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THE MACKEY-ARENS AND HAHN-BANACH THEOREMS

FOR SPACES OVER VALUED FIELDS

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Astract. Characterizations of the spherical completeness of a non-archimedean complete non-trivially valued field in terms of classical theorems of Functional Analysis are obtained.

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Spherical completeness

Throughout this paper $K = (K, | \, . \, |)$ will denote a non-archimedean complete valued field with a non-trivial valuation $| \, . \, |$. It is well-known that the absolute value function $| \cdot |$ of the field of the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ satisfies the following properties :

(i) $0 \leq |x|$, $|x| = 0$ iff $x = 0$,

(ii) $|x + y| \leq |x| + |y|$,

(iii) $|xy| = |x||y|$, $x, y \in \mathbb{R}$ or $x, y \in \mathbb{C}$.

If K is a field, then by a *valuation* on K we will mean a map $| \, . \, |$ of K into $\mathbb{R}$ satisfying the above properties; in this case $(K, | \, . \, |)$ will be called a *valued field*. We will assume that K is complete with respect to the natural metric of K.

It turns out that if K is not isomorphic to $\mathbb{R}$ or $\mathbb{C}$, then its valuation satisfies the following *strong triangle inequality*, cf. e.g. [12],

(ii') $|x + y| \leq \max\{|x|, |y|\}$, $x, y \in K$.

A valued field K whose valuation satisfies (ii') will be called *non-archimedean* and its valuation *non-archimedean*.

Let us first recall the following well-known result of Cantor

Theorem 0 *Let (X, ρ) be a metric space. Then it is complete iff every shrinking sequence of closed balls whose radii tend to zero has non-empty intersection.*

Consider the set $\mathbb{N}$ of the natural numbers endowed with the following metric ρ defined by $\rho(m,n) = 0$ if $m = n$ and $1 + \max(\frac{1}{m}, \frac{1}{n})$ if $m \neq n$.

Then the metric ρ is non-archimedean, i.e. $\rho(m,n) = 0$ iff either $m = n$, or $\rho(m,n) \leq \max\{\rho(m,k), \rho(k,n)\}$, for all $m,n,k \in \mathbb{N}$.

It is easy to see that every shrinking sequence of balls in $\mathbb{N}$ whose radii tend to zero has non-empty intersection; note that every ball whose radius is smaller than 1 contains exactly one point. On the other hand, the balls $B_{1+\frac{1}{1}}(1), B_{1+\frac{1}{2}}(2), \ldots$, form a decreasing sequence and their intersection is empty. This suggests the following, see Ingleton [3] :

A non-archimedean metric space (X, ρ) will be said to be *spherically complete* if the intersection of every shrinking sequence of its balls is non-empty.

Clearly spherical completeness implies completeness; the converse fails : The space $(\mathbb{N}, \rho)$ is complete but not spherically complete. We refer to [11] and [12] for more infomation concerning this property.

Theorem 1 *Let (X, ρ) be a non-archimedean metric space. Then (X, ρ) is spherically complete iff given an arbitrary family $\mathcal{B}$ of balls in X, no two of which are disjoint, then the intersection of the elements of $\mathcal{B}$ is non-empty.*

The aim of this note is to collect a few characterizations of the spherical completeness of K in terms of the Mackey-Arens, Hahn-Banach and weak Schauder basis theorems, respectively, see [5], [6], [7], [12].

The Mackey-Arens and Hahn-Banach theorems

The terms "*K-space*", "*topology*","*seminorm or norm*" will mean a Hausdorff locally convex space (lcs) over K, a locally convex topology (in the sense of Monna) and a non-archimedean seminorm (norm), respectively. A seminorm on a vector space E over K is *non-archimedean* if it satisfies condition (ii'). Clearly the topology τ generated by a norm is *locally convex*. Recall that a *topological vector space* (tvs) $E = (E, \tau)$ over K is *locally convex* [10] if τ has a basis of absolutely convex neighbourhoods of zero. A subset U of E is *absolutely convex* (in the sense of Monna [10]) if $\alpha x + \beta y \in U$, whenever $x, y \in U$, $\alpha, \beta \in$, $|\alpha| \leq 1, |\beta| \leq 1$. For the basic notions and properties concerning tvs and lcs over K we refer to [10], [11], [13].

A locally convex (lc) topology γ on (E, τ) is called *compatible* with τ, if τ and γ have the same continuous linear functionals; $(E,\tau)^* = (E,\gamma)^*$. (E,τ) is *dual-separating* if $(E,\tau)^*$ separates points of E. If G is a vector subspace of E, $\tau|G$ and τ/G denote the topology τ restricted to G and the quotient topology of the quotient space E/G, respectively. If α is a finer l.c. topology on E/G, we denote by $\gamma := \tau \vee \alpha$ the weakest l.c. topology on E such that $\tau \leq \gamma$, $\gamma/G = \alpha$, $\gamma|G = \tau|G$, cf. e.g. [1]. The sets $U \cap q^{-1}(V)$ compose a basis of neighbourhoods of zero for γ, where U, V run over bases of neighbourhoods of zero for τ and α, respectively, $q := EE/G$ is the quotient map. By $\sup\{\tau, \alpha\}$ we denote the weakest l.c. topology on E which is finer than τ and α.

By the *Mackey topology* $\mu(E, E^*)$ associated with a lcs $E = (E, \tau)$ we mean the finest locally convex topology on E compatible with τ. In [14] Van Tiel showed that every lcs over spherically complete K admits the Mackey topology.

In [3] Ingleton obtained a non-archimedean variant of the Hahn-Banach theorem for normed spaces, where K is spherically complete.

Theorem 2 *If $E = (E, \|\cdot\|)$ is a normed space over K and K is spherically complete and D is a subspace of E, then for every continuous linear functional $g \in D^*$ there exists a continuous linear extension $f \in E^*$ of g such that $\|g\| = \|f\|$.*

This suggests the following : A lcs E will be said to have the *Hahn-Banach Extension Property* (HBEP) [9] if for every subspace D every $g \in D^*$ can be extended to $f \in E^*$. It is known that every lcs over spherically complete K has the HBEP, cf. e.g. [11].

The following theorem characterizes the spherical completeness of K in terms of classical theorems of Functional Analysis; cf. also [5], [6] and [12], Theorem 4.15. The proof of our Theorem 3 uses some ideas of [4] extended to the non-archimedean case.

l^∞ (resp. c_0) denotes the space of the bounded sequences (resp. the sequences of limit 0) with coefficients in K.

Theorem 3 *The following conditions on K are equivalent :*
(i) *K is spherically complete.*
(ii) *There exists $g \in (l^\infty)^*$ such that $g(x) = \sum_n x_n$ for every $x \in c_0$.*
(iii) $(l^\infty/c_0)^* \neq 0$.
(iv) *Every lcs over K admits the Mackey topology.*
(v) *Every lcs over K (resp. K-normed space) has the HBEP.*
(vi) *The completion of a dual-separating lcs over K (resp. K-normed space) is dual-separating.*
(vii) *Every closed subspace of a dual-separating lcs over K (resp. K-normed space) is weakly closed.*
(viii) *For every lcs over K (resp. K-normed space) every weakly convergent sequence is convergent.*
(ix) *Every weak Schauder basis in a lcs over K (resp. K-normed space) is a Schauder basis.*

Proof By Theorem 4.15 of [12] conditions (i), (ii), (iii) are equivalent. (i) *implies* (iv) : [14], Theorem 4.17. (i) *implies* (v) : [3], [11]. The implications (v) *implies* (vi), (v) *implies* (vii) are obvious. (i) *implies* (viii) : see [7], Theorem 3, [2], Proposition 4.3. (viii) *implies* (ix) is obvious.

(iv) *implies* (i) : Assume that K is not spherically complete and consider the space l^∞ of K-valued bounded sequences endowed with the topology τ generated by the norm $\|x\| = \sup_n |x_n|$, $x = (x_n) \in l^\infty$. Let f be a non-zero linear function on l^∞ with $f|_{c_0} = 0$. Set $E := l^\infty$ and $F := c_0$. Define a linear functional h on the quotient space E/F by $h(q(x)) = f(x)$, where $q : E \to E/F$ is the quotient map. Let α be the quotient topology of E/F. Since $(E/F,\alpha)^* = 0$, see (iii) *implies* (i), F is dense in the weak topology $\sigma(E,E^*)$ (recall that $E^* = F$, [12], Theorem 4.17). Observe that on E/F there exists a K-normed topology β such that $(E/F,\alpha)$ and $(E/F,\beta)$ are isomorphic and h is continuous in the topology $\sup\{\alpha,\beta\}$. Indeed, choose $x_0 \in E/F$ such that $h(x_0) = 2$ and define a linear map $T : E/F \to E/F$ by $T(x) := x - h(x)x_0$, $x \in E/F$. Then $T^2 = id$. Define $\beta := T(\alpha)$ (the image topology). Then h is continuous in the topology $sup\{\alpha,\beta\}$.

Set $\gamma_\alpha := \sigma(E,E^*) \vee \alpha$, $\quad \gamma_\beta := \sigma(E,E^*) \vee \beta$. Then γ_α and γ_β are compatible with $\sigma(E,E^*)$, hence with τ. Assume that E admits the finest locally convex topology μ compatible with τ. Then $\sigma(E,E^*) \leq \sup\{\gamma_\alpha,\gamma_\beta\} \leq \mu$.

On the other hand $\sup\{\gamma_\alpha,\gamma_\beta\}/F = \sup\{\alpha,\beta\}$. Therefore f is continuous in $\sup\{\gamma_\alpha,\gamma_\beta\}$. Since f is not continuous in $\sigma(E,E^*)$ we get a contradiction. The proof is complete.

(vi) *implies* (i) : Assume that K is not spherically complete. By the Baire category theorem we find a dense subspace G of E with dim(E/G)=dim(E/F), where E and F are defined as above. Indeed, let $\{x_s\}_{s\in S}$ be a Hamel basis of E and (S_n) a partition of S such that $S = \bigcup_{n\in\mathbb{N}} S_n$ and card S_n=card S, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, we denote by G_n the vector space generated by the elements x_s when s runs in $\bigcup_{k=1}^{n} S_k$. Then we have $E = \bigcup_{n\in\mathbb{N}} G_n$ and $dim\ G_n = dim\ (E/G_n) = dim\ E, n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that G_m is dense in E. Hence we obtain a subspace G as required. Let α be a K-normed topology on E/G such that the spaces $(E/G,\alpha)$ and $(E/F,\tau/F)$ are isomorphic. Then the topology $\gamma := \tau \vee \alpha$ is compatible with τ and strictly finer than τ. Let E_0 be the completion of the dual-separating K-normed space (E,γ). Choose $x \in E_0 \backslash E$. There exists a sequence (x_n) in E and $y \in E$ such that $x_n \to x$ in E_0 and $x_n \to y$ in (E,τ). Then $f(x-y) = 0$ for all $f \in E_o^*$ but $x - y \neq 0$. This completes the proof.

(vii) *implies* (i) : Assume that K is not spherically complete. The space G constructed in the previous case is closed in (E,γ) and dense in $(E,\sigma(E,E^*))$, where $E^* := (E,\gamma)^*$.

(v) *implies* (i) : Assume that K is not spherically complete. Let (e_n) be the sequence of the unit vectors in E, where E is as above. Then $e_n \to 0$ in $\sigma(E,E^*)$, [13]. Clearly (e_n) is a normalized Schauder basis in F. If $x = (x_n) \in F$, then $x = \sum_n x_n e_n$. Set $g(x) := \sum_n x_n$. Then g is a well-defined continuous linear functional on F. Suppose that g has a continuous linear extension f to the whole space E. Then $f(e_n) \to 0$ but $g(e_n) = 1$ for all $n \in \mathbb{N}$, a contradiction.

(viii) *implies* (i) : See the proof of the previous implication.

(ix) *implies* (i) : Assume that K is not spherically complete. The sequence (e_n) is a Schauder basis in $(E,\sigma(E,E^*))$ but it is not a Schauder basis in the original topology of E. The second part of this sentence follows from the fact that E is not of countable type, cf. e.g. [12]. On the other hand, by Theorem 4.17 of [12] (and its proof) the space E is reflexive and for every $g \in E^*$ there exists $(a_n) \in F$ such that $g(x) = \sum_n x_n a_n$ for every

$x = (x_n) \in E$. Since $(E, \sigma(E, E^*))$ is a sequentially complete lcs [12], Theorem 9.6, then $\sum_{k=1}^{n} x_k e_k$ weakly converges to $x = (x_n)$.

Remark In [9] Martinez-Maurica and Perez-Garcia proved that whenever K is spherically complete, then the local convexity is a *three space property*, i.e. if E is an A-Banach tvs over K and F its subspace such that F and E/F are locally convex, then E is locally convex. Is the converse also true?

By $L(E, F)$ we denote the space of all continuous linear maps between lcs E and F. A topology α on E will be called *compatible* with the pair $(E, L(E, F))$ if $L((E, \alpha), F) = L(E, F)$; if $F =$, as usual we shall say that α is compatible with the dual pair (E, E^*), where $E^* := L(E, K)$.

A lcs space F will be said to have the *Mackey-Arens property* (MA-property) if for every lcs space E the finest topology $\mu(E, L(E, F))$ compatible with $(E, L(E, F))$ exists, [7].

As we have already mentioned Van Tiel [14] proved that if K is spherically complete, then K has the MA-property, i.e. every K-space E over spherically complete K admits the finest topology $\mu(E, E^*)$ compatible with the dual pair (E, E^*). We have already proved the converse : If K is not spherically complete, then ℓ^∞ does not admit the Mackey topology $\mu(\ell^\infty, (\ell^\infty)^*)$. Hence

Corollary *K is spherically complete iff it has the MA-property.*

On the other hand one has the following

Theorem 4 *Every spherically complete normed K-space $F = (F, \|.\|)$ has the MA-property.*

We shall need the following

Lemma 1 *Let E, F be two vector spaces over K, where F is endowed with a norm $\|.\|$ and p, q are seminorms on E. Let $T : E \to F$ be a linear map such that $\|(T(x))\| \leq \max(p(x), q(x))$. If F is spherically complete, then there exists two linear maps $T_i : E \to F$, $i = 1, 2$, such that $T = T_1 + T_2$ and $\|(T_1(x))\| \leq p(x)$, $\|(T_2(x))\| \leq q(x)$, $x \in E$.*

Proof Set $P(x, x) = T(x)$, $U(x, y) = \max\{p(x), q(y)\}$, $x, y \in E$. Then $U(x, y)$ is a seminorm on $E \times E$ and $\|(P(x, x))\| = \|(T(x))\| \leq \max\{p(x), q(x)\} = U(x, x)$. Since F is spherically complete, then by Ingleton theorem, cf. e.g. [6], Theorem 4.18, there exists a linear map $P_0 : E \times E \to F$ extending P such that $\|(P_0(x, y))\| \leq U(x, y)$, $x, y \in E$. To complete the proof it is enough to put $T_1(x) = P_0(x, 0)$, $T_2(x) = P_0(0, x)$.

We shall also need the following lemma. Its proof uses some ideas of [1] and [4].

Lemma 2 *Let E, F be two dual-separating K-spaces over non-spherically complete K and such that F is complete and E is an infinite dimensional metrizable and complete. Then E admits two topologies τ_1 and τ_2 strictly finer than the original one of E and compatible with the pair $(E, L(E, F))$ and such that the topology $\sup\{\tau_1, \tau_2\}$ is not compatible with $(E, L(E, F))$.*

Proof : Observe that E contains a dense subspace G with $\dim(E/G)=\dim(l^\infty/c_0)$. Let h be a non-zero linear functional on E vanishing on G. As above we construct on E two topologies τ_1 and τ_2 strictly finer than the original one τ of E such that $\tau_j|G = \tau|G$ and $(E/G, \tau_j/G)$ is isomorphic to the quotient space l^∞/c_0, $j = 1, 2$, and h is continuous in $\sup\{\tau_1, \tau_2\}$. We show that the topologies τ_j, $j = 1, 2$, are compatible with the pair $(E, L(E, F))$. Fix $j \in \{1, 2\}$ and non-zero $T \in L((E, \tau_j), F)$. There exists $x_0 \in E$ and $f \in F^*$ such that $f(T(x_0)) \neq 0$. Suppose that $T|G = \{0\}$. Then the map $q(x) \to f(Tx))$ defines a non-zero continuous linear functional on $(E/G, \tau_j/G)$, $q : E \to E/G$ is the quotient map. Since $(l^\infty/c_0)^* = \{0\}$, [12], Corollary 4.3, we get a contradiction. Hence $T|G$ is non-zero. Since G is dense in E and τ and τ_j coincide on G, there exists a continuous linear extension W of T to E. It is easy to see that $T = W$. Hence $T \in L(E, F)$. Finally the map $x \to h(x)y$, for fixed $y \in F$, defines a τ-discontinuous linear map H of E into F such that $H \in L((E, \sup\ \tau_1, \tau_2), F)$.

Proof of Theorem 4 Let $E = (E, \tau)$ be a lcs and $\mathcal{F}$ the family of all topologies on E compatible with $(E, L(E, F))$. It is enough to show that the topology $\mu := \sup \mathcal{F}$ belongs to $\mathcal{F}$. Let $T : (E, \mu) \to F$ be a continuous linear map. There exist seminorms p_j on E, $j = 1, \ldots, n$, continuous in topologies γ_j ($\gamma_j \in \mathcal{F}$), respectively, and $M > 0$ such that $|(Tx)\| \leq M \max_{1 \leq j \leq n} p_j(x)$ for every $x \in E$. Using Lemma 1 one shows that T is τ-continuous.

Remarks (1) There exist complete normed K-spaces having the MA-property which are not spherically complete. In fact, assume that K is spherically complete; then ℓ^∞ is spherically complete [12], p. 97; hence ℓ^∞ has the MA-property (by our Theorem 4). On the other hand there exists on the space ℓ^∞ another norm ν which is equivalent with the usual norm, such that (ℓ^∞, ν) is not spherically complete [12], p. 50 and p. 98. On the other hand the space (ℓ^∞, ν) has the MA-property.

(2) Let E be an infinite dimensional normed and complete K-space. Since $F := \prod_n E_n / \bigoplus_n E_n$, where $E_n = E$ for every $n \in \mathbb{N}$, is spherically complete for any K [12], Theorem 4.1, then by our Theorem 4 the space F has the MA-property. For concrete spaces put $E = \ell^\infty$; then $F = \ell^\infty/c_0$. If K is not spherically complete, then by Lemma 2 the space ℓ^∞ does not admit the Mackey topology $\mu(\ell^\infty, (\ell^\infty)^*)$ but ℓ^∞/c_0 has the MA-property. In particular there exists on ℓ^∞ the finest topology μ compatible with $(\ell^\infty, L(\ell^\infty, \ell^\infty/c_0))$.

(3) Let E and F be K-spaces and assume that E admits the Mackey topology $\mu = \mu(E, E^*)$. Then the finest topology on E compatible with $((E, \mu), L((E, \mu), F))$ exists and equals μ.

(4) In [13], Corollary 7.9, Schikhof proved that for polarly barrelled or polarly bornological K-spaces (E, τ) where K is not spherically complete, the finest *polar* topology $\mu(E, E^*)$ compatible with (E, E^*) exists and equals τ.

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