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# $(\mathrm{GL}_k \times \mathrm{Sym}_n)$-modules and Nabla of hook-indexed Schur functions

François Bergeron

Abstract The aim of this paper is to describe structural properties of spaces of diagonal rectangular harmonic polynomials in several sets (say $k$) of $n$ variables, both as $\mathrm{GL}_k$-modules and $\mathrm{Sym}_n$-modules. We construct explicit such modules associated to any hook shape partitions. For the two sets of variables case, we conjecture that the associated graded Frobenius characteristic corresponds to the effect of the operator Nabla on the corresponding hook-indexed Schur function, up to a usual renormalization. We prove identities that give indirect support to this conjecture, and show that its restriction to one set of variables holds. We further give indications on how the several sets context gives a better understanding of questions regarding the structures of these modules and the links between them.

## 1. Introduction

Our aim in this paper is to describe, for all hook-shape partitions $\rho$, new $(\mathrm{GL}_k \times \mathbb{S}_n)$-modules of $k$-variate diagonal harmonic polynomials, here denoted by $\mathcal{S}_\rho^{\langle k\rangle}$, whose $\mathbb{N}^k$-graded Frobenius characteristic specializes to $\nabla(\widehat{s}_\rho)$ when one sets $k=2$. Here $\widehat{s}_\rho$ stands for the normalized Schur function[(1)]

$$\widehat{s}_\rho := \left(\tfrac{-1}{qt}\right)^{\iota_\rho} s_\rho, \qquad \text{with} \qquad \iota_\rho := \sum_{\rho_i > i} \rho_i - i. \tag{1}$$

We also recall that $\nabla$ is the operator (introduced in [7]) on symmetric polynomials (with coefficients in $\mathbb{Q}(q,t)$) for which the "combinatorial Macdonald polynomials" $\widetilde{H}_\lambda(q,t;\boldsymbol{z})$ are joint eigenfunctions, with eigenvalue $T_\lambda = T_\lambda(q,t) := \prod_{(i,j)} q^i t^j$. Here the product is over the cells $(i,j)$ of $\lambda$. For more background on these notions, see [8, 20].

For hook-shape partitions $(a+1, 1^b)$, of $n = a+b+1$, we will use the **Frobenius notation** $(a\,|\,b)$ (see Figure 1). For example, we have

$$(4\,|\,0) = 5, \qquad (3\,|\,1) = 41, \qquad (2\,|\,2) = 311, \qquad (1\,|\,3) = 2111, \qquad (0\,|\,4) = 11111.$$

Observe that, for $\rho = (a\,|\,b)$, the value of $\iota(\rho)$ is simply equal to $a$.

Our graded modules $\mathcal{S}_\rho^{\langle k\rangle}$ are associated to modules $\mathcal{M}_\rho^{\langle k\rangle}$. These occur in a bi-filtration of $\mathrm{GL}_k \times \mathbb{S}_n$-modules (over the field $\mathbb{Q}$), with rows indexed by hooks going

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(1) We essentially use Macdonald's notations (see [21]), but with French conventions.

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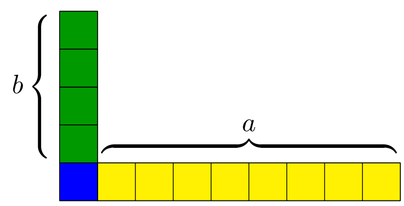


Figure 1. The hook shape $(a\,|\,b)$.

from $(n-1\,|\,0)$ to $(0\,|\,n-1)$, hence in increasing number of 1's; columns correspond to integers $k \in \mathbb{N}^+$ (as numbers of "sets" of variables)

$$\begin{tikzcd}
\mathcal{M}^{\langle 1\rangle}_{(n-1\,|\,0)} \arrow[r, hook] \arrow[d, hook] & \mathcal{M}^{\langle 2\rangle}_{(n-1\,|\,0)} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathcal{M}^{\langle k\rangle}_{(n-1\,|\,0)} \arrow[r, hook, dotted, "k\to\infty"] \arrow[d, hook] & \mathcal{M}_{(n-1\,|\,0)} \arrow[d, hook] \\
\mathcal{M}^{\langle 1\rangle}_{(n-2\,|\,1)} \arrow[r, hook] \arrow[d, hook] & \mathcal{M}^{\langle 2\rangle}_{(n-2\,|\,1)} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathcal{M}^{\langle k\rangle}_{(n-2\,|\,1)} \arrow[r, hook, dotted, "k\to\infty"] \arrow[d, hook] & \mathcal{M}_{(n-2\,|\,1)} \arrow[d, hook] \\
\vdots \arrow[d, hook] & \vdots \arrow[d, hook] & & \vdots \arrow[d, hook] & \vdots \arrow[d, hook] \\
\mathcal{M}^{\langle 1\rangle}_{(1\,|\,n-2)} \arrow[r, hook] \arrow[d, hook] & \mathcal{M}^{\langle 2\rangle}_{(1\,|\,n-2)} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathcal{M}^{\langle k\rangle}_{(1\,|\,n-2)} \arrow[r, hook, dotted, "k\to\infty"] \arrow[d, hook] & \mathcal{M}_{(1\,|\,n-2)} \arrow[d, hook] \\
\mathcal{M}^{\langle 1\rangle}_{(0\,|\,n-1)} \arrow[r, hook] & \mathcal{M}^{\langle 2\rangle}_{(0\,|\,n-1)} \arrow[r, hook] & \cdots \arrow[r, hook] & \mathcal{M}^{\langle k\rangle}_{(0\,|\,n-1)} \arrow[r, hook, dotted, "k\to\infty"] & \mathcal{M}_{(0\,|\,n-1)}
\end{tikzcd}$$

Each row stabilizes when $k$ becomes large enough; and we have the inductive limits

$$\mathcal{M}_{(a\,|\,b)} \coloneqq \lim_{k\to\infty} \mathcal{M}^{\langle k\rangle}_{(a\,|\,b)},$$

which are $\mathrm{GL}_\infty \times \mathbb{S}_n$-modules. It is convenient to set $\mathcal{M}_{(n\,|\,-1)} \coloneqq 0$, and then consider the quotient modules

$$\mathcal{S}_{(a\,|\,b)} = \mathcal{M}_{(a\,|\,b)}/\mathcal{M}_{(a+1\,|\,b-1)},$$

for all hooks $(a\,|\,b)$. Explicitly $\mathcal{M}_{(a\,|\,b)}$ is the smallest module which is:

- closed for "polarization" (see Equation 5 below),
- closed for derivation with respect to all variables except the $\theta_i$'s, and
- contains the determinant:

$$\boldsymbol{D}_{(a\,|\,b)}(\boldsymbol{x}) \coloneqq \det \begin{pmatrix} \theta_1 & 1 & x_1 & \cdots & \widehat{x_1^a} & \cdots & x_1^{n-1} \\ \theta_i & 1 & x_2 & \cdots & \widehat{x_2^a} & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \theta_n & 1 & x_n & \cdots & \widehat{x_n^a} & \cdots & x_n^{n-1} \end{pmatrix}, \tag{2}$$

where $\widehat{(-)}$ indicates that entries of that column are removed. Only the first column involves the variables $\theta_i$, which are said to be **inert** and considered to be of 0-degree. Observe that, if we remove the first column (the $\theta$-column) and keep all others, the result is the classical Vandermonde determinant. The inclusions occurring in the columns are easily obtained if one observes that

$$\sum_{i=1}^n \partial x_i\, \boldsymbol{D}_{(a\,|\,b)} = \begin{cases} (a+1)\, \boldsymbol{D}_{(a+1\,|\,b-1)}, & \text{if } 0 \leqslant a < n-1, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, $\mathcal{M}_\rho$ is a (multi-)homogeneous sub-module of the $\mathbb{N}^\infty$-graded ring $\boldsymbol{R} \coloneqq \mathbb{Q}[\boldsymbol{X}]$ of polynomials in the set of variables $\boldsymbol{X}$, consisting of a denumerable number of sets of $n$-variables.[2] The variables $\boldsymbol{X}$ are conveniently presented as an $\infty \times n$ matrix

$$\boldsymbol{X} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} \tag{3}$$

in which the first row is the "set" $\boldsymbol{x}$: of variables $x_i$, for $1 \leqslant i \leqslant n$. The grading of a polynomial $F$ in $\boldsymbol{R}$ is the sequence of degrees $\deg(F) = (\deg_{\boldsymbol{x}}(F), \deg_{\boldsymbol{y}}(F), \dots)$, respective to each of the rows of $\boldsymbol{X}$. The natural commuting actions of $\mathrm{GL}_\infty$ and $\mathbb{S}_n$, on polynomials $F(\boldsymbol{X})$ in $\boldsymbol{R}$, are jointly described by setting

$$F(\boldsymbol{X}) \mapsto F(g\boldsymbol{X}\sigma) \qquad \text{for} \qquad g \in \mathrm{GL}_\infty \quad \text{and} \quad \sigma \in \mathbb{S}_n, \tag{4}$$

where elements of the symmetric group $\mathbb{S}_n$ are here considered as $n \times n$ permutation matrices. For each pair $\boldsymbol{u} = (u_i)_i$ and $\boldsymbol{v} = (v_i)_i$ of rows of $\boldsymbol{X}$, and integer $r \geqslant 1$, the **(higher) polarization operator** $E^{(r)}_{\boldsymbol{uv}}$ is:

$$E^{(r)}_{\boldsymbol{uv}} \coloneqq \sum_{i=1}^{n} v_i \, \frac{\partial^r}{\partial u_i^r}. \tag{5}$$

We often drop the super index "$(r)$" when $r = 1$. We may formulate the definition of $\mathcal{M}_{(a\,|\,b)}$ as

$$\mathcal{M}_{(a\,|\,b)} \coloneqq \mathbb{Q}(\{\partial x_i\}_{x_i \in \boldsymbol{x}}; \{E^{(r)}_{\boldsymbol{uv}}\}_{r,\boldsymbol{u},\boldsymbol{v}})\, \boldsymbol{D}_{(a\,|\,b)}(\boldsymbol{x}).$$

The $\mathbb{N}^\infty$-graded Frobenius characteristic of such a module $\mathcal{M}_\rho$, for a hook $\rho = (a\,|\,b)$, is the generating function of the characters of its graded components, defined as:

$$\begin{aligned} \mathcal{M}_\rho(\boldsymbol{q}; \boldsymbol{z}) &\coloneqq \sum_{\boldsymbol{d} \in \mathbb{N}^\infty} \boldsymbol{q}^{\boldsymbol{d}} \sum_{\mu \vdash n} \chi^\rho_{\boldsymbol{d}}(\mu) \frac{p_\mu(\boldsymbol{z})}{z_\mu} \\ &= \sum_{\mu \vdash n} \Big( \sum_{f \in \mathcal{B}^\mu_\rho} \boldsymbol{q}^{\deg(f)} \Big) s_\mu(\boldsymbol{z}), \end{aligned} \tag{6}$$

where $\boldsymbol{q}^{\boldsymbol{d}} \coloneqq q_1^{d_1} q_2^{d_2} \cdots$, for $\boldsymbol{d} = (d_1, d_2, \dots)$, and $\chi^\rho_{\boldsymbol{d}}$ is the character of the $\boldsymbol{d}$-homogeneous component of $\mathcal{M}_\rho$. We recall that, with respect to the Frobenius map, irreducible $\mathbb{S}_n$-representations are precisely encoded by Schur functions $s_\mu(\boldsymbol{z})$, with $\boldsymbol{z} = (z_i)_i$ a set of formal variables. The expansion on the right-hand side of Equation 6 corresponds to the decomposition of $\mathcal{M}_\rho$ into $\mathbb{S}_n$-isotypic components $\mathcal{M}^\mu_\rho$, one for each partition $\mu$ of $n$. Indeed, these $\mathcal{M}^\mu_\rho$'s are clearly graded, and they afford bases of homogeneous polynomials $\mathcal{B}^\mu_\rho$. Hence, considering $\boldsymbol{q}$ as a formal diagonal matrix in $\mathrm{GL}_\infty$, we may express this homogeneity of $F \in \mathcal{B}^\mu_\rho$ as

$$F(\boldsymbol{q}\boldsymbol{X}) = \boldsymbol{q}^{\deg(F)} F(\boldsymbol{X}).$$

The coefficients of each $s_\mu(\boldsymbol{z})$ in Equation 6 may thus be considered, either as the Hilbert series of the corresponding $\mathbb{S}_n$-isotypic components, or as $\mathrm{GL}_\infty$-characters of (polynomial) representations. Recall that the characters of polynomial irreducible $\mathrm{GL}_\infty$-representations are also Schur functions (here in the variables $\boldsymbol{q}$). It follows that we have

$$\mathcal{M}_\rho(\boldsymbol{q}; \boldsymbol{z}) = \sum_{\mu \vdash n} \sum_{\lambda} a^\rho_{\lambda\mu} s_\lambda(\boldsymbol{q}) s_\mu(\boldsymbol{z}), \tag{7}$$

---

[2]The modules $\mathcal{M}^{\langle k \rangle}_\rho$ are likewise defined with variables restricted to the first $k$ rows of $\boldsymbol{X}$.

where each integer $a^\rho_{\lambda\mu}$ gives the number of copies of the $\mathrm{GL}_\infty$-irreducible having character $s_\lambda(\boldsymbol{q})$ in the $\mathbb{S}_n$-isotypic component $\mathcal{M}^\mu_\rho$.

In light of the discussion above, for $\rho = (a\,|\,b)$, we have

$$\begin{aligned} \mathcal{S}_\rho(\boldsymbol{q};\boldsymbol{z}) &:= \mathcal{M}_{(a\,|\,b)}(\boldsymbol{q};\boldsymbol{z}) - \mathcal{M}_{(a+1\,|\,b-1)}(\boldsymbol{q};\boldsymbol{z}) \\ &= \sum_{\mu\vdash n}\sum_\lambda c^\rho_{\lambda\mu} s_\lambda(\boldsymbol{q}) s_\mu(\boldsymbol{z}), \end{aligned} \tag{8}$$

with the **multiplicities** $c^\rho_{\lambda\mu} \in \mathbb{N}$ equal to $a^{(a\,|\,b)}_{\lambda\mu} - a^{(a+1\,|\,b-1)}_{\lambda\mu}$. It is convenient to express this in a "tensor" variable-free format

$$\mathcal{S}^\otimes_\rho = \sum_{\mu\vdash n}\sum_\lambda c^\rho_{\lambda\mu} s_\lambda \otimes s_\mu, \tag{9}$$

with the tensor product keeping track of the distinction between $\mathrm{GL}_\infty$-characters (left-hand side) and Frobenius of $\mathbb{S}_n$-irreducibles (right-hand side). The multiplicities $c^\rho_{\lambda\mu}$ are only non-vanishing when the partition $\lambda$ has at most $n-1$ parts, with size bounded by $\binom{n}{2} + b$ (for $\rho = (a\,|\,b)$). We denote by $\boldsymbol{c}_{\rho\mu}$ the **coefficient** of $s_\mu$ in $\mathcal{S}^\otimes_\rho$, and also write $\mathcal{A}_\rho$ for $\boldsymbol{c}_{\rho,1^n}$. In formulas:

$$\boldsymbol{c}_{\rho\mu} = \sum_\lambda c^\rho_{\lambda\mu} s_\lambda, \qquad \text{and} \qquad \mathcal{A}_\rho = \sum_\lambda c^\rho_{\lambda,1^n} s_\lambda. \tag{10}$$

We also consider the "scalar product" such that $\langle f \otimes s_\nu, s_\mu\rangle = f$ if $\nu = \mu$ and $0$ otherwise, so that $\boldsymbol{c}_{\rho\mu} = \langle \mathcal{S}^\otimes_\rho, s_\mu\rangle$. The **length restriction operator** $L_{\leqslant k}$ effect on $\mathcal{S}^\otimes_\rho$ is set to be:

$$L_{\leqslant k}(\mathcal{S}^\otimes_\rho) := \sum_{\mu\vdash n}\sum_{\ell(\lambda)\leqslant k} c^\rho_{\lambda\mu} s_\lambda \otimes s_\mu. \tag{11}$$

1.1. Effect of $\nabla$ on Schur functions indexed by hooks. To better express our main conjecture, we consider the following "variable free" $s_\lambda \otimes s_\mu$-expansion of $\nabla(\widehat{s}_\rho)$ (with $\rho$ a hook as above):

$$\nabla(\widehat{s}_\rho)^\otimes := \sum_{\mu\vdash n}\sum_\lambda b^\rho_{\lambda\mu} s_\lambda \otimes s_\mu \tag{12}$$

as an equivalent encoding of

$$\nabla(\widehat{s}_\rho)(q,t;\boldsymbol{z}) = \sum_{\mu\vdash n}\sum_\lambda b^\rho_{\lambda\mu} s_\lambda(q,t) s_\mu(\boldsymbol{z}).$$

To illustrate,

$$\nabla(\widehat{s}_{111})^\otimes = 1 \otimes s_3 + (s_1 + s_2) \otimes s_{21} + (s_{11} + s_3) \otimes s_{111},$$

encodes

$$\nabla(\widehat{s}_{111})(q,t;\boldsymbol{z}) = s_3(\boldsymbol{z}) + (q + t + q^2 + qt + t^2) s_{21}(\boldsymbol{z}) + (qt + q^3 + q^2t + qt^2 + t^3) s_{21}(\boldsymbol{z}).$$

Now, let

$$\delta^{(n)} := (n-1, n-2, \cdots, 2, 1, 0) \tag{13}$$

be the $n$**-staircase** partition (see Figure 2). A **Dyck path** $\gamma$ may be identified to a partition contained in $\delta^{(n)}$ (equivalently it lies below the **diagonal** going from $(0,n)$ to $(n,0)$), as is illustrated in Figure 2. For each row $\gamma_i$ of $\gamma \subseteq \delta^{(n)}$, one considers the **row area** $a_i = \delta^{(n)}_i - \gamma_i$. In other words, $a_i$ is the number of cells lying on the row $i$ between the Dyck path and the diagonal, and $\gamma = \delta^{(n)} - \alpha(\gamma)$, with $\alpha(\gamma) := (a_1, a_2, \ldots, a_n)$. It follows from results of [17], together with the compositional shuffle theorems of [12, 22], that we have the following combinatorial formula.

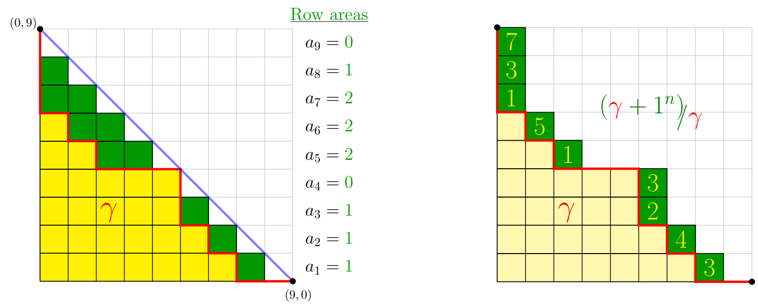


Figure 2. Dyck path $\boldsymbol{\gamma} = 765521000$, and one of its associated skew-shaped SSYT's.

Proposition 1.1 (Shuffle formula).

$$\nabla(\widehat{s}_{(a\,|\,b)})(q,t;\boldsymbol{z}) = \sum_{\gamma \subseteq \Gamma_a} t^{\mathrm{area}(\gamma)-a}\, \mathbb{L}_\gamma(q;\boldsymbol{z}), \tag{14}$$

*where $\gamma$ runs over the set of Dyck paths contained in*

$$\Gamma_a \coloneqq \delta^{(n)} - (0,\ldots,0,\underbrace{1,\ldots,1}_{a \text{ copies}},0); \tag{15}$$

*and $\mathbb{L}_\gamma$ is the associated LLT-polynomial.*

We recall that the LLT-polynomial $\mathbb{L}_\gamma(q;\boldsymbol{z})$, of a Dyck path $\gamma$, is an instance of vertical-strip LLT-polynomial (see [2], which includes a short survey of generalized LLT-polynomials). It is obtained as a weighted sum over the set $\mathrm{SSYT}((\gamma+1^n)/\gamma)$ of semi-standard Young tableaux[(3)] of skew shape $(\gamma+1^n)/\gamma$:

$$\mathbb{L}_\gamma(q;\boldsymbol{z}) \coloneqq \sum_{\tau \in \mathrm{SSYT}(\gamma+1^n)} t^{\mathrm{dinv}(\tau)} \boldsymbol{z}_\tau, \tag{16}$$

with $\boldsymbol{z}_\tau$ equal to the product of $z_i$ over entries $i$ of $\tau$. For details of the dinv-statistic for skew shape semi-standard Young tableaux, see [15]. It has been shown (see [16]) that $\mathbb{L}_\gamma$ is Schur-positive.

Until now, the combinatorial description Equation 14 of $\nabla(\widehat{s}_{(a\,|\,b)})$ lacked a representation theory counterpart, i.e. a module for which it is the graded Frobenius. We now propose the following.

Conjecture 1.2 (Modules). *For all hook-indexed shape $\rho = (a\,|\,b)$, $\mathcal{S}_\rho^\otimes$ is such that*

$$L_{\leqslant 2}(\mathcal{S}_\rho^\otimes) = \nabla(\widehat{s}_\rho)^\otimes. \tag{17}$$

*In other words, $c^\rho_{\lambda\mu} = b^\rho_{\lambda\mu}$ for all $\rho$, $\mu$, and partitions $\lambda$ having length at most two.*

For sure, to calculate $L_{\leqslant 2}(\mathcal{S}_\rho^\otimes)$, we need only use two sets of variables (the first two rows of $\boldsymbol{X}$). Hence, Conjecture 1.2 gives but a partial picture of $\mathcal{S}_\rho^\otimes$. As we will see below, the information contained in the "other" terms of $\mathcal{S}_\rho^\otimes$ plays an important role in understanding the global picture.

(3) Whose shape is the set of cells sitting immediately to the right a vertical step of the Dyck path $\gamma$. See Fig. 2.

For example, consider the hook $(a\,|\,b)=(2\,|\,0)$, so that $\mathcal{S}_{(2\,|\,0)}=\mathcal{M}_{(2\,|\,0)}$. Then,

$$\begin{aligned}\boldsymbol{D}_{(2\,|\,0)}(\boldsymbol{x}) &= \det\begin{pmatrix}\theta_1 & 1 & x_1\\ \theta_2 & 1 & x_2\\ \theta_3 & 1 & x_3\end{pmatrix}\\ &= \theta_1(x_3-x_2)-\theta_2(x_3-x_1)+\theta_3(x_2-x_1).\end{aligned}$$

One readily checks that the module $\mathcal{S}_{(2\,|\,0)}$ is spanned by the set of polynomials $\theta_1(u_3-u_2)-\theta_2(u_3-u_1)+\theta_3(u_2-u_1)$, one for each row $(u_1,u_2,u_3)$ of $\boldsymbol{X}$, together with the polynomials $\theta_1-\theta_2$ and $\theta_1-\theta_3$. Thus, its $\mathbb{N}^\infty$-graded Frobenius characteristic is equal to $\mathcal{S}_{(2\,|\,0)}(\boldsymbol{q};\boldsymbol{z})=s_{21}(\boldsymbol{z})+(q_1+q_2+\cdots)\,s_3(\boldsymbol{z})$; and we have the equality

$$\mathcal{S}^{\otimes}_{(2\,|\,0)}=\nabla(\widehat{s}_3)^{\otimes}=1\otimes s_{21}+s_1\otimes s_3. \tag{18}$$

The following table gives explicit calculated values, which complete the picture[(4)] for all cases with $n\leqslant 4$. It confirms that Conjecture 1.2 holds in these instances; and, we see that the smallest case for which $\mathcal{S}^{\otimes}_\rho$ is "larger" than $\nabla(\widehat{s}_\rho)^{\otimes}$ is for $\rho=1111$.

$$\begin{aligned}
&\mathcal{S}^{\otimes}_{1}=\nabla(\widehat{s}_1)^{\otimes}=1\otimes s_1;\\
&\mathcal{S}^{\otimes}_{2}=\nabla(\widehat{s}_2)^{\otimes}=1\otimes s_{11},\\
&\mathcal{S}^{\otimes}_{11}=\nabla(\widehat{s}_{11})^{\otimes}=1\otimes s_2+s_1\otimes s_{11};\\
&\mathcal{S}^{\otimes}_{21}=\nabla(\widehat{s}_{21})^{\otimes}=s_1\otimes s_{21}+s_2\otimes s_{111},\\
&\mathcal{S}^{\otimes}_{111}=\nabla(\widehat{s}_{111})^{\otimes}=1\otimes s_3+(s_1+s_2)\otimes s_{21}+(s_{11}+s_3)\otimes s_{111};\\
&\mathcal{S}^{\otimes}_{4}=\nabla(\widehat{s}_4)^{\otimes}=1\otimes s_{31}+s_1\otimes s_{22}+(s_1+s_2)\otimes s_{211}+(s_{11}+s_3)\otimes s_{1111},\\
&\mathcal{S}^{\otimes}_{31}=\nabla(\widehat{s}_{31})^{\otimes}=s_1\otimes s_{31}+s_2\otimes s_{22}+(s_{11}+s_2+s_3)\otimes s_{211}+(s_{21}+s_4)\otimes s_{1111},\\
&\mathcal{S}^{\otimes}_{211}=\nabla(\widehat{s}_{211})^{\otimes}=s_2\otimes s_{31}+(s_{11}+s_3)\otimes s_{22}+(s_{21}+s_3+s_4)\otimes s_{211}+(s_{31}+s_5)\otimes s_{1111}\\
&\mathcal{S}^{\otimes}_{1111}=1\otimes s_4+(s_1+s_2+s_3)\otimes s_{31}+(s_2+s_{21}+s_4)\otimes s_{22}\\
&\qquad+(s_{11}+s_{21}+s_{31}+s_3+s_4+s_5)\otimes s_{211}\\
&\qquad+(s_{111}+s_{31}+s_{41}+s_6)\otimes s_{1111}=\nabla(\widehat{s}_{1111})^{\otimes}+s_{111}\otimes s_{1111}.
\end{aligned}$$

We observe, for values in this table, that we have

Conjecture 1.3 (Skew). *For all $n$,*

$$(\mathrm{Id}\otimes e_1^{\perp})\,\mathcal{S}^{\otimes}_{(n)}=\sum_{a=0}^{n-2}\mathcal{S}^{\otimes}_{(a\,|\,n-a-2)}, \tag{19}$$

$$(e_1^{\perp}\otimes \mathrm{Id})\,\mathcal{S}^{\otimes}_{1^n}=\sum_{a=1}^{n-1}\mathcal{S}^{\otimes}_{(a\,|\,n-a-1)}. \tag{20}$$

These identities have been checked to hold for all $n\leqslant 6$. In particular, using Equations (1.7) and (1.10) of [18] and assuming a conjecture of [5] recalled further below as **??**, we get that

Theorem 1.4. *The length 2 restriction of Equation 19 holds, and Conjecture 1.2 implies that the length 2 restriction of Equation 20 is also true.*

**Proof.** To show the first equality, we observe that

$$L_{\leqslant 2}((\mathrm{Id}\otimes e_1^{\perp})\,\mathcal{S}^{\otimes}_{(n)})=(\mathrm{Id}\otimes e_1^{\perp})\,L_{\leqslant 2}(\mathcal{S}^{\otimes}_{(n)}),$$

(4) More values may be found in the appendix.

since length restriction on the left-hand side of a tensor $s_\lambda \otimes s_\mu$ is clearly independent from operators acting on the right-hand side. Moreover, from the point of view of representation theory, the operator $(\mathrm{Id} \otimes e_1^\perp)$ corresponds to restriction of the $\mathbb{S}_n$-action to the subgroup $\mathbb{S}_{n-1}$, of permutations that fix $n$. As discussed in [6], $(\mathrm{Id} \otimes e_1^\perp)\, L_{\leqslant 2}(\mathcal{S}_{(n)}^\otimes)$ may be identified with the derivation-polarization span of the determinant

$$\boldsymbol{D}^-(\boldsymbol{x}) := \det \begin{pmatrix} \theta_1 & x_1 & \cdots & x_1^{n-2} \\ \theta_i & x_2 & \cdots & x_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{n-1} & x_{n-1} & \cdots & x_{n-1}^{n-2} \end{pmatrix}, \tag{21}$$

since we can get a basis without involving the variable $x_n$. This is made clear in the discussion preceding (I.4) in [6]. In other terms, $(\mathrm{Id} \otimes e_1^\perp)\, L_{\leqslant 2}(\mathcal{S}_{(n)}^\otimes)$ is isomorphic to $\mathcal{M}_{(1\,|\,n-2)}^{\langle 2 \rangle}$. Thus, we get

$$(\mathrm{Id} \otimes e_1^\perp)\, L_{\leqslant 2}(\mathcal{S}_{(n)}^\otimes) = \sum_{a=0}^{n-2} L_{\leqslant 2}(\mathcal{S}_{(a\,|\,n-a-2)}^\otimes),$$

simply unfolding the definition $\mathcal{S}_{(a\,|\,b)} := \mathcal{M}_{(a\,|\,b)}/\mathcal{M}_{(a+1\,|\,b-1)}$. Next, assuming Conjecture 1.2, Equation 20 corresponds to

$$e_1^\perp \nabla(\widehat{s}_n)^\otimes = \sum_{a=0}^{n-2} \nabla(\widehat{s}_{(a\,|\,n-a-2)})^\otimes. \tag{22}$$

This is shown to hold as follows. Using Formulas (I.12) from [8, page 368], we get the operator identity $e_1^\perp \nabla = \nabla\,(e_1^\perp \Delta_{e_1} - \Delta_{e_1} e_1^\perp)$, with some rewriting. Hence Equation 22 is equivalent to

$$(e_1^\perp \Delta_{e_1} - \Delta_{e_1} e_1^\perp)\, \widehat{s}_n = \sum_{a=0}^{n-2} \widehat{s}_{(a\,|\,n-a-2)}.$$

But this follows easily from

$$\Delta_{e_1} \widehat{s}_n = \sum_{a=0}^{n-1} \widehat{s}_{(a\,|\,n-a-1)},$$

which, up to a multiplicative factor, is Prop. 6.5 of [9].

Now, the length-2 restriction of the second identity corresponds, modulo Conjecture 1.2, to the equality:

$$L_{\leqslant 2}((e_1^\perp \otimes \mathrm{Id})\, \mathcal{S}_{1^n}^\otimes) = \sum_{a=1}^{n-1} \nabla(\widehat{s}_{(a\,|\,b)})^\otimes,$$

but **??** states that $\Delta'_{e_{n-2}} e_n = L_{\leqslant 2}((e_1^\perp \otimes \mathrm{Id})\, \mathcal{S}_{1^n}^\otimes)$. Hence the second statement also holds. □

1.2. Links between the $\mathcal{M}_\rho$'s. We first recall that the Garsia-Haiman module $\mathcal{G}_\mu$ gives a representation theoretical interpretation for the combinatorial Macdonald polynomials $\widetilde{H}_\mu$. For any **diagram** (a finite subset $\boldsymbol{d}$ of $\mathbb{N} \times \mathbb{N}$), one may consider the determinant

$$\boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x}, \boldsymbol{y}) := \det(x_i^k y_i^\ell),$$

with $1 \leqslant i \leqslant n = \operatorname{card} \boldsymbol{d}$, and $(k, \ell) \in \boldsymbol{d}$. To make the sign of $\boldsymbol{D}_{\boldsymbol{d}}$ unambiguous, cells of $\boldsymbol{d}$ are ordered so that $(k', \ell') \prec (k, \ell)$, if $\ell' > \ell$, or if $\ell' = \ell$ and $k' < k$. For any $\boldsymbol{d}$ we then consider the derivation closure

$$\mathcal{G}_{\boldsymbol{d}} = \mathbb{Q}(\partial x_1, \ldots, \partial x_n; \partial y_1, \ldots, \partial y_n) \boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x}, \boldsymbol{y}),$$

where $\boldsymbol{d}$ is the Ferrers diagram (with "French" convention) of a partition $\mu$. We recall that the following holds.

Theorem 1.5 ($n$!-Theorem, Haiman [19]). *The combinatorial Macdonald polynomial* $\widetilde{H}_\mu(q,t;\boldsymbol{z})$ *is the bigraded Frobenius characteristic of the module* $\mathcal{G}_\mu$.

Beside this case of Ferrers diagrams, modules associated to Ferrers diagrams "missing" one cell are (conjecturally) described in [6]. Observe that the determinant in Equation 2 is obtained by replacing in $\boldsymbol{D}_{\boldsymbol{d}}$ the variables $\boldsymbol{y}$ by the inert variables $\boldsymbol{\theta}$, for the diagram[(5)]

$$\boldsymbol{d} = \boldsymbol{d}(a,b) \coloneqq \{(i,0) \mid 0 \leqslant i \leqslant a+b, \text{ and } i \neq a\} \cup \{(0,1)\},$$

which is illustrated in Figure 3. Thus, the module $\mathcal{M}^{\langle 1\rangle}_{(a\,|\,b)}$ corresponds to the top $\boldsymbol{y}$-degree of $\mathcal{G}_{\boldsymbol{d}(a,b)}$; and this top degree is equal to 1. Its (simply)-graded[(6)] Frobenius characteristics is thus

$$\mathcal{M}_{(a\,|\,b)}(q;\boldsymbol{z}) = \mathcal{G}_{\boldsymbol{d}(a,b)}(q,t;\boldsymbol{z})\big|_{\text{coeff } t}.$$

In particular, the diagram $\boldsymbol{d}(n-1,0)$ happens to be the hook-shape $(n-2\,|\,1)$. In

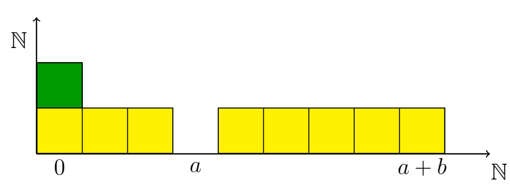


Figure 3. The diagram $\boldsymbol{d}(a,b)$.

view of the $n$!-Theorem above we get

$$\begin{aligned}\mathcal{M}_{(n-1\,|\,0)}(q;\boldsymbol{z}) &= \widetilde{H}_{(n-2\,|\,1)}(q,t;\boldsymbol{z})\big|_{\text{coeff } t}\\ &= \boldsymbol{H}_{(n-2\,|\,1)}(q;\boldsymbol{z}),\end{aligned}\tag{23}$$

where $\big|_{\text{coeff } t}$ means that we take the coefficient of $t$, with the right-hand side of the above identity being an instance, with $\mu = (n-2\,|\,1)$, of symmetric polynomials that we denote by $\boldsymbol{H}_\mu(q;\boldsymbol{z})$. These are directly related to the dual Hall-Littlewood symmetric functions $Q'_\mu$ (following Macdonald's [21, Exer.7, page 234] notation; see also [18]), which are such that

$$Q'_\mu(q;\boldsymbol{z}) = \sum_\lambda K_{\lambda\mu}(q)\, s_\lambda(\boldsymbol{z}),$$

where the $K_{\lambda\mu}(q) \in \mathbb{N}[q]$ are the Kostka-Foulkes polynomials. More precisely, we have $\boldsymbol{H}_\mu = \omega Q'_{\mu'}$, so that

$$\boldsymbol{H}_\mu(q;\boldsymbol{z}) = \sum_\lambda K_{\lambda'\mu'}(q)\, s_\lambda(\boldsymbol{z}).\tag{24}$$

Now, letting $\pi$ stand for the operator $\pi \coloneqq \sum_{i=1}^n \partial x_i$, it is clear that $\pi\boldsymbol{D}_{(a\,|\,b)}(\boldsymbol{x}) = \boldsymbol{D}_{(a+1\,|\,b-1)}(\boldsymbol{x})$. It follows that we have the following projection, and associated degree 1 reducing isomorphism:

$$\mathcal{M}^{\langle 1\rangle}_{(a\,|\,b)} \xrightarrow{\ \pi\ } \mathrel{\mkern-14mu}\rightarrow \mathcal{M}^{\langle 1\rangle}_{(a+1\,|\,b-1)}, \qquad \text{and} \qquad \mathcal{S}^{\langle 1\rangle}_{(a\,|\,b)} \xrightarrow{\ \sim\ } \mathcal{S}^{\langle 1\rangle}_{(a+1\,|\,b-1)}.$$

(5) Careful, this is not the Ferrers diagram of the hook $(a\,|\,b)$.

(6) Obtained by setting $q_1 = q$, and all other $q_i$ equal to 0.

This last isomorphism translates into the equality $\mathcal{S}_{(a\,|\,b)}(q;\boldsymbol{z}) = q\,\mathcal{S}_{(a+1\,|\,b-1)}(q;\boldsymbol{z})$, and we conclude by Equation 23 that

$$\mathcal{M}_{(a\,|\,b)}(q;\boldsymbol{z}) = (1+q+\cdots+q^b)\,\boldsymbol{H}_{(n-2\,|\,1)}(q;\boldsymbol{z}), \tag{25}$$

where $a+b=n-1$. It is interesting to notice that, together with Conjecture 1.2, a particular case of Identity 1.7 of [18] also gives

$$\mathcal{M}_{(1\,|\,n-2)}(q;\boldsymbol{z}) = \Delta'_{e_{n-2}}(q,0;\boldsymbol{z}). \tag{26}$$

Moreover, using Equation 14, we have the LLT-polynomial expression

$$\nabla(\widehat{s}_{(a\,|\,b)})(q,0;\boldsymbol{z}) = \mathbb{L}_{\Gamma_a}(q;\boldsymbol{z}) \tag{27}$$

since $\Gamma_a$ is the only Dyck path for which $\mathrm{area}(\gamma)-a=0$. From all this, we get the following.

Proposition 1.6. *Conjecture* 1.2 *holds when* $t=0$.

1.3. Link to intersection of Garsia-Haiman modules. The main objective of paper [7] is to describe the decomposition of families of Garsia-Haiman modules indexed by partitions of $n$ (covered by a given partition $\mu$ of $n+1$), with respect to their relative intersections. In the particular case when $\mu=(n-1\,|\,1)$, one may thus consider the two hook partitions $(n-1,1)=(n-2\,|\,1)$ and $(n)=(n-1\,|\,0)$. A special case of the conjectures stated therein, asserts that the bi-graded Frobenius of the intersection $\mathcal{I}_n \coloneqq \mathcal{G}_{\boldsymbol{d}(n-2,1)} \cap \mathcal{G}_{\boldsymbol{d}(n-1,0)}$ is given by the formula

$$\mathcal{I}_n(q,t;\boldsymbol{z}) = \frac{q^{n-1}\widetilde{H}_{(n-1,1)} - t\widetilde{H}_{(n)}}{q^{n-1}-t}. \tag{28}$$

Moreover, still assuming conjectures of [7], the module $\mathcal{G}_{\boldsymbol{d}(n-2,1)}$ decomposes as $\mathcal{G}_{\boldsymbol{d}(n-2,1)} = \mathcal{I}_n \oplus \mathcal{I}_n^{\perp}$, with

$$\mathcal{I}_n^{\perp} = \{f(\partial\boldsymbol{x})\boldsymbol{D}_{\boldsymbol{d}(n-2,1)}(\boldsymbol{x}) \mid f(\partial\boldsymbol{x}) \in \mathcal{I}_n\}.$$

It may be shown that both $\mathcal{I}_n$ and $\mathcal{I}_n^{\perp}$ are of dimension $n!/2$. It follows that

$$\begin{aligned}
\mathcal{I}_n^{\perp}(q,t;\boldsymbol{z}) &= \widetilde{H}_{(n-1,1)}(q,t;\boldsymbol{z}) - \frac{q^{n-1}\widetilde{H}_{(n-1,1)}(q,t;\boldsymbol{z}) - t\widetilde{H}_{(n)}(q,t;\boldsymbol{z})}{q^{n-1}-t}\\
&= \frac{t}{q^{n-1}-t}(\widetilde{H}_{(n-1,1)}(q,t;\boldsymbol{z}) - \widetilde{H}_{(n)}(q,t;\boldsymbol{z}))\\
&= t\,\boldsymbol{H}_{(n-1,1)}(q;\boldsymbol{z}).
\end{aligned}$$

Moreover, $\mathcal{I}_n(q,t;\boldsymbol{z}) = q^{\binom{n-1}{2}}\omega\boldsymbol{H}_{(n-1,1)}(1/q;\boldsymbol{z})$, so that we get

$$\widetilde{H}_{(n-1,1)}(q,t;\boldsymbol{z}) = q^{\binom{n-1}{2}}\omega\boldsymbol{H}_{(n-1,1)}(1/q;\boldsymbol{z}) + t\,\boldsymbol{H}_{(n-1,1)}(q;\boldsymbol{z}). \tag{29}$$

For example, with $n=5$, we get

$$\begin{aligned}
\widetilde{H}_{41}(q,t;\boldsymbol{z}) =&(q^6 s_{2111} + (q^4+q^5)s_{221} + (q^3+q^4+q^5)s_{311} + (q^2+q^3+q^4)s_{32}\\
&\quad + (q+q^2+q^3)s_{41} + s_5)\\
&+ t\,(s_{41} + (q^2+q)s_{32} + (q^3+q^2+q)s_{311} + (q^4+q^3+q^2)s_{221}\\
&\quad + (q^5+q^4+q^3)s_{2111} + q^6 s_{11111}),
\end{aligned}$$

where each term $q^a s_\nu$ in the top portion corresponds to a term $tq^{6-a}s_{\nu'}$ in the bottom portion. All Schur functions are in the variables $\boldsymbol{z}$. Setting $q=1$, we may check that Equation 29 specializes to

$$\widetilde{H}_{(n-1,1)}(1,t;\boldsymbol{z}) = h_{21^{n-2}}(\boldsymbol{z}) + t\,e_{21^{n-2}}(\boldsymbol{z}).$$

We may use these observations to explicitly construct a basis of $\mathcal{S}^{\langle 1\rangle}_{(a\,|\,b)}$ in the following manner. Let $\mathcal{B}_n$ be a basis of $\mathcal{I}_n$, then

Lemma 1.7. *For each $a \geqslant 1$, with $b = n - a - 1$ and $\boldsymbol{d} := \boldsymbol{d}(a,b)$, the set $\{\varphi(\partial\boldsymbol{x})\boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x}) \mid \varphi(\boldsymbol{x}) \in \mathcal{B}_n\}$ forms a basis of $\mathcal{S}^{\langle 1\rangle}_{(a\,|\,b)}$.*

*Proof.* We only check that the proposed set is linearly independent. By hypothesis, we already know that any $\varphi(\boldsymbol{x})$ in the span of $\mathcal{B}_n$ is of the form $\varphi(\boldsymbol{x}) = \psi(\partial\boldsymbol{x})\boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x})$, for some polynomial $\psi(\boldsymbol{x})$. We may verify that $\varphi(\partial\boldsymbol{x})\boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x}) \neq 0$, as follows. By commutation,

$$\psi(\partial\boldsymbol{x})\,(\varphi(\partial\boldsymbol{x})\boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x})) = \varphi(\partial\boldsymbol{x})\,(\psi(\partial\boldsymbol{x})\boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x})) = \varphi(\partial\boldsymbol{x})\varphi(\boldsymbol{x}).$$

This last expression does not vanish, since its constant term is the sum of the square of coefficients of $\varphi(\boldsymbol{x})$, hence is not equal to 0. It follows that the map sending $\varphi(\boldsymbol{x})$ to $\varphi(\partial\boldsymbol{x})\boldsymbol{D}_{\boldsymbol{d}}(\boldsymbol{x})$ is injective, and we get the expected linear independence. □

Recall that $\mathcal{M}^{\langle 1\rangle}_{(n)}$, which is the classical module of $\mathbb{S}_n$-harmonic polynomials, is known to decompose into irreducibles given by (homogeneous) higher Specht modules (see [3]). One may readily exploit this to construct a basis $\mathcal{B}_n$ which reflects the fact that $\mathcal{I}_n$ is a graded sub-module of $\mathcal{M}^{\langle 1\rangle}_{(n)}$.

1.4. Hook components conjecture. Our second conjecture describes a link between the alternant isotypic component of $\mathcal{S}_\rho$ and the isotypic components corresponding to hooks $(a\,|\,b)$. We have:

Conjecture 1.8 (Hook Components). *For all $\rho$, and all $0 \leqslant a \leqslant n-1$, the hook-component coefficients are obtained as*

$$\boldsymbol{c}_{\rho,(a\,|\,b)} = e_a^{\perp}\mathcal{A}_\rho. \tag{30}$$

Now, observe that the equality $\mathcal{A}_{(n+1)} = \mathcal{A}_{1^n}$ follows readily from the definition of the module $\mathcal{S}_\rho$. Thus we may deduce, using Equation 30, that for all $a$

$$\boldsymbol{c}_{(n\,|\,0),(a\,|\,b+1)} = \boldsymbol{c}_{(0\,|\,n-1),(a\,|\,b)}. \tag{31}$$

1.5. Length conjecture. One of the interesting implications of this theorem, together with Conjecture 1.9 below, is that we can reconstruct $\mathcal{A}_\rho$ from (very) partial knowledge of the values of the $\langle \mathcal{S}^{\otimes}_\rho, s_\mu\rangle$. To see how this goes, let us first state the following conjecture, defining the **length** $\ell(f)$ of a symmetric function $f$, to be the maximum number of parts $\ell(\lambda)$ in a partition $\lambda$ that index a Schur function $s_\lambda$ occurring with non-zero coefficients $a_\lambda$ in its Schur expansion $f = \sum_\lambda a_\lambda f_\lambda$. In formula:

$$\ell(f) = \max_{a_\lambda \neq 0} \ell(\lambda).$$

The following conjecture extends to all hooks $\rho$, a similar conjecture (see [5, Conj. 3]) for the $\mathcal{S}^{\otimes}_{1^n} = \mathcal{E}_n$.

Conjecture 1.9 (Coefficients-Length). *If $\rho = (a\,|\,b)$ with $a \geqslant 1$, then we have*

$$\ell(\boldsymbol{c}_{\rho\mu}) \leqslant n - \mu_1, \tag{32}$$

*for all partitions $\mu$ of $n$.*

In particular, when $\mu = (n-2\,|\,1)$, the length of $\langle \mathcal{S}^{\otimes}_\rho, s_\mu\rangle$ is conjectured to be bounded by 1. As it happens, we have

$$\langle \mathcal{S}^{\otimes}_{(a\,|\,b)}, s_{(n-1\,|\,1)}\rangle = \langle \nabla(\widehat{s}_{(a\,|\,b)})^{\otimes}, s_{(n-1\,|\,1)}\rangle = 0, \qquad \text{and} \tag{33}$$

$$\langle \mathcal{S}^{\otimes}_{(a\,|\,b)}, s_{(n-2\,|\,1)}\rangle = \langle \nabla(\widehat{s}_{(a\,|\,b)})^{\otimes}, s_{(n-2\,|\,1)}\rangle = s_b. \tag{34}$$

Indeed (see [5]), we already know that

$$\langle \mathcal{S}_{1^n}^{\otimes}, s_{(n-1\,|\,0)}\rangle = \langle \nabla(e_n)^{\otimes}, s_{(n-1\,|\,0)}\rangle = 1, \qquad \text{and} \tag{35}$$

$$\langle \mathcal{S}_{1^n}^{\otimes}, s_{(n-2\,|\,1)}\rangle = \langle \nabla(e_n)^{\otimes}, s_{(n-2\,|\,1)}\rangle = s_1 + s_2 + \cdots + s_{n-1}. \tag{36}$$

Thus we obtain formulas in Equation 33 and Equation 34 (together with the above), by respectively taking coefficients of $s_{(n-1\,|\,0)}$ and $s_{(n-2\,|\,1)}$ on both sides of Equation 20.

1.6. Reconstruction of Hilbert series of alternants. Let us illustrate, assuming the Hook Component Conjecture and the Coefficient-Length Conjectures (Conjecture 1.8 and 1.9, respectively), how we may reconstruct(7) $\mathcal{A}_\rho$, when $\rho = (a\,|\,b)$ for $a \geqslant 1$. First, we have

$$e_{n-1}^{\perp}\mathcal{A}_\rho = 0, \tag{37}$$

so that $\mathcal{A}_\rho$ contains no terms(8) of length larger or equal to $n-1$. Next, using Equation 34, we get that

$$e_{n-2}^{\perp}\,\mathcal{A}_\rho = \boldsymbol{c}_{(a\,|\,b),(n-2\,|\,1)} = s_b, \tag{38}$$

from which we infer that

$$\mathcal{A}_\rho = s_{(b\,|\,n-3)} + \underbrace{\cdots}_{\text{lower length terms}}, \tag{39}$$

Likewise, all terms of length $n-3$ of $\mathcal{A}_\rho$ are imposed by the identity

$$e_{n-3}^{\perp}\,\mathcal{A}_\rho = \langle \mathcal{S}_\rho^{\otimes}, s_{(n-3\,|\,2)}\rangle = \langle \nabla(\widehat{s}_\rho)^{\otimes}, s_{(n-3\,|\,2)}\rangle, \tag{40}$$

since $\langle \mathcal{S}_\rho^{\otimes}, s_{(n-3\,|\,2)}\rangle$ is of at most length 2, hence its value is entirely characterized by that of $\nabla(\widehat{s}_\rho)^{\otimes}$. For instance, for hooks of size 6, we may calculate explicitly that

$$\begin{aligned}
&\langle \nabla(\widehat{s}_{(6)})^{\otimes}, s_{411}\rangle = s_1 + s_2 + s_3 + s_4,\\
&\langle \nabla(\widehat{s}_{51})^{\otimes}, s_{411}\rangle = s_{11} + s_{21} + s_{31} + s_2 + s_3 + s_4 + s_5,\\
&\langle \nabla(\widehat{s}_{411})^{\otimes}, s_{411}\rangle = s_{21} + s_{31} + s_{41} + s_{22} + s_3 + s_4 + s_5 + s_6,\\
&\langle \nabla(\widehat{s}_{3111})^{\otimes}, s_{411}\rangle = s_{31} + s_{41} + s_{51} + s_{32} + s_4 + s_5 + s_6 + s_7,\\
&\langle \nabla(\widehat{s}_{21111})^{\otimes}, s_{411}\rangle = s_{41} + s_{51} + s_{61} + s_{32} + s_{42} + s_5 + s_6 + s_7 + s_8;
\end{aligned}$$

from which we deduce all terms of $\mathcal{A}_\rho$ of length larger or equal to 3. This gives

$$\begin{aligned}
&\mathcal{A}_{(6)} = s_{1111} + s_{311} + s_{411} + s_{511} + \langle \nabla(\widehat{s}_{(6)})^{\otimes}, e_6\rangle,\\
&\mathcal{A}_{51} = s_{2111} + s_{321} + s_{421} + s_{411} + s_{511} + s_{611} + \langle \nabla(\widehat{s}_{51})^{\otimes}, e_6\rangle,\\
&\mathcal{A}_{411} = s_{3111} + s_{331} + s_{421} + s_{521} + s_{511} + s_{611} + s_{711} + \langle \nabla(\widehat{s}_{411})^{\otimes}, e_6\rangle,\\
&\mathcal{A}_{3111} = s_{4111} + s_{431} + s_{521} + s_{621} + s_{611} + s_{711} + s_{811} + \langle \nabla(\widehat{s}_{3111})^{\otimes}, e_6\rangle,\\
&\mathcal{A}_{21111} = s_{5111} + s_{431} + s_{531} + s_{621} + s_{721} + s_{711} + s_{811} + s_{911} + \langle \nabla(\widehat{s}_{21111})^{\otimes}, e_6\rangle
\end{aligned}$$

in which the first terms correspond to Equation 39. Observe that some of the terms in $\langle \nabla(\widehat{s}_\rho)^{\otimes}, s_{411}\rangle$ are already obtained by skewing by $e_3$ the length-4 terms in the $\mathcal{A}_\rho$'s. Hence, we only need to add the necessary length-3 terms to account for the "missing" terms. We can then conclude the entire construct by adding $\langle \nabla(\widehat{s}_\rho)^{\otimes}, e_6\rangle$, since it contains precisely the terms of length less or equal to 2 that should appear in $\mathcal{A}_\rho$.

(7)A similar reconstruction, for the case when $\rho = 1^n$, is described in [5].

(8)Recall that the Schur expansion of $\mathcal{A}_\rho$ only has positive integer coefficients.

## 2. The $e$-positivity phenomenon

As discussed in [4], most of the symmetric functions constructed via the elliptic Hall algebra approach exhibit a $e$-positivity when specialized at $t=1$. We consider here the case of $\mathcal{S}_\rho$, for which we get the specialization of any one of the (infinitely many) parameters $q_i$ to the value 1. This is obtained via the plethystic evaluation at $1+\boldsymbol{q}$ of the $\mathrm{GL}_\infty$-coefficients $\boldsymbol{c}_{\rho\mu}$ of $\mathcal{S}_\rho^\otimes$. Noteworthy is the fact that this operation is invertible, as long as there are infinitely many parameters.

It is worth underlining the difference between

$$\begin{aligned} &p_j[1+q_1+q_2+\cdots+q_k+\cdots] = 1+q_1^j+q_2^j+\cdots+q_k^j+\cdots, \qquad \text{and}\\ &p_j[q_1+q_2+\cdots+q_k+\cdots]\Big|_{q_1\to 1+q_1} = (1+q_1)^j+q_2^j+\cdots+q_k^j+\cdots \end{aligned}$$

We see here the difference between the two possible orders in which we may apply the operators $p_j[-]$, and substitution of $1+q_1$ for $q_1$. The $e$-positivity phenomenon considered below is for the first of these, in contrast with similar results that appeared in [1, 2, 13, 14], in which the second order of application of the operators is considered.

For the sake of discussion, let us write

$$\mathcal{T}_\rho := \mathcal{S}_\rho^\otimes[1+\boldsymbol{q};\boldsymbol{z}], \tag{41}$$

and write

$$\mathcal{T}_\rho = \sum_{\mu\vdash n} \boldsymbol{c}_{\rho\mu}[1+\boldsymbol{q}]\otimes s_\mu(\boldsymbol{z});$$

or equivalently in $\otimes$-format:

$$\mathcal{T}_\rho = \sum_{\nu\vdash n} \boldsymbol{d}_{\rho\nu}\otimes e_\nu, \tag{42}$$

with $\boldsymbol{d}_{\rho\nu}$ the coefficients of $e_\nu(\boldsymbol{z})$ in $\mathcal{T}_\rho$. Then, as far as we can check experimentally, all of the $\boldsymbol{d}_{\rho\nu}$ are **Schur-positive**. For instance, we have

$$\begin{aligned} \mathcal{T}_{41} = {} & (s_{211}+s_{32}+s_{41}+s_{51}+s_7)\otimes e_5\\ & + (s_{111}+s_{22}+s_{11}+s_{21}+s_3+2s_{31}+s_4+s_{41}+s_5+s_6)\otimes e_{41}\\ & + (2s_{21}+s_{31}+s_3+s_4+s_5)\otimes e_{32} + (s_{11}+s_{21}+s_1+2s_2+s_3+s_4)\otimes e_{311}\\ & + (s_{11}+s_1+2s_2+s_3)\otimes e_{221} + (1+s_1)\otimes e_{2111}. \end{aligned}$$

By definition, the $\boldsymbol{c}_{\rho\mu}$ are related to the $\boldsymbol{d}_{\rho\nu}$ by the identity

$$\boldsymbol{c}_{\rho\mu}[1+\boldsymbol{q}] = \sum_{\nu\vdash n} K_{\mu'\nu}\boldsymbol{d}_{\rho\nu}, \tag{43}$$

where the $K_{\mu\lambda}$ are the usual Kostka numbers.

There are close ties between this $e$-positivity phenomenon and our conjectures. To see this, recall that the coefficient of $e_n$ in the $e$-expansion of $s_\mu$ vanishes for all $\mu$ except hooks; and it is known to be equal to $(-1)^k$ when $\mu=(k\,|\,j)$, with $n=k+j+1$. Since the forgotten symmetric functions $f_\nu$ are dual to the $e_\nu$, we may write this as

$$\langle s_\mu, f_n\rangle = \begin{cases} (-1)^k, & \text{if } \mu=(k\,|\,j),\\ 0 & \text{otherwise.}\end{cases}$$

We may then calculate, using Equation 30, that

$$\begin{aligned}\boldsymbol{d}_{\rho,(n)} &= \langle \mathcal{S}^{\otimes}_{\rho}[1+\boldsymbol{q};\boldsymbol{z}], f_n\rangle = \sum_{\mu\vdash n} \boldsymbol{c}_{\rho\mu}[1+\boldsymbol{q}]\,\langle s_\mu, f_n\rangle \\ &= \left(\sum_{k=0}^{n-1}(-1)^k\,\boldsymbol{c}_{\rho,(k\,|\,j)}\right)[1+\boldsymbol{q}] = \left(\sum_{k\geqslant 0}(-1)^k\,e_k^{\perp}\mathcal{A}_n\right)[1+\boldsymbol{q}].\end{aligned}$$

For any symmetric function $F$, one has $\sum_{k\geqslant 0}(-1)^k\,e_k^{\perp}F(\boldsymbol{q}) = F[\boldsymbol{q}-1]$. Thus, we find that $\boldsymbol{d}_{\rho,(n-1\,|\,0)} = (\mathcal{A}_\rho[\boldsymbol{q}-1])[1+\boldsymbol{q}] = \mathcal{A}_\rho$, and we conclude the following.

Proposition 2.1. *The Hook Components Conjecture* 1.8 *implies that, for all $\rho$, the coefficient $\boldsymbol{d}_{\rho,(n)}$ of $e_n$ in $\mathcal{T}_\rho$ is Schur positive.*

To get more, let $\mu$ be any partition of $n$ which is largest in dominance order among those such that $\boldsymbol{c}_{\rho\mu} \neq 0$. Then it is easy to see that

$$\boldsymbol{d}_{\rho\mu'} = \boldsymbol{c}_{\rho\mu}[1+\boldsymbol{q}]. \tag{44}$$

We thus automatically have Schur-positivity of $\boldsymbol{d}_{\rho\mu'}$. Experiments suggest that, for all hooks $\rho = (a\,|\,b)$ and $\mu = (k\,|\,j)$, if $m := \min(j,k)$ then we have

$$\boldsymbol{d}_{\rho\mu} = \sum_{i=0}^{m} \boldsymbol{c}_{(a\,|\,b-i),(k-i\,|\,j)}, \tag{45}$$

except when $\rho = \mu = 1^n$, in which case we simply have $\boldsymbol{d}_{\rho\mu} = 1$.

## 2.1. Trivariate shuffle conjecture.

The trivariate shuffle conjecture of [10], corresponding below to $\rho = (0\,|\,n-1)$, may at least be extended to other cases as follows. Recall the definition of $\Gamma_a$ in Equation 15.

Conjecture 2.2 (Trivariate shuffle). *For hooks $\rho = (a\,|\,b)$, with $a$ equal to either* 0*,* 1*, or $n-1$, we have*

$$\mathcal{S}_\rho(q,t,1;\boldsymbol{z}) = \sum_{\Gamma_a\leqslant\alpha\leqslant\beta} q^{d(\alpha,\beta)}\mathbb{L}_\beta(t;\boldsymbol{z}), \tag{46}$$

*where the Dyck path $\alpha$ lies below the Dyck path $\beta$ in the Tamari poset, and $d(\alpha,\beta)$ is the length of the longest strict chain going from $\alpha$ to $\beta$ in this poset.*

Again, we underline that the case $a = 0$ already appears in [10], and that the case $a = n-1$ is more or less implicit in [11]. We expect that some variant of this formula should hold for other hooks, maybe with some tweak to the LLT-polynomial part. It would also be nice to have similar expressions involving $r$, for $\mathcal{S}_\rho(q,t,r;\boldsymbol{z})$, but this is not known.

## 2.2. More observed properties.

Recall that Identity 4.17 in Theorem 4.2 of [8] states (in our notations) that for $a+b=n$

$$\langle \nabla(\widehat{s}_{n+1})^{\otimes}, s_{(a\,|\,b)}\rangle = \langle \nabla(e_n)^{\otimes}, s_{(a\,|\,b-1)}\rangle.$$

This equality appears to lift to the following similar multivariate identity:

$$\langle \mathcal{S}^{\otimes}_{(n+1)}, s_{(a\,|\,b)}\rangle = \langle \mathcal{S}^{\otimes}_{1^n}, s_{(a\,|\,b-1)}\rangle, \qquad \text{for all} \qquad a+b=n. \tag{47}$$

# 3. Appendix

### 3.1. The $s_\lambda\otimes s_\mu$-expansions of $\nabla(\widehat{s}_\rho)^\otimes$ and $\mathcal{S}_\rho^\otimes$ for hooks of size 5.

$$\begin{aligned}\nabla(\widehat{s}_5)^\otimes = 1\otimes s_{41} &+ (s_1+s_2)\otimes s_{32} + (s_1+s_2+s_3)\otimes s_{311}\\ &+ (s_{11}+s_{21}+s_2+s_3+s_4)\otimes s_{221}\\ &+ (s_{11}+s_{21}+s_{31}+s_3+s_4+s_5)\otimes s_{2111} + (s_{31}+s_{41}+s_6)\otimes s_{11111}\end{aligned}$$

$$\begin{aligned}\nabla(\widehat{s}_{41})^\otimes = s_1\otimes s_{41} &+ (s_{11}+s_2+s_3)\otimes s_{32} + (s_{11}+s_{21}+s_2+s_3+s_4)\otimes s_{311}\\ &+ (2s_{21}+s_{31}+s_3+s_4+s_5)\otimes s_{221}\\ &+ (s_{21}+s_{22}+2s_{31}+s_{41}+s_4+s_5+s_6)\otimes s_{2111}\\ &+ (s_{32}+s_{41}+s_{51}+s_7)\otimes s_{11111}\end{aligned}$$

$$\begin{aligned}\nabla(\widehat{s}_{311})^\otimes = s_2\otimes s_{41} &+ (s_{21}+s_3+s_4)\otimes s_{32} + (s_{21}+s_{31}+s_3+s_4+s_5)\otimes s_{311}\\ &+ (s_{22}+2s_{31}+s_{41}+s_4+s_5+s_6)\otimes s_{221}\\ &+ (s_{31}+s_{32}+2s_{41}+s_{51}+s_5+s_6+s_7)\otimes s_{2111}\\ &+ (s_{42}+s_{51}+s_{61}+s_8)\otimes s_{11111}\end{aligned}$$

$$\begin{aligned}\nabla(\widehat{s}_{2111})^\otimes = s_3\otimes s_{41} &+ (s_{21}+s_{31}+s_4+s_5)\otimes s_{32}\\ &+ (s_{22}+s_{31}+s_{41}+s_4+s_5+s_6)\otimes s_{311}\\ &+ (s_{32}+s_{31}+2s_{41}+s_{51}+s_5+s_6+s_7)\otimes s_{221}\\ &+ (s_{32}+s_{42}+s_{41}+2s_{51}+s_{61}+s_6+s_7+s_8)\otimes s_{2111}\\ &+ (s_{33}+s_{52}+s_{61}+s_{71}+s_9)\otimes s_{11111}\end{aligned}$$

$$\begin{aligned}\mathcal{S}_5^\otimes &= \nabla(\widehat{s}_5)^\otimes + s_{111}\otimes s_{11111},\\ \mathcal{S}_{41}^\otimes &= \nabla(\widehat{s}_{41})^\otimes + s_{111}\otimes s_{2111} + s_{211}\otimes s_{11111},\\ \mathcal{S}_{311}^\otimes &= \nabla(\widehat{s}_{311})^\otimes + s_{111}\otimes s_{221} + s_{211}\otimes s_{2111} + s_{311}\otimes s_{11111},\\ \mathcal{S}_{2111}^\otimes &= \nabla(\widehat{s}_{2111})^\otimes + s_{211}\otimes s_{221} + s_{311}\otimes s_{2111} + s_{411}\otimes s_{11111},\\ \mathcal{S}_{11111}^\otimes &= \nabla(\widehat{s}_{11111})^\otimes + (s_{211}+s_{311})\otimes s_{221} + (s_{111}+s_{211}+s_{311}+s_{411})\otimes s_{2111}\\ &\quad + (s_{1111}+s_{311}+s_{411}+s_{511})\otimes s_{11111}.\end{aligned}$$

(The value of $\nabla(\widehat{s}_{11111})^\otimes$) may be found in [5].)

### 3.2. The expansions of $\mathcal{S}_\rho^\otimes$ for hooks of size 6.

$$\begin{aligned}\mathcal{S}_6^\otimes = 1\otimes s_{51} &+ (s_1+s_2+s_3)\otimes s_{42} + (s_1+s_2+s_3+s_4)\otimes s_{411}\\ &+ (s_{21}+s_2+s_4)\otimes s_{33}\\ &+ (s_{22}+s_{11}+2s_{21}+2s_{31}+s_{41}+s_2+2s_3+2s_4+2s_5+s_6)\otimes s_{321}\\ &+ (s_{32}+s_{11}+s_{21}+2s_{31}+s_{41}+s_{51}+s_3+s_4+2s_5+s_6+s_7)\otimes s_{3111}\\ &+ (s_{211}+s_{32}+s_{21}+s_{31}+s_{41}+s_{51}+s_4+s_5+s_7)\otimes s_{222}\\ &+ (s_{111}+s_{211}+s_{311}+s_{22}+s_{32}+s_{42}+s_{21}+2s_{31}+3s_{41}+2s_{51}+s_{61}\\ &\qquad + s_4+s_5+2s_6+s_7+s_8)\otimes s_{2211}\\ &+ (s_{111}+s_{211}+s_{311}+s_{411}+s_{33}+s_{32}+s_{42}+s_{52}\\ &\qquad + s_{31}+2s_{41}+2s_{51}+2s_{61}+s_{71}+s_6+s_7+s_8+s_9)\otimes s_{21111}\\ &+ (s_{1111}+s_{311}+s_{411}+s_{511}+s_{43}+s_{42}+s_{62}+s_{61}+s_{71}+s_{81}+s_{10.})\otimes s_{111111}\end{aligned}$$

$$
\begin{aligned}
\mathcal{S}^{\otimes}_{51} = {} & s_1 \otimes s_{51} + (s_{11} + s_{21} + s_2 + s_3 + s_4) \otimes s_{42} \\
& + (s_{11} + s_{21} + s_{31} + s_2 + s_3 + s_4 + s_5) \otimes s_{411} \\
& + (s_{21} + s_{31} + s_3 + s_5) \otimes s_{33} \\
& + (s_{111} + s_{211} + 2s_{22} + s_{32} + 2s_{21} + 4s_{31} + 3s_{41} + s_{51} \\
& \qquad + s_3 + 2s_4 + 2s_5 + 2s_6 + s_7) \otimes s_{321} \\
& + (s_{111} + s_{211} + s_{311} + s_{21} + s_{22} + 2s_{32} + s_{42} + 2s_{31} + 3s_{41} + 2s_{51} + s_{61} \\
& \qquad + s_4 + s_5 + 2s_6 + s_7 + s_8) \otimes s_{3111} \\
& + (s_{211} + s_{311} + s_{22} + s_{32} + s_{42} + s_{31} + 2s_{41} + s_{51} + s_{61} \\
& \qquad + s_5 + s_6 + s_8) \otimes s_{222} \\
& + (s_{221} + 2s_{211} + 2s_{311} + s_{411} + s_{33} + s_{22} + 3s_{32} + 2s_{42} + s_{52} + s_{31} + 3s_{41} \\
& \qquad + 4s_{51} + 2s_{61} + s_{71} + s_5 + s_6 + 2s_7 + s_8 + s_9) \otimes s_{2211} \\
& + (s_{1111} + s_{221} + s_{321} + s_{211} + 2s_{311} + 2s_{411} + s_{511} + s_{33} + s_{43} + s_{32} + 3s_{42} \\
& \qquad + 2s_{52} + s_{62} + s_{41} + 2s_{51} + 3s_{61} + 2s_{71} + s_{81} \\
& \qquad + s_7 + s_8 + s_9 + s_{(10)}) \otimes s_{21111} \\
& + (s_{2111} + s_{321} + s_{421} + s_{411} + s_{511} + s_{611} + s_{43} + s_{53} + s_{52} + s_{62} \\
& \qquad + s_{71} + s_{72} + s_{81} + s_{91} + s_{(11)}) \otimes s_{111111}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}^{\otimes}_{411} = {} & s_2 \otimes s_{51} + (s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{42} \\
& + (s_{22} + s_{21} + s_{31} + s_{41} + s_3 + s_4 + s_5 + s_6) \otimes s_{411} \\
& + (s_{211} + s_{22} + s_{31} + s_{41} + s_4 + s_6) \otimes s_{33} \\
& + (2s_{211} + s_{311} + s_{22} + 3s_{32} + s_{42} + 2s_{31} + 4s_{41} + 3s_{51} + s_{61} \\
& \qquad + s_4 + 2s_5 + 2s_6 + 2s_7 + s_8) \otimes s_{321} \\
& + (s_{221} + s_{211} + s_{311} + s_{411} + s_{33} + 2s_{32} + 2s_{42} + s_{52} + s_{31} + 2s_{41} \\
& \qquad + 3s_{51} + 2s_{61} + s_{71} + s_5 + s_6 + 2s_7 + s_8 + s_9) \otimes s_{3111} \\
& + (s_{221} + s_{311} + s_{411} + s_{33} + s_{32} + s_{42} + s_{52} \\
& \qquad + s_{41} + 2s_{51} + s_{61} + s_{71} + s_6 + s_7 + s_9) \otimes s_{222} \\
& + (s_{1111} + s_{221} + s_{321} + 3s_{311} + 2s_{411} + s_{511} \\
& \qquad + s_{33} + s_{43} + s_{32} + 4s_{42} + 2s_{52} + s_{62} \\
& \qquad + s_{41} + 3s_{51} + 4s_{61} + 2s_{71} + s_{81} \\
& \qquad + s_6 + s_7 + 2s_8 + s_9 + s_{(10)}) \otimes s_{2211} \\
& + (s_{2111} + 2s_{321} + s_{421} + s_{311} + 2s_{411} + 2s_{511} + s_{611} + s_{33} + 2s_{43} + s_{53} \\
& \qquad + s_{42} + 3s_{52} + 2s_{62} + s_{72} + s_{51} + 2s_{61} + 3s_{71} + 2s_{81} + s_{91} \\
& \qquad + s_8 + s_9 + s_{(10)} + s_{(11)}) \otimes s_{21111} \\
& + (s_{3111} + s_{331} + s_{421} + s_{521} + s_{511} + s_{611} + s_{711} + s_{44} + s_{53} + s_{63} \\
& \qquad + s_{62} + s_{72} + s_{82} + s_{81} + s_{91} + s_{(10,1)} + s_{(12)}) \otimes s_{111111}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}^{\otimes}_{3111} = {} & s_3 \otimes s_{51} + \big(s_{22} + s_{31} + s_{41} + s_4 + s_5 + s_6\big) \otimes s_{42} \\
& + \big(s_{32} + s_{31} + s_{41} + s_{51} + s_4 + s_5 + s_6 + s_7\big) \otimes s_{411} \\
& + \big(s_{32} + s_{41} + s_{51} + s_5 + s_7\big) \otimes s_{33} \\
& + \big(s_{221} + 2s_{311} + s_{411} + s_{33} + 2s_{32} + 3s_{42} + s_{52} + 2s_{41} + 4s_{51} \\
& \qquad + 3s_{61} + s_{71} + s_5 + 2s_6 + 2s_7 + 2s_8 + s_9\big) \otimes s_{321} \\
& + \big(s_{321} + s_{311} + s_{411} + s_{511} + s_{33} + s_{43} + 2s_{42} + 2s_{52} + s_{62} \\
& \qquad + s_{41} + 2s_{51} + 3s_{61} + 2s_{71} + s_{81} \\
& \qquad + s_6 + s_7 + 2s_8 + s_9 + s_{(10)}\big) \otimes s_{3111} \\
& + \big(s_{321} + s_{311} + s_{411} + s_{511} + s_{43} + 2s_{42} + s_{52} + s_{62} \\
& \qquad + s_{51} + 2s_{61} + s_{71} + s_{81} + s_7 + s_8 + s_{(10)}\big) \otimes s_{222} \\
& + \big(s_{2111} + 2s_{321} + s_{421} + 3s_{411} + 2s_{511} + s_{611} + s_{33} + 2s_{43} + s_{53} \\
& \qquad + s_{42} + 4s_{52} + 2s_{62} + s_{72} \\
& \qquad + s_{51} + 3s_{61} + 4s_{71} + 2s_{81} + s_{91} \\
& \qquad + s_7 + s_8 + 2s_9 + s_{(10)} + s_{11.}\big) \otimes s_{2211} \\
& + \big(s_{3111} + s_{331} + 2s_{421} + s_{521} + s_{411} + 2s_{511} + 2s_{611} + s_{711} \\
& \qquad + s_{44} + s_{43} + 2s_{53} + s_{63} \\
& \qquad + s_{52} + 3s_{62} + 2s_{72} + s_{82} \\
& \qquad + s_{61} + 2s_{71} + 3s_{81} + 2s_{91} + s_{(10,1)} \\
& \qquad + s_9 + s_{(10)} + s_{(11)} + s_{(12)}\big) \otimes s_{21111} \\
& + \big(s_{4111} + s_{431} + s_{521} + s_{621} + s_{611} + s_{711} + s_{811} \\
& \qquad + s_{54} + s_{63} + s_{73} + s_{72} + s_{82} \\
& \qquad + s_{91} + s_{92} + s_{(10,1)} + s_{(11,1)} + s_{(13)}\big) \otimes s_{111111}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}^{\otimes}_{21111} = s_4 \otimes s_{51} &+ (+s_{32}s_{31} + s_{41} + s_{51} + s_5 + s_6 + s_7) \otimes s_{42}\\
&+ (s_{32} + s_{42} + s_{41} + s_{51} + s_{61} + s_5 + s_6 + s_7 + s_8) \otimes s_{411}\\
&+ (s_{311} + s_{22} + s_{42} + s_{41} + s_{51} + s_{61} + s_6 + s_8) \otimes s_{33}\\
&+ (s_{221} + s_{321} + s_{311} + 2s_{411} + s_{511} + s_{33} + s_{43}\\
&\qquad + s_{32} + 3s_{42} + 3s_{52} + s_{62}\\
&\qquad + s_{41} + 3s_{51} + 4s_{61} + 3s_{71} + s_{81}\\
&\qquad + s_6 + 2s_7 + 2s_8 + 2s_9 + s_{(10)}) \otimes s_{321}\\
&+ (s_{321} + s_{421} + s_{411} + s_{511} + s_{611} + s_{33} + s_{43} + s_{53}\\
&\qquad + s_{42} + 3s_{52} + 2s_{62} + s_{72}\\
&\qquad + s_{51} + 2s_{61} + 3s_{71} + 2s_{81} + s_{91}\\
&\qquad + s_7 + s_8 + 2s_9 + s_{(10)} + s_{(11)}) \otimes s_{3111}\\
&+ (s_{2111} + s_{321} + s_{421} + s_{33} + s_{411} + s_{511} + s_{611}\\
&\qquad + s_{43} + s_{53} + 2s_{52} + s_{62} + s_{72}\\
&\qquad + s_{51} + s_{61} + 2s_{71} + s_{81} + s_{91} + s_8 + s_9 + s_{(11)}) \otimes s_{222}\\
&+ (s_{3111} + s_{331} + s_{321} + 2s_{421} + s_{521} + s_{411} + 3s_{511} + 2s_{611} + s_{711}\\
&\qquad + s_{44} + 2s_{43} + 2s_{53} + s_{63}\\
&\qquad + s_{42} + 2s_{52} + 4s_{62} + 2s_{72} + s_{82}\\
&\qquad + 2s_{61} + 3s_{71} + 4s_{81} + 2s_{91} + s_{(10,1)}\\
&\qquad + s_8 + s_9 + 2s_{(10)} + s_{(11)} + s_{(12)}) \otimes s_{2211}\\
&+ (s_{4111} + s_{331} + s_{431} + s_{421} + 2s_{521} + s_{621}\\
&\qquad + s_{511} + 2s_{611} + 2s_{711} + s_{811}\\
&\qquad + s_{54} + s_{43} + 2s_{53} + 2s_{63} + s_{73}\\
&\qquad + 2s_{62} + 3s_{72} + 2s_{82} + s_{92}\\
&\qquad + s_{71} + 2s_{81} + 3s_{91} + 2s_{(10,1)} + s_{(11,1)}\\
&\qquad + s_{(10)} + s_{(11)} + s_{(12)} + s_{(13)}) \otimes s_{21111}\\
&+ (s_{5111} + s_{431} + s_{531} + s_{621} + s_{721} + s_{711} + s_{811} + s_{911}\\
&\qquad + s_{64} + s_{63} + s_{73} + s_{83}\\
&\qquad + s_{82} + s_{92} + s_{(10,2)}\\
&\qquad + s_{(10,1)} + s_{(11,1)} + s_{(12,1)} + s_{(14)}) \otimes s_{111111}
\end{aligned}
$$

(The value of $\mathcal{S}^{\otimes}_{111111} = \mathcal{E}_6$ may be found in [5].)

### 3.3. The $e$-expansions of the $\mathcal{T}_\rho$'s for hooks of size $\leqslant 4$.

$\mathcal{T}_1 = 1 \otimes e_1;$

$\mathcal{T}_2 = 1 \otimes e_2,$

$\mathcal{T}_{11} = 1 \otimes e_{11} + s_1 \otimes e_2;$

$\mathcal{T}_3 = 1 \otimes e_{21} + s_1 \otimes e_3,$

$\mathcal{T}_{21} = 1 \otimes e_{21} + s_1 \otimes e_{21} + s_2 \otimes e_3,$

$\mathcal{T}_{111} = 1 \otimes e_{111} + (2s_1 + s_2) \otimes e_{21} + (s_{11} + s_3) \otimes e_3;$

$\mathcal{T}_4 = 1 \otimes e_{211} + s_1 \otimes e_{22} + (s_1 + s_2) \otimes e_{31} + (s_{11} + s_3) \otimes e_4,$

$\mathcal{T}_{31} = (1 + s_1) \otimes e_{211} + s_2 \otimes e_{22} + (s_{11} + s_1 + s_2 + s_3) \otimes e_{31} + (s_{21} + s_4) \otimes e_4,$

$$\begin{aligned}\mathcal{T}_{211} &= (1+s_1+s_2)\otimes e_{211} + (s_{11}+s_1+s_3)\otimes e_{22} + (s_{21}+s_2+s_3+s_4)\otimes e_{31}\\ &\quad + (s_{31}+s_5)\otimes e_4,\\ \mathcal{T}_{1111} &= 1\otimes e_{1111} + (3s_1+2s_2+s_3)\otimes e_{211} + (s_{11}+s_{21}+s_2+s_4)\otimes e_{22}\\ &\quad + (2s_{11}+s_{21}+s_{31}+2s_3+s_4+s_5)\otimes e_{31} + (s_{111}+s_{31}+s_{41}+s_6)\otimes e_4.\end{aligned}$$

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