by assuming suitable conditions on $\sigma^{\pm}$, as for example a growth condition of order 1. Given $R > 0$, we define the random density W by

$$W(\omega, x, \xi) = \begin{cases} W^-(\xi) & \text{if } x \in \bigcup_{i\in\mathbb{N}} B_R(\omega_i), \\ W^+(\xi) & \text{otherwise} \end{cases}$$

and the random integral functional $\Phi_\varepsilon : L^2(\Omega) \to]-\infty, +\infty]$ by

$$\Phi_\varepsilon(\omega, u) = \begin{cases} \displaystyle\int_\Omega W\left(\omega, \frac{x}{\varepsilon}, \frac{\mathrm{d}u}{\mathrm{d}x}(x)\right)\mathrm{d}x & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to show that $\Phi_\varepsilon(\omega, \cdot)$ is a proper convex lsc functional with domain $H_0^1(\Omega)$ and that for all $\omega \in \Sigma$, its subdifferential (actually its derivative) is given by

$$\begin{cases} \operatorname{dom}(\partial\Phi_\varepsilon(\omega, \cdot)) = \left\{u \in H_0^1(\Omega) : \left(W'_\xi\left(\omega, \frac{\cdot}{\varepsilon}, \frac{\mathrm{d}u}{\mathrm{d}x}\right)\right)' \in L^2(\Omega)\right\} \\ \partial\Phi_\varepsilon(\omega, \cdot) = -\left(W'_\xi\left(\omega, \frac{\cdot}{\varepsilon}, \frac{\mathrm{d}u}{\mathrm{d}x}\right)\right)'. \end{cases}$$

On the other hand, we set

$$a(\omega, x) = \begin{cases} a^-(x) & \text{if } x \in \bigcup_{i\in\mathbb{N}} B_R(\omega_i), \\ a^+(x) & \text{otherwise} \end{cases}$$

and we define the random quadratic integral functional $\Psi_\varepsilon : L^2(\Omega) \to]-\infty, +\infty]$ by

$$\Psi_\varepsilon(\omega, u) = \begin{cases} \displaystyle\frac{1}{2}\int_\Omega a\left(\omega, \frac{x}{\varepsilon}\right)\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 \mathrm{d}x & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

The subdifferential of $\Psi_\varepsilon(\omega, \cdot)$ (actually its derivative) is given by

$$\begin{cases} \operatorname{dom}(\partial\Psi_\varepsilon(\omega, \cdot)) = \left\{u \in H_0^1(\Omega) : \left(a\left(\omega, \frac{\cdot}{\varepsilon}\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right)' \in L^2(\Omega)\right\} \\ \partial\Psi_\varepsilon(\omega, \cdot) = -\left(a\left(\omega, \frac{\cdot}{\varepsilon}\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right)'. \end{cases}$$

Condition (2.1) is uniformly satisfied: take $\alpha_{\Psi_\varepsilon} = \frac{1}{2}\min(a^-, a^+)$. In the lemma below we state that (2.2) is uniformly fulfilled for $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$.

Lemma 6.2. *For $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$, the subdifferentials $\partial\Phi_\varepsilon(\omega,\cdot)$ and $\partial\Psi_\varepsilon(\omega,\cdot)$ are connected as follows:*

$$\begin{cases} \mathrm{dom}(\partial\Phi_\varepsilon(\omega,\cdot)) \subset \mathrm{dom}(\partial\Psi_\varepsilon(\omega,\cdot)), \\ \langle \partial\Phi_\varepsilon(\omega,u), \partial\Psi_\varepsilon(\omega,u)\rangle \geq \|\partial\Psi_\varepsilon(\omega,u)\|^2_{L^2(\Omega)} \textit{ for all } u \in \mathrm{dom}(\partial\Phi_\varepsilon(\omega,\cdot)). \end{cases}$$

Proof. For $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$ set $\Omega^-_\varepsilon(\omega) := \Omega \cap [\frac{x}{\varepsilon} \in \bigcup_{i\in\mathbb{N}} B_R(\omega_i)]$ and $\Omega^+_\varepsilon(\omega) := \Omega \cap [\frac{x}{\varepsilon} \notin \bigcup_{i\in\mathbb{N}} B_R(\omega_i)]$. Let $u \in \mathrm{dom}(\partial\Phi_\varepsilon(\omega,\cdot))$, we have

$$\begin{aligned} \int_\Omega \left(a\left(\omega, \frac{\cdot}{\varepsilon}\right) \frac{\mathrm{d}u}{\mathrm{d}x}\right)^{\prime 2} \mathrm{d}x &= \int_{\Omega^-_\varepsilon(\omega)} (a^-)^2 \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \int_{\Omega^+_\varepsilon(\omega)} (a^+)^2 \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \\ &\leq \int_{\Omega^-_\varepsilon(\omega)} \left(\sigma^{-\prime}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \int_{\Omega^+_\varepsilon(\omega)} \left(\sigma^{+\prime}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \\ &= \int_\Omega \left(W'_\xi\left(\omega, \frac{\cdot}{\varepsilon}, \frac{\mathrm{d}u}{\mathrm{d}x}\right)\right)^{\prime 2} \mathrm{d}x < +\infty \end{aligned}$$

so that $u \in \mathrm{dom}(\Psi_\varepsilon(\omega,\cdot))$.

Fix now $u \in \mathrm{dom}(\partial\Phi_\varepsilon(\omega,\cdot))$. From (6.4) we have

$$\begin{aligned} \langle \partial\Phi_\varepsilon(\omega,u), \partial\Psi_\varepsilon(\omega,u)\rangle &= \int_\Omega \left(W'_\xi\left(\omega, \frac{x}{\varepsilon}, \frac{\mathrm{d}u}{\mathrm{d}x}\right)\right)' \left(a\left(\omega,\frac{x}{\varepsilon}\right)\frac{\mathrm{d}u}{\mathrm{d}x}\right)' \mathrm{d}x \\ &= \int_{\Omega^-_\varepsilon(\omega)} \left(\sigma^-\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right)' a^- \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{d}x + \int_{\Omega^+_\varepsilon(\omega)} \left(\sigma^+\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right)' a^+ \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{d}x \\ &= \int_{\Omega^-_\varepsilon(\omega)} \sigma^{-\prime} a^- \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \int_{\Omega^+_\varepsilon(\omega)} \sigma^{+\prime} a^+ \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \\ &\geq \int_{\Omega^-_\varepsilon(\omega)} a^{-2} \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \int_{\Omega^+_\varepsilon(\omega)} a^{+2} \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \\ &= \|\partial\Psi_\varepsilon(\omega,u)\|^2_{L^2(\Omega)}. \end{aligned}$$

This completes the proof. □

Let K be a kernel as defined in Section 2 and a reaction functional as in the previous section with $N = 1$, i.e.

$$F_\varepsilon(\omega,t,u)(x) = r\left(\omega,t,\frac{x}{\varepsilon}\right) \cdot g(u(x)) + q\left(\omega,t,\frac{x}{\varepsilon}\right),$$

fulfilling the same conditions. Consider the random integrodifferential reaction-diffusion problem defined for $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$ by

$$(\mathcal{P}_\varepsilon(\omega)) \begin{cases} \dfrac{\mathrm{d}u_\varepsilon(\omega)}{\mathrm{d}t}(t) + \partial\Phi_\varepsilon(\omega, u_\varepsilon(\omega,t)) + \displaystyle\int_0^t K(t-s)\partial\Psi_\varepsilon(\omega, u_\varepsilon(\omega,s))\,\mathrm{d}s \\ \qquad\qquad\qquad\qquad = F_\varepsilon(\omega, t, u_\varepsilon(\omega,t)) \text{ for a.e. } t \in (0,T) \\ u_\varepsilon(\omega,0) = u^0_\varepsilon(\omega),\ u^0_\varepsilon(\omega) \in \mathrm{dom}(\partial\Phi_\varepsilon(\omega)). \end{cases}$$

From Proposition 4.2, $(\mathcal{P}_\varepsilon(\omega))$ admits a unique solution. When $\omega \in \Sigma$ and ε are fixed, the problem $(\mathcal{P}_\varepsilon(\omega))$ is nothing but the problem treated in [12, Example 2] in term of well posedness. Here we consider sequences of such problems with, additionally, a reaction source and in a stochastic homogenization framework. A straightforward application of Theorem 5.3 yields

Corollary 6.3. *Assume that for $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$*

(HOM$_1$) $\sup_{\varepsilon>0} \Phi_\varepsilon(u^0_\varepsilon(\omega)) < +\infty$*;*

(HOM$_2$) $u^0_\varepsilon(\omega) \to u^0(\omega)$ *strongly in* $L^2(\Omega)$.

Then for $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$, the solution $u_\varepsilon(\omega,\cdot)$ of $(\mathcal{P}_\varepsilon(\omega))$ converges to $u(\omega,\cdot)$ in $C([0,T], L^2(\Omega))$, solution of the homogenized problem

$$(\mathcal{P}(\omega)) \begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(\omega,t) - (\partial W_{\mathrm{hom}}(u(\omega,t)))' \\ \qquad - \displaystyle\int_0^t K(t-s) a_{\mathrm{hom}} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(\omega,s)\,\mathrm{d}s \ni F(t, u(\omega,t)) \text{ for a.e. } t \in (0,T) \\ u(\omega,0) = u^0(\omega),\ u^0(\omega) \in \mathrm{dom}(\partial\Phi). \end{cases}$$

where W_{hom} is deterministic, given by

$$\begin{aligned} W_{\mathrm{hom}}(\xi) &= \lim_{n\to+\infty} \inf\left\{ \frac{1}{n^N}\int_{nY} W\Big(\omega, y, \xi + \frac{\mathrm{d}u}{\mathrm{d}y}(y)\Big)\mathrm{d}y : u \in H^1_0(nY)\right\} \\ &= \inf_{n\in\mathbb{N}^*} \mathbb{E}_\theta \inf\left\{ \frac{1}{n^N}\int_{nY} W\Big(\cdot, y, \xi + \frac{\mathrm{d}u}{\mathrm{d}y}(y)\Big)\mathrm{d}y : u \in H^1_0(nY)\right\}, \end{aligned}$$

the coefficient a_{hom} is given by

$$\begin{cases} a_{\mathrm{hom}} = \dfrac{a^- a^+}{\Theta a^- + (1-\Theta)a^+}, \\ \Theta = 1 - \exp(2\theta R) \end{cases}$$

and, with the preamble convention, $\partial\Phi_{\text{hom}}$, possibly multivalued, is given by

$$\begin{cases} \text{dom}(\partial\Phi_{\text{hom}}) = \left\{v \in H_0^1(\Omega) : \left(W'_{\text{hom}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\right)' \in L^2(\Omega)\right\} \\ \partial\Phi_{\text{hom}} = -\left(W'_{\text{hom}}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\right)'. \end{cases}$$

The homogenized reaction functional is given for every $u \in L^2(\Omega)$, and all $(t,x) \in [0,T] \times \mathbb{R}$ by

$$F_{\text{hom}}(t,u)(x) = r_{\text{hom}}(t) \cdot g(u(x)) + q_{\text{hom}}(t),$$

where

$$r_{\text{hom}}(t) = \mathbb{E}_\theta\left(\int_{(0,1)^N} r(\,\cdot\,, t, y)\,\mathrm{d}y\right),$$
$$q_{\text{hom}}(t) = \mathbb{E}_\theta\left(\int_{(0,1)^N} q(\,\cdot\,, t, y)\,\mathrm{d}y\right).$$

Assume further that the Fenchel conjugate of $W^\pm$ satisfies the following condition: there exists $\gamma^* > 0$ such that $\left\langle \xi_1^* - \xi_2^*, \xi^1 - \xi^2 \right\rangle \geq \gamma^* |\xi_1 - \xi_2|^2$ for all $(\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}$ and all $(\xi_1^*, \xi_2^*) \in \partial W^{\pm *} \times \partial W^{\pm *}$. Then W_{hom} is univalent and is the $\mathbb{P}_\theta$-almost sure pointwise limit of $W'_n(\omega, \cdot)$ where $W_n(\omega, \xi) = \inf\left\{\frac{1}{n^N}\int_{nY} W(\omega, y, \xi + \frac{\mathrm{d}u}{\mathrm{d}y}(y))\mathrm{d}y : u \in H_0^1(nY)\right\}$.

Proof. The weak limit of the reaction term is obtained as in the proof of Corollary 6.1. In order to apply Theorem 5.3, it is enough to establish that for $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$, the following variational convergences hold: $\Phi_\varepsilon(\omega) \xrightarrow{M} \Phi$ and $\Psi_\varepsilon(\omega)_{\lfloor H_0^1(\Omega)} \xrightarrow{\Gamma_{w-H_0^1}} \Psi$ where

$$\Phi(u) = \begin{cases} \displaystyle\int_\Omega W_{\text{hom}}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\Psi(u) = \frac{1}{2}\int_\Omega a_{\text{hom}} \left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 \mathrm{d}x.$$

The first convergence is well known (see for instance [6, Theorem 12.4.7]. For the second convergence, note that for quadratic functionals in one dimension

$$F_\varepsilon(u) = \int_\Omega a_\varepsilon(x)\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 \mathrm{d}x,$$

with $0 \leq \alpha \leq a_\varepsilon \leq \beta$, one has : $F_\varepsilon \xrightarrow{\Gamma_{w-H_0^1}} F$ iff $\frac{1}{a_\varepsilon}$ converges to $\frac{1}{a}$ for the $\sigma(L^\infty, L^1)$ topology, and F has the integral representation

$$F(u) = \int_\Omega a(x)\left|\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right|^2 \mathrm{d}x.$$

For a proof, see [6, Theorem 12.3.1]. Hence it remains to establish that for $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$ the following convergence holds

$$\frac{1}{a(\omega, \frac{\cdot}{\varepsilon})} \rightharpoonup \frac{\Theta a^- + (1-\Theta)a^+}{a^+a^-} \quad \sigma(L^\infty, L^1).$$

This result is a direct consequence of the additive ergodic theorem (see [6, Theorem 12.4.2]) which states that for $\mathbb{P}_\theta$-a.e. $\omega \in \Sigma$

$$\frac{1}{a(\omega, \frac{\cdot}{\varepsilon})} \rightharpoonup \mathbb{E}_\theta \int_{(0,1)} \frac{1}{a(\cdot, y)} \mathrm{d}y.$$

An easy calculation gives

$$\mathbb{E}_\theta \int_{(0,1)} \frac{1}{a(\cdot, y)} \mathrm{d}y = \frac{\Theta a^- + (1-\Theta)a^+}{a^+a^-}.$$

The last claim follows straightforwardly from [6, Proposition 17.4.6]. This completes the proof. □

Appendix A. Proof of Lemma 3.2

Step 1: Proof of (3.6) and (3.7). Observe that from Lemma 3.1, for all $t \in [0, T]$, $u_\lambda(t) \in \mathrm{dom}(\partial\Phi) \subset \mathrm{dom}(\partial\Psi)$. For a.e. $t \in [0, \widetilde{T}]$, form the scalar product in X of $\partial\Psi(u_\lambda(t))$ with the approximate equation (3.1) and integrate over $[0, \widetilde{T}]$. This yields

$$\int_0^{\tilde{T}} \frac{\mathrm{d}}{\mathrm{d}t}\Psi(u_\lambda(t))\mathrm{d}t + \int_0^{\tilde{T}} \langle \partial\Phi(u_\lambda(t)), \partial\Psi(u_\lambda(t))\rangle \,\mathrm{d}t$$
$$+ \int_0^{\tilde{T}} \langle K \star \partial\Psi_\lambda(u_\lambda)(t), \partial\Psi(u_\lambda(t))\rangle \,\mathrm{d}t = \int_0^{\tilde{T}} \langle F(t, u_\lambda(t)), \partial\Psi(u_\lambda(t))\rangle \,\mathrm{d}t. \quad \text{(A.1)}$$

We have used the fact that from (2.2), $\partial\Psi(u_\lambda) \in L^2(0, T, X)$, hence

$$\left\langle \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t), \partial\Psi(u_\lambda(t))\right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Psi(u_\lambda(t))$$

(cf. [6, Proposition 17.2.5]). An easy calculation gives

$$\|K \star \partial\Psi_\lambda(u_\lambda)\|_{L^2(0,\tilde{T},X)} \le \widetilde{T}^{\frac{1}{2}} \|K\|_{L^2(0,T)} \|\partial\Psi_\lambda(u_\lambda)\|_{L^2(0,\tilde{T},X)}. \quad \text{(A.2)}$$

Since for all $\lambda > 0$,

$$\|\partial\Psi_\lambda(u_\lambda(t))\|_X \le \|\partial\Psi(u_\lambda(t))\|_X \quad \text{(A.3)}$$

(see [6, Proposition 17.2.2]), we infer that

$$\left| \int_0^{\tilde{T}} \langle K \star \partial\Psi_\lambda(u_\lambda)(t), \partial\Psi(u_\lambda(t))\rangle \,\mathrm{d}t \right| \le \widetilde{T}^{\frac{1}{2}} \|K\|_{L^2(0,T)} \|\partial\Psi(u_\lambda)\|^2_{L^2(0,\tilde{T},X)}. \quad \text{(A.4)}$$

On the other hand from (C_1) and (2.1)

$$\|F(\cdot,u_\lambda)\|_{L^2(0,\tilde{T},X)} \leq \|F(\cdot,0)\|_{L^2(0,T,X)} + \|L\|_{L^2(0,\tilde{T})}\|u_\lambda\|_{L^2(0,\tilde{T},X)}$$
$$\leq \|F(\cdot,0)\|_{L^2(0,T,X)} + \alpha_\Psi\|L\|_{L^2(0,\tilde{T})}\|\partial\Psi(u_\lambda)\|_{L^2(0,\tilde{T},X)}. \quad (A.5)$$

Hence from (A.5)

$$\int_0^{\tilde{T}} \langle F(t,u_\lambda(t)), \partial\Psi(u_\lambda(t))\rangle \,\mathrm{d}t \leq \|F(\cdot,0)\|_{L^2(0,T,X)}\|\partial\Psi(u_\lambda)\|_{L^2(0,\tilde{T},X)}$$
$$+ \alpha_\Psi\|L\|_{L^2(0,\tilde{T})}\|\|\partial\Psi(u_\lambda)\|^2_{L^2(0,\tilde{T},X)}. \quad (A.6)$$

Combining (A.1), (2.2), and (A.4), (A.6) yields that

$$\Big[\alpha_{\Phi,\Psi} - \big(\widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)} + \alpha_\Psi\|L\|_{L^2(0,\tilde{T})}\big)\Big]\|\partial\Psi(u_\lambda)\|^2_{L^2(0,\tilde{T},X)}$$
$$\leq T\beta_{\Phi,\Psi} + \Psi(u_0) - \inf_X \Psi + \|F(\cdot,0)\|_{L^2(0,\tilde{T},X)}\|\partial\Psi(u_\lambda)\|_{L^2(0,\tilde{T},X)}$$

from which we deduce (3.6) provided that $\widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)} + \alpha_\Psi\|L\|_{L^2(0,\tilde{T})} < \alpha_{\Phi,\Psi}$. Estimate (3.7) follows by combining (3.6) with (2.1).

Step 2: Proof of (3.8)*,* (3.9) *and* (3.10). For a.e. $t \in (0,\widetilde{T})$, form the scalar product in X of $\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(t)$ with the approximate equation and integrate over $(0,\widetilde{T})$. This yields

$$\left\|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right\|^2_{L^2(0,\tilde{T},X)} \leq \Phi(u_0) - \inf_X \Phi$$
$$+ \big(\|K\star\partial\Psi_\lambda(u_\lambda)\|_{L^2(0,\tilde{T},X)} + \|F(\cdot,u_\lambda)\|_{L^2(0,\tilde{T},X)}\big)\left\|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right\|_{L^2(0,\tilde{T},X)}$$

and (3.8) follows from (3.6), (3.7), (A.2), and (A.5).

Estimate (3.9) follows straightforwardly from the approximate equation, (3.6), (3.7), (3.8), and (A.2). Estimate (3.10) is obtained from (3.8), according to

$$\|u_\lambda(t)\|_X \leq \|u_0\|_X + \widetilde{T}^{\frac{1}{2}}\left\|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right\|_{L^2(0,\tilde{T},X)}.$$

Step 3. Proof of (3.11), (3.12) and (3.13)] First observe that

$$\sup_{\lambda>0}\|K\star\partial\Psi_\lambda(u_\lambda)\|_{W^{1,2}(0,\tilde{T},X)} < +\infty, \quad (A.7)$$

which follows from (3.6) and the two inequalities:

$$\|K\star\partial\Psi_\lambda(u_\lambda)\|_{L^2(0,\tilde{T},X)} \leq \widetilde{T}^{\frac{1}{2}}\|K\|_{L^2(0,T)}\|\partial\Psi(u_\lambda)\|_{L^2(0,\tilde{T},X)},$$
$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}K\star\partial\Psi_\lambda(u_\lambda)\right\|_{L^2(0,\tilde{T},X)} \leq (K(0) + \widetilde{T}^{\frac{1}{2}}\|K'\|_{L^2(0,T)})\|\partial\Psi(u_\lambda)\|_{L^2(0,\tilde{T},X)}$$

(the second inequality follows from (3.4)). Next, from (3.5) and (3.8), we have

$$\sup_\lambda \left\| \frac{\mathrm{d}F(\,\cdot\,, u_\lambda)}{\mathrm{d}t} \right\|_{L^1(0,\tilde{T},X)} < +\infty. \tag{A.8}$$

From (A.7) and (A.8) we deduce that

$$\sup_\lambda \left\| \frac{\mathrm{d}G_\lambda}{\mathrm{d}t} \right\|_{L^1(0,\tilde{T},X)} < +\infty.$$

Hence (3.11) is a straightforward consequence of 3.8 and [2, Lemma 3.3] which states that for each $t \in \,]0, \widetilde{T}]$

$$\left\| \frac{\mathrm{d}^+ u_\lambda}{\mathrm{d}t}(t) \right\|_X \leq \frac{1}{t} \int_0^t \left\| \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(s) \right\|_X \mathrm{d}s + \int_0^t \left\| \frac{\mathrm{d}G_\lambda}{\mathrm{d}t}(s) \right\|_X \mathrm{d}s.$$

To establish (3.12), for each $t \in \,]0, \widetilde{T}]$ form the scalar product of the approximate equation S_λ

$$\frac{\mathrm{d}^+ u_\lambda}{\mathrm{d}t}(t) + \partial\Phi(u_\lambda(t)) = G_\lambda(t, u_\lambda)$$

with $\partial\Psi(u_\lambda(t))$. This yields from (2.2)

$$\left\langle \frac{\mathrm{d}^+ u_\lambda}{\mathrm{d}t}(t), \partial\Psi(u_\lambda(t)) \right\rangle + \alpha_{\Phi,\Psi} \|\partial\Psi(u_\lambda(t))\|_X^2 \leq \langle G_\lambda(t, u_\lambda), \partial\Psi(u_\lambda(t)) \rangle - \beta_{\Phi,\Psi}$$

from which we deduce

$$\alpha_{\Phi,\Psi} \|\partial\Psi(u_\lambda(t))\|_X^2 \leq \left(\left\| \frac{\mathrm{d}^+ u_\lambda}{\mathrm{d}t}(t) \right\|_X + \|G_\lambda(t, u_\lambda)\|_X \right) \|\partial\Psi(u_\lambda(t))\|_X - \beta_{\Phi,\Psi}.$$

The claim follows from (3.6), (3.10), (3.11), and $\sup_\lambda \|G_\lambda(t, u_\lambda(t))\|_X < +\infty$ obtained according to

$$\|G_\lambda(t, u_\lambda)\|_X \leq \|K\|_{L^2(0,T)} \|\partial\Psi(u_\lambda)\|_{L^2(0,\tilde{T},X)} + \|F(t, 0)\|_X + L(t)\|u_\lambda(t)\|_X.$$

For $t = 0$, $\partial\Psi(u_\lambda(t)) = \partial\Psi(u_0)$ which does not depend on λ. To obtain (3.13), take the scalar product of the approximate equation with $\partial\Phi(u_\lambda(t))$ and follows the same calculation. □

Appendix B. **A Grönwall type inequality**

The following lemma generalizes the result stated in [9, Lemma A.5].

Lemma B.1. *Let* $T > 0$*,* $\mathbf{m} \in L^1(0, T)$ *such that* $\mathbf{m} \geq 0$ *a.e. in* $(0, T)$*, and* $a \geq 0$*. Let* $p \in [1, +\infty)$ *and* $\phi : [0, T] \to [0, +\infty)$ *be a continuous function satisfying*

$$\frac{1}{p}\phi^p(t) \leq \frac{1}{p}a^p + \int_0^t \phi^{p-1}(s)\mathbf{m}(s)\,\mathrm{d}s \quad \textit{for all } t \in [0, T].$$

Then

$$\phi(t) \leq a + \int_0^t \mathbf{m}(s)\,\mathrm{d}s \quad \textit{for all } t \in [0,T].$$

Proof. We assume that $a > 0$, otherwise substitute $a + \varepsilon$ for a and make $\varepsilon \to 0$ in the last inequality of the proof. Set $\psi(t) = \frac{1}{p}a^p + \int_0^t \phi^{p-1}(s)\mathbf{m}(s)\,\mathrm{d}s$ so that $\psi > 0$ and

$$\phi(s) \leq p^{\frac{1}{p}}\psi^{\frac{1}{p}}(s) \quad \text{for all } s \in [0,T]. \tag{B.1}$$

Hence, since ψ is absolutely continuous

$$\frac{d\psi}{\mathrm{d}t}(s) = \mathbf{m}(s)\phi^{p-1}(s) \leq \mathbf{m}(s)p^{\frac{1}{q}}\psi^{\frac{1}{q}}(s) \quad \text{for a.e. } s \in (0,T) \tag{B.2}$$

where q is the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. It follows from (B.2) that for a.e. $s \in (0,T)$

$$p^{-\frac{1}{q}}\frac{\mathrm{d}\psi}{\mathrm{d}t}(s)\psi^{-\frac{1}{q}}(s) \leq \mathbf{m}(s),$$

that is

$$p^{\frac{1}{p}}\frac{\mathrm{d}}{\mathrm{d}t}(\psi^{\frac{1}{p}}(s)) \leq \mathbf{m}(s).$$

Integrating over $(0,t)$, we infer that for all $t \in [0,T]$

$$p^{\frac{1}{p}}\psi^{\frac{1}{p}}(t) \leq p^{\frac{1}{p}}\psi^{\frac{1}{p}}(0) + \int_0^t \mathbf{m}(s)\,\mathrm{d}s,$$

that is, according to (B.1), $\phi(t) \leq a + \int_0^t \mathbf{m}(s)\,\mathrm{d}s$ for all $t \in [0,T]$. □

Appendix C. Graph-convergence

Let us recall the classical notion of the Kuratowski–Painlevé convergence for sequence of sets: let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a metric space (X,d), or more generally of a topological space. The lower limit of the sequence $(A_n)_{n\in\mathbb{N}}$ is the subset of X denoted by $\liminf A_n$ and defined by

$$\liminf A_n = \{x \in X : \exists\, x_n \to x,\ x_n \in A_n \text{ for all } n \in \mathbb{N}\}$$

The upper limit of the sequence $(A_n)_{n\in\mathbb{N}}$ is the subset of X denoted by $\limsup A_n$ and defined by

$$\limsup A_n = \left\{x \in X : \exists\, (n_k)_{k\in\mathbb{N}},\ \exists\, (x_k)_{k\in\mathbb{N}},\ \forall\, k, x_k \in A_{n_k},\ x_k \to x\right\}.$$

The sets $\liminf A_n$ and $\limsup A_n$ are clearly two closed subsets of (X,d) satisfying

$$\liminf A_n \subset \limsup A_n.$$

The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be convergent if

$$\liminf A_n = \limsup A_n.$$

The common value A is called the limit of $(A_n)_{n\in\mathbb{N}}$ in the Kuratowski–Painlevé sense and denoted by K-$\lim A_n$. Therefore, by definition $A := K$-$\lim A_n$ if and only if

$$\limsup A_n \subset A \subset \liminf A_n.$$

We may also write $A_n \to A$ or $A = \lim A_n$, adding the metric or topology which is used in X.

From now on $(V, \|\cdot\|)$ is a Banach space and V^* its topological dual space whose dual norm is denoted by $\|\cdot\|_*$ and we recall that for $(u, u^*) \in V \times V^*$, we write $\langle u^*, u\rangle$ for $u^*(u)$. Given a multivalued operator $A : V \to 2^{V^*}$, for any $v \in V$ we write Av instead of $A(v)$. Let us recall some basic definitions

$$\begin{aligned} &\mathrm{dom}(A) = \{v \in X : Av \neq \emptyset\} && \text{denotes the domain of } A;\\ &G(A) := \{(v, v^*) \in V \times V^* : v^* \in Av\} && \text{denotes the graph of } A;\\ &R(A) := \{v^* \in V^* : \exists\, v \in V \;\; v^* \in Av\} && \text{denotes the range of } A. \end{aligned}$$

We define the inverse operator $A^{-1} : V^* \to V$ of A by

$$A^{-1}(v^*) = \{v \in V : v^* \in Av\}\,.$$

Note that $\mathrm{dom}(A^{-1}) = R(A)$. Consider another multivalued operator $B : V \to 2^{V^*}$. The range of A with respect to B is the set

$$R_B(A) := \{v^* \in V^* : \exists\, v \in \mathrm{dom}(B) \;\; v^* \in Av\}\,.$$

Note that we have $R_B(A) = A(\mathrm{dom}(A) \cap \mathrm{dom}(B))$.

Definition C.1. An operator $A : V \to 2^{V^*}$ is said to be monotone, if $\langle u^* - v^*, u - v\rangle \geq 0$ whenever $(u, u^*) \in G(A)$ and $(v, v^*) \in G(A)$. It is maximal monotone, if it is monotone and if its graph is maximal among all the monotone operators mapping V to V^* when $V \times V^*$ is ordered by inclusion. An element (u, u^*) of $V \times V^*$ is said to be monotonically related to a monotone operator A provided

$$\langle u^* - v^*, u - v\rangle \geq 0 \;\text{ for all } (v, v^*) \in G(A).$$

A useful form of the definition of maximality for a monotone operator A is the following condition whose proof follows straightforwardly from Definition C.1.

Proposition C.2. *Let $A : V \to 2^{V^*}$ be a monotone operator. Then A is maximal monotone if and only if whenever (u, u^*) is monotonically related to A then $u \in \mathrm{dom}\, A$ and $u^* \in Au$.*

The most basic class of maximal monotone operators is the class of subdifferentials of convex functions (see [6, Theorem 17.4.1]). Given a sequence of operators, one can consider the lim inf and lim sup of the sequence of their graphs as subsets of $V \times V^*$. This leads to the following definition.

Definition C.3. A sequence $(A_n)_{n\in\mathbb{N}}$ of operators mapping V to V^* is said to be graph convergent to $A : V \to 2^{V^*}$, if the sequence $(G(A_n))_{n\in\mathbb{N}}$ converges to the graph $G(A)$ of A in the sense of Kuratowski–Painlevé when $V \times V^*$ is endowed with the product norm.

From now on we identify the operators with their graphs so that we write A instead of $G(A)$ and $A = G\text{-}\lim A_n$ or $A_n \overset{G}{\to} A$ instead of $G(A) = K\text{-}\lim_{n\to+\infty} G(A_n)$. When considering sequences of maximal monotone operators, thus subdifferentials, the definition of the graph convergence is reduced to:

Proposition C.4. *Let $(A_n, A)_{n\in\mathbb{N}}$ be a sequence of maximal monotone operators mapping V to V^*. Then we have*

$$A = G\text{-}\lim_{n\to+\infty} A_n \iff A \subset \liminf_{n\to+\infty} A_n. \tag{C.1}$$

Proof. The only implication we have to establish is

$$A \subset \liminf_{n\to+\infty} A_n \Longrightarrow A = G\text{-}\lim_{n\to+\infty} A_n,$$

the converse being trivial. Thus, it remains to show that $\limsup A_n \subset A$ is automatically satisfied. Let $(u, u^*) \in \limsup A_n$, then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of integers and $(u_k, u_k^*) \in A_{n_k}$ such that $(u_k, u_k^*) \to (u, u^*)$ in $V \times V^*$ whenever $k \to +\infty$.

In the other hand, since $A \subset \liminf A_n$, for all $(v, v^*) \in A$, there exists $(v_n, v_n^*) \in A_n$ such that $(v_n, v_n^*) \to (v, v^*)$ in $V \times V^*$. Passing to the limit in

$$\left\langle u_k^* - v_{n_k}^*, u_k - v_{n_k} \right\rangle \geq 0$$

when $k \to +\infty$ (recall that A_{n_k} is monotone), we infer

$$\langle u^* - v^*, u - v\rangle \geq 0 \text{ for all } (v, v^*) \in A.$$

Therefore (u, u^*) is monotonically related to A and, according to Proposition C.2, $(u, u^*) \in A$, which completes the proof. □

Denote by $A_n \overset{G_{s,s}}{\to} A$ the graph convergence in $V \times V^*$ of A_n to A when $V \times V^*$ is equipped with the strong product topology, and by $A_n \overset{G_{w,s}}{\to} A$ the graph convergence in $V \times V^*$ of A_n to A when $V \times V^*$ is equipped with the weak-strong product topology. On the other hand denote by $\Psi_n \overset{\Gamma_{s\text{-}V}}{\to} \Psi$ and $\Psi_n \overset{\Gamma_{w\text{-}V}}{\to} \Psi$ the sequential Γ-convergence of the functional $\Phi_n : V \to \mathbb{R} \cup \{+\infty\}$ toward the functional $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ when V is equipped with its strong and weak topology respectively. The following theorem states the link between the variational convergence of convex functionals and the graph convergence of their subdifferentials. For a proof, refer to [5, Theorems 3.66, 3.67] or [6, Theorem 17.4.4]

Theorem C.5. *Let* $\Psi_n, \Psi : V \to \mathbb{R} \cup \{+\infty\}$ *be a sequence of lsc convex proper functions. Then the following implications hold:*

$$\Psi_n \overset{\Gamma_{s\text{-}V}}{\longrightarrow} \Psi \Longrightarrow \partial\Psi_n \overset{G_{s,s}}{\longrightarrow} \partial\Psi,$$

$$\Psi_n \overset{\Gamma_{w\text{-}V}}{\longrightarrow} \Psi \Longrightarrow \partial\Psi_n \overset{G_{w,s}}{\longrightarrow} \partial\Psi.$$

Note that converse implications hold true, up to normalization.

Appendix D. Γ-convergence versus Mosco-convergence

Definition D.1 (Mosco convergence)**.** Let $(V, \|\cdot\|)$ be a Banach space, and $(\Phi_n)_{n\in\mathbb{N}}$ a sequence of extended real-valued functions $\Phi_n : V \to \mathbb{R} \cup \{+\infty\}$. The sequence $(\Phi_n)_{n\in\mathbb{N}}$ Mosco converges to the extended real-valued function $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ and we write $\Phi_n \overset{M}{\to} \Phi$ if

$$\Phi = \Gamma_{w\text{-}V}\text{-}\Phi_n = \Gamma_{s\text{-}V}\text{-}\Phi_n.$$

The argument, which naturally led to introduce the Mosco convergence notion, is the bicontinuity of the Fenchel duality transformation in the context of convex functions (see [6, 20, 21]). This Appendix is devoted to the following Proposition.

Proposition D.2. *Let X and V be two Banach spaces with* $V \overset{c}{\hookrightarrow} X$*, and* $\Psi_n, \Psi : X \to]-\infty, +\infty]$ *lsc convex proper functions such that* $\mathrm{dom}(\Psi_n) = \mathrm{dom}(\Psi) = V$*. Assume that for all* $r \in \mathbb{R}$*, there exists a weakly compact subset* K_r *of* V *such that for all* $n \in \mathbb{N}$

$$[\Psi_{n\lfloor V} \leq r] \subset K_r.$$

Then

$$\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\longrightarrow} \Psi_{\lfloor V} \Longrightarrow \Psi_n \overset{M}{\longrightarrow} \Psi.$$

Proof. Assume that $\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\to} \Psi_{\lfloor V}$. Let $u_n \in X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X, and assume that $\liminf_{n\to+\infty} \Psi_n(u_n) < +\infty$. Then

$$\liminf_{n\to+\infty} \Psi_n(u_n) = \liminf_{n\to+\infty} \Psi_{n\lfloor V}(u_n).$$

According to the equi-coerciveness hypothesis and to the compact embedding $V \overset{c}{\hookrightarrow} X$ we can extract a subsequence of $(u_n)_{n\in\mathbb{N}}$ which weakly converges in V and strongly in X to some $v \in V$. Thus $v = u$ and $u_n \rightharpoonup u$ in V. Hence from $\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\to} \Psi_{\lfloor V}$ we infer that

$$\Psi(u) = \Psi_{\lfloor V}(u) \leq \liminf_{n\to+\infty} \Psi_{n\lfloor V}(u_n) = \liminf_{n\to+\infty} \Psi_n(u_n). \tag{D.1}$$

Let $u \in X$ and assume that $\Psi(u) < +\infty$ so that $\Psi(u) = \Psi_{\lfloor V}(u)$. From $\Psi_{n\lfloor V} \overset{\Gamma_{w\text{-}V}}{\to} \Psi_{\lfloor V}$ and the compact embedding $V \overset{c}{\hookrightarrow} X$, we can derive that for a subsequence of $(\Psi_{n\lfloor V})_{n\in\mathbb{N}}$

(not relabeled) there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of V such that $u_n \rightharpoonup u$ in V, $u_n \to u$ strongly in X, which satisfies

$$\lim_{n\to+\infty} \Psi_n(u_n) = \lim_{n\to+\infty} \Psi_{n\lfloor V}(u_n) = \Psi(u). \tag{D.2}$$

From (D.1) and (D.2) we deduce that there exists a subsequence of $(\Psi_n)_{n\in\mathbb{N}}$ such that $\Psi_n \overset{\mathrm{M}}{\to} \Psi$. This conclusion being valid for any subsequence of $(\Psi_n)_{n\in\mathbb{N}}$, we conclude that $\Psi_n \overset{\mathrm{M}}{\to} \Psi$, which completes the proof. □

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