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Jacobi radial stable processes

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Summary: *We compute the Lévy measures of the stable processes associated to the Jacobi hypergroups on $[0,\infty[$. Equivalents of their densities near zero and infinity are obtained. The domains of attraction are deduced from a new Tauberian theorem.*

Introduction.

N.H Bingham wrote at the end of his survey ([B2]) that he was introduced to Tauberian theory by the following:

Kingman's Theorem. *Let $\alpha > -\frac{1}{2}$ such that $d = 2\alpha + 2$ is an integer, μ a probability on $\mathbb{R}^+$ and l a slowly varying function near infinity. Set*

$$j_\alpha(r) = \Gamma(\alpha+1)\frac{J_\alpha(r)}{(r/2)^\alpha}$$

where J_α denotes a Bessel function of the first kind. Then if s belongs to $]0,2[$, the properties

$$\mu([x,+\infty]) \sim \frac{l(x)}{x^s}.\frac{2^\alpha\Gamma(1+\alpha+s/2)}{\Gamma(1+\alpha)\Gamma(1-s/2)} \quad \textit{near infinity}$$

and

$$1 - \int_0^{+\infty} j_\alpha(px)d\mu(x) \sim p^\alpha l(1/p) \quad \textit{near zero}$$

are equivalent.

This Theorem is equivalent to the Fourier Tauberian Theorem on $\mathbb{R}^d$ for radial probabilities. A well-known byproduct is the characterization of the domain of attraction of stable radial Euclidean processes of index s in $]0,1[$. On the other hand, when α is a real number $> -\frac{1}{2}$, the Chébli Trimèche theory of hypergroups generalizes in an analytical setting the usual geometric interpretation of the convolutions on groups. Kingman's theorem remains true in this analytical sense.
A natural question is to give a generalization to a wide class of Tauberian theorems. N.H. Bingham used in ([B1]) Tauberian theory to prove some results of Klosowska ([Kl]) on domains of attraction for stable laws with respect to generalized convolution in Urbanik's sense. Hence he obtained some new Tauberian theorems when the spherical functions $j_\alpha(xt)$ are replaced by $\Omega(xt)$ for some kernels Ω on $\mathbb{R}^+$.

The second order differential operator associated to the Bessel-Kingman hypergroup

$$\frac{\partial^2}{\partial r^2} + \frac{(2\alpha+1)}{r}\frac{\partial}{\partial r}$$

has obvious companions, the Jacobi operators

$$\mathcal{G}_{\alpha,\beta} = \frac{\partial^2}{\partial r^2} + (2\alpha+1)\coth(r)\frac{\partial}{\partial r} + (2\beta+1)\tanh(r)\frac{\partial}{\partial r}.$$

Although they give rise to some of the most celebrated hypergroups, not much is known about the associated Tauberian theorems. We were even unable to find in Gangolli's papers his promised study of stable processes on symmetric spaces of rank 1, which ought to contain the geometric case of our Tauberian theorem.
We will see in the sequel that the Jacobi case is not covered in [B1] because the space variable and the parameter p do not play symmetric roles in the spherical functions.

1. Background and notations.

From now on, we fix the indices α and β such that $\alpha \geq \beta \geq -\frac{1}{2}$.
For some specific choices of α and β displayed in the following chart

	"field"	α	β
$\mathbb{H}^d(\mathbb{R})$	real hyperbolic	$\frac{d}{2}-1$	$-\frac{1}{2}$
$\mathbb{H}^d(\mathbb{C})$	complex hyperbolic	$d-1$	0
$\mathbb{H}^d(\mathbb{H})$	quaternionic	$2d-1$	1
$\mathbb{H}^2(Cay)$	Cayley octave	7	3

$\mathcal{G}_{\alpha,\beta}$ is the radial part of the Laplacian on a symmetric space ([K]) . Hence we can define our radial Jacobi stable processes with index $0 < 2\gamma < 2$ as the radial parts of the symmetric stable Levy processes with pseudo-differential generators $(-\Delta/2)^{\gamma}$.
In the general case, when no group structure is available, the harmonic analysis still bears an analytic meaning. Moreover, the hypergroup theory explains the connection with ordinary probabilities. In fact, the condition $\alpha \geq \beta \geq -\frac{1}{2}$ is necessary and sufficient to define a Jacobi hypergroup structure on $\mathbb{R}^+$ ([B-H]). We will not consider in the sequel the degenerate case $\alpha = -\frac{1}{2}$.
Set $\rho := \alpha + \beta + 1$ and let $dV(r) = \sinh^{2\alpha+1}(r)\cosh^{2\beta+1}(r)\,dr$ be "the volume element". Remark that ρ^2 is the positive bottom of the spectrum of $-\mathcal{G}_{\alpha,\beta}$.

Definition. *If p is nonnegative, the spherical function $\phi_p(r)$ is the unique solution of*

$$\mathcal{G}_{\alpha,\beta} f(r) = -\left(\rho^2 + p^2\right) f(r)$$

such that $\phi_p(0) = 1$ and $\phi_p'(0) = 0$.

From [K] (2.7), we have

$$\phi_p(r) = \cosh^{-(\alpha+\beta+1+ip)}(r) \cdot {}_2F_1\left(\frac{\alpha+\beta+1+ip}{2}, \frac{\alpha-\beta+1+ip}{2}, \alpha+1, \tanh^2(r)\right).$$

where ${}_2F_1$ denotes the hypergeometric function. Following [K], we define the associated Fourier transform

$$\hat{f}(p) = \int_0^\infty f(r)\phi_p(r)dV(r).$$

By [K] (2.25), there exists a spectral density $m_{\alpha,\beta}(p)$

$$m_{\alpha,\beta}(p) = \frac{\left|\Gamma(\frac{ip+\rho}{2})\Gamma(\frac{ip+\alpha-\beta+1}{2})\right|^2}{\Gamma(\alpha+1)^2|\Gamma(ip)|^2}$$

such that the following inversion formula for rapidly decreasing Fourier transforms $\hat{f}$ holds:

$$f(r) = \frac{1}{\pi}\int_0^\infty \hat{f}(p)\phi_p(r)\, m_{\alpha,\beta}(p)\, dp. \qquad (\dagger)$$

Definition. Let $(R_t, t \geq 0)$ be the linear diffusion on $\mathbb{R}^+$ generated by $\frac{1}{2}\,\mathcal{G}_{\alpha,\beta}$ with a reflecting boundary at 0. For instance, when $\beta = -\frac{1}{2}$ we obtain an hyperbolic Bessel process.

Let $(\tau_t, t \geq 0)$ be a stable subordinator with index γ in $]0,1[$ characterized by its Laplace transform

$$\mathbb{E}\left(\exp(-\lambda\tau_t)\right) = \exp(-\lambda^\gamma t).$$

Then the stable radial processes are constructed by subordination:

Definition. Let the index 2γ be in $[0,2]$. Then $(X_t^{(2\gamma)}, t \geq 0)$ is the subordinated process $(R(\tau_t), t \geq 0)$.

Getoor ([Ge]) defined only the real hyperbolic subordinators.

Let $q_u(.)$ be the density of $(R_u, u \geq 0)$ with respect to the volume element and $p_t^{(\gamma)}(.)$ be the Lebesgue density of the subordinator. Of course, the density of $(X_t^{(2\gamma)}, t \geq 0)$ is $\int_0^\infty q_u(r)p_t^{(\gamma)}(u)\,du$, but contrary to the Euclidean case (see [Be] for instance), due to the lack of scaling properties, this formula doesn't lead easily to the asymptotics when r increases to infinity. Moreover the explicit computation of the densities $q_u(r)$ gives rise to complicated formulae.

Remark. By the help of the estimate deduced from Lévy's definition of his measure,

$$p_t^{(\gamma)}(u) \sim \frac{\gamma}{\Gamma(1-\gamma)} \frac{t}{u^{\gamma+1}} \quad (t \longrightarrow 0)$$

and a lot of care, we could compute the Lévy measure of the process $(X_t^{(2\gamma)}, t \geq 0)$.

Definition. The normalized Gaussian density is defined by its Fourier transform $\hat{f}(p) = \exp(-t(p^2 + \rho^2)/2)$.

By a standard martingale argument, $\hat{q}_u(p) = \exp(-\frac{u}{2}(p^2 + \rho^2))$. Hence, $(X_t^{(2\gamma)}, t \geq 0)$ has the Fourier transform $\exp\left(-t\left(\frac{p^2+\rho^2}{2}\right)^\gamma\right)$.

We now examine the densities of the Jacobi processes when the time goes to infinity:

Proposition 1. *When t increases to infinity, the density of $X_t^{(2\gamma)}$ with respect to the volume element is equivalent to* $K\phi_0(r)\exp\left(-(\rho^2/2)^\gamma.t\right) t^{-3/2}$ *where K is*

$$\frac{\sqrt{\pi}}{4} \frac{\Gamma^2(\frac{\rho}{2})\Gamma^2(\frac{\alpha-\beta+1}{2})}{\Gamma^2(\alpha+1)} \left(\gamma\rho^{2\gamma-2}2^{-\gamma}\right)^{-3/2}.$$

Proof. Apply Laplace's method to the inversion formula (†) and observe that the spectral measure has a density equivalent to const.p^2 near 0. □

Every infinitely divisible r.v. with respect to the hypergroup structure has a Fourier transform such that $\Phi(p) = \exp(-\Psi(p))$. We will call Ψ the exponent function.
In the group case, Gangolli [G] has obtained the following Lévy Khintchine formula.

The Lévy Khintchine formula. *The exponent function Ψ of every radial Lévy process on a symmetric space of non compact type has the unique representation*

$$\Psi(p) = P(p) + \int_0^{+\infty} (1 - \phi_p(r))\, dL(r)$$

where the unique positive measure L is such that $\int_0^\infty \inf(1, r^2)\, dL(r)$ is finite and the unique Gaussian part P is a polynomial function proportional to $(\rho^2 + p^2)$.

Note that the spherical functions $\phi_{\pm iu}$ where u belongs to $]0, \rho[$ are lacking in this formula. Indeed the set of characters doesn't carry an hypergroup dual structure.
We were unable to find in the literature a proof of this identity for arbitrary α and β. In fact, the general results are mainly concerned with strong hypergroups (see [BH] for instance). Obviously, the representation of our exponents $\Psi(p) = \left(\frac{p^2+\rho^2}{2}\right)^\gamma$ remains true. Hence we will still call $dL(r)$ a Lévy measure.

Theorem 2. *The Lévy measure of the process with exponent 2γ has a density $m^{(\gamma,\alpha,\beta)}(r)$ with respect to the volume element, given by the integral formula*

$$m^{(\gamma,\alpha,\beta)}(r) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \frac{q_s(r)}{s^{1+\gamma}}\, ds.$$

Remark. The Euclidean counterpart is hidden by the scaling property.
Proof. Since $\lim_{p\longrightarrow+\infty} \frac{\Psi(p)}{p^2} = 0$, there is no Gaussian part.
We may use Fubini theorem since $1-\phi_p(r)$ is positive. From

$$\int_0^\infty \phi_p(r)\, q_s(r)\, dV(r) = \exp(-\frac{s}{2}(p^2+\rho^2)) \quad \text{and} \quad \int_0^\infty q_s(r)\, dV(r) = 1,$$

we obtain

$$\int_0^\infty \int_0^\infty (1-\phi_p(r))\, \frac{q_s(r)}{s^{1+\gamma}} ds\, dV(r) = \int_0^\infty \left(1-\exp(-\frac{s}{2}(p^2+\rho^2))\right) \frac{ds}{s^{1+\gamma}}.$$

But for every positive a

$$\int_0^\infty \frac{1-\exp(-sa)}{s^{1+\gamma}}\, ds = \frac{\Gamma(1-\gamma)}{\gamma}\, a^\gamma. \ \square$$

Examples.
a) In the three dimensional real hyperbolic space $\mathbb{H}^3$, the Lévy densities with respect to $\sinh^2(r)dr$ are computed from Theorem 2:

$$m^{(\gamma,-\frac{1}{2},\frac{1}{2})}(r) = \frac{2\gamma}{\Gamma(1-\gamma)} \sqrt{\frac{2}{\pi}} \frac{1}{\sinh(r) r^{1/2+\gamma}} K_{3/2+\gamma}(r)$$

where K_ν denotes the modified Bessel function with index ν.
b) When $\gamma = \frac{1}{2}$ and $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$ (ie the three dimensional real hyperbolic space), the density of $X_t^{(2\gamma)}$ with respect to the volume element is

$$p_t^{(\frac{1}{2},-\frac{1}{2},\frac{1}{2})}(r) = \frac{2\sqrt{2}}{\pi} \frac{t}{t^2+2r^2} \frac{r}{\sinh(r)} K_2(\sqrt{t^2/2+r^2}).$$

Hence we obtain again $m^{(\frac{1}{2},-\frac{1}{2},\frac{1}{2})}(r)$ as

$$\lim_{t\to 0+} \frac{1}{t} p_t^{(\frac{1}{2},-\frac{1}{2},\frac{1}{2})}(r) = m^{(\frac{1}{2},-\frac{1}{2},\frac{1}{2})}(r) = \frac{\sqrt{2}}{\pi} \frac{K_2(r)}{r\sinh(r)}.$$

Remark. We do not consider other examples because the classical formulae for the hyperbolic heat kernels with $d \neq 3$ give rise to cumbersome identities. Unfortunately we were unable to use our new integral representation valid for any hyperbolic Jacobi hypergroup ([Gr]).

Proof of a). Apply the well known identity ([G-R] 3.471.9),

$$\int_0^{+\infty} \exp(-as)\exp(-b/s)s^{\lambda-1}\,ds = 2\left(\frac{b}{a}\right)^{\lambda/2} K_\lambda(2\sqrt{ab})$$

where a and b are positive.

Proof of b). Recall the density with respect to $\sinh^2(r)dr$ of the radial part of the 3-dimensional hyperbolic Brownian motion

$$q_s(r) = \sqrt{\frac{2}{\pi s^3}}\exp(-s/2)\,\frac{r}{\sinh(r)}\exp(-r^2/2s)$$

and the Lebesgue density of the stable subordinator

$$p_t^{(\frac{1}{2})}(u) = \frac{t}{2\sqrt{\pi u^3}}\exp(-t^2/4u).$$

We know from Gangolli ([G]) that the measures $\frac{1}{t}\,\mathbb{P}(X_t^{(2\gamma)} \in dr)$ converge vaguely on $\mathbb{R}_*^+$ to the Lévy measure when t goes to 0. Hence the limit in our example is identified as the density of the Lévy measure.

Remark. All the computations are easy in $\mathbb{H}^3$. For instance, $\phi_p(r) = \dfrac{\sin(pr)}{p\sinh(r)}$.

2. The tail near infinity.

Although Mizony ([M]) already defined a Laplace transform, we will consider here a different Laplace transform which looks like the one on the real line, whence the Tauberian theorem becomes very easy to prove. We could perhaps consider a true Tauberian theorem with respect to the spherical functions ϕ_p, $p \geq 0$, closer to a Tauberian result for an oscillatory Fourier transform.

We will use instead the positive exponentially decreasing spherical functions ϕ_{-iu} if u belongs to $]0,\rho[$. For instance, we obtain in $\mathbb{H}^3$ $\phi_{-iu}(r) = \frac{\operatorname{sh}(ur)}{u\operatorname{sh}(r)}$. By analytic continuation,

$$\hat{q}_t(iu) = \exp\left(-\frac{t}{2}(\rho^2 - u^2)\right).$$

Hence $X^{(2\gamma)}$ has a Fourier transform such that

$$\hat{p}_t(iu) = \exp\left(-t\left(\frac{\rho^2 - u^2}{2}\right)^\gamma\right).$$

Theorem 3. *The tail* $\int_x^{+\infty} dL(r)$ *of the Lévy measure* dL *of* $X^{(2\gamma)}$ *is equivalent to* $\dfrac{\rho^\gamma}{\Gamma(1-\gamma)}x^{-\gamma}$ *when* x *increases to infinity.*

Remark. This result is in accordance with the computation of example b) (case $\gamma = \frac{1}{2}$). Indeed, multiplying by $\sinh^2(r)$, since $K_2(r) \sim \sqrt{\frac{\pi}{2r}}\exp(-r)$ near infinity, the Levy measure has a Lebesgue density equivalent to $\frac{1}{2\sqrt{\pi}}r^{-3/2}$.

Proof. Let $\epsilon = \rho - u$. Hence by the usual integral representation of hypergeometric functions ([G-R], 9.111),

$$\phi_{-iu}(r) = \frac{1}{\Gamma(\epsilon/2)} \frac{1}{\cosh^{2\rho-\epsilon}(r)} \int_0^1 \frac{t^{\epsilon/2+\alpha}(1-t)^{\epsilon/2-1}}{(1-t\tanh^2(r))^{-\epsilon/2+\rho}}\, dt.$$

Use now the new variable $x = \sinh^2(r)(1-t)$. We obtain

$$\phi_{-iu}(r) = \frac{\sinh^{-\epsilon}(r)}{\Gamma(\epsilon/2)} \int_0^{\sinh^2(r)} x^{\epsilon/2-1}\left(1 - \frac{x}{\sinh^2(r)}\right)^{\epsilon/2+\alpha} (1+x)^{\epsilon/2-\rho}\, dx.$$

If $\epsilon < \rho$, let $K_{r,\epsilon}$ be

$$\frac{\sinh^{-\epsilon}(r)}{\Gamma(\epsilon/2)} \int_0^{+\infty} x^{\epsilon/2-1}(1+x)^{\epsilon/2-\rho}\, dx$$

or

$$\frac{1}{\sinh^{\epsilon}(r)} \frac{\Gamma(\rho-\epsilon)}{\Gamma(\rho-\epsilon/2)}.$$

The difference between $\phi_{-iu}(r)$ and $K_{r,\epsilon}$ is bounded by the sum of two terms:

$$A(r,\epsilon) := \frac{1}{\Gamma(\epsilon/2)} \sinh^{-\epsilon}(r) \int_{\sinh^2(r)}^{+\infty} x^{\epsilon/2-1}(1+x)^{\epsilon/2-\rho}\, dx,$$

which is $O(\epsilon.\exp((\epsilon - 2\rho)r)$ uniformly for $r \geq 1$ and small ϵ.
The second term is

$$B(r,\epsilon) := \frac{1}{\Gamma(\epsilon/2)} \sinh^{-\epsilon}(r) \int_0^{\sinh^2(r)} x^{\epsilon/2-1} \left| \left(1 - \frac{x}{\sinh^2(r)}\right)^{\epsilon/2+\alpha} - 1 \right| (1+x)^{\epsilon/2-\rho}\, dx.$$

But from the inequality $1-(1-h)^{\epsilon/2+\alpha} \leq \max((\epsilon/2+\alpha),1)\, h$ when h belongs to $[0,1]$, we deduce that $B(r,\epsilon)$ is $O(\epsilon.\exp((\epsilon-2\rho)r)$ uniformly for $r \geq 1$ and small ϵ. Moreover $\frac{\Gamma(\rho-\epsilon)}{\Gamma(\rho-\epsilon/2)} = 1 + O(\epsilon)$.
By the very definition of ϵ, we obtain

$$\Psi(-i(\rho-\epsilon)) = \int_0^{+\infty} (1 - \phi_{-i(\rho-\epsilon)}(r))\, dL(r) \sim (\rho\epsilon)^{\gamma}$$

when ϵ decreases to zero.
Integrating by parts, the tail $x \longrightarrow \int_x^{+\infty} dL(r)$ has an ordinary Laplace transform $\mathcal{L}(\epsilon)$ equivalent to

$$\rho^{\gamma}\epsilon^{\gamma-1} \quad \text{when} \quad \epsilon \longrightarrow 0.$$

We conclude by the usual Tauberian Theorem on the half line. □

3. Domains of attraction.

Since $\phi_{-iu}(r)$ has the same properties as const$\times\exp(-(\rho-u)r)$ when $r\geq 1$ and u goes to ρ, the Karamata Tauberian theorem turns out to be in the Jacobi setting:

Theorem 4. *Let μ be a probability on $\mathbb{R}^+$ and l be a slowly varying function at zero and λ in $]0,1[$. Then each of the relations are equivalent*

$$\mu([r,+\infty[)\sim\frac{1}{\Gamma(1-\lambda)}r^{-\lambda}l(\frac{1}{r})\quad when\ \ r\longrightarrow+\infty$$

$$\int_0^\infty\phi_{-iu}(r)d\mu(r)\sim l(\rho-u)(\rho-u)^\lambda\quad when\ \ u\longrightarrow\rho_-.$$

We are now in a position to characterize the domains of attraction by the usual Laplace transform method ([F] for instance).

Theorem 5. *A probability μ belongs to the domain of attraction of the radial stable process with exponent 2γ in $]0,2[$ if and only if one of the two equivalent conditions in the last theorem is satisfied with $\lambda=\gamma$. Hence the exponent $-\gamma$ is not at all -2γ as in the Euclidean case.*

4. The equivalent at the origin

Theorem 6. *The Lebesgue density of the Lévy measure $m^{(\gamma,\alpha,\beta)}(r)$ is equivalent to $\dfrac{2\gamma2^\gamma\Gamma(1+\alpha+\gamma)}{\Gamma(1+\alpha)\Gamma(1-\gamma)}\dfrac{1}{r^{2\gamma+1}}$ near 0. The same equivalent would be obtained for the stable radial process with index 2γ in the Euclidean $2\alpha+2$ dimensional case, (when the dimension is not an integer, we consider the Bessel-Kingman hypergroup with exponent α).*

Proof. Fix $\epsilon>0$ and split the integral of Theorem 2 in two parts. Then $\int_\epsilon^\infty\frac{q_u(r)}{u^{1+\gamma}}du$ has a finite limit when r decreases to 0. Now the density $q_u(r)$ is close to the Bessel density with index α. More precisely, the Girsanov theorem implies that $q_u(r)$ is equal to

$$\frac{2\pi^{\alpha+1}}{\Gamma(1+\alpha)}\frac{r^{\alpha+\frac{1}{2}}}{\sinh^{\alpha+\frac{1}{2}}(r)\cosh^{\beta+\frac{1}{2}}(r)}(2\pi u)^{-(\alpha+1)}\exp(-r^2/2u)\mathbb{E}_{0,u}\left(\exp(\int_0^u f(X_s)\,ds)\right)$$

where $(X_s,s\leq u)$ is a Bessel bridge started at 0 such that $X_u=r$ and f is a bounded function. See the details in [LR]. Here $\frac{2.\pi^{\alpha+1}}{\Gamma(\alpha+1)}$ is the "area" of the unit sphere in dimension $2\alpha+2$.

Hence we get the existence of a function $k(\epsilon)$ such that $\lim_{\epsilon\to0}k(\epsilon)=0$ and for every u and r in $]0,\epsilon[$,

$$\left|\sinh^{\alpha+\frac{1}{2}}(r)\cosh^{\beta+\frac{1}{2}}(r)\,q_u(r)-\frac{2}{\Gamma(\alpha+1)}(2u)^{-(\alpha+1)}\exp(-r^2/2u)\right|$$

is dominated by $k(\epsilon)(2\pi u)^{-(\alpha+1)}\exp(-r^2/2u)$. Then the result follows from Theorem 2.

Thus the tail is the same as in the Euclidean case with real dimension $2\alpha+2$, *ie* the Bessel-Kingman hypergroup of index α. Asymptotically, our exponent function is the Euclidean exponent $\left(\frac{p^2}{2}\right)^\gamma$, and the density of the Euclidean Levy measure is readily computed from the following Hankel formula ([B1] for instance):

$$\int_0^{+\infty} (1 - j_\alpha(x)) \frac{dx}{x^{1+2\gamma}} = \frac{\Gamma(1+\alpha)\Gamma(1-\gamma)}{2\gamma 2^{2\gamma}\Gamma(1+\alpha+\gamma)}. \quad \square$$

Remark. The results near the origin have obviously an Euclidean $2\alpha+2$ dimensional flavor. For instance, Voit ([V]) obtained a central limit result with the help of a bound on the difference between the Jacobi spherical functions and the Euclidean ones.

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