

ANN VERDOODT

The construction of normal bases for the space of continuous functions on V_q , with the aid of operators

Annales mathématiques Blaise Pascal, tome 2, n° 1 (1995), p. 299-305

http://www.numdam.org/item?id=AMBP_1995__2_1_299_0

© Annales mathématiques Blaise Pascal, 1995, tous droits réservés.

L'accès aux archives de la revue « Annales mathématiques Blaise Pascal » (<http://math.univ-bpclermont.fr/ambp/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

THE CONSTRUCTION OF NORMAL BASES FOR THE SPACE OF CONTINUOUS FUNCTIONS ON V_q , WITH THE AID OF OPERATORS

Ann Verdoodt

Abstract. Let a and q be two units of $\mathbb{Z}_p$, q not a root of unity, and let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, \dots\}$. K is a non-archimedean valued field, K contains $\mathbb{Q}_p$, and K is complete for the valuation $|\cdot|$, which extends the p -adic valuation. $C(V_q \rightarrow K)$ is the Banach space of continuous functions from V_q to K , equipped with the supremum norm. Let $\mathcal{E}$ and D_q be the operators on $C(V_q \rightarrow K)$ defined by $(\mathcal{E}f)(x) = f(qx)$ and $(D_q f)(x) = (f(qx) - f(x))/(x(q-1))$. We will find all linear and continuous operators that commute with $\mathcal{E}$ (resp. with D_q), and we use these operators to find normal bases $(r_n(x))$ for $C(V_q \rightarrow K)$. If f is an element of $C(V_q \rightarrow K)$, then there exist elements α_n of K such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients α_n .

1991 Mathematics subject classification : 46S10

1. Introduction

Let p be a prime, $\mathbb{Z}_p$ the ring of the p -adic integers, $\mathbb{Q}_p$ the field of the p -adic numbers. K is a non-archimedean valued field, $K \supset \mathbb{Q}_p$, and we suppose that K is complete for the valuation $|\cdot|$, which extends the p -adic valuation. Let a and q be two units of $\mathbb{Z}_p$ (i.e. $|a| = |q| = 1$), q not a root of unity. Let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, \dots\}$. We denote by $C(V_q \rightarrow K)$ (resp. $C(\mathbb{Z}_p \rightarrow K)$) the set of all continuous functions $f : V_q \rightarrow K$ (resp. $f : \mathbb{Z}_p \rightarrow K$) equipped with the supremum norm. If f is an element of $C(V_q \rightarrow K)$ then we define the operators $\mathcal{E}$ and D_q as follows :

$$(\mathcal{E}f)(x) = f(qx)$$

$$(D_q f)(x) = \frac{f(qx) - f(x)}{x(q-1)}$$

We remark that the operator $\mathcal{E}$ does not commute with D_q . Furthermore, the operator D_q lowers the degree of a polynomial with one, whereas the operator $\mathcal{E}$ does not.

If $\mathcal{L}$ is a non-archimedean Banach space over a non-archimedean valued field L , and $e_1, e_2, \dots$ is a finite or infinite sequence of elements of $\mathcal{L}$, then we say that this sequence is orthogonal if $\|\epsilon_1 e_1 + \dots + \epsilon_k e_k\| = \max\{\|\epsilon_i e_i\| : i = 1, \dots, k\}$ for all k in $\mathbb{N}$ (or for all k that do not exceed the length of the sequence) and for all $\epsilon_1, \dots, \epsilon_k$ in L . An orthogonal sequence $e_1, e_2, \dots$ is called orthonormal if $\|e_i\| = 1$ for all i . A family (e_i) of elements of $\mathcal{L}$ forms a(n) (ortho)normal basis of $\mathcal{L}$ if the family (e_i) is orthonormal and also a basis. We will call a sequence of polynomials $(p_n(x))$ a polynomial sequence if p_n is exactly of degree n for all natural numbers n .

The aim here is to find normal bases for $C(V_q \rightarrow K)$, which consist of polynomial sequences. Therefore we will use linear, continuous operators which commute with D_q or with $\mathcal{E}$. If $(r_n(x))$ is such a polynomial sequence, and if f is an element of $C(V_q \rightarrow K)$, there exist coefficients α_n in K such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients α_n .

We remark that all the results (with proofs) in this paper can be found in [5], except for theorem 5.

2. Notations.

Let V_q , K and $C(V_q \rightarrow K)$ be as in the introduction. The supremum norm on $C(V_q \rightarrow K)$ will be denoted by $\|\cdot\|$. We introduce the following :

$$A_0(x) = 1, A_n(x) = (x - aq^{n-1})A_{n-1}(x) \quad (n \geq 1),$$

$$B_n(x) = A_n(x)/A_n(aq^n), \quad C_n(x) = a^n q^{n(n-1)/2} (q-1)^n B_n(x)$$

It is clear that $(A_n(x))$, $(B_n(x))$ and $(C_n(x))$ are polynomial sequences. The sequence $(C_n(x))$ forms a basis for $C(V_q \rightarrow K)$ and the sequence $(B_n(x))$ forms a normal basis for $C(V_q \rightarrow K)$. From this it follows that $\|B_n\| = 1$ and $\|C_n\| = |(q-1)^n|$. Let $\mathcal{E}$ and D_q be as in the introduction. Then we introduce the following :

Definition. Let f be a function from V_q to K . We define the following operators :

$$(D_q^n f)(x) = (D_q(D_q^{n-1} f))(x)$$

$$(\mathcal{E}^n f)(x) = f(q^n x)$$

$$\mathcal{D}f(x) = \mathcal{D}^{(1)}f(x) = f(qx) - f(x) = ((\mathcal{E} - 1)f)(x)$$

$$\mathcal{D}^{(n)}f(x) = ((\mathcal{E} - 1) \dots (\mathcal{E} - q^{n-1})f)(x), \quad \mathcal{D}^{(0)}f(x) = f(x)$$

The operator D_q does not commute with $\mathcal{D}$. The following properties are easily verified :

$D_q^j C_k(x) = C_{k-j}(x)$ if $k \geq j$, $D_q^j C_k(x) = 0$ if $j > k$. So D_q^j lowers the degree of a polynomial with j

$\mathcal{D}^{(j)} B_k(x) = (x/a)^j q^{j(j-k)} B_{k-j}(x)$ if $j \leq k$, $\mathcal{D}^{(j)} B_k(x) = 0$ if $j > k$

If $p(x)$ is a polynomial of degree n , then $(\mathcal{D}^{(j)} p)(x)$ is a polynomial of degree n if n is at least j , and $(\mathcal{D}^{(j)} p)(x)$ is the zero-polynomial if n is strictly smaller than j .

If f is an element of $C(V_q \rightarrow K)$, then we also have

$$\text{i) } (\mathcal{D}^{(n)} f)(x) = x^n q^{n(n-1)/2} (q-1)^n (D_q^n f)(x)$$

$$\text{ii) } (q-1)^n D_q^n f(x) \rightarrow 0 \text{ uniformly}$$

$$\text{iii) } \mathcal{D}^{(n)} f(x) \rightarrow 0 \text{ uniformly}$$

(i) can be found in [1] , p. 60 , ii) can be found in [3] , p. 124-125 , iii) follows from i) and ii)).

3. Linear Continuous Operators which Commute with $\mathcal{E}$ or with D_q

Let us start this section with the following known result :

If f is an element of $C(\mathbb{Z}_p \rightarrow K)$, then the translation operator E on $C(\mathbb{Z}_p \rightarrow K)$ is the operator defined by $Ef(x) = f(x+1)$.

If we put $G_n(x) = \binom{x}{n}$ (the binomial polynomials), then L. Van Hamme ([4]) proved the following theorem :

A linear , continuous operator Q on $C(\mathbb{Z}_p \rightarrow K)$ commutes with the translation operator E if and only if the sequence (g_n) is bounded , where $g_n = QG_n(0)$.

Such an operator Q can be written in the following way : $Q = \sum_{i=0}^{\infty} g_i \Delta^i$, where Δ is the operator defined as follows : $(\Delta f)(x) = f(x+1) - f(x)$

We can prove analogous theorems for the operators $\mathcal{E}$ and D_q on $C(V_q \rightarrow K)$:

Theorem 1 *An operator Q on $C(V_q \rightarrow K)$ is continuous , linear and commutes with $\mathcal{E}$ if and only if the sequence (b_n) is bounded, where $b_n = (QB_n)(a)$.*

From the proof of the theorem it follows that Q can be written in the form $Q = \sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)}$.

If f is an element of $C(V_q \rightarrow K)$, then $(Qf)(x) = \sum_{i=0}^{\infty} b_i (\mathcal{D}^{(i)} f)(x)$ and the series on the right-hand-side is uniformly convergent (since $\mathcal{D}^{(n)} f(x) \rightarrow 0$ uniformly) . Clearly we have

$$b_n = (QB_n)(a) , \text{ since } (QB_n)(a) = \left(\sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)} B_n \right)(a) = \left(\sum_{i=0}^n b_i (x/a)^i q^{i(i-n)} B_{n-i} \right)(a) = b_n.$$

Furthermore , Qx^n is a K -multiple of x^n .

If $b_0 = \dots = b_{N-1} = 0$, $b_N \neq 0$, and if $p(x)$ is a polynomial , then x^N divides $(Qp)(x)$.

Some examples

1) For the operator $\mathcal{E}$ we have : $(\mathcal{E} B_n)(x) = B_n(qx)$, so $(\mathcal{E} B_0)(a) = 1$, $(\mathcal{E} B_1)(a) = 1$, and $(\mathcal{E} B_n)(a) = 0$ if $n \geq 2$. This gives us $\mathcal{E} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)}$.

2) The operator $\mathcal{E} \circ \mathcal{D} = \mathcal{E}\mathcal{D}$ clearly commutes with $\mathcal{E}$. We have $((\mathcal{E}\mathcal{D})B_0)(a) = 0$, and since $(n \geq 1)$ $((\mathcal{E}\mathcal{D})B_n)(x) = (\mathcal{E}(\frac{x}{a}q^{1-n}B_{n-1}))(x) = \frac{qx}{a}q^{1-n}B_{n-1}(qx)$, we find $((\mathcal{E}\mathcal{D})B_1)(a) = q$, $((\mathcal{E}\mathcal{D})B_2)(a) = 1$ and $((\mathcal{E}\mathcal{D})B_n)(a) = 0$ if $n \geq 3$. We conclude that $\mathcal{E}\mathcal{D} = q\mathcal{D}^{(1)} + \mathcal{D}^{(2)}$.

Analogous to theorem 1 we have :

Theorem 2 An operator Q on $C(V_q \rightarrow K)$ is continuous, linear and commutes with D_q if and only if the sequence $(c_n/(q-1)^n)$ is bounded, where $c_n = (QC_n)(a)$.

Such an operator Q can be written in the form $Q = \sum_{i=0}^{\infty} c_i D_q^i$, and if f is an element of $C(V_q \rightarrow K)$ it follows that $(Qf)(x) = \sum_{i=0}^{\infty} c_i (D_q^i f)(x)$, where the series on the right-hand-side converges uniformly (since $(q-1)^n D_q^n f(x) \rightarrow 0$ uniformly). Furthermore, we have $c_n = (QC_n)(a)$ since

$$(QC_n)(a) = \left(\sum_{i=0}^{\infty} c_i D_q^i C_n \right)(a) = \sum_{i=0}^n c_i C_{n-i}(a) = c_n.$$

Remarks

1) Let R and Q be linear, continuous operators on $C(V_q \rightarrow K)$, with R of the form $R = \sum_{i=1}^{\infty} b_i D^{(i)}$ (i.e. R commutes with $\mathcal{E}$, $b_0 = 0$), and Q of the form $Q = \sum_{i=1}^{\infty} c_i D_q^i$ (i.e. Q commutes with D_q , $c_0 = 0$). The main difference between the operators Q and R is that Q lowers the degree of each polynomial with at least one, where R does not necessarily lower the degree of a polynomial.

2) If Q_1 and Q_2 both commute with D_q and if $Q_1 = \sum_{i=0}^{\infty} c_{1;i} D_q^i$,

$$Q_2 = \sum_{i=0}^{\infty} c_{2;i} D_q^i, \text{ then } (Q_1 \circ Q_2)(f) = (Q_2 \circ Q_1)(f) = \sum_{k=0}^{\infty} D_q^k f \left(\sum_{j=0}^k c_{1;j} c_{2;k-j} \right).$$

If we take two formal power series $q_1(t) = \sum_{i=0}^{\infty} c_{1;i} t^i$, $q_2(t) = \sum_{i=0}^{\infty} c_{2;i} t^i$, then

$$q_1(t) \cdot q_2(t) = \sum_{k=0}^{\infty} t^k \left(\sum_{j=0}^k c_{1;j} c_{2;k-j} \right), \text{ so the composition of two operators which commute}$$

with D_q , corresponds with multiplication of power series.

This is not the case if we take two operators which commute with $\mathcal{E}$: Take e.g. $\mathcal{E} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)}$ and $\mathcal{D}^{(1)}$, then $\mathcal{E} \circ \mathcal{D}^{(1)} = \mathcal{E} \mathcal{D}^{(1)} = q \mathcal{D}^{(1)} + \mathcal{D}^{(2)}$, whereas for power series this gives $q_1(t) = 1 + t$, $q_2(t) = t$ and $q_1(t) \cdot q_2(t) = t + t^2$.

4. Normal bases for $C(V_q \rightarrow K)$

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \rightarrow K)$. If Q is an operator as found in theorem 1, with b_0 equal to zero, we associate a (unique) polynomial sequence $(p_n(x))$ with Q . We remark that the operator $R = \sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)}$ does not necessarily lowers the degree of a polynomial.

Proposition 1 Let $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ ($N \geq 1$) with $|b_N| > |b_n|$ if $n > N$. There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

In the same way as in proposition 1 we have.

Proposition 2 Let $Q = \sum_{i=N}^{\infty} c_i \mathcal{D}_q^{(i)}$ ($N \geq 1$), $c_N \neq 0$, $(c_n/(q-1)^n)$ bounded.

Then there exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \rightarrow K)$. If f is an element of $C(V_q \rightarrow K)$, there exist coefficients α_n such that $f(x) = \sum_{n=0}^{\infty} \alpha_n p_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases, it is also possible to give an expression for the coefficients α_n .

Theorem 3 Let $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ ($N \geq 1$) with $|b_n| < |b_N| = 1$ if $n > N$

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \rightarrow K)$ and the norm of Q equals one.

2) If f is an element of $C(V_q \rightarrow K)$, then f can be written as a uniformly convergent series $f(x) = \sum_{n=0}^{\infty} \beta_n p_n(x)$, $\beta_n = ((D^{(i)}(x^{-N}Q)^k)f)(a)$ if $n = i + kN$ ($0 \leq i < N$), with

$\|f\| = \max_{0 \leq k; 0 \leq i < N} |((D^{(i)}(x^{-N}Q)^k)f)(a)|$, where $x^{-N}Q$ is a linear continuous operator with norm equal to one.

And analogous to theorem 3 we have

Theorem 4 Let $Q = \sum_{i=N}^{\infty} c_i D_q^i$ ($N \geq 1$) with $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ if $n > N$.

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \rightarrow K)$ and the norm of Q equals one.

2) If f is an element of $C(V_q \rightarrow K)$, there exists a unique, uniformly convergent expansion of the form $f(x) = \sum_{n=0}^{\infty} \gamma_n p_n(x)$, where $\gamma_n = a^i (q-1)^i q^{i(i-1)/2} (D_q^i Q^k f)(a)$ if $n = i + kN$ ($0 \leq i < N$), with $\|f\| = \max_{0 \leq k; 0 \leq i < N} \{|(q-1)^i (D_q^i Q^k f)(a)|\}$.

Remark. Here we have $|c_n| \leq |c_N|$, in contrast with theorem 3, where we need $|b_n| < |b_N|$ ($n > N$).

An example

Let us consider the following operator $Q = (q-1)D_q$. Then $c_1 = (q-1)$ and $c_k = 0$ if $k \neq 1$. The polynomials $p_k(x)$ are given by $p_k(x) = C_k(x)/(q-1)^k$, and they form a normal basis for $C(V_q \rightarrow K)$. The expansion $f(x) = \sum_{k=0}^{\infty} ((q-1)^k D_q^k f)(a) p_k(x) = \sum_{k=0}^{\infty} (D_q^k f)(a) C_k(x)$ is known as Jackson's interpolation formula ([2],[3]).

If Q is an operator as found in theorem 4, with N equal to one, then we can prove a theorem analogous to theorem 2:

Theorem 5 Let Q be an operator such that $Q = \sum_{i=1}^{\infty} c_i D_q^i$, with $|c_1| = |(q-1)|$,

$|c_n| \leq |(q-1)^n|$ if $n > 1$, and let $p_n(x)$ be the polynomial sequence as found in theorem 4.

An operator T on $C(V_q \rightarrow K)$ is continuous, linear and commutes with D_q if and only if T is of the form $T = \sum_{i=0}^{\infty} d_i Q^i$, where the sequence (d_n) is bounded, where $d_n = (Tp_n)(a)$.

Remark. In theorem 2 the sequence $(c_n/(q-1)^n)$ must be bounded, whereas here the sequence (d_n) must be bounded. This follows from the fact that the norm of the operator D_q equals $|q-1|^{-1}$, whereas the norm of the operator Q equals 1.

5. More Normal Bases

We want to make more normal bases, using the ones we found in theorems 3 and 4. For operators which commute with $\mathcal{E}$ we can prove the following theorem:

Theorem 6 Let $(p_n(x))$ be a polynomial sequence which forms a normal basis for $C(V_q \rightarrow K)$, and let $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ ($N \geq 0$) with $1 = |b_N| > |b_k|$ if $k > N$. If $Qp_n(x) = x^N r_{n-N}(x)$ ($n \geq N$), then the polynomial sequence $(r_k(x))$ forms a normal basis for $C(V_q \rightarrow K)$.

And analogous for operators which commute with the operator D_q we have :

Theorem 7 Let $(p_n(x))$ be a polynomial sequence which forms a normal basis for $C(V_q \rightarrow K)$, and let $Q = \sum_{i=N}^{\infty} c_i D_q^i$ ($N \geq 0$) with $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ if $n > N$. If $(Qp_n)(x) = r_{n-N}(x)$ ($n \geq N$), then the polynomial sequence $(r_k(x))$ forms a normal basis for $C(V_q \rightarrow K)$.

We remark that analogous results can be found on the space $C(\mathbb{Z}_p \rightarrow K)$ for linear continuous operators which commute with the translation operator E . The result analogous to theorems 3 and 4 for the case N equal to one, was found by L. Van Hamme (see [4]), and the extensive version of theorems 3 and 4, and the analogons of theorems 5, 6 and 7 can be found with proofs similar to the proofs of the theorems in this paper.

REFERENCES

- [1] F.H. Jackson , *Generalization of the Differential Operative Symbol with an Extended Form of Boole's Equation* , Messenger of Mathematics , vol. 38 , 1909 , p. 57-61.
- [2] F.H. Jackson , *q-form of Taylor's Theorem*, Messenger of Mathematics , vol 38 , (1909) p. 62-64.
- [3] L. Van Hamme , *Jackson's Interpolation Formula in p-adic Analysis* Proceedings of the Conference on p-adic Analysis , report nr . 7806 , Nijmegen, June 1978, p. 119-125.
- [4] L. Van Hamme, *Continuous Operators which commute with Translations , on the Space of Continuous Functions on $\mathbb{Z}_p$* , in " p-adic Functional Analysis " , Bayod / Martinez-Maurica / De Grande - De Kimpe (Editors) , p. 75-88 , Marcel Dekker , 1992.
- [5] A. Verdoordt , *The Use of Operators for the Construction of Normal Bases for the Space of Continuous Functions on V_q* , Bulletin of the Belgian Mathematical Society - Simon Stevin, vol 1, 1994, p.685-699.

Vrije Universiteit Brussel,
Faculty of Applied Sciences,
Pleinlaan 2,
B 1050 Brussels,
Belgium