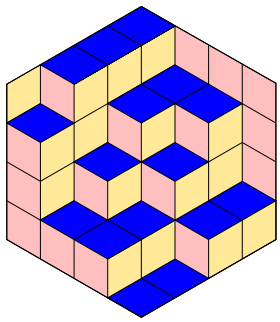


ALGEBRAIC COMBINATORICS

David Einstein & James Propp

Combinatorial, piecewise-linear, and birational homomesy for products of two chains

Volume 4, issue 2 (2021), p. 201-224.

<http://alco.centre-mersenne.org/item/ALCO_2021__4_2_201_0>

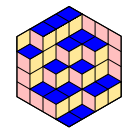
© The journal and the authors, 2021.
Some rights reserved.

This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
http://creativecommons.org/licenses/by/4.0/

Access to articles published by the journal *Algebraic Combinatorics* on the website http://alco.centre-mersenne.org/ implies agreement with the Terms of Use (http://alco.centre-mersenne.org/legal/).

Algebraic Combinatorics is member of the
Centre Mersenne for Open Scientific Publishing
www.centre-mersenne.org

Algebraic Combinatorics
Volume 4, issue 2 (2021), p. 201–224
https://doi.org/10.5802/alco.139



Combinatorial, piecewise-linear, and birational homomesy for products of two chains

David Einstein & James Propp

Abstract This article illustrates the dynamical concept of *homomesy* in three kinds of dynamical systems – combinatorial, piecewise-linear, and birational – and shows the relationship between these three settings. In particular, we show how the rowmotion and promotion operations of Striker and Williams [16] can be lifted to (continuous) piecewise-linear operations on the order polytope of Stanley [14], and then lifted to birational operations on the positive orthant in $\mathbb{R}^{|P|}$ and indeed to a dense subset of $\mathbb{C}^{|P|}$. When the poset P is a product of a chain of length a and a chain of length b, these lifted operations have order $a+b$, and exhibit the homomesy phenomenon: the time-averages of various quantities are the same in all orbits. One important tool is a concrete realization of the conjugacy between rowmotion and promotion found by Striker and Williams; this *recombination map* allows us to use homomesy for promotion to deduce homomesy for rowmotion.

1. Introduction

Many authors [2, 3, 6, 12, 16] have studied an operation ρ on the set of order ideals of a poset P that, following Striker and Williams, we call *rowmotion*. In exploring the properties of rowmotion, Striker and Williams also introduced and studied a closely related operation π they call *promotion* on account of its ties with promotion of Young tableaux, which depends on the choice of an *rc embedding* (a particular kind of embedding of P into the poset $\mathbb{Z}\times\mathbb{Z}$ that sharpens the idea of a Hasse diagram). In this article (an expanded version of a 2014 FPSAC presentation [5]) we mostly focus on a very particular case, where P is of the form $[a]\times[b]$ and the rc embedding sends $(i,j)\in P$ to $(j-i, i+j-2)\in\mathbb{Z}^2$ (the standard Hasse embedding; see Figure 2), and we explore how the cardinality of an order ideal I behaves as one iterates rowmotion and promotion. Indeed, we find regularities for the average cardinality of sets of the form $I\cap S_\ell$ as I varies over the elements of a rowmotion-orbit or promotion-orbit, where $\{S_1, S_2, \ldots, S_{a+b-1}\}$ is a partition of $[a]\times[b]$ into special sets called *files* (which Striker and Williams call columns).

Let $\mathcal{I}(P)$ denote the set of order ideals of a poset P (usually written as $J(P)$ in the literature). It has long been known [2] that the order of π or ρ acting on $\mathcal{I}([a]\times[b])$ is $a+b$. Propp and Roby [13] showed that the average of $|I|$ as I varies over an orbit in $\mathcal{I}([a]\times[b])$ is $ab/2$, and sketched a proof of a more detailed claim:

Manuscript received 2nd September 2018, revised 1st July 2020, accepted 26th July 2020.

Keywords. Dynamics, homomesy, order ideal, order polytope, piecewise-linear, promotion, recombination, rowmotion, toggle group, tropicalization.

Acknowledgements. The authors were partially supported by NSF Grant #1001905.

Theorem 1.1. *Fix $a, b \geqslant 1$, let $n = a + b$, let $P = [a] \times [b]$, and for $1 \leqslant \ell \leqslant n-1$ let $S_\ell = \{(i,j) \in P \mid j - i + a = \ell\}$. Then for every order ideal I in $\mathcal{I}(P)$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} |\pi^k(I) \cap S_\ell| = \frac{1}{n}\sum_{k=0}^{n-1} |\rho^k(I) \cap S_\ell| = \begin{cases} b\ell/n & \text{if } \ell \leqslant a, \\ a(n-\ell)/n & \text{if } \ell \geqslant a. \end{cases}$$

Summing over ℓ, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} |\pi^k(I)| = \frac{1}{n}\sum_{k=0}^{n-1} |\rho^k(I)| = ab/2.$$

(Here as elsewhere in the article, overlap between cases is intentional; it is easily checked that the answers given in borderline cases are consistent.)

It is no accident that the same averages are seen for the promotion operation π and the rowmotion operation ρ; the recombination principle discussed in Section 6 explains why we get same averages for both actions. In some cases we will only state our results for promotion, but in every case considered here (specifically, in Theorems 3.4, 3.5, 4.4, and 5.3) one may replace promotion by rowmotion without changing the common value of the orbit-averages.

The notion of looking at the average of a quantity over an orbit was an outgrowth of the second author's work on chip-firing and rotor-routing [8, 9]; see in particular Proposition 3 of [13]. Further inspiration came from conjectures of Panyushev [12] (later proved by Armstrong, Stump, and Thomas [1]).

This article presents a new proof of Theorem 1.1 (see Section 7) which, although less direct than the Propp–Roby proof, indicates that the constant-averages-over-orbits phenomenon (also called the homomesy phenomenon) applies not just for actions on order ideals but also for dynamical systems of a different sort. Specifically, we define (continuous) piecewise-linear maps from the order polytope of P to itself (*piecewise-linear rowmotion* and *promotion*) that exhibit homomesy, and birational maps from a dense open subset of $\mathbb{C}^{ab}$ to itself (*birational rowmotion* and *promotion*) that exhibit a multiplicative version of homomesy. (See Subsection 2.1 for definitions of these terms.)

Our main result is Theorem 5.3, whose proof involves three main ingredients. The first is the main result of Grinberg and Roby (Theorem 30 in [7]), showing that birational rowmotion on the poset $[a] \times [b]$ has period $a + b$. The second ingredient is the recombination operation explained in Section 6. Recombination equivariantly takes birational rowmotion to birational promotion, giving a concrete way of chopping up rowmotion-orbits and reassembling the pieces to obtain promotion-orbits, and vice versa; the recombination picture tells us that birational promotion has the same orbit structure (and hence the same period) as birational rowmotion. Recombination has its roots in the work of Striker and Williams (see Theorem 5.4 in [16]), but the more detailed combinatorial picture presented here is required if we want to prove not just results about periodicity but also results about homomesy. The third ingredient is Lemma 5.1 in which the specific nature of promotion and the specific structure of the poset $[a] \times [b]$ play crucial roles. The lemma concretizes and exploits the intuition that, viewed from the correct perspective, promotion can be seen as a form of rotation, but in a different manner than in the work of Grinberg–Roby.

The plan of the article is as follows. In Section 2, after introducing needed preliminaries and notation, including the definition of (additive) homomesy, we review some of the background on the rowmotion and promotion operations $\rho, \pi : \mathcal{I}(P) \to \mathcal{I}(P)$. We then define (in Section 3) piecewise-linear maps $\rho_{\mathcal{P}}, \pi_{\mathcal{P}} : \mathbb{R}^{|P|} \to \mathbb{R}^{|P|}$ and show that $\rho_{\mathcal{P}}$ and $\pi_{\mathcal{P}}$ specialize to ρ and π if one restricts attention to the vertices of the order polytope $\mathcal{O}(P)$ (replacing order ideals by filters as required). Changing variables, we obtain slightly different piecewise-linear maps $\overline{\rho}_{\mathcal{P}}, \overline{\pi}_{\mathcal{P}} : \mathbb{R}^{|P|} \to \mathbb{R}^{|P|}$ that

are homogeneous versions of $\rho_{\mathcal{P}}$, $\pi_{\mathcal{P}}$. Then we show (in Section 4) how $\overline{\rho}_{\mathcal{P}}$, $\overline{\pi}_{\mathcal{P}}$ can in turn each be viewed as a tropicalization of a birational map $\overline{\rho}_{\mathcal{B}}$, $\overline{\pi}_{\mathcal{B}}$ from a dense open subset U of $\mathbb{C}^{ab}$ to itself; we call the elements of U *P-arrays*. In Section 5, invoking Grinberg and Roby's theorem about birational rowmotion and using the recombination method to show that birational promotion (like birational rowmotion) is of order $a+b$, we give a proof of Theorem 4.4: for $v = (v_1, \ldots, v_{ab}) \in U$, the product of the coordinates of v associated with elements of the file S_ℓ (denoted by $|v|_\ell$) has the property that $|v|_\ell\,|\pi(v)|_\ell\,|\pi^2(v)|_\ell \cdots |\pi^{n-1}(v)|_\ell = 1$. In Section 6, we describe recombination and prove its basic properties. In Section 7, we use tropicalization to deduce from Theorem 4.4 a piecewise-linear analogue (Theorem 3.5) that by an affine change of variables yields the homomesy result for the action of promotion on $\mathcal{O}(P)$ (Theorem 3.4). This last result then yields homomesy for the action of promotion on $\mathcal{I}(P)$ (Theorem 1.1). That is, ignoring the use of recombination for passing back and forth between rowmotion and promotion, the logic of the argument is that we first prove birational homomesy, we then deduce piecewise-linear homomesy by tropicalization, and we finally deduce combinatorial homomesy by specialization. For completeness, in Section 8 we use the reciprocity principle of Grinberg–Roby [7] to prove that the function that sends f to $f(x)f(x')$ with $x = (i,j)$ and $x' = (a+1-i, b+1-j)$ is multiplicatively homomesic.

The philosophy of lifting combinatorial actions to piecewise-linear actions and thence to birational actions (called "geometric actions" by some authors, as in the phrase "geometric Robinson–Schensted–Knuth") is not original, and in particular Kirillov and Berenstein's work on operations on Gel′fand–Tsetlin patterns [10] has some parallels with our constructions. For more background on homomesy, including several examples different in nature from the ones considered here but philosophically similar, see [13].

The authors are grateful to Arkady Berenstein, Darij Grinberg, Michael Joseph, Tom Roby, Richard Stanley, and Jessica Striker for helpful conversations and detailed comments on the manuscript.

2. Background

2.1. Homomesy. Given a set X, an operation $T : X \to X$ of finite order n (so that T^n is the identity map on X), and a function F from X to a field $\mathbb{K}$ of characteristic 0, we say that F is additively *homomesic* relative to (or under the action of) T, or that the triple (X, T, F) exhibits additive *homomesy*, if, for all $x \in X$, the average of F on the T-orbit of x equals some constant c (independent of x); that is, if $(F(x) + F(T(x)) + F(T^2(x)) + \cdots + F(T^{n-1}(x)))/n = c$. We also say in this situation that the function F (which in this context we will sometimes call a *statistic* on X) is *c-mesic* relative to the map T. We will apply this notion in situations where T is piecewise-linear (or when X is finite).

In situations where T is birational, we will use a multiplicative analogue of this notion. If F is positive throughout X, then we say F is multiplicatively homomesic if its geometric mean is the same on every orbit. More generally, F is multiplicatively homomesic if $(F(x)\,F(T(x))\,F(T^2(x)) \cdots F(T^{n-1}(x)))^{1/n}$ is independent of x. (In some settings it is more natural to relax the assumption of positivity and merely assert that $F(x)\,F(T(x))\,F(T^2(x)) \cdots F(T^{n-1}(x))$ is independent of x, but we will not take that course here.) We say F is multiplicatively c-mesic if the geometric mean of F on every orbit is c.

We will usually omit the qualifiers "additive" and "multiplicative", since the context should make clear which meaning is intended (additive homomesy in the combinatorial and piecewise-linear realms, multiplicative homomesy in the birational realm).

2.2. Posets and toggling. We assume readers are familiar with the definition of a finite poset $(P, \leqslant)$, as for instance given in Ch. 3 of [15]. For the most part, we are studying the case $P = [a] \times [b] = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \ 1 \leqslant i \leqslant a, \ 1 \leqslant j \leqslant b\}$ with ordering defined by $(i,j) \leqslant (i',j')$ iff $i \leqslant i'$ and $j \leqslant j'$. We put $n = a + b$.

We write $x \lessdot y$ ("x is covered by y") or equivalently $y \gtrdot x$ ("y covers x") when $x < y$ and no $z \in P$ satisfies $x < z < y$. We call $S \subseteq P$ a *filter* (or *upset* or *dual order ideal*) of P when $x \in S$ and $y \geqslant x$ imply $y \in S$. We call $S \subseteq P$ an *order ideal* (or *ideal* or *downset*) of P when $x \in S$ and $y \leqslant x$ imply $y \in S$. The set of filters (resp. order ideals) of P is denoted by $\mathcal{F}(P)$ (resp. $\mathcal{I}(P)$).

Following Cameron and Fon-Der-Flaass [3] and Striker and Williams [16] we define *toggle* operations on $\mathcal{F}(P)$ and $\mathcal{I}(P)$. We treat $\mathcal{I}(P)$ first.

Given $x \in P$ and $I \in \mathcal{I}(P)$, let $\tau_x(I)$ ("I toggled at x" in Striker and Williams' terminology) denote the set $I \triangle \{x\}$ if this set is in $\mathcal{I}(P)$ and I otherwise (where $X \triangle Y$ denotes the symmetric difference of the sets X and Y). Equivalently, $\tau_x(I)$ is I unless $y \in I$ for all $y \lessdot x$ and $y \notin I$ for all $y \gtrdot x$, in which case $\tau_x(I)$ is $I \triangle \{x\}$. (We will sometimes say "toggling x turns I into $\tau_x(I)$.") Clearly τ_x is an involution. It is also easy to show that τ_x and τ_y commute unless $x \lessdot y$ or $x \gtrdot y$. Cameron and Fon-Der-Flaass proved that if $x_1, x_2, \ldots, x_{|P|}$ is any *linear extension* of P (that is, a listing of the elements of P such that $x_i < x_j$ implies $i < j$), then the composition $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_{|P|}}$ ("toggling from top to bottom") is independent of the choice of linear ordering; we denote it by ρ. In the case where the poset P is graded (that is, where the elements can be partitioned into integer-indexed ranks such that $x \lessdot y$ implies that the rank of x is 1 less than the rank of y), one natural way to linearly extend P is to list the elements by rank, starting with the lowest rank and working upward. Given the right-to-left order of composition of $\tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_{|P|}}$, this corresponds to toggling the top rank first, then the next-to-top rank, and so on, lastly toggling the bottom rank. Note that when x, y belong to the same rank of P, the toggle operations τ_x and τ_y commute, so even without availing ourselves of the theorem of Cameron and Fon-Der-Flaass we can see that this composite operation on $\mathcal{I}(P)$ ("toggling by ranks from top to bottom") is well-defined. Striker and Williams, in their theory of rc posets, use the term "row" as a synonym for "rank", and they refer to ρ as *rowmotion*.

For example, let $P = [2] \times [2]$, and write $(1,1), (2,1), (1,2), (2,2)$ as w, x, y, z for short, with $w < x < z$ and $w < y < z$ in P, and with the rc embedding shown in Figure 1. Under the action of τ_z, τ_y, τ_x, and τ_w, the order ideal $\{w, x\}$ gets successively mapped to $\{w, x\}$, $\{w, x, y\}$, $\{w, y\}$, and $\{w, y\}$. Thus $\rho(\{w, x\}) = \{w, y\}$.

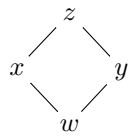


Figure 1. The poset $[2] \times [2]$.

We will not go into the general theory of rc posets, as most of our work concerns the special case of rc posets of the form $[a] \times [b]$. We define the *rank* of $(i,j) \in [a] \times [b]$ to be $i + j - 1$, so that in particular the bottom element $(1,1)$ has rank 1 and the top element (a,b) has rank $a + b - 1 = n - 1$ (later we will introduce an extension of P whose bottom element $\hat{0}$ has rank 0 and whose top element $\hat{1}$ has rank n). As an rc poset $[a] \times [b]$ admits an embedding in the plane that maps (i,j) to the point $(j - i, i + j - 2)$; all poset-elements of rank m are mapped into the horizontal line at height $m - 1$ above the origin. We refer to elements of P that lie on a common

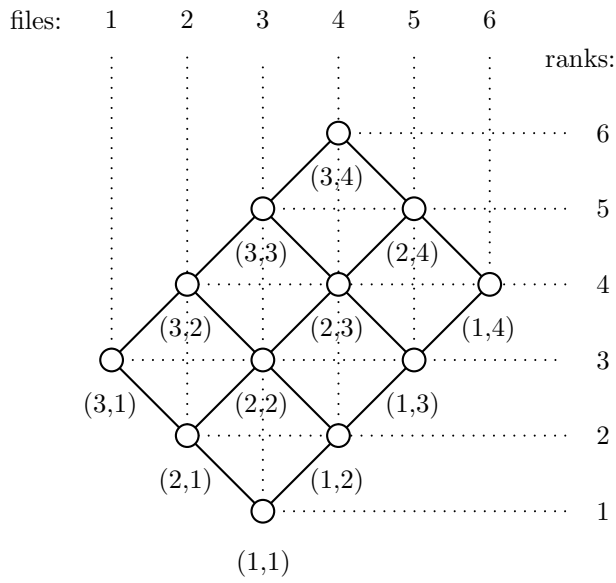


FIGURE 2. The standard rc embedding of the poset $P = [3] \times [4]$, and the associated partitions of P into ranks and files. Diagonal edges are associated with the covering relation in P.

vertical line as belonging to the same *file*. In particular, we say (i, j) belongs to the $(j - i + a)$th$^{(1)}$ file of P. See Figure 2. Note that if x and y belong to the same file, the toggle operations τ_x and τ_y commute, since neither of x, y can cover the other. Thus the composite operation of toggling the elements of P from left to right is well-defined; Striker and Williams call this operation *promotion*, and we denote it by π.

We have explained rowmotion and promotion as toggling by ranks (from top to bottom) and by files (from left to right). A different way to understand these operations is by toggling *fibers*$^{(2)}$ of $[a] \times [b]$, that is, sets of the form $[a] \times \{j\}$ or $\{i\} \times [b]$. (This view of rowmotion and promotion goes back to Striker and Williams; see the proof of Theorem 5.4 in [16].) We refer to fibers as being positive or negative according to whether their slope in the rc embedding is $+1$ or -1.

We illustrate these ideas in the context of $P = [3] \times [3]$. We consider two orders (both linear extensions of P) in which one can toggle all the elements of P to obtain ρ. The first order is just the standard order for rowmotion (from top to bottom rank by rank, from left to right within each rank). The second order toggles the elements in the topmost positive fiber from top to bottom, then the elements in the middle positive fiber from top to bottom, and then the elements in the bottommost positive fiber from top to bottom. Since the element of P marked 3 in the left frame neither covers nor is covered by the element of P marked 4, the associated toggles commute, and the same goes for the two toggles associated with the elements of P marked 6 and 7. Hence the composite operation on the left ("rowmotion by ranks") coincides with the composite operation on the right ("rowmotion by fibers"). This holds for all a, b by the theorem of Cameron and Fon-Der-Flaass [3].

---

(1)Note that as i ranges from 1 to a and j ranges from 1 to b, $j - i + a$ ranges from 1 to $a + b - 1 = n - 1$; this is slightly different from the indexing in [13].

(2)We prefer to avoid words like "row", "column" and "diagonal" since each of these words has two different meanings according to whether one imbeds $[a] \times [b]$ as an rc poset or as a subposet of the first quadrant.

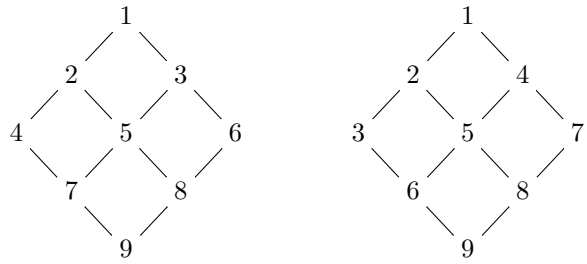


Figure 3. Rowmotion by ranks and fibers.

There is a similar picture for promotion. Here we consider two other orders (not linear extensions of P) in which one can toggle all the elements of P. The first is just the standard order for promotion (from left to right file by file, from top to bottom within each file). The second toggles the elements in the topmost positive fiber from left to right, then the elements in the middle positive fiber from left to right, and then the elements in the bottommost positive fiber from left to right. As before, the 3 and the 4 can be swapped, as can the 6 and the 7. Hence the composite operation on the left ("promotion by files") coincides with the composite operation on the right ("promotion by fibers").

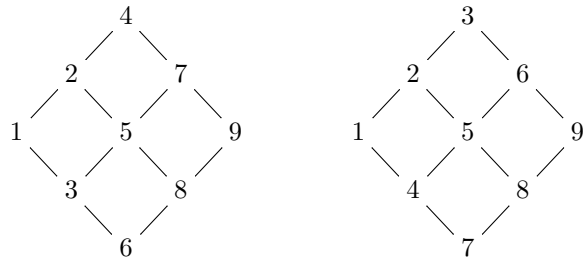


Figure 4. Promotion by ranks and fibers.

Note that for both rowmotion and promotion we divide the poset into positive fibers and toggle them from top to bottom; the only difference is whether we toggle the elements within each fiber from left to right (promotion) or right to left (rowmotion). The proof that promotion by files and promotion by fibers coincide for all a, b uses the same commutation results as in the case of rowmotion; it is just a matter of turning the picture on its side.

So far we have limited ourselves to toggling, rowmotion, and promotion of order ideals; the definitions for filters are very similar. Given $x \in P$ and $F \in \mathcal{F}(P)$, let $\tau_x(F)$ be $F \triangle \{x\}$ if this set is in $\mathcal{F}(P)$ and F otherwise. Rowmotion is defined as toggling from top to bottom and promotion is defined as toggling from left to right. When we wish to be ultra-clear about context we may write order-ideal rowmotion and promotion as $\rho_{\mathcal{I}}$ and $\pi_{\mathcal{I}}$ and write filter rowmotion and promotion as $\rho_{\mathcal{F}}$ and $\pi_{\mathcal{F}}$, but often we will omit the subscripts.

Figure 5 shows what $\rho_{\mathcal{F}}$ looks like as a map on the six-element set $\mathcal{F}$ (we represent each filter by its indicator function).

The matching $\|$: and :$\|$ symbols, borrowed from music notation, indicate the orbit structure; applying the map ρ to the last listed filter (preceding the :$\|$) brings us back to the first (following the $\|$:).

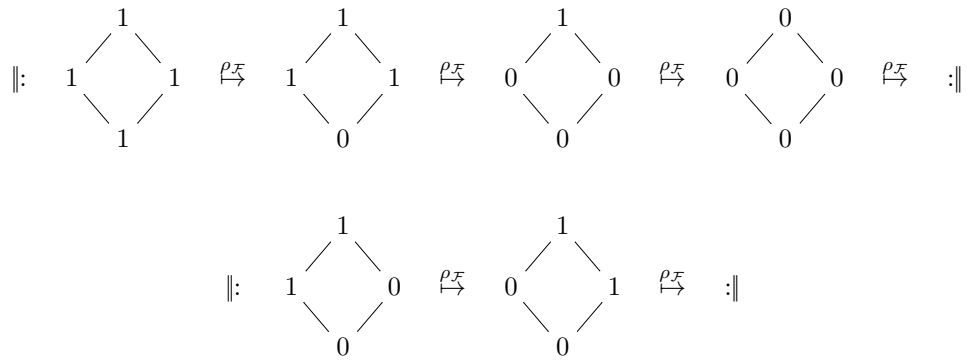


FIGURE 5. The orbits of $\rho_{\mathcal{F}}$ on the filters of $[2] \times [2]$.

3. Piecewise-linear toggling

Given a poset P with elements $x_1, \ldots, x_{|P|}$, let $\mathbb{R}^P$ denote the set of functions $f : P \to \mathbb{R}$; we can represent such an f as a *P-array* (or *array* for short) in which the values of $f(x)$ (for all $x \in P$) are arranged according to a Hasse diagram for P. We will assume that the elements of P are listed in order of rank (from lowest to highest), and when P has an rc embedding, we will assume for convenience that the elements within each rank are listed from left to right. We will sometimes identify $\mathbb{R}^P$ with $\mathbb{R}^{|P|}$, associating $f \in \mathbb{R}^P$ with $v = (f(x_1), \ldots, f(x_{|P|}))$. (We will use "$f$" when we want to view P-arrays as functions that take arguments and return values, and "v" when we want to view P-arrays as points in Euclidean space.) Let $\hat{P}$ denote the augmented poset obtained from P by adding two extra elements $\hat{0}$ and $\hat{1}$ (which we sometimes denote by x_0 and $x_{|P|+1}$) satisfying $\hat{0} < x < \hat{1}$ for all $x \in P$. The *order polytope* $\mathcal{O}(P) \subset \mathbb{R}^P$ (see [14]) is the set of vectors $(\hat{f}(x_1), \ldots, \hat{f}(x_{|P|}))$ arising from functions $\hat{f} : \hat{P} \to \mathbb{R}$ that satisfy $\hat{f}(\hat{0}) = 0$ and $\hat{f}(\hat{1}) = 1$ and are *order-preserving* ($x \leqslant y$ in $\hat{P}$ implies $\hat{f}(x) \leqslant \hat{f}(y)$ in $\mathbb{R}$). Note that the vertices of the order polytope are the indicator functions of filters. In some cases it is better to work with the augmented vector $(\hat{f}(x_0), \hat{f}(x_1), \ldots, \hat{f}(x_{|P|}), \hat{f}(x_{|P|+1}))$. In either case we have a convex compact polytope.

For example, if $P = [2] \times [2] = \{w, x, y, z\}$ as depicted in Subsection 2.2, so that $(x_1, x_2, x_3, x_4) = (w, x, y, z)$, then $\mathcal{O}(P)$ is the set of vectors $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ such that $0 \leqslant v_1$, $v_1 \leqslant v_2 \leqslant v_4$, $v_1 \leqslant v_3 \leqslant v_4$, and $v_4 \leqslant 1$. It can also be written as the convex hull of the vectors $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$, and $(1,1,1,1)$, which are precisely the vectors associated with the filters of P. It is shown in [14] that for any poset P, the vertices of $\mathcal{O}(P)$ correspond to the indicator functions of the filters of P.

We begin our discussion of toggling informally; for formal definitions, see Definitions 3.1, 3.2, and 3.3. Given a convex compact polytope K in $\mathbb{R}^{|P|}$ (we are only concerned with the case $K = \mathcal{O}(P)$ here but the definition makes sense more generally), we define the *piecewise-linear toggle operation* ϕ_i ($1 \leqslant i \leqslant |P|$) as the unique map from K to itself whose action on the 1-dimensional cross-sections of K in the ith coordinate direction is the affine map that switches the two endpoints of the cross-section. (We sometimes call these cross-sections fibers, e.g. in our use of the term fiber-toggle below, though this use of the word "fiber" should not be confused with the use found in Subsection 2.2.) That is, given $v = (v_1, \ldots, v_{|P|}) \in K$, we define

$$\phi_i(v) = (v_1, \ldots, v_{i-1}, L_i(v) + R_i(v) - v_i, v_{i+1}, \ldots, v_{|P|}), \tag{1}$$

where the real numbers $L_i(v)$ and $R_i(v)$ (usually L and R for short) are respectively the left and right endpoints of the set $\{t \in \mathbb{R} : \ (v_1, \ldots, v_{i-1}, t, v_{i+1}, \ldots, v_{|P|}) \in K\}$, which is a bounded interval because K is convex and compact.[(3)] Since $L + R - (L + R - v_i) = v_i$, each toggle operation is an involution.

Similar involutions were studied by Kirillov and Berenstein [10] in the context of Gel′fand–Tsetlin triangles. Indeed, one can view their action as an instance of our framework, where instead of looking at the rectangle posets $[a] \times [b]$ one looks at the triangle posets with elements $\{(i, j) : \ 1 \leqslant i \leqslant j \leqslant n\}$ and covering-relations $(i, j-1) \lessdot (i, j)$ $(1 \leqslant i \leqslant j \leqslant n)$ and $(i+1, j+1) \lessdot (i, j)$ $(1 \leqslant i \leqslant j \leqslant n-1)$; Kirillov and Berenstein, in their Definition 0.1, use the term "elementary transformations" for what we call "fiber-toggles".

In the case where K is the order polytope of P, and a particular element $x \in P$ has been indexed as x_i, we write ϕ_i as ϕ_x. The L and R that appear in (1) are given by

$$L = \max\{v_j : \ 0 \leqslant j \leqslant |P| + 1, \ x_j \lessdot x_i\} \tag{2}$$

and

$$R = \min\{v_j : \ 0 \leqslant j \leqslant |P| + 1, \ x_j \gtrdot x_i\}. \tag{3}$$

Using these definitions of L and R,[(4)] we see that equation (1) defines an involution on all of $\mathbb{R}^P$, not just $\mathcal{O}(P)$. It is easy to show that ϕ_x and ϕ_y commute unless $x \lessdot y$ or $x \gtrdot y$. These piecewise-linear toggle operations ϕ_x are analogous to the combinatorial toggle operations τ_x (and indeed ϕ_x generalizes τ_x in a sense to be made precise below), so it is natural to define piecewise-linear rowmotion $\rho_{\mathcal{P}} : \mathcal{O}(P) \to \mathcal{O}(P)$ as the composite operation accomplished by toggling from top to bottom (much as filter rowmotion $\rho_{\mathcal{F}} : \mathcal{F}(P) \to \mathcal{F}(P)$ can be defined as the composite operation obtained by toggling from top to bottom). Likewise, if P comes equipped with an rc embedding (as is the case for $P = [a] \times [b]$), we can define piecewise-linear promotion $\pi_{\mathcal{P}} : \mathcal{O}(P) \to \mathcal{O}(P)$ as the composite operation accomplished by toggling from left to right.

Continuing the example $P = [2] \times [2] = \{w, x, y, z\}$ from Subsection 2.2, let $v = (.1, .2, .3, .4) \in \mathcal{O}(P)$, corresponding to the order-preserving function f that maps w, x, y, z to $.1, .2, .3, .4$, respectively. Under the action of ϕ_z, ϕ_y, ϕ_x, and ϕ_w, the vector $v = (.1, .2, .3, .4)$ gets successively mapped to

$$\begin{aligned}
\phi_z v &= (.1, \ .2, \ .3, \ \max(.2, .3) + 1 - .4) \\
&= (.1, \ .2, \ .3, \ .9), \\
\phi_y \phi_z v &= (.1, \ .2, \ .1 + .9 - .3, \ .9) \\
&= (.1, \ .2, \ .7, \ .9), \\
\phi_x \phi_y \phi_z v &= (.1, \ .1 + .9 - .2, \ .7, \ .9) \\
&= (.1, \ .8, \ .7, \ .9),
\end{aligned}$$

(3) It is in some ways unfortunate that for an interval $[a, b]$, the greatest lower bound a (resp. least upper bound b) is called the left endpoint (resp. right endpoint) rather than the lower endpoint (resp. upper endpoint); in the Hasse diagram of P as we have drawn it, L should be thought as being associated with the downward direction, while R should be thought as being associated with the upward direction.

(4) For all v in $\mathcal{O}(P)$ one also has $L = \max\{v_j : x_j < x_i\}$ and $R = \min\{v_j : x_j > x_i\}$, but the formulas (2) and (3)) turn out to be the right ones to use in the complement of the order polytope in $\mathbb{R}^P$, as well as the right ones vis-a-vis lifting the action to the birational setting.

and

$$\begin{aligned}\phi_w\phi_x\phi_y\phi_z v &= (0 + \min(.8, .7) - .1,\ .8,\ .7,\ .9)\\ &= (.6,\ .8,\ .7,\ .9)\\ &= \rho_{\mathcal{P}} v.\end{aligned}$$

Other examples of piecewise-linear rowmotion can be seen in Figure 5 (since combinatorial rowmotion of filters is just the special case of piecewise-linear rowmotion in which all entries in the P-array are equal to 0 or 1, and the associated filter is the preimage of the value 1).

Grinberg and Roby [7] have shown that $\rho_{\mathcal{P}}$ on $P = [a] \times [b]$ is of order $n = a + b$, and by applying recombination (see Section 6) we will conclude that $\pi_{\mathcal{P}}$ is of order n as well.

The vertices of $\mathcal{O}(P)$ are precisely the 0,1-valued functions f on P with the property that $x \leqslant y$ in P implies $f(x) \leqslant f(y)$ in $\{0, 1\}$. Each toggle operation acts as a permutation on the set of vertices of $\mathcal{O}(P)$. Indeed, if we think of each vertex $\mathcal{O}(P)$ as determining a cut of the poset P into an upset (filter) S_{up} and a downset (order ideal) S_{down} (the pre-image of 1 and 0, respectively, under the order-preserving map from P to $\{0, 1\}$), then the effect of the toggle operation ϕ_x ($x \in P$) is just to move x from S_{up} to S_{down} (if x is in S_{up}) or from S_{down} to S_{up} (if x is in S_{down}) unless this would violate the property that S_{up} must remain an upset and S_{down} must remain a downset. In particular, we can see that when our point $v \in \mathcal{O}(P)$ is a vertex associated with the cut $(S_{\text{up}}, S_{\text{down}})$, the effect of ϕ_x on S_{down} is just toggling the order ideal S_{down} at the element $x \in P$.[(5)]

It is not hard to show that each toggle operation preserves $\min\{f(y) - f(x) : x, y \in \hat{P},\ x \lessdot y\}$. Therefore $\rho_{\mathcal{P}}$ (and $\pi_{\mathcal{P}}$, when P comes with an rc embedding) preserve this quantity as well.

In our formal definitions we generalize the above construction by allowing $f(\hat{0})$ and $f(\hat{1})$ to have fixed values in $\mathbb{R}$ other than 0 and 1 respectively.

Definition 3.1. *Suppose P is a poset with $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$, and fix α, ω in $\mathbb{R}$. We view each $f \in \mathbb{R}^P$ as an element of $f \in \mathbb{R}^{\hat{P}}$ by setting $f(\hat{0}) = \alpha$ and $f(\hat{1}) = \omega$. For $x \in P$, define $\phi_x f$ as the unique element of $\mathbb{R}^P$ such that $(\phi_x f)(y) = f(y)$ for all $y \neq x$ in P and*

$$(\phi_x f)(x) = \max\{f(y) : y \in \hat{P},\ y \lessdot x\} + \min\{f(y) : y \in \hat{P},\ y \gtrdot x\} - f(x).$$

(Note that the sets in Definition 3.1 are guaranteed to be nonempty, so that the max and min are well-defined.)

Definition 3.2. *With P and $\hat{P}$ as in Definition* 3.1*, and for f in $\mathbb{R}^P$, let $\rho_{\mathcal{P}}(f)$ be the element of $\mathbb{R}^P$ obtained by applying to f, in succession, the toggle operations $\phi_{|P|}, \ldots, \phi_2, \phi_1$, where $x_1, x_2, \ldots, x_{|P|}$ is some linear extension of P and $\phi_i = \phi_{x_i}$. An easy adaptation of the proof of Cameron and Fon-Der-Flaass* [3] *shows that this operation is independent of the linear ordering.*

For example, returning to the example $P = [2] \times [2] = \{w, x, y, z\}$ from Subsection 2.2, let $v = (.1, .2, .3, .4) \in \mathcal{O}(P)$ as before, but now set $\alpha = 1$ and $\omega = 0$ (rather than $\alpha = 0$ and $\omega = 1$). Under the action of ϕ_z, ϕ_y, ϕ_x, and ϕ_w, the vector

(5) This point of view is quite similar to the monotone Boolean functions point of view seen in the original literature on what is now called rowmotion.

$v = (.1, .2, .3, .4)$ gets successively mapped to

$$\begin{aligned}\phi_z v &= (.1,\ .2,\ .3,\ \max(.2,.3)+0-.4)\\ &= (.1,\ .2,\ .3,\ -.1),\\ \phi_y\phi_z v &= (.1,\ .2,\ .1+(-.1)-.3,\ -.1)\\ &= (.1,\ .2,\ -.3,\ -.1),\\ \phi_x\phi_y\phi_z v &= (.1,\ .1+(-.1)-.2,\ -.3,\ -.1)\\ &= (.1,\ -.2,\ -.3,\ -.1),\end{aligned}$$

and

$$\begin{aligned}\phi_w\phi_x\phi_y\phi_z v &= (1+\min(-.2,-.3)-.1,\ -.2,\ -.3,\ -.1)\\ &= (.6,\ -.2,\ -.3,\ -.1)\\ &= \rho_{\mathcal{P}} v.\end{aligned}$$

Definition 3.3. *Take P and $\hat{P}$ as in Definition* 3.1*, and suppose we are given an rc embedding of P. For f in $\mathbb{R}^P$, let $\pi_{\mathcal{P}}(f)$ be the element of $\mathbb{R}^P$ obtained by applying to f, in succession, the toggle operations from left to right.*

We call $\alpha = \omega = 0$ the *homogeneous case* and call $\alpha = 0$, $\omega = 1$ the *order polytope case*. In the homogeneous case, we write the rowmotion operation as $\overline{\rho}_{\mathcal{P}}$ (and likewise we write the homogeneous piecewise-linear promotion operation as $\overline{\pi}_{\mathcal{P}}$ when P comes with an rc embedding). We can always translate by α (replacing $f(x)$ by $f(x)-\alpha$ for all $x \in P$) to reduce to the case $\alpha = 0$. In the case where P is graded, the maps $\rho_{\mathcal{P}}$ and $\overline{\rho}_{\mathcal{P}}$ are related by an affine change of variables, as are the maps $\pi_{\mathcal{P}}$ and $\overline{\pi}_{\mathcal{P}}$ in the rc case. Suppose that $\hat{P}$ has $r+1$ ranks, numbered 0 (bottom) through r (top). Given an arbitrary f in $\mathbb{R}^P$, define $\widetilde{f}(x) = f(x) - \frac{r-m}{r}\alpha - \frac{m}{r}\omega$ for x belonging to rank m ($0 \leqslant m \leqslant r$). Then $\widetilde{f}$ sends $\hat{0}$ and $\hat{1}$ to 0, and each function from $\hat{P}$ to $\mathbb{R}$ that sends $\hat{0}$ and $\hat{1}$ to 0 arises as $\widetilde{f}$ for a unique f in $\mathbb{R}^P$. Furthermore, the map $f \mapsto \widetilde{f}$ commutes with rowmotion,[(6)] so every homomesy for homogeneous rowmotion gives rise to a (rank-adjusted) homomesy for order-polytope rowmotion, and vice versa; the same goes for promotion in the rc case.

Using the homogenization/dehomogenization trick one can show that the following two theorems are equivalent:

Theorem 3.4. *Fix $a, b \geqslant 1$, let $n = a+b$, let $P = [a] \times [b]$, for $1 \leqslant \ell \leqslant n-1$ let $|v|_\ell = \sum_x f(x)$ where x ranges over all $(i,j) \in [a]\times[b]$ satisfying $j - i + a = \ell$, and let $|v|$ be $\sum_{\ell=1}^{n-1} |v|_\ell$, the sum of all the entries of v. Take $\alpha = 0$, $\omega = 1$. Then for every v in $\mathbb{R}^P$, and for each k between 1 and $n-1$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} |\pi_{\mathcal{P}}^k(v)|_\ell = \begin{cases} a\ell/n & \text{if } \ell \leqslant b,\\ b(n-\ell)/n & \text{if } \ell \geqslant b.\end{cases}$$

Summing over ℓ, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} |\pi_{\mathcal{P}}^k(v)| = ab/2.$$

(The reason we have different formulas in Theorems 1.1 and 3.4 is that the former concerns promotion of order ideals while the latter concerns promotion in the order polytope. If we replaced order ideals by filters in the statement of Theorem 1.1, the formula for the orbit-average would become what we see in Theorem 3.4.)

(6) More generally, this way of relating $\rho_{\mathcal{P}}$ and $\overline{\rho}_{\mathcal{P}}$ works whenever the poset P is graded.

THEOREM 3.5. *Fix $a, b \geqslant 1$, let $n = a + b$, let $P = [a] \times [b]$, for $1 \leqslant \ell \leqslant n-1$ let $|v|_\ell = \sum_x f(x)$ where x ranges over all $(i,j) \in [a] \times [b]$ satisfying $j - i + a = \ell$, and let $|v|$ be $\sum_{\ell=1}^{n-1} |v|_\ell$, the sum of all the entries of v. Take $\alpha = \omega = 0$. Then for every v in $\mathbb{R}^P$, and for each k between 1 and $n-1$,*

$$\sum_{k=0}^{n-1} |\overline{\pi}_{\mathcal{P}}^k(v)|_\ell = 0.$$

Summing over ℓ, we obtain

$$\sum_{k=0}^{n-1} |\overline{\pi}_{\mathcal{P}}^k(v)| = 0.$$

We will obtain both Theorem 3.4 and Theorem 3.5 as consequences of Theorem 4.4. (We do not state a fully general version of Theorem 3.4 with arbitrary values of α and ω, but it is easy to obtain such a result by tropicalizing Theorem 5.3.)

Note that if a or b is 1, so that $[a] \times [b]$ is a chain with $n-1$ elements, then our piecewise-linear maps are all affine, and if we set $v_0 = v_n = 0$, then the effect of the map ϕ_i $(1 \leqslant i \leqslant n-1)$ is to swap the ith and $i+1$st elements of the difference-vector $(v_1 - v_0, v_2 - v_1, \ldots, v_n - v_{n-1})$, a vector of length n whose entries sum to 0. Consequently $\overline{\pi}_{\mathcal{P}}$ is a cyclic shift of a vector whose entries sum to 0, and the claim of Theorem 3.5 follows in this special case.

It might be possible to prove Theorem 3.4 in the general case by figuring out how the map $\pi_{\mathcal{P}}$ dissects the order polytope into pieces and re-arranges them via affine maps. Likewise, it might be possible to prove Theorem 3.5 by giving a precise analysis of the piecewise-linear structure of the map $\overline{\pi}_{\mathcal{P}}$. However, we will take a different approach, proving the result in the piecewise-linear setting by proving it in the birational setting and then tropicalizing.

4. Birational toggling

The definition of the toggling operation involves only addition, subtraction, min, and max. One can define birational transformations on $(\mathbb{R}^+)^P$ that have some formal resemblance to the toggle operations on $\mathcal{O}(P)$. This transfer makes use of a dictionary in which 0, addition, subtraction, max, and min are respectively replaced by 1, multiplication, division, addition, and parallel addition (defined below), resulting in a subtraction-free rational expression.[7] Parallel addition can be expressed in terms of the other operations, but taking a symmetrical view of the two forms of addition turns out to be fruitful. Indeed, in setting up the correspondence we have a choice to make: by series-parallel duality, one could equally well use a dictionary that switches the roles of addition and parallel addition.

For x, y satisfying $x + y \neq 0$, we define the parallel sum of x and y as $x \parallel y = xy/(x+y)$. In the case where x, y and $x+y$ are all nonzero, $xy/(x+y)$ is equal to $1/(\frac{1}{x} + \frac{1}{y})$, which clarifies the choice of notation and terminology: if two electrical resistors of resistance x and y are connected in parallel, the resulting circuit element has effective resistance $x \parallel y$. Note that if x and y are in $\mathbb{R}^+$, then $x+y$ and $x \parallel y$ are in $\mathbb{R}^+$ as well. Also note that $\parallel$ is commutative and associative, so that a compound parallel sum $x \parallel y \parallel z \parallel \cdots$ is well-defined; it is equal to product $xyz\cdots$ divided by the sum of all products that omit exactly one of the variables, and in the case where

[7] The authors are indebted to Arkady Berenstein for pointing out the details of this transfer of structure from the piecewise-linear setting to the birational setting. A key signpost pointing in the correct direction was his remark that $\min(x, y) = -\max(-x, -y)$.

$x, y, z, \ldots$ are all nonzero, it can also be written as $1/(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\cdots)$. We thus have the reciprocity relation

$$(x \parallel y \parallel z \parallel \cdots)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\cdots\right)=1. \tag{4}$$

The identity

$$(x \parallel y)(x+y)=xy \tag{5}$$

will also play an important role.

As in the previous section, we begin informally, preparing for Definitions 4.1, 4.2, and 4.3. Recall formulas (1), (2) and (3) above. Instead of taking the maximum of the v_j's satisfying $x_j \lessdot x_i$, we take their ordinary (or "series") sum, and instead of taking the minimum of the v_j's satisfying $x_j \gtrdot x_i$, we take their parallel sum. Proceeding formally, given a nonempty multiset $S=\{s_1, s_2, \ldots\}$, let $\sum^{+} S$ denote $s_1+s_2+\cdots$ and $\sum^{\parallel} S$ denote $s_1 \parallel s_2 \parallel \cdots$. (To see why we must think of S as a multiset and take multiplicity into account, consider for instance the series sum and parallel sum in equations (7) and (8) respectively; if we happen to have $v_{j_1}=v_{j_2}$ for some pair $j_1 \neq j_2$, both of the terms must be included.) Then for $v=(v_0, v_1, \ldots, v_{|P|}, v_{|P|+1}) \in (\mathbb{R}^+)^{\hat{P}}$ with $v_0=\alpha=1$ and $v_{|P|+1}=\omega=1$ (recall that $x_0=\hat{0}$ and $x_{|P|+1}=\hat{1}$) and for $1 \leqslant i \leqslant |P|$ we take

$$\phi_i(v)=(v_0, v_1, \ldots, v_{i-1}, LR/v_i, v_{i+1}, \ldots, v_{|P|}, v_{|P|+1}), \tag{6}$$

with

$$L=\sum\nolimits^{+}\{v_j:\ 0 \leqslant j \leqslant |P|+1,\ x_j \lessdot x_i\} \tag{7}$$

and

$$R=\sum\nolimits^{\parallel}\{v_j:\ 0 \leqslant j \leqslant |P|+1,\ x_j \gtrdot x_i\} \tag{8}$$

where as in Section 3 the sets in question are guaranteed to be nonempty. We call the maps ϕ_i given by (6) *birational toggle operations*, as opposed to the piecewise-linear toggle operations treated in the previous section.[8] As the 0th and $|P|+1$st coordinates of v are not affected by any of the toggle operations we can just omit those coordinates, reducing our toggle operations to actions on $(\mathbb{R}^+)^P$. Since $LR/(LR/v_i)=v_i$, each birational toggle operation is an involution on the orthant $(\mathbb{R}^+)^P$. The birational toggle operations are analogous to the piecewise-linear toggle operations (in a sense to be made precise below), so it is natural to define *birational rowmotion* $\rho_{\mathcal{B}}: (\mathbb{R}^+)^P \to (\mathbb{R}^+)^P$ as the composite operation accomplished by toggling from top to bottom, and to define *birational promotion* $\pi_{\mathcal{B}}: (\mathbb{R}^+)^P \to (\mathbb{R}^+)^P$ as the composite operation accomplished by toggling from left to right in the rc case.

It is not hard to show that each birational toggle operation preserves

$$\sum\nolimits^{+}\{f(x)/f(y):\ x, y \in \hat{P},\ x \lessdot y\}$$

(or, if one prefers, the reciprocal quantity $\sum^{\parallel}\{f(y)/f(x):\ x, y \in \hat{P},\ x \lessdot y\}$). Therefore $\rho_{\mathcal{B}}$ also preserves this quantity, as does $\pi_{\mathcal{B}}$ in the rc case. (We are indebted to Arkady Berenstein for this observation.) One could define the *birational toggle group* as the group generated by all birational toggles, and the *piecewise-linear toggle group* analogously. These are related to the (combinatorial) "toggle group" earlier authors have studied (the group induced by the action of all the toggles on $\mathcal{I}(P)$), but unlike

[8] We use the same symbol ϕ_i for both, but this should cause no confusion.

the combinatorial toggle group, these groups are in general infinite. It seems possible that, at least in some cases, the birational toggle group contains all birational transformations that preserve $\sum^{+}\{f(x)/f(y) : x, y \in \hat{P},\, x \lessdot y\}$.

Continuing our running example $P = [2] \times [2] = \{w, x, y, z\}$ with $f(\hat{0}) = f(\hat{1}) = 1$, let $v = (1, 2, 3, 4) \in (\mathbb{R}^{+})^{P}$, corresponding to the positive function f that maps w, x, y, z to $1, 2, 3, 4$, respectively. Under the action of ϕ_z, ϕ_y, ϕ_x, and ϕ_w, the vector $v = (1,\ 2,\ 3,\ 4)$ gets successively mapped to

$$\begin{aligned}
\phi_z v &= (1,\ 2,\ 3,\ (2+3)(1)/4) \\
&= (1,\ 2,\ 3,\ \tfrac{5}{4}), \\
\phi_y\phi_z v &= (1,\ 2,\ (1)(\tfrac{5}{4})/3,\ \tfrac{5}{4}) \\
&= (1,\ 2,\ \tfrac{5}{12},\ \tfrac{5}{4}), \\
\phi_x\phi_y\phi_z v &= (1,\ (1)(\tfrac{5}{4})/2,\ \tfrac{5}{12},\ \tfrac{5}{4}) \\
&= (1,\ \tfrac{5}{8},\ \tfrac{5}{12},\ \tfrac{5}{4}),
\end{aligned}$$

and

$$\begin{aligned}
\phi_w\phi_x\phi_y\phi_z &= ((1)(\tfrac{5}{8} \parallel \tfrac{5}{12})/1,\ \tfrac{5}{8},\ \tfrac{5}{12},\ \tfrac{5}{4}) \\
&= (\tfrac{1}{4},\ \tfrac{5}{8},\ \tfrac{5}{12},\ \tfrac{5}{4}) \\
&= \rho_{\mathcal{B}}(v).
\end{aligned}$$

We can check that the quantity $\sum^{+}\{f(x)/f(y) : x, y \in \hat{P}, x \lessdot y\}$ retains the value $\frac{85}{12}$ throughout the process.

For most of the purposes of this article, it suffices to take $\pi_{\mathcal{B}}$ to be a map from $(\mathbb{R}^{+})^{P}$ to itself; since the variables take on only positive values and since the expressions in those variables are all subtraction-free, all of the denominators are non-zero, so none of the rational functions blow up. Alternatively, as in the work of Grinberg and Roby, one can replace $\mathbb{R}^{+}$ by a ring of rational functions in formal indeterminates indexed by the elements of P, thereby avoiding the singularity issue. A third approach is to extend $\rho_{\mathcal{B}}$ and $\pi_{\mathcal{B}}$ to maps from a dense open subset U of $\mathbb{C}^{P}$ to itself by avoiding the points where denominators vanish. This is slightly subtle, since one needs to exclude all points whose orbits intersect the singular set. That is, we need to avoid not merely those points where the map blows up, but also points where the kth iterate of the map blows up. When the map is of finite order, this means avoiding a finite union of Zariski-closed proper subsets. It should be possible to characterize the resulting dense open set U, but we do not do this here; we merely point out that such a dense open set must exist since there are only finitely many denominators.

We now give formal definitions. Fix α, ω in $\mathbb{R}^{+}$.

Definition 4.1. *Suppose P is a poset with $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$. For f in $(\mathbb{R}^{+})^{P}$ (extended to an element of $(\mathbb{R}^{+})^{\hat{P}}$ by putting $f(\hat{0}) = \alpha$ and $f(\hat{1}) = \omega$) and for $x \in P$, define $\phi_x f$ as the unique element of $(\mathbb{R}^{+})^{P}$ such that $(\phi_x f)(y) = f(y)$ for all $y \neq x$ in P and*

$$(\phi_x f)(x) = \left(\sum\nolimits^{+}\{f(y) : y \in \hat{P},\ y \lessdot x\}\right)\left(\sum\nolimits^{\parallel}\{f(y) : y \in \hat{P},\ y \gtrdot x\}\right)/f(x).$$

(Note that the sets in Definition 4.1 are guaranteed to be nonempty, so that the sum and parallel sum are well-defined.)

Definition 4.2. *With P and $\hat{P}$ as in Definition* 4.1*, and for f in $(\mathbb{R}^{+})^{P}$, let $\rho_{\mathcal{B}}(f)$ be the element of $(\mathbb{R}^{+})^{P}$ obtained by applying to f, in succession, the toggle operations $\phi_{|P|}, \ldots, \phi_2, \phi_1$, where $x_1, x_2, \ldots, x_{|P|}$ is some linear extension of P and $\phi_i = \phi_{x_i}$.*

An easy adaptation of the proof of Cameron and Fon-Der-Flaass [3] *shows that this operation is independent of the linear ordering.*

Definition 4.3. *Take P and $\hat{P}$ as in Definition* 4.1*, and suppose we are given an rc embedding of P. For f in $(\mathbb{R}^+)^P$, let $\pi_{\mathcal{B}}(f)$ be the element of $(\mathbb{R}^+)^P$ obtained by applying to f, in succession, the toggle operations from left to right.*

We focus mostly on the *monic* case $f(\hat{0}) = f(\hat{1}) = 1$, since (just as in the piecewise-linear setting) no generality is lost, if P is graded and we restrict to vectors v in the positive orthant so that rth roots are globally well-defined, where $r+1$ is the number of ranks of $\hat{P}$. Given an arbitrary $f : \hat{P} \to \mathbb{R}^+$, let $\alpha = f(\hat{0})$ and $\omega = f(\hat{1})$, and define $\widetilde{f}$ by $\widetilde{f}(x) = f(x)/(\alpha^{1-m/r}\omega^{m/r})$ for x belonging to rank m ($0 \leqslant m \leqslant r$). Then $\widetilde{f}$ sends $\hat{0}$ and $\hat{1}$ to 1, and for any choice of $\alpha, \omega \in \mathbb{R}^+$, each function from $\hat{P}$ to $\mathbb{R}^+$ that sends $\hat{0}$ and $\hat{1}$ to 1 arises as $\widetilde{f}$ for a unique $f : \hat{P} \to \mathbb{R}^+$ sending $\hat{0}$ to α and $\hat{1}$ to ω. Furthermore, the map $f \mapsto \widetilde{f}$ commutes with rowmotion (and with promotion in the rc case), so homomesy results for $\widetilde{f}$ yield homomesy results for f as immediate consequences. When we wish to emphasize that we are in the monic case $\alpha = \omega = 1$, we will write $\rho_{\mathcal{B}}$ and $\pi_{\mathcal{B}}$ as $\overline{\rho}_{\mathcal{B}}$ and $\overline{\pi}_{\mathcal{B}}$.

As in the piecewise-linear case, our proof-method will enable us to demonstrate (multiplicative) homomesy of the action when P is a product of two chains.

Theorem 4.4. *Fix $a, b \geqslant 1$, let $n = a + b$, let $P = [a] \times [b]$, for $1 \leqslant \ell \leqslant n-1$ let $|v|_\ell = \prod_x f(x)$ where x ranges over all $(i,j) \in [a] \times [b]$ satisfying $j - i + a = \ell$, and let $|v|$ be $\prod_{\ell=1}^{n-1} |v|_\ell$, the product of all the entries of v. Then for every v in $(\mathbb{R}^+)^P$ corresponding to an $f : \hat{P} \to \mathbb{R}^+$ with $f(\hat{0}) = f(\hat{1}) = 1$,*

$$\prod_{k=0}^{n-1} |\overline{\pi}_{\mathcal{B}}^k(v)|_\ell = 1.$$

Multiplying over ℓ, we obtain

$$\prod_{k=0}^{n-1} |\overline{\pi}_{\mathcal{B}}^k(v)| = 1.$$

In other words, the geometric mean of the values of $|\overline{\pi}_{\mathcal{B}}(v)|_\ell$ as v traces out an orbit in $(\mathbb{R}^+)^P$ is equal to 1 for every orbit. Theorem 4.4 also applies to a dense open subset of $\mathbb{R}^P$, and indeed to a dense open subset of $\mathbb{C}^P$, but the paraphrase in terms of geometric means does not hold in general since $z \mapsto z^{1/n}$ is not single-valued on $\mathbb{C}$.

We will derive Theorem 4.4 from Theorem 5.3.

5. File-toggling and promotion

In this section we assume $P = [a] \times [b]$. We work in the birational realm, though we switch to considering promotion rather than rowmotion for reasons that will soon be clear. Earlier we defined S_ℓ, the ℓth file in $[a] \times [b]$ (with $1 \leqslant \ell \leqslant n-1$), as the set of all $(i,j) \in [a] \times [b]$ with $j - i + a = \ell$.

We start in the monic case ($\alpha = \omega = 1$), though later in the section we will consider the general not-necessarily-monic case. Given $f : \hat{P} \to \mathbb{R}^+$ with $f(\hat{0}) = f(\hat{1}) = 1$, let $p_\ell = |v|_\ell$ ($1 \leqslant \ell \leqslant n-1$) be the product of the numbers $f(x)$ with x belonging to the ℓth file of P, let $p_0 = p_n = 1$, and for $1 \leqslant \ell \leqslant n$ let $q_\ell = p_\ell / p_{\ell-1}$. Call $q_1, \ldots, q_n$ the *quotient sequence* associated with f, and denote it by $Q(f)$. This is analogous to the difference sequence introduced in [13]. Note that the product $q_1 \cdots q_n$ telescopes to $p_n/p_0 = 1$. For ℓ between 1 and $n-1$, let ϕ_ℓ^* be the product of the commuting involutions ϕ_x for all x belonging to the ℓth file. Lastly, given

a sequence of n numbers $w = (w_1, \ldots, w_n)$, and given $1 \leqslant \ell \leqslant n-1$, define $\sigma_\ell(w) = (w_1, \ldots, w_{\ell-1}, w_{\ell+1}, w_\ell, w_{\ell+2}, \ldots, w_n)$; that is, σ_ℓ switches the ℓth and $\ell+1$st entries of w.

LEMMA 5.1. *For all* $1 \leqslant \ell \leqslant n-1$,

$$Q(\phi_\ell^* f) = \sigma_\ell Q(f).$$

That is, toggling the ℓth file of f swaps the ℓth and $\ell+1$st entries of the quotient sequence of f.

Proof. Let $f' = \phi_\ell^* f$, let p'_ℓ ($1 \leqslant \ell \leqslant n-1$) be the product of the numbers $f'(x)$ with x belonging to the ℓth file of P, let $p'_0 = p'_n = 1$, and for $1 \leqslant \ell \leqslant n$ let $q'_\ell = p'_\ell / p'_{\ell-1}$. We have $p'_{\ell'} = p_{\ell'}$ for all $\ell' \neq \ell$ (since only the values associated with elements of P of the ℓth file are affected by ϕ_ℓ), so we have $q'_{\ell'} = p'_\ell / p'_{\ell-1} = p_\ell / p_{\ell-1} = q_{\ell'}$ for all ℓ' other than ℓ and $\ell+1$. The product $q'_1 \cdots q'_n$ telescopes to 1 as before. The lemma asserts that $q'_\ell = q_{\ell+1}$ and $q'_{\ell+1} = q_\ell$. It suffices to prove just one of the two assertions, since the two previous sentences tell us that $q'_\ell q'_{\ell+1} = q_\ell q_{\ell+1}$. Expressed in terms of the $p_{\ell'}$'s, the assertion $q'_\ell = q_{\ell+1}$ amounts to the claim $p'_\ell / p'_{\ell-1} = p_{\ell+1}/p_\ell$, or equivalently $p_\ell p'_\ell = p'_{\ell-1} p_{\ell+1}$, which is just the claim $p_\ell p'_\ell = p_{\ell-1} p_{\ell+1}$.

We write $p_\ell p'_\ell$ as the product of $f(x)f'(x)$ as x varies over the ℓth file of P. For each x in the ℓth file of P, $f(x)f'(x) = L_x R_x$ where $L_x = \sum^+ \{f(w) : \ w \lessdot x\}$ and $R_x = \sum^\# \{f(y) : \ y \gtrdot x\}$ (with $w, y \in \hat{P}$). Now we note a key property of the structure of $P = [a] \times [b]$: if x_+ and x_- are vertically adjacent elements of the ℓth file, with x_+ above x_-, the w's that contribute to L_{x_+} are precisely the y's that contribute to R_{x_-}. So, when we take the product of $f(x)f'(x) = L_x R_x$ over all x in the ℓth file, the factors L_{x_+} and R_{x_-} combine to give $\prod_w f(w)$ where w varies over the (two) elements satisfying $x_- \lessdot w \lessdot x_+$ (here we are using the identity (5)). The only factors that do not combine in this way are L_x where x is the bottom element of the ℓth file and R_x where x is the top element of the ℓth file. Both of these factors can be written in the form $f(z)$ where z is a single element of $\hat{P}$ belonging to either the $\ell-1$st or $\ell+1$st file. By examining cases, it is easy to check that every element of the $\ell-1$st file or $\ell+1$st file makes a single multiplicative contribution, so that $p_\ell p'_\ell$ is the product of $f(z)$ as z varies over the union of the $\ell-1$st and $\ell+1$st files of P. But this product is precisely $p_{\ell-1} p_{\ell+1}$. So we have proved that $p_\ell p'_\ell = p_{\ell-1} p_{\ell+1}$, which concludes the proof. □

COROLLARY 5.2. *$Q(\overline{\pi}_{\mathcal{B}} f)$ is the leftward cyclic shift of $Q(f)$.*

Proof. Recall that $\overline{\pi}_{\mathcal{B}}$ is the composition $\phi_{n-1}^* \circ \cdots \circ \phi_1^*$. □

Proof of Theorem 4.4. For each ℓ between 1 and n, view q_ℓ as a function of f. Corollary 5.2 tells us that the numbers $q_\ell(\overline{\pi}_{\mathcal{B}}^0 f), q_\ell(\overline{\pi}_{\mathcal{B}}^1 f), q_\ell(\overline{\pi}_{\mathcal{B}}^2 f), \ldots, q_\ell(\overline{\pi}_{\mathcal{B}}^{n-1} f)$ are respectively equal to the numbers $q_\ell(f), q_{\ell+1}(f), q_{\ell+2}(f), \ldots, q_{\ell-1}(f)$, which multiply to 1. Therefore q_ℓ (viewed as a function of f) is multiplicatively homomesic under the action of $\overline{\pi}_{\mathcal{B}}$ (with average value 1 on all orbits), for all $1 \leqslant \ell \leqslant n$. Hence the same is true of $p_1 = q_1$, $p_2 = q_1 q_2$, $p_3 = q_1 q_2 q_3$, etc., so that for all $\ell \leqslant n-1$, $p_\ell = |v|_\ell$ is multiplicatively 1-homomesic, as claimed. □

In fact, switching now to the general (not necessarily monic) case, we have:

THEOREM 5.3. *Fix $a, b \geqslant 1$, let $n = a+b$, let $P = [a] \times [b]$, and for $1 \leqslant \ell \leqslant n-1$ let $|v|_\ell = \prod_x f(x)$ where x ranges over all $(i,j) \in [a] \times [b]$ satisfying $j - i + a = \ell$. Also take $\alpha, \omega \in \mathbb{R}^+$. Then for every v in $(\mathbb{R}^+)^P$ corresponding to an $f : \hat{P} \to \mathbb{R}^+$ with*

$f(\hat{0}) = \alpha$ *and* $f(\hat{1}) = \omega$,

$$\prod_{k=0}^{n-1} |\pi_{\mathcal{B}}^k(v)|_\ell = \begin{cases} \alpha^{b\ell}\omega^{a\ell} & \text{if } \ell \leqslant \min(a,b), \\ \alpha^{a(n-\ell)}\omega^{a\ell} & \text{if } a \leqslant \ell \leqslant b, \\ \alpha^{b\ell}\omega^{b(n-\ell)} & \text{if } b \leqslant \ell \leqslant a, \\ \alpha^{a(n-\ell)}\omega^{b(n-\ell)} & \text{if } \ell \geqslant \max(a,b). \end{cases}$$

Multiplying over ℓ, we obtain

$$\prod_{k=0}^{n-1} |\pi_{\mathcal{B}}^k(v)| = \alpha^{nab/2}\omega^{nab/2}.$$

Proof of Theorem 5.3. Suppose $a \geqslant b$ (the proof for the case $a \leqslant b$ is similar). Define $\widetilde{f}$ by $\widetilde{f}(x) = f(x)/\alpha^{(n-m)/n}\omega^{m/n}$ for x belonging to rank m. Since the statistic $\widetilde{f} \mapsto |\widetilde{f}|_\ell$ is multiplicatively 1-mesic under the action of $\overline{\pi}_{\mathcal{B}}$, and since the map $f \mapsto \widetilde{f}$ intertwines $\overline{\pi}_{\mathcal{B}}$ and $\pi_{\mathcal{P}}$, the statistic $f \mapsto |f|_\ell$ is multiplicatively c_ℓ-mesic under the action of $\overline{\pi}_{\mathcal{B}}$, where c_ℓ is the nth root of the product of $\alpha^{n+1-i-j}\omega^{i+j-1}$ as (i,j) ranges over S_ℓ. Hence the product in the formula above is equal to c_ℓ^n. It is slightly tedious but not hard to check that for $1 \leqslant \ell \leqslant b \leqslant a$ we have $c_\ell^n = \alpha^{b\ell}\omega^{a\ell}$, for $b \leqslant \ell \leqslant a$ we have $c_\ell^n = \alpha^{b\ell}\omega^{b(n-\ell)}$, and for $b \leqslant a \leqslant \ell \leqslant n-1$ we have $c_\ell^n = \alpha^{a(n-\ell)}\omega^{b(n-\ell)}$. We omit the proof of the aggregate formula (obtained by multiplying over ℓ). □

Note that Theorem 5.3 immediately implies Theorem 4.4 via the substitution $\alpha = \omega = 1$.

Also note that, although the theorem asserts homomesy only for v's in $(\mathbb{R}^+)^P$, the same holds for every v in a dense open subset of $\mathbb{R}^P$, whose complement consists of points for which the orbit $v, \pi_{\mathcal{B}}(v), \ldots, \pi_{\mathcal{B}}^{n-1}(v)$ is not well-defined because of some denominator vanishing. The same is true for $\mathbb{C}^P$.

6. Recombination

The ideas of this section apply in all three realms (combinatorial, piecewise-linear, and birational) and apply to many posets, though we will restrict ourselves to the the case $P = [a] \times [b]$. For the sake of generality, we treat the general birational case, where P-arrays f tacitly have $f(\hat{0}) = \alpha$ and $f(\hat{1}) = \omega$.

The essential idea behind recombination is presented in Figure 6. This Figure shows (at top) the four elements of a particular $\rho_{\mathcal{P}}$ orbit for $P = [2] \times [2]$ and (at bottom) the four elements of a particular $\pi_{\mathcal{P}}$ orbit for $P = [2] \times [2]$, with each element of the first orbit being mapped via $\mathfrak{R}$ (the recombination operation) to a corresponding element of the second orbit: The sixteen numbers that appear in the $\rho_{\mathcal{P}}$ orbit are the same as the sixteen numbers that appear in the $\pi_{\mathcal{P}}$ orbit, in a different order. Specifically, if f denotes a P-array, then the P-array $\mathfrak{R}f$ consists of the bottom and left entries of f along with the right and top entries of $\rho_{\mathcal{P}} f$.

Recombination is implicit in the work of Striker and Williams [16]; they show (in the combinatorial realm) that ρ and π are conjugate to one another. However, the study of homomesies requires that the Striker–Williams conjugation map be expressed in a more explicit form, as is done here. The discussion of recombination here is focused on the case $P = [a] \times [b]$, but the notion applies more generally; a version of it is discussed in [4] (see Section 6), and Vorland's article [17] treats a multidimensional version suitable for rowmotion and promotion in the case where P is a product of more than two chains.

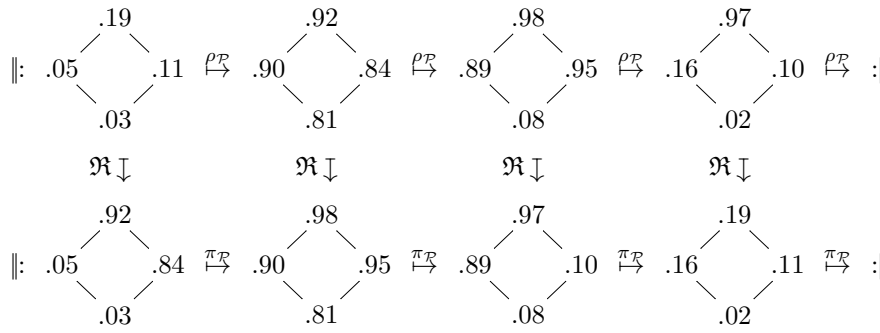


FIGURE 6. Recombination in $[2] \times [2]$.

Note that it is not immediately obvious that recombination as defined above carries the order polytope to itself, or that it sends filters to filters and order ideals into order ideals.

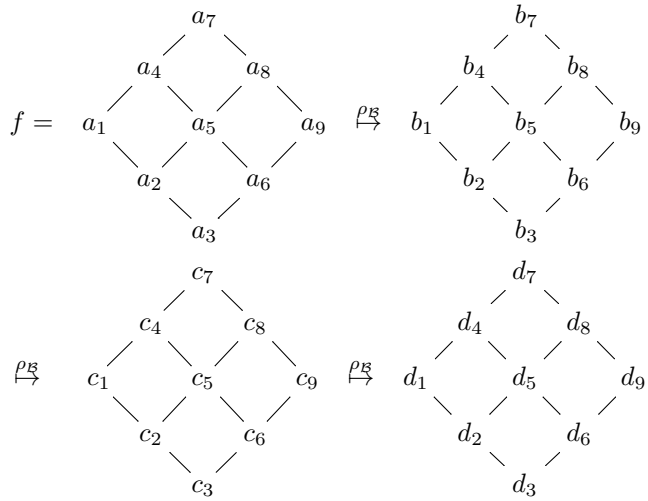


FIGURE 7. A partial orbit of $\rho_{\mathcal{B}}$ for $P = [3] \times [3]$.

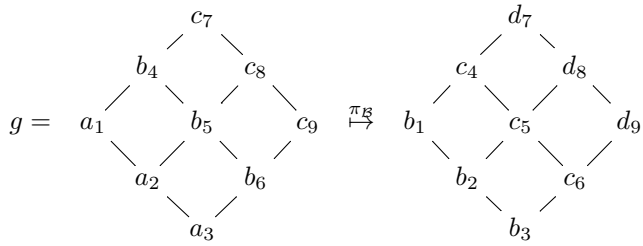


FIGURE 8. Recombination.

The link between rowmotion and promotion is seen mostly clearly if we look at the negative fibers. It is helpful to consider the specific example $P = [3] \times [3]$, where we implement rowmotion by toggling the P-array row by row from top to bottom and

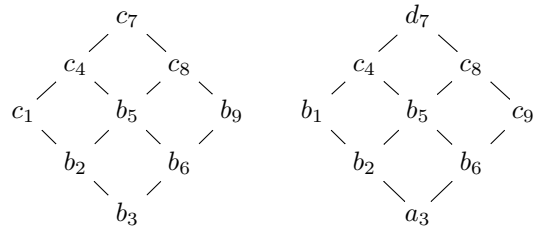


FIGURE 9. Key example: Toggling the middle entry of $\tau_{(3,1)}\tau_{(2,3)}\tau_{(3,2)}\tau_{(3,3)}\rho_{\mathcal{B}}(f)$ (left) versus toggling the middle entry of $\tau_{(3,3)}\tau_{(2,1)}\tau_{(3,2)}\tau_{(3,1)}g$ (right).

within each row from left to right (at $(3,3)$, $(3,2)$, $(2,3)$, $(3,1)$, $(2,2)$, $(1,3)$, $(2,1)$, $(1,2)$, and $(1,1)$, respectively), and we implement promotion by toggling the P-array file by file from left to right and within each file from top to bottom (at $(3,1)$, $(3,2)$, $(2,1)$, $(3,3)$, $(2,2)$, $(1,1)$, $(2,3)$, $(1,2)$, and $(1,3)$, respectively).

Figure 7 shows a partial orbit of $\rho_{\mathcal{B}}$ for $P = [3] \times [3]$, depicting a generic P-array f along with the associated arrays $\rho_{\mathcal{B}}f$, $\rho_{\mathcal{B}}^2 f$, and $\rho_{\mathcal{B}}^3 f$. Consider the array $g = \mathfrak{R}f$ formed by recombining the bottom negative fiber of f (consisting of a_1, a_2, a_3), the middle negative fiber of $\rho_{\mathcal{B}}f$ (consisting of b_4, b_5, b_6), and the top negative fiber of $\rho_{\mathcal{B}}^2 f$ (consisting of c_7, c_8, c_9), as shown in the left frame of Figure 8. We claim that $\pi_{\mathcal{B}}g$ coincides with the array formed by recombining the bottom negative fiber of $\rho_{\mathcal{B}}f$ (consisting of b_1, b_2, b_3), the middle negative fiber of $\rho_{\mathcal{B}}^2 f$ (consisting of c_4, c_5, c_6), and the top negative fiber of $\rho_{\mathcal{B}}^3 f$ (consisting of d_7, d_8, d_9), as shown in the right frame of Figure 8. For example, consider what happens when we compute the middle (i.e. $(2,2)$) entry of $\pi_{\mathcal{B}}g$, assuming that our claim applies to the previously-computed entries. The left panel of Figure 9 shows the array four-ninths of the way through the process of applying rowmotion to $\rho_{\mathcal{B}}(f)$ to compute $\rho_{\mathcal{B}}^2(f)$; it consists of the five entries b_5, b_9, b_2, b_6, b_3 from $\rho_{\mathcal{B}}(f)$ (seen in the upper right panel of Figure 7) and the four entries c_7, c_4, c_8, c_1 from $\rho_{\mathcal{B}}^2(f)$ (seen in the lower left panel of Figure 7). The right panel of Figure 9 shows the array four-ninths of the way through the process of applying promotion to g to compute $\pi_{\mathcal{B}}(g)$; it consists of the five entries b_5, a_3, c_8, b_6, c_9 from g (seen in the left panel of Figure 8) and the four entries b_1, c_4, b_2, d_7 from $\pi_{\mathcal{B}}(g)$ (seen in the right panel of Figure 8). In both cases, when one toggles the middle entry and replaces b_5 by c_5, with c_5 expressed in terms of previously-computed entries, the governing relation is $b_5 c_5 = (b_2 + b_6)(c_4 \mathbin{\#} c_8)$. (Indeed, the proof of Theorem 6.2 was found by thinking hard about why the four entries that adjoin b_5 are the same in both panels, and generalizing.)

DEFINITION 6.1. *Given $a, b \geqslant 1$ and $P = [a] \times [b]$, and given a P-array $f : P \to \mathbb{R}^+$, let*

$$(\mathfrak{R}f)(i,j) = (\rho_{\mathcal{B}}^{j-1} f)(i,j). \tag{9}$$

That is, $\mathfrak{R}f$ is the P-array whose b negative fibers, read from bottom to top, have the same values as the corresponding negative fibers in f, $\rho_{\mathcal{B}}f$, $\rho_{\mathcal{B}}^2 f$, ..., $\rho_{\mathcal{B}}^{b-1} f$. The operation $\mathfrak{R}$ can also be expressed as a product of toggles. Specifically, for $1 \leqslant j \leqslant b$ let T_j be the operation $\tau_{(1,j)} \circ \tau_{(2,j)} \circ \cdots \circ \tau_{(a-1,j)} \circ \tau_{(a,j)}$ that toggles the elements of the jth negative fiber from left to right (or equivalently from top to bottom), and for $1 \leqslant j \leqslant b$ let U_j be the operation $T_{b-j+1} \circ T_{b-j+2} \circ \cdots \circ T_{b-1} \circ T_b$ that toggles the top j negative fibers from top to bottom; then $\mathfrak{R} = U_1 \circ U_2 \circ \cdots \circ U_{b-2} \circ U_{b-1}$. It is no accident that a similar product of toggles can be found in Theorem 5.4 of [16]. Note

that this representation of recombination as a composition of involutions makes it clear that $\mathfrak{R}$ is invertible. For a concrete formula for the inverse map, see Lemma 6.5.

Now we come to the main result of this section, a birational analogue of Striker and Williams' main result.

THEOREM 6.2. *Fix $a, b \geqslant 1$, let $n = a + b$, and let $P = [a] \times [b]$. Then for all $f : P \to \mathbb{R}^+$,*

$$\mathfrak{R}\rho_{\mathcal{B}} f = \pi_{\mathcal{B}}\mathfrak{R} f. \tag{10}$$

Proof. We will show that

$$(\mathfrak{R}\rho_{\mathcal{B}} f)(i,j) = (\pi_{\mathcal{B}}\mathfrak{R} f)(i,j) \tag{11}$$

for all $(i,j) \in [a] \times [b]$ by left-to-right induction (starting with $(a,1)$ and ending with $(1,b)$). The reader may find it helpful to consult Figure 10) which shows the vicinity of a typical element $x = (i,j)$ in the poset $[a] \times [b]$ (away from the boundary).

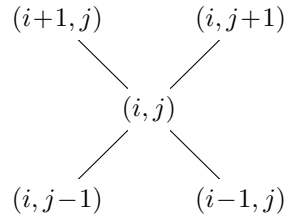


FIGURE 10. The local picture.

In the boundary cases where one or more of the ordered pairs $(i+1,j), (i,j+1), (i,j-1), (i-1,j)$ does not belong to $[a] \times [b]$, the out-of-bounds ordered pair(s) may be ignored.

Note that $(i,j-1)$ and $(i+1,j)$ are to the left of (i,j). If we now assume that (11) holds for $(i,j-1)$ (which we take to be vacuously true if $(i,j-1) \notin [a] \times [b]$) and for $(i+1,j)$ (which we take to be vacuously true if $(i+1,j) \notin [a] \times [b]$), then

$$(\mathfrak{R}\rho_{\mathcal{B}} f)(i,j-1) = (\pi_{\mathcal{B}}\mathfrak{R} f)(i,j-1) \tag{12}$$

and

$$(\mathfrak{R}\rho_{\mathcal{B}} f)(i+1,j) = (\pi_{\mathcal{B}}\mathfrak{R} f)(i+1,j). \tag{13}$$

Note that $\rho_{\mathcal{B}}$ can be described directly via the recurrence

$$(\rho_{\mathcal{B}} f)(x) = \frac{1}{f(x)} \sum\nolimits^{+} \{f(y) : y \in x^{-}\} \sum\nolimits^{\sharp} \{(\rho_{\mathcal{B}} f)(y) : y \in x^{+}\} \tag{14}$$

for all $x \in P$, where in lieu of including $\hat{0}$ and $\hat{1}$ we interpret $\sum^{+}$ and $\sum^{\sharp}$ of the empty set as α and ω respectively. We rewrite (14) for $x = (i,j)$ as

$$\begin{aligned}(\rho_{\mathcal{B}} f)(i,j) = \frac{1}{f(i,j)} \times \sum\nolimits^{+} \{f(i-1,j), f(i,j-1)\} \\ \times \sum\nolimits^{\sharp} \{(\rho_{\mathcal{B}} f)(i+1,j), (\rho_{\mathcal{B}} f)(i,j+1)\},\end{aligned} \tag{15}$$

where terms $f(\cdot,\cdot)$ and $(\rho_{\mathcal{B}} f)(\cdot,\cdot)$ are to be ignored if the arguments do not belong to $[a] \times [b]$. Likewise, promotion can be described by the recurrence

$$\begin{aligned}(\pi_{\mathcal{B}} f)(i,j) = \frac{1}{f(i,j)} \times \sum\nolimits^{+} \{f(i-1,j), (\pi_{\mathcal{B}} f)(i,j-1)\} \\ \times \sum\nolimits^{\sharp} \{(\pi_{\mathcal{B}} f)(i+1,j), f(i,j+1)\}.\end{aligned} \tag{16}$$

From (9) we obtain $(\mathfrak{R}f)(i,j{+}1) = (\rho_{\mathcal{B}}^{j}f)(i,j{+}1)$ and $(\mathfrak{R}f)(i,j{-}1) = (\rho_{\mathcal{B}}^{j-2}f)(i,j{-}1)$, which we will use presently.

One the one hand we have

$$\begin{aligned}
(\mathfrak{R}\rho_{\mathcal{B}}f)(i,j) &= (\rho_{\mathcal{B}}^{j-1}\rho_{\mathcal{B}}f)(i,j) \text{ (by (9))} \\
&= (\rho_{\mathcal{B}}\rho_{\mathcal{B}}^{j-1}f)(i,j) \\
&= \frac{1}{(\rho_{\mathcal{B}}^{j-1}f)(i,j)} \times \sum\nolimits^{+}\{(\rho_{\mathcal{B}}^{j-1}f)(i-1,j),(\rho_{\mathcal{B}}^{j-1}f)(i,j-1)\} \\
&\qquad \times \sum\nolimits^{\#}\{(\rho_{\mathcal{B}}\rho_{\mathcal{B}}^{j-1}f)(i+1,j),(\rho_{\mathcal{B}}\rho_{\mathcal{B}}^{j-1}f)(i,j+1)\} \text{ (by (15))} \\
&= \frac{1}{(\rho_{\mathcal{B}}^{j-1}f)(i,j)} \times \sum\nolimits^{+}\{(\rho_{\mathcal{B}}^{j-1}f)(i-1,j),(\rho_{\mathcal{B}}^{j-1}f)(i,j-1)\} \\
&\qquad \times \sum\nolimits^{\#}\{(\rho_{\mathcal{B}}^{j}f)(i+1,j),(\rho_{\mathcal{B}}^{j}f)(i,j+1)\}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
(\pi_{\mathcal{B}}\mathfrak{R}f)(i,j) &= \frac{1}{(\mathfrak{R}f)(i,j)} \times \sum\nolimits^{+}\{(\mathfrak{R}f)(i-1,j),(\pi_{\mathcal{B}}\mathfrak{R}f)(i,j-1)\} \\
&\qquad \times \sum\nolimits^{\#}\{(\pi_{\mathcal{B}}\mathfrak{R}f)(i+1,j),(\mathfrak{R}f)(i,j+1)\} \text{ (by (16))} \\
&= \frac{1}{(\rho_{\mathcal{B}}^{j-1}f)(i,j)} \times \sum\nolimits^{+}\{(\rho_{\mathcal{B}}^{j-1}f)(i-1,j),(\pi_{\mathcal{B}}\mathfrak{R}f)(i,j-1)\} \\
&\qquad \times \sum\nolimits^{\#}\{(\pi_{\mathcal{B}}\mathfrak{R}f)(i+1,j),(\rho_{\mathcal{B}}^{j}f)(i,j+1)\} \text{ (by (9))} \\
&= \frac{1}{(\rho_{\mathcal{B}}^{j-1}f)(i,j)} \times \sum\nolimits^{+}\{(\rho_{\mathcal{B}}^{j-1}f)(i-1,j),(\mathfrak{R}\rho_{\mathcal{B}}f)(i,j-1)\} \\
&\qquad \times \sum\nolimits^{\#}\{(\mathfrak{R}\rho_{\mathcal{B}}f)(i+1,j),(\rho_{\mathcal{B}}^{j}f)(i,j+1)\} \text{ (by (12) and (13))} \\
&= \frac{1}{(\rho_{\mathcal{B}}^{j-1}f)(i,j)} \times \sum\nolimits^{+}\{(\rho_{\mathcal{B}}^{j-1}f)(i-1,j),(\rho_{\mathcal{B}}^{j-2}\rho_{\mathcal{B}}f)(i,j-1)\} \\
&\qquad \times \sum\nolimits^{\#}\{(\rho_{\mathcal{B}}^{j-1}\rho_{\mathcal{B}}f)(i+1,j),(\rho_{\mathcal{B}}^{j}f)(i,j+1)\} \text{ (by (9))} \\
&= \frac{1}{(\rho_{\mathcal{B}}^{j-1}f)(i,j)} \times \sum\nolimits^{+}\{(\rho_{\mathcal{B}}^{j-1}f)(i-1,j),(\rho_{\mathcal{B}}^{j-1}f)(i,j-1)\} \\
&\qquad \times \sum\nolimits^{\#}\{(\rho_{\mathcal{B}}^{j}f)(i+1,j),(\rho_{\mathcal{B}}^{j}f)(i,j+1)\}.
\end{aligned}$$

Comparing the final expressions in the two equation blocks, we conclude that $(\mathfrak{R}\rho_{\mathcal{B}}f)(i,j) = (\pi_{\mathcal{B}}\mathfrak{R}f)(i,j)$, which was to be proved. □

Corollary 6.3. *For all $k \geqslant 0$,*

$$\mathfrak{R}\rho_{\mathcal{B}}^{k}f = \pi_{\mathcal{B}}^{k}\mathfrak{R}f. \tag{17}$$

Proof. Immediate. □

Corollary 6.4. *The map $\pi_{\mathcal{B}}$ (for $P = [a] \times [b]$) is of order $a + b$.*

Proof. This follows from Grinberg–Roby periodicity, Corollary 6.3, and the fact that $\mathfrak{R}$ is bijective. □

Lemma 6.5. *The map $\mathfrak{D}$ defined by*

$$(\mathfrak{D}g)(i,j) = (\pi_{\mathcal{B}}^{n+1-j}g)(i,j) \tag{18}$$

satisfies $\mathfrak{D}\mathfrak{R}f = f$ and hence coincides with $\mathfrak{R}^{-1}$.

Proof.

$$\begin{aligned}(\mathfrak{D}\mathfrak{R}f)(i,j) &= (\pi_{\mathcal{B}}^{n+1-j}(\mathfrak{R}f))(i,j)\\ &= (\mathfrak{R}\rho_{\mathcal{B}}^{n+1-j}f)(i,j) && \text{(by Corollary 6.3 with } k=n+1-j)\\ &= (\rho_{\mathcal{B}}^{j-1}\rho_{\mathcal{B}}^{n+1-j}f)(i,j) && \text{(by (9) applied to } \rho_{\mathcal{B}}^{n+1-j}f)\\ &= (\rho_{\mathcal{B}}^{n}f)(i,j)\end{aligned}$$

which equals $f(i,j)$ by the periodicity of $\rho_{\mathcal{B}}$. □

THEOREM 6.6. *Fix $a,b \geqslant 1$, let $n=a+b$, and let $P=[a]\times[b]$. For any statistic F on $(\mathbb{R}^+)^P$ of the form $Ff=\prod_{x\in P} f(x)^{a_x}$ with $a_x\in\mathbb{Z}$ for all $x\in P$,*

$$\prod_{k=0}^{n-1} F(\rho_{\mathcal{B}}^k f) = \prod_{k=0}^{n-1} F(\pi_{\mathcal{B}}^k \mathfrak{R} f).$$

In particular, if F is homomesic with respect to promotion, F is homomesic with respect to rowmotion.

Proof. Recall that $\rho_{\mathcal{B}}^n$ and $\pi_{\mathcal{B}}^n$ are both the identity map. It will suffice to consider F of the form $F_x f = f(x)^{a_x}$ for some particular $x\in P$, say $x=(i,j)$. Theorem 6.2 implies $\rho_{\mathcal{B}}^k f = \mathfrak{R}^{-1}\pi_{\mathcal{B}}^k \mathfrak{R} f$. Then setting $g_j = \pi_{\mathcal{B}}^{n+1-j}\mathfrak{R}f$ we have

$$\begin{aligned}F_x(\rho_{\mathcal{B}}^k f) &= ((\rho_{\mathcal{B}}^k f)(x))^{a_x}\\ &= ((\mathfrak{R}^{-1}\pi_{\mathcal{B}}^k\mathfrak{R}f)(x))^{a_x}\\ &= ((\pi_{\mathcal{B}}^{n+1-j}\pi_{\mathcal{B}}^k\mathfrak{R}f)(x))^{a_x} \quad \text{(by Lemma 6.5)}\\ &= ((\pi_{\mathcal{B}}^k\pi_{\mathcal{B}}^{n+1-j}\mathfrak{R}f)(x))^{a_x}\\ &= F_x(\pi_{\mathcal{B}}^k g_j)\end{aligned}$$

so that

$$\begin{aligned}\prod_{k=0}^{n-1} F_x(\rho_{\mathcal{B}}^k f) &= \prod_{k=0}^{n-1} F_x(\pi_{\mathcal{B}}^k g_j)\\ &= \prod_{k=0}^{n-1} F_x(\pi_{\mathcal{B}}^{k+n+1-j}\mathfrak{R}f)\\ &= \prod_{k=0}^{n-1} F_x(\pi_{\mathcal{B}}^k\mathfrak{R}f)\end{aligned}$$

(by reindexing and appealing to periodicity). Since this holds for all x, the claim follows by multiplication. □

7. TROPICALIZATION

Theorem 4.4 is nothing more than a complicated identity involving the operations of multiplication, division, addition, and parallel addition. As such, Theorem 4.4 can be tropicalized to yield an identity involving the operations of addition, subtraction, min, and max. The resulting identity is Theorem 3.5. Here we provide details of the tropicalization process.

LEMMA 7.1. *Suppose $E_1(t_1,\ldots,t_r)$ and $E_2(t_1,\ldots,t_r)$ are subtraction-free rational functions, expressed in terms of the number 1 and the operations $+$, $\parallel$, $\times$, and $/$, such that $E_1(t_1,\ldots,t_r)=E_2(t_1,\ldots,t_r)$ for all $t_1,\ldots,t_r$ in $\mathbb{R}^+$. Let $e_i(t_1,\ldots,t_r)$ (for $1\leqslant i\leqslant 2$) be the result of replacing 1, $+$, $\parallel$, $\times$, and $/$ by 0, $\min$, $\max$, $+$, and $-$, respectively. Then $e_1(t_1,\ldots,t_r)=e_2(t_1,\ldots,t_r)$ for all $t_1,\ldots,t_r$ in $\mathbb{R}$.*

Proof. Write $e_i(t_1, \ldots, t_m) = -\lim_{N\to\infty} \frac{1}{N} \log E_i(e^{-Nt_1}, \ldots, e^{-Nt_m})$.[9] □

It is worth observing that the converse of the lemma is false. That is, there are tropical identities like $\max(t, t) = t$ whose "detropicalization" is false. Thus, while Theorem 3.5 is a consequence of Theorem 4.4, Theorem 4.4 cannot be derived as a consequence of Theorem 3.5, at least not using any methods we are aware of.

At the same time, we should mention that in a certain sense, Theorem 3.5 can be proved without relying on Theorem 4.4. Specifically, one can tropicalize each of the steps in the proof of Theorem 4.4, using (for instance) the identity

$$\min(x, y) + \max(x, y) = x + y \tag{19}$$

in place of the identity (5).

Proof of Theorem 3.5. This is an immediate consequence of Theorem 4.4 via tropicalization. □

Proof of Theorem 3.4. Putting $\alpha = 1$ in Theorem 5.3, we get

$$\prod_{k=0}^{n-1} |\pi_{\mathcal{B}}^k(v)|_\ell = \begin{cases} \omega^{a\ell} & \text{if } 1 \leqslant \ell \leqslant b, \\ \omega^{b(n-\ell)} & \text{if } b \leqslant \ell \leqslant n. \end{cases}$$

Tropicalizing this result gives Theorem 3.4. □

Proof of Theorem 1.1. Putting $\omega = 1$ in Theorem 5.3, we get

$$\prod_{k=0}^{n-1} |\pi_{\mathcal{B}}^k(v)|_\ell = \begin{cases} \alpha^{b\ell} & \text{if } \ell \leqslant a, \\ \alpha^{a(n-\ell)} & \text{if } a \leqslant \ell n. \end{cases}$$

Tropicalizing this result gives a homomesy result for the promotion on the reverse order polytope (the set of order-reversing maps from $[a] \times [b]$ to $[0, 1]$). Specializing to the vertices of this polytope gives the desired homomesy for order ideals. □

8. Other homomesies

Recall that P_i $(1 \leqslant i \leqslant n-1)$ denotes the ith file of $P = [a] \times [b]$. We have shown that for $P = [a] \times [b]$, the functions

$$p_i : f \mapsto \sum_{x \in P_i} f(x)$$

(and their analogues in the birational setting) are homomesic under the action of rowmotion and promotion. These are not, however, the only combinations of the local evaluation operations $f \mapsto f(x)$ $(x \in P)$ that exhibit homomesy. Specifically, we now show that for all a, b, the functions $f \mapsto f(x) + f(x')$ are homomesic under rowmotion and promotion, where $x = (i, j)$ and $x' = (i', j')$ are opposite elements of P, that is, $i + i' = a + 1$ and $j + j' = b + 1$. We will prove the birational version, since the PL version is an easy consequence. In the birational setting, the statistic is $f \mapsto f(x)f(x')$.

Theorem 8.1. *Fix $a, b \geqslant 1$, let $n = a + b$, and let $P = [a] \times [b]$. Fix $(i, j), (i', j')$ in P satisfying $i + i' = a + 1$ and $j + j' = b + 1$. For $f : P \to \mathbb{R}^+$ let $F(f) = f(i, j)f(i', j')$. Then F is multiplicatively $\alpha\omega$-mesic under $\rho_{\mathcal{B}}$.*

[9] We are indebted to Colin McQuillan and Will Sawin for clarifying this point; see [11].

Proof. Theorem 32 in [7] says that $f(i',j')f'(i,j) = \alpha\omega$ with $f' = \rho_{\mathcal{B}}^{i+j-1} f$, so

$$\begin{aligned}
(\alpha\omega)^n &= \prod_{k=1}^{n} (\rho_{\mathcal{B}}^k f)(i',j')(\rho_{\mathcal{B}}^k f')(i,j) \\
&= \prod_{k=1}^{n} (\rho_{\mathcal{B}}^k f)(i',j') \prod_{k=1}^{n} (\rho_{\mathcal{B}}^k f')(i,j) \\
&= \prod_{k=1}^{n} (\rho_{\mathcal{B}}^k f)(i',j') \prod_{k=1}^{n} (\rho_{\mathcal{B}}^k f)(i,j) \qquad \text{(since } \rho_{\mathcal{B}} \text{ has period } n\text{)} \\
&= \prod_{k=1}^{n} F(\rho_{\mathcal{B}}^k f)
\end{aligned}$$

as claimed. □

Acknowledgements. The authors are grateful to Arkady Berenstein, Darij Grinberg, Michael Joseph, Tom Roby, Richard Stanley, and Jessica Striker for helpful conversations and detailed comments on the manuscript. The two anonymous referees also had many good suggestions.

References

[1] Drew Armstrong, Christian Stump, and Hugh Thomas, *A uniform bijection between nonnesting and noncrossing partitions*, Trans. Amer. Math. Soc. **365** (2013), no. 8, 4121–4151.

[2] Andries Brouwer and Alexander Schrijver, *On the period of an operator, defined on antichains*, Math. Centrum report ZW **24/74** (1974).

[3] Peter Cameron and Dmitry G. Fon-Der-Flaass, *Orbits of antichains revisited*, European J. Combin. **16** (1995), no. 6, 545–554.

[4] David Einstein, Miriam Farber, Emily Gunawan, Michael Joseph, Matthew Macauley, James Propp, and Simon Rubinstein-Salzedo, *Noncrossing partitions, toggles, and homomesies*, Electron. J. Combin. **23** (2016), no. 3, Paper no. 3.52 (26 pages).

[5] David Einstein and James Propp, *Piecewise-linear and birational toggling*, in 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014), Discrete Math. Theor. Comput. Sci. Proc., AT, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2014, pp. 513–524.

[6] Dmitry G. Fon-Der-Flaass, *Orbits of antichains in ranked posets*, European J. Combin. **14** (1993), no. 1, 17–22.

[7] Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. Combin. **22** (2015), no. 3, Paper no. 3.40 (49 pages).

[8] Alexander E. Holroyd, Lionel Levine, Karola Mészáros, Yuval Peres, James Propp, and David B. Wilson, *Chip-firing and rotor-routing on directed graphs*, in In and out of equilibrium. 2, Progr. Probab., vol. 60, Birkhäuser, Basel, 2008, pp. 331–364.

[9] Alexander E. Holroyd and James Propp, *Rotor walks and Markov chains*, in Algorithmic probability and combinatorics, Contemp. Math., vol. 520, Amer. Math. Soc., Providence, RI, 2010, pp. 105–126.

[10] Anatol N. Kirillov and Arkady Berenstein, *Groups generated by involutions, Gel′fand–Tsetlin patterns, and combinatorics of Young tableaux*, St. Petersburg Math. J. **7** (1996), 77–127, Originally published in *Algebra i Analiz* **7** (1995), 92–152; also available at `http://math.uoregon.edu/~arkadiy/bk1.pdf`.

[11] MathOverflow, *Do all subtraction-free identities tropicalize?*, `http://mathoverflow.net/questions/127108/do-all-subtraction-free-identities-tropicalize`, 2013.

[12] Dmitri I. Panyushev, *On orbits of antichains of positive roots*, European J. Combin. **30** (2009), no. 2, 586–594.

[13] James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22** (2015), no. 3, Paper no. 3.4 (29 pages).

[14] Richard P. Stanley, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23.

[15] ———, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

[16] Jessica Striker and Nathan Williams, *Promotion and rowmotion*, European J. Combin. **33** (2012), no. 8, 1919–1942.
[17] Corey Vorland, *Homomesy in products of three chains and multidimensional recombination*, https://arxiv.org/abs/1705.02665, 2017.

David Einstein, University of Massachusetts Lowell, Department of Mathematical Sciences
E-mail : deinst@gmail.com

James Propp, University of Massachusetts Lowell, Department of Mathematical Sciences
Url : https://jamespropp.org