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## Convolution of Nörlund methods in non-archimedean fields

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### Abstract

In the present paper we obtain a few inclusion theorems for the convolution of Nörlund methods in the form  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$  in complete, non-trivially valued, non-archimedean fields.

Throughout the present paper  $K$  denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices and sequences, which are considered in the sequel, have entries in  $K$ . If  $A = (a_{nk})$ ,  $a_{nk} \in K$ ,  $n, k = 0, 1, 2, \dots$  is an infinite matrix, the  $A$ -transform  $Ax = \{(Ax)_n\}$  of the sequence  $x = \{x_k\}$ ,  $x_k \in K$ ,  $k = 0, 1, 2, \dots$  is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0, 1, 2, \dots,$$

where it is assumed that the series on the right converge. If  $\lim_{n \rightarrow \infty} (Ax)_n = s$ , we say that  $\{x_k\}$  is  $A$ -summable to  $s$ , written as  $x_k \rightarrow s(A)$  or  $A\text{-lim } x_k = s$ . If  $\lim_{n \rightarrow \infty} (Ax)_n = s$  whenever  $\lim_{k \rightarrow \infty} x_k = s$ , we say that  $A$  is regular. The following result is well-known (see [2], [4]).

**Theorem 1**  $A = (a_{nk})$  is regular if and only if

$$\sup_{n,k} |a_{nk}| < \infty; \quad (1)$$

$$\lim_{n \rightarrow \infty} a_{nk} = 0, \quad \text{for every fixed } k; \quad (2)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} = 1. \quad (3)$$

Any matrix  $A$  for which (1) holds is called a  $K_r$ -matrix. If  $A$  and  $B$  are two infinite matrices such that  $x_k \rightarrow s(A)$  implies  $x_k \rightarrow s(B)$ , we say that  $A$  is included in  $B$ , written as  $A \subseteq B$ .  $A$  is said to be row-finite if for  $n = 0, 1, 2, \dots$ , there exists a positive integer  $k_n$  such that  $a_{nk} = 0, k > k_n$ .

Given two infinite matrices  $A$  and  $B$ , their convolution is defined as the matrix  $C = (c_{nk})$ , where

$$c_{nk} = \sum_{i=0}^k a_{ni} b_{n,k-i}, \quad n, k = 0, 1, 2, \dots \quad (4)$$

In such a case we write  $C = A * B$ .

The following properties of the convolution can be easily proved.

1. If  $A$  and  $B$  are both row-finite or both  $K_r$ , then their convolution  $C$  is row-finite or  $K_r$  respectively and their row sums satisfy

$$\sum_{k=0}^{\infty} c_{nk} = \left( \sum_{k=0}^{\infty} a_{nk} \right) \left( \sum_{k=0}^{\infty} b_{nk} \right), \quad n = 0, 1, 2, \dots \quad (5)$$

2. If  $A, B$  are both regular, then  $C$  is regular too.

The Nörlund method of summability i.e.,  $(N, p_n)$  method in  $K$  is defined as follows (see [5]):  $(N, p_n)$  is defined by the infinite matrix  $(a_{nk})$  where

$$\begin{aligned} a_{nk} &= \frac{p_{n-k}}{P_n}, \quad k \leq n; \\ &= 0, \quad k > n, \end{aligned}$$

where  $p_0 \neq 0, |p_0| > |p_j|, j = 1, 2, \dots$  and  $P_n = \sum_{k=0}^n p_k, n = 0, 1, 2, \dots$ . It is to be noted that  $|P_n| = |p_0| \neq 0$  so that  $P_n \neq 0, n = 0, 1, 2, \dots$ .

The following result is very useful in the sequel.

**Theorem 2** (See [5], Theorem 1.)  $(N, p_n)$  is regular if and only if

$$p_n \rightarrow 0, \quad n \rightarrow \infty. \quad (6)$$

The purpose of the present paper is to prove a few inclusion theorems for the convolution of Nörlund methods in the form  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$ .

We need to define  $\{\bar{p}_n\}$  by

$$p_0\bar{p}_0 = 1, \quad p_0\bar{p}_n + p_1\bar{p}_{n-1} + \cdots + p_n\bar{p}_0 = 0, \quad n \geq 1 \quad (7)$$

$$\text{i.e., } \bar{p}(x) = \sum_{n=0}^{\infty} \bar{p}_n x^n = \frac{1}{\sum_{n=0}^{\infty} p_n x^n} = \frac{1}{p(x)}, \text{ assuming that these series converge.}$$

The following result is an easy consequence of Kojima-Schur theorem (see [2], [4]).

**Lemma 1** *Let  $A = (a_{nk})$  be a row-finite matrix and  $(N, p_n)$  be a regular Nörlund method. Then  $A$ -lim  $x_k$  exists whenever  $(N, p_n)$ -lim  $x_k$  exists if and only if*

$$\sup_{0 \leq \gamma \leq k_n} |P_\gamma \sum_{k=\gamma}^{k_n} a_{nk} \bar{p}_{k-\gamma}| = O(1), \quad n \rightarrow \infty; \quad (8)$$

$$\lim_{n \rightarrow \infty} \sum_{k=\gamma}^{k_n} a_{nk} \bar{p}_{k-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma; \quad (9)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{k_n} a_{nk} = \delta. \quad (10)$$

**Corollary 1**  $(N, p_n) \subseteq A$  if and only if (8), (9) and (10) hold with  $\delta_\gamma = 0$ ,  $\gamma = 0, 1, 2, \dots$  and  $\delta = 1$ .

**Corollary 2** If  $(N, p_n)$  and  $(N, q_n)$  are regular Nörlund methods, then  $(N, p_n) \subseteq (N, q_n)$  if and only if  $h_n \rightarrow 0$ ,  $n \rightarrow \infty$  where

$$h(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{\sum_{n=0}^{\infty} q_n x^n}{\sum_{n=0}^{\infty} p_n x^n} = \frac{q(x)}{p(x)}$$

(see [5]).

Let  $(N, p_n)$ ,  $(N, q_n)$ ,  $(N, r_n)$  be regular Nörlund methods. Let  $p_n(x) = \sum_{i=n}^{\infty} p_i x^i$ ,  $p_0(x) = p(x)$  with similar expressions for  $q_n(x)$ ,  $r_n(x)$ . Let

$$\begin{aligned} \frac{p(x)q(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \theta_\gamma x^\gamma; \\ \frac{p_{n+1}(x)q(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^\gamma; \\ \frac{p(x)q_{n+1}(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \beta_{n\gamma} x^\gamma, \end{aligned} \quad (11)$$

and

$$\frac{1}{r(x)} \{p(x)q(x) - p_{n+1}(x)q(x) - p(x)q_{n+1}(x)\} = \sum_{\gamma=0}^{\infty} \varphi_{n\gamma} x^\gamma. \quad (12)$$

It now follows that

$$\varphi_{n\gamma} = \theta_\gamma - \alpha_{n\gamma} - \beta_{n\gamma} \quad (13)$$

and

$$\alpha_{n\gamma} = \beta_{n\gamma} = 0, \quad 0 \leq \gamma \leq n. \quad (14)$$

Taking  $C = (N, p_n) * (N, q_n)$ ,  $C$  is a row-finite matrix with

$$c_{nk} = \frac{1}{P_n Q_n} \sum_{i=\max(0, k-n)}^{\min(k, n)} p_{n-i} q_{n-k+i} \quad \text{with } k_n = 2n. \quad (15)$$

We write

$$f_{n\gamma} = \sum_{k=\max(0, 2n-\gamma)}^{2n} c_{nk} \bar{r}_{k+\gamma-2n}, \quad n, \gamma \geq 0. \quad (16)$$

**Lemma 2**

$$P_n Q_n f_{n\gamma} = \varphi_{n\gamma}, \quad 0 \leq \gamma \leq 2n + 1. \quad (17)$$

**Proof.** The result follows as in [6].

**Theorem 3**  $(N, p_n) * (N, q_n)$ -lim  $x_k$  exists whenever  $(N, r_n)$ -lim  $x_k$  exists if and only if

$$\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \rightarrow \infty; \quad (18)$$

and

$$\lim_{n \rightarrow \infty} \frac{\varphi_{n, 2n-\gamma}}{P_n Q_n} = \delta_\gamma, \quad \text{for every fixed } \gamma. \quad (19)$$

**Proof.** Let  $(N, p_n) * (N, q_n)$ -lim  $x_k$  exist whenever  $(N, r_n)$ -lim  $x_k$  exists. Applying Lemma 1 with  $(N, p_n) = (N, r_n)$  and  $A = (N, p_n) * (N, q_n) = (c_{nk})$ , we have,

$$\sup_{0 \leq \gamma \leq 2n} |R_\gamma \sum_{k=\gamma}^{2n} c_{nk} \bar{r}_{k-\gamma}| = O(1), \quad n \rightarrow \infty \quad (20)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=\gamma}^{2n} c_{nk} \bar{r}_{k-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma. \quad (21)$$

Using (16) and (20), we get

$$\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} f_{n\gamma}| = O(1), \quad n \rightarrow \infty. \quad (22)$$

Using (17), we note that  $|f_{n\gamma}| = \frac{|\varphi_{n\gamma}|}{|p_0| |q_0|}$  since  $|P_n| = |p_0|$  and  $|Q_n| = |q_0|$ . Consequently, in view of (22), we get

$$\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \rightarrow \infty.$$

In view of (16) and (21), we have

$$\lim_{n \rightarrow \infty} f_{n, 2n-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma.$$

Now, using (17), we get

$$\lim_{n \rightarrow \infty} \frac{\varphi_{n, 2n-\gamma}}{P_n Q_n} = \delta_\gamma, \quad \text{for every fixed } \gamma.$$

Thus (18) and (19) hold. Conversely (18) and (19) imply (20) and (21) respectively. Using (5), we have,  $\lim_{n \rightarrow \infty} \sum_{k=0}^{2n} c_{nk} = 1$ . Using Lemma 1, the result follows, completing the proof of the theorem.

**Corollary 3**  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$  if and only if (18) and (19) hold with  $\delta_\gamma = 0$ .

**Corollary 4** If  $\lim_{n \rightarrow \infty} \bar{r}_n = 0$ , then  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$  if and only if (18) holds.

**Proof.** The result follows using (9) and the fact that  $(N, p_n) * (N, q_n)$  is regular.

**Theorem 4** If

$$\varphi_{n, 2n-\gamma} = o(1), \quad n \rightarrow \infty, \quad \text{for every fixed } \gamma, \quad (23)$$

and either

$$\varphi_{n\gamma} = O(1), \quad n, \gamma \rightarrow \infty, \quad (24)$$

or

$$\theta_\gamma, \alpha_{n\gamma}, \beta_{n\gamma} = O(1), \quad n, \gamma \rightarrow \infty, \quad (25)$$

then

$$(N, r_n) \subseteq (N, p_n) * (N, q_n).$$

**Proof.** Using (23), (19) follows with  $\delta_\gamma = 0$  since  $|P_n| = |p_0|$  and  $|Q_n| = |q_0|$ . Because of (13) and (25), (24) holds. So if (24) or (25) holds, (18) holds since  $R_n = O(1)$ ,  $n \rightarrow \infty$ ,  $(N, r_n)$  being a regular method. The result now follows from Corollary 3.

We shall now take up an application of Theorem 4.

**Theorem 5** Let  $\bar{p}_n, \bar{q}_n \rightarrow 0$ ,  $n \rightarrow \infty$  and  $t_n = p_0q_n + p_1q_{n-1} + \cdots + p_nq_0$ ,  $n = 0, 1, 2, \dots$ . Then

$$(N, t_n) \subseteq (N, p_n) * (N, q_n)$$

and

$$(N, p_n) \subseteq (N, t_n), \quad (N, q_n) \subseteq (N, t_n).$$

**Proof.** We apply Theorem 4 with  $r_n = t_n$ . With the usual notation we have  $t(x) = p(x)q(x)$  and  $\bar{t}(x) = \bar{p}(x)\bar{q}(x)$ . Since  $\bar{p}_n, \bar{q}_n \rightarrow 0$ ,  $n \rightarrow \infty$ ,  $\bar{t}_n \rightarrow 0$ ,  $n \rightarrow \infty$  (see [3], Theorem 1). Consequently (23) follows using (9). In view of (11), we have,

$$\sum_{\gamma=0}^{\infty} \theta_{\gamma} x^{\gamma} = \frac{p(x)q(x)}{t(x)} = 1,$$

so that

$$\theta_0 = 1 \text{ and } \theta_{\gamma} = 0, \quad \gamma \geq 1;$$

$$\sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^{\gamma} = \frac{p_{n+1}(x)q(x)}{t(x)} = p_{n+1}(x)\bar{p}(x),$$

so that

$$\begin{aligned} \alpha_{n\gamma} &= \sum_{\lambda=0}^{\gamma-(n+1)} \bar{p}_{\lambda} p_{\gamma-\lambda}, \quad \gamma \geq n+1; \\ &= 0, \quad 0 \leq \gamma \leq n. \end{aligned}$$

Consequently  $\alpha_{n\gamma} = O(1)$ ,  $n, \gamma \rightarrow \infty$ . Similarly  $\beta_{n\gamma} = O(1)$ ,  $n, \gamma \rightarrow \infty$ . In view of Theorem 4,  $(N, t_n) \subseteq (N, p_n) * (N, q_n)$ . Now  $\frac{t(x)}{p(x)} = q(x)$  and  $q_n \rightarrow 0$ ,  $n \rightarrow \infty$ ,  $(N, q_n)$  being regular, by (6). So by Corollary 2,  $(N, p_n) \subseteq (N, t_n)$ . Similarly  $(N, q_n) \subseteq (N, t_n)$ . The proof of the theorem is now complete.

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