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Short time existence of the classical solution to the fractional mean curvature flow

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Abstract

We establish short-time existence of the smooth solution to the fractional mean curvature flow when the initial set is bounded and $C^{1,1}$-regular. We provide the same result also for the volume preserving fractional mean curvature flow.

Keywords: Fractional mean curvature flow; Short time existence; Classical solution; Fractional perimeter

1. Introduction

In this paper we study the motion of sets by their fractional mean curvature from the classical point of view. In the problem we are given a bounded regular set $E_0 \subset \mathbb{R}^{n+1}$ and we let it evolve to a family of smooth sets $(E_t)_{t\in(0,T]}$ according to the law where the normal velocity at a point equals its fractional mean curvature. More precisely this can be written as

$$V_t = -H^s_{E_t} \qquad \text{on } \partial E_t, \tag{1.1}$$

where V_t denotes the normal velocity and H^s_E is the fractional mean curvature of $E \subset \mathbb{R}^{n+1}$ with $s \in (0, 1)$, given by

$$H^s_E(x) := \text{p.v.} \left(\int_{E^c} \frac{dy}{|y-x|^{n+1+s}} - \int_E \frac{dy}{|x-y|^{n+1+s}} \right). \tag{1.2}$$

One motivation to study (1.1) is that it can be interpreted as the gradient flow of the fractional perimeter and is therefore the evolutionary counterpart to the problem of minimizing the fractional perimeter. The stationary problem has received a lot of attention since the work [5], where the authors study the regularity of sets which locally minimize the fractional perimeter. By [5] and [3] we know that the local perimeter minimizers are smooth up to a small singular set which precise dimension is not known (see [29] and [14] and the references therein). The global isoperimetric problem in the Euclidian space is also well understood. It is proven in [18] that the ball is the solution to the fractional

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isoperimetric problem, and we even have the sharp quantification of the isoperimetric inequality [17,19]. A related result is the generalization of the Alexandroff theorem [4,13], where the authors prove that the ball is the only smooth compact set with constant fractional mean curvature. Note that one does not need to assume the set to be connected, which is in contrast to the classical Alexandroff theorem.

Concerning the evolutionary problem, the first existence result is due to Imbert [24] who defines the *viscosity solution* of (1.1) and proves that it exists for all times and is unique. This means that the flow (1.1) has a well defined weak solution which is unique up to fattening. In [6] Caffarelli-Souganidis construct weak solution by using threshold dynamics, and in [9] Chambolle-Morini-Ponsiglione use the gradient flow structure to construct weak solution by using the minimizing movement scheme (see also [8]).

The issue with the weak solution is, as observed in [11] (see also [7]), that the flow may develop singularities even in the planar case. Cinti-Sinestrari-Valdinoci [12] avoid singularities by studying the volume preserving fractional mean curvature flow

$$V_t = -(H^s_{E_t} - \bar{H}^s_{E_t}) \qquad \text{on } \partial E_t, \tag{1.3}$$

for convex initial sets, where $\bar{H}^s_{E_t} = \fint_{\partial E_t} H^s_{E_t}\, d\mathcal{H}^n$. By [10], the fractional mean curvature flow with a forcing term preserves convexity and thus one expects Huisken's result [22] to hold also in the fractional case, i.e., the flow (1.3) remains convex, does not develop singularities and converges to the sphere. This is indeed the main result in [12] under an additional regularity assumption. We remark that also (1.1) preserves the convexity and therefore one may expect a result similar to [21] to hold also for (1.1), but to the best of our knowledge no result in this direction exists. Finally we refer to [28] for interesting analysis of the smooth solution of (1.1).

Here we are interested in the classical solution of (1.1) which means that the sets $(E_t)_t$ are smooth and diffeomorphic to E_0. As we mentioned we may expect the classical solution to exist only for a short time interval $(0, T)$ as the flow will vanish in finite time or it may develop singularities before that. We prove the short time existence of the classical solution under the assumption that the initial set is $C^{1,1}$-regular, or $C^{1+s+\alpha}$-close to a $C^{1,1}$-regular set. The same result holds also for the volume preserving flow (1.3) and we choose to state the result for both cases simultaneously.

Main Theorem. *Let $E_0 \subset \mathbb{R}^{n+1}$ be a bounded set such that ∂E_0 is a $C^{1,1}$-regular hypersurface and let $\alpha \in (0, 1)$. There exists $T \in (0, 1)$ such that the fractional mean curvature flow* (1.1)*, and the volume preserving flow* (1.3)*, has a unique classical solution $(E_t)_{t\in(0,T]}$ starting from E_0. The flow becomes instantaneously smooth, i.e., each surface ∂E_t with $t \in (0, T]$ is C^∞-hypersurface. Moreover, there is $\varepsilon > 0$ with the property that if the $C^{1+s+\alpha}$-distance between E_0 and E is less than ε, then the flow* (1.1) *(and the flow* (1.3)*) starting from E also exists for the time interval* $(0, T]$.

We give a more quantitative statement of the main theorem in the last section in Theorem 5.1 and in Theorem 5.3. We expect the smooth solution of (1.1) to agree with the viscosity solution on the time interval $(0, T]$ but we do not prove this.

The proof of the main theorem is based on Schauder estimates on parabolic equations. As in [23] we first parametrize the flow (1.1) by using the 'height' function over a smooth reference surface Σ, which is close to the initial surface. This leads us to solve the nonlinear nonlocal parabolic equation

$$\partial_t u = \Delta^{\frac{1+s}{2}} u + Q(x, u, \nabla u) - H^s_\Sigma \qquad \text{on } \Sigma \times (0, T], \tag{1.4}$$

where $\Delta^{\frac{1+s}{2}}$ denotes the fractional Laplacian on Σ and Q is nonlinear. Following the idea in [16] we prove that the nonlinear term Q in (1.4) remains small for a short time interval and the equation can thus be seen as a small perturbation of the fractional heat equation. We then use Schauder estimates and a standard fixed point argument to obtain the existence of a solution which is $C^{1+s+\alpha}$-regular in space and obtain the smoothness by differentiating the equation. The right hand side of (1.4) is essentially the parametrization of the fractional mean curvature. Similarly as in [3] and [14], the main difficulty in our analysis is due to the complicated structure of the nonlinear term Q, which makes it challenging to estimate its $C^{k+\alpha}$-norm in a quantitative way. In order to differentiate the equation (1.4) multiple times we need effective notation and basic tools from differential geometry. We also point out that the $C^{1,1}$-regularity of the initial set causes additional difficulties, because some of the constants in (1.4) depend on

the C^α-norm of the curvature of Σ, which we cannot bound uniformly if we want Σ to be close to ∂E_0. Finally another technical issue, although a minor one, is that there is no comprehensive Schauder theory for the fractional heat equation on compact hypersurfaces in the literature, and therefore we have to prove the appropriate estimates ourselves (see Theorem 2.2).

The paper is organized as follows. After a short preliminary section (Section 2) we derive the equation (1.4) in Section 3 by using the parametrization by the height function (Proposition 3.2). In Section 4 we study the spatial regularity properties of the operator on the right-hand-side of (1.4) and give the proof of the main theorem in Section 5. The Appendix contains the Schauder estimates for the fractional heat equation with a forcing term on compact hypersurfaces, which might be of independent interest.

2. Preliminaries

Throughout the paper we assume that $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface, i.e., there is a smooth bounded set $G \subset \mathbb{R}^{n+1}$ such that $\partial G = \Sigma$. We choose G as our reference set and define the classical solution of (1.1) (and (1.3)) as in the case of the classical mean curvature flow [26], i.e., we say that $(E_t)_{t\in(0,T]}$ is a classical solution of (1.1) starting from E_0 if there exists a map $\Psi \in C(\mathbb{R}^{n+1} \times [0,T]; \mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1} \times (0,T]; \mathbb{R}^{n+1})$ such that $\Psi(\cdot,t)$ is a C^{1+s}-diffeomorphism for $t \in [0,T]$ and smooth diffeomorphism for $t \in (0,T]$, with $E_t = \Psi(G,t)$ for $t \in (0,T]$, $E_0 = \Psi(G,0)$ and E_t satisfies (1.1) (or (1.3) in the volume preserving case).

2.1. Geometric preliminaries

We recall some basic analysis related to Riemannian manifolds. For an introduction to the topic we refer to [25], from where we also adopt our notation.

Since Σ is embedded in $\mathbb{R}^{n+1}$ it has a natural metric g induced by the Euclidian metric. Then (Σ, g) is a Riemannian manifold and we denote the inner product on each tangent space $X, Y \in T_x\Sigma$ by $\langle X, Y\rangle$. We extend the inner product in a natural way for tensors. We denote by $\mathcal{T}(\Sigma)$ the smooth vector fields on Σ and recall that for $X \in \mathcal{T}(\Sigma)$ and $u \in C^\infty(\Sigma)$ the notation Xu means the derivative of u in direction of X. We emphasize that we assume every vector field to be smooth.

We denote the Riemannian connection on Σ by ∇ and recall that for a function $u \in C^\infty(\Sigma)$ the covariant derivative ∇u is a 1-tensor field defined for $X \in \mathcal{T}(\Sigma)$ as

$$\nabla u(X) = \nabla_X u = Xu,$$

i.e., the derivative of u in the direction of X. The covariant derivative of a smooth k-tensor field $F \in \mathcal{T}^k(\Sigma)$, denoted by ∇F, is a $(k+1)$-tensor field and is defined for $Y_1, \dots, Y_k, X \in \mathcal{T}(\Sigma)$ as

$$\nabla F(Y_1, \dots, Y_k, X) = (\nabla_X F)(Y_1, \dots, Y_k),$$

where

$$(\nabla_X F)(Y_1, \dots, Y_k) = XF(Y_1, \dots, Y_k) - \sum_{i=1}^{k} F(Y_1, \dots, \nabla_X Y_i, \dots, Y_k).$$

Here $\nabla_X Y$ is the covariant derivative of Y in the direction of X (see [25]) and since ∇ is the Riemannian connection it holds $\nabla_X Y = \nabla_Y X + [X, Y]$ for every $X, Y \in \mathcal{T}(\Sigma)$.

We denote the kth order covariant derivative of a smooth function $u \in C^\infty(\Sigma)$ by $\nabla^k u$, which is a k-tensor field defined recursively as $\nabla^k u = \nabla(\nabla^{k-1} u)$. Let $X_1, \dots, X_k \in \mathcal{T}(\Sigma)$ be vector fields on Σ. Then $\nabla^k u(X_1, \dots, X_k)$ denotes the covariant derivative applied to $X_1, \dots, X_k$ and we often use the notation

$$\nabla_{X_k} \cdots \nabla_{X_1} u = \nabla^k u(X_1, \dots, X_k).$$

We use the fact that Σ is embedded in $\mathbb{R}^{n+1}$ and define the sup-norm and the α-Hölder norm, for $\alpha \in (0,1)$, of a function $u \in C(\Sigma)$ in a standard way,

$$\|u\|_{C^0(\Sigma)} := \sup_{x\in\Sigma} |u(x)|$$

and

$$\|u\|_{C^\alpha(\Sigma)} := \sup_{\substack{x,y\in\Sigma\\ x\neq y}} \frac{|u(y)-u(x)|}{|y-x|^\alpha} + \|u\|_{C^0(\Sigma)}.$$

Moreover, we set $\|u\|_{C^k(\Sigma)} := \sum_{l=0}^k \|\nabla^l u\|_{C^0(\Sigma)}$ for all $k \in \mathbb{N}$. We define further the α-Hölder norm of a k-tensor field $F \in \mathcal{T}^k(\Sigma)$ by

$$\|F\|_{C^\alpha(\Sigma)} := \sup\{\|F(X_1,\dots,X_k)\|_{C^\alpha(\Sigma)} : X_i \in \mathcal{T}(\Sigma) \text{ with } \|X_i\|_{C^1(\Sigma)} \le 1,\ i = 1\dots,k\}.$$

It is straightforward to check that this agrees with the more standard definition via partition of unity. Using this we define the $C^{k+\alpha}$- norm of function u, with $k \in \mathbb{N}$ and $\alpha \in [0,1)$, as

$$\|u\|_{C^{k+\alpha}(\Sigma)} := \sum_{l=0}^k \|\nabla^l u\|_{C^\alpha(\Sigma)}.$$

We use the notation $u \in C^{k+\alpha}(\Sigma)$ for a function u with bounded $C^{k+\alpha}(\Sigma)$-norm when $\alpha \in (0,1)$ and $u \in C^{k,1}(\Sigma)$ when $\alpha = 1$. We assume that $\Sigma = \partial G$ is uniformly $C^{1,1}$-regular surface and define its $C^{1,1}$-norm as the smallest number R such that G satisfies the interior and the exterior ball condition with radius $1/R$. Note that this norm also bounds $\|\nu\|_{C^1(\Sigma)}$. If a constant depends on the $C^{1,1}$-norm of Σ we choose not to mention it and call such a constant *uniform*.

We recall the following interpolation inequality. The proof is essentially the same as [20, Lemma 6.32] (see also [2]).

Lemma 2.1. *Assume Σ is a compact $C^{1,1}$-hypersurface and let $s \in (0,1)$ and $\alpha \in (0, 1-s)$. For every $\delta \in (0,1)$ there is $C_\delta > 0$ such that for $u \in C^{1+s+\alpha}(\Sigma)$ it holds*

$$\|u\|_{C^{1+\alpha}(\Sigma)} \le \delta \|u\|_{C^{1+s+\alpha}(\Sigma)} + C_\delta \|u\|_{C^0(\Sigma)}.$$

Observe that $\nabla^2 u$ is symmetric, i.e., $\nabla^2 u(X,Y) = \nabla^2 u(Y,X)$ for every vector fields X and Y, while $\nabla^k u$ for $k \ge 3$ is not. From the definition we see that for $X_1,\dots,X_k \in \mathcal{T}(\Sigma)$ with $\|X_i\|_{C^{k+2}(\Sigma)} \le 1$, $i = 1,\dots,k$ it holds

$$\nabla^k u(X_1,\dots,X_i,\dots,X_j,\dots,Y_k) = \nabla^k u(X_1,\dots,X_j,\dots,X_i,\dots,Y_k) + \partial^{k-1} u. \tag{2.1}$$

Here the notation $\partial^l u$ stands for a function which satisfies

$$\|\partial^l u\|_{C^\gamma(\Sigma)} \le C_{l,\gamma} \|u\|_{C^{l+\gamma}(\Sigma)} \tag{2.2}$$

for any $\gamma \in (0,2)$. Note also that in general $\nabla_Y \nabla_X u \neq \nabla_Y(\nabla_X u)$ for $X, Y \in \mathcal{T}(\Sigma)$ since $\nabla_Y(\nabla_X u) = Y(Xu) = YXu$ and $\nabla_Y \nabla_X u = YXu - (\nabla_Y X)u$. On the other hand if $X_1,\dots,X_k$ are vector fields with $\|X_i\|_{C^{k+2}(\Sigma)} \le 1$, $i = 1,\dots,k$, then it holds

$$\nabla_{X_k} \cdots \nabla_{X_1} u = X_k \cdots X_1 u + \partial^{k-1} u, \tag{2.3}$$

where $\partial^{k-1} u$ denotes a function which satisfies (2.2). It is then straightforward to check that for any $\gamma \in (0,2)$ it holds

$$\begin{aligned} &\sup\{\|X_k \cdots X_1 u\|_{C^\gamma(\Sigma)} : X_i \in \mathcal{T}(\Sigma),\ \|X_i\|_{C^{k+2}(\Sigma)} \le 1,\ i = 1,\dots,k\} \\ &\qquad \ge \frac{1}{C_k} \|u\|_{C^{k+\gamma}(\Sigma)} - C_k \|u\|_{C^{k-1+\gamma}(\Sigma)}, \end{aligned} \tag{2.4}$$

where C_k depends on k.

We may use the fact that Σ is embedded in $\mathbb{R}^{n+1}$ to extend any function $F \in C^1(\Sigma;\mathbb{R}^m)$ to $\tilde{F} \in C^1(\mathbb{R}^{n+1};\mathbb{R}^m)$ such that $\tilde{F} = F$ on Σ. We define the *tangential differential* of F by

$$D_\tau F(x) = D\tilde{F}(x)(I - \nu(x) \otimes \nu(x)),$$

where ν denotes the unit outer normal of $\Sigma = \partial G$ (outer with respect to G). We denote ν_Σ if we want to be emphasize that the normal is related to Σ and denote by $\nu_E(x)$ the normal of a generic set E. It is clear that $D_\tau F(x)$ does not

depend on the chosen extension. With a slight abuse of notation we denote the tangential gradient of $u \in C^\infty(\Sigma)$ at x also by $D_\tau u(x)$, even if it is a vector in $\mathbb{R}^{n+1}$.

We may use the embedding to associate the tangent space $T_x\Sigma$ with the linear subspace $\{p \in \mathbb{R}^{n+1} : p \cdot \nu(x) = 0\}$ by the relation

$$v(u) = D_\tau u(x) \cdot p \qquad \text{for all } u \in C^\infty(\Sigma),$$

where $v \in T_x\Sigma$, i.e., a derivation at x, and $p \in \mathbb{R}^{n+1}$ with $p \cdot \nu(x) = 0$. The components of the vector p are then given by $p_i = v(x_i)$. Indeed, by 'tangent space' we usually mean the geometric tangent space, i.e., a linear subspace of $\mathbb{R}^{n+1}$, but for clarity we use '$\cdot$' for the standard inner product of two vectors in $\mathbb{R}^{n+1}$ while '$\langle\cdot,\cdot\rangle$' denotes the inner product on the tangent space. Similarly we may associate a smooth vector field $X \in \mathcal{T}(\Sigma)$ with the vector valued function $\tilde{X} \in C^\infty(\Sigma, \mathbb{R}^{n+1})$ which satisfies $\tilde{X}(x) \cdot \nu(x) = 0$ for all $x \in \Sigma$ and

$$\nabla_X u = D_\tau u \cdot \tilde{X} \qquad \text{for all } u \in C^\infty(\Sigma).$$

Therefore, by a vector field X we usually mean a vector valued function which values are on the (geometric) tangent space, $X \cdot \nu = 0$, with the convention that Xu denotes the derivative of u in direction of X. It is also clear that the tangential gradient of $u \in C^\infty(\Sigma)$ is equivalent to its covariant derivative and for every $\alpha \in (0, 1)$ it holds

$$\frac{1}{C}\|u\|_{C^{1+\alpha}(\Sigma)} \le \|D_\tau u\|_{C^\alpha(\Sigma)} + \|u\|_{C^0(\Sigma)} \le C\|u\|_{C^{1+\alpha}(\Sigma)}.$$

We denote the divergence of a vector field $X \in \mathcal{T}(\Sigma)$ by $\operatorname{div} X$ and the divergence theorem states

$$\int_\Sigma \operatorname{div} X \, d\mathcal{H}^n = 0.$$

For clarity we denote the divergence of a vector valued function $F \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ in $\mathbb{R}^{n+1}$ by $\operatorname{div}_{\mathbb{R}^{n+1}} F$. We may extend the definition of divergence to vector valued functions $\tilde{X} \in C^\infty(\Sigma, \mathbb{R}^{n+1})$ by $\operatorname{div} \tilde{X} := \operatorname{Trace}(D_\tau \tilde{X})$. Then the divergence theorem generalizes to

$$\int_\Sigma \operatorname{div} \tilde{X} \, d\mathcal{H}^n = \int_\Sigma H_\Sigma \, \tilde{X} \cdot \nu \, d\mathcal{H}^n,$$

where H_Σ denotes the mean curvature of Σ, which is the sum of the principal curvatures.

2.2. *Fractional Laplacian*

We define the fractional Laplacian on Σ as

$$\Delta^{\frac{s+1}{2}} u(x) := 2 \int_\Sigma \frac{u(y) - u(x)}{|y - x|^{n+1+s}} \, d\mathcal{H}^n_y.$$

This should be understood in principal valued sense, but from now on we assume this without further mention. It is not difficult to see, and it actually follows from Proposition 4.9, that if $u \in C^\infty(\Sigma)$ then $\Delta^{\frac{s+1}{2}} u$ is a smooth function on Σ. It is well known [14,17] that by linearizing the fractional mean curvature at Σ one obtains the following Jacobi operator

$$L[u](x) := \Delta^{\frac{s+1}{2}} u(x) + c_s^2(x)\, u(x), \tag{2.5}$$

where

$$c_s^2(x) = \int_\Sigma \frac{|\nu(y) - \nu(x)|^2}{|y - x|^{n+1+s}} \, d\mathcal{H}^n_y.$$

We note that since Σ is a smooth surface, $c_s^2(\cdot)$ defines a smooth function on Σ. Again this is not difficult to see and it follows from our analysis in Section 4. Moreover, since we assume Σ is uniformly $C^{1,1}$-regular, the α-Hölder norm of c_s^2, for small α, is uniformly bounded (see Lemma 4.3).

As we mentioned in the introduction, the proof of the main theorem is based on regularity estimates for nonlinear nonlocal parabolic equation. To this aim we need standard Schauder estimates for the fractional heat equation with a forcing term

$$\begin{cases} \partial_t u = \Delta^{\frac{s+1}{2}} u + f(x,t) + g(x) & \text{on } \Sigma \times (0,T] \\ u(x,0) = u_0(x) & \text{for } x \in \Sigma. \end{cases} \tag{2.6}$$

We prove the following Schauder estimate. We give the proof in the Appendix.

Theorem 2.2. *Assume that $f : \Sigma \times [0,T] \to \mathbb{R}$ and $u_0, g : \Sigma \to \mathbb{R}$ are smooth and fix $\alpha \in (0, 1-s)$. Then (2.6) has a unique smooth solution and it holds*

$$\sup_{0<t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le C(1+T)\big(\|u_0\|_{C^{1+s+\alpha}(\Sigma)} + \sup_{0<t<T} \|f(\cdot,t)\|_{C^{\alpha}(\Sigma)} + T\|g\|_{C^{1+s+\alpha}(\Sigma)}\big)$$

and

$$\sup_{0<t<T} \|u(\cdot,t)\|_{C^{0}(\Sigma)} \le \|u_0\|_{C^{0}(\Sigma)} + T\big(\sup_{0<t<T} \|f(\cdot,t)\|_{C^{0}(\Sigma)} + \|g\|_{C^{0}(\Sigma)}\big)$$

The second statement is in fact a simple consequence of the maximum principle.

3. Parametrization of the flow (1.1)

In this section we follow [23] (see also [26]) and parametrize the equation (1.1) by using the height function over a smooth reference surface. Note first that since ∂E_0 is a compact $C^{1,1}$-hypersurface we find for any $\varepsilon > 0$ a smooth compact hypersurface Σ such that we may write ∂E_0 as a graph over Σ,

$$\partial E_0 = \{x + h_0(x)\nu(x) : x \in \Sigma\}$$

with $\|h_0\|_{C^0(\Sigma)} < \varepsilon$ and $\|h_0\|_{C^2(\Sigma)} \le C$. Indeed, we may first fix a smooth surface Σ_0 such that

$$\partial E_0 = \{x + \tilde{h}(x)\nu_{\Sigma_0}(x) : x \in \Sigma_0\},$$

where $\tilde{h} \in C^{1,1}(\Sigma_0)$ (note that $\|\tilde{h}\|_{C^0(\Sigma_0)}$ is not necessarily small). By standard mollification argument we find $\tilde{h}_\varepsilon \in C^\infty(\Sigma_0)$ with

$$\|\tilde{h}_\varepsilon - \tilde{h}\|_{C^0(\Sigma_0)} \le \varepsilon \quad \text{and} \quad \|\tilde{h}_\varepsilon\|_{C^2(\Sigma_0)} \le C.$$

Thus we may define $\Sigma = \{x + \tilde{h}_\varepsilon(x)\nu_{\Sigma_0}(x) : x \in \Sigma_0\}$.

From now on we assume that α is a positive number such that $\alpha < (1-s)/2$. We note that because ∂E_0 is only $C^{1,1}$-regular we have $\|\nu_\Sigma\|_{C^1(\Sigma)} \le C$ but $\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\varepsilon^{-s-\alpha}$. This means that we have to be careful in our analysis whenever we have terms which depend on the norm $\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$, because we cannot bound it uniformly if we want Σ to be close to ∂E_0. Note that

$$\|h_0\|_{C^\alpha(\Sigma)} \le \|h_0\|^{1-\alpha}_{C^0(\Sigma)} \|h_0\|^{\alpha}_{C^1(\Sigma)} \le \varepsilon^{1-\alpha}$$

and therefore even if $\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$ is large it still holds

$$\|h_0\|_{C^\alpha(\Sigma)} \|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\varepsilon^{1-s-2\alpha} \tag{3.1}$$

for $\alpha < (1-s)/2$. Therefore for any $\delta > 0$ we may choose ε small such that

$$\|h_0\|_{C^{1+s+\alpha}(\Sigma)} + \|h_0\|_{C^\alpha(\Sigma)} \|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le \delta. \tag{3.2}$$

In particular, this implies $\|h_0\, \nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\delta$.

Our goal is to write the family of sets $(E_t)_{t\in(0,T]}$, which is a solution of (1.1), as a graph over the reference surface Σ. To be more precise, we look for a function $h \in C(\Sigma \times [0,T]) \cap C^\infty(\Sigma \times (0,T])$ such that the family of sets E_t given by

$$\partial E_t = \{x + h(x,t)\nu(x) : x \in \Sigma\} \quad \text{and} \quad h(x,0) = h_0(x)$$

is a solution of (1.1). In this section we provide the calculations which show that this leads to the equation

$$\partial_t h = L[h] + P(x,h,\nabla h) - H^s_\Sigma(x), \tag{3.3}$$

where H^s_Σ is the fractional mean curvature of the reference surface Σ and $L[\cdot]$ is the linear operator defined in (2.5). The precise formula for the remainder term P is given in Proposition 3.2. Our goal in the next section is then to show that for $\delta > 0$ small the function $x \mapsto P(x,u,\nabla u)$ satisfies

$$\|P(\cdot,u,\nabla u)\|_{C^\alpha(\Sigma)} \le C\delta \|u\|_{C^{1+s+\alpha}(\Sigma)},$$

when $\|u\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$. This means that we may treat (3.3) as a small perturbation of the fractional heat equation, i.e., (2.6) with $f = 0$ and $g = 0$.

In order to calculate (3.3) we define the class of sets $\mathfrak{h}_\delta(\Sigma)$ such that $E \in \mathfrak{h}_\delta(\Sigma)$ if its boundary can be written as

$$\partial E = \{x + h_E(x)\nu(x) : x \in \Sigma\} \qquad \text{and} \quad \|h_E\|_{C^{1+s+\alpha}(\Sigma)} \le \delta. \tag{3.4}$$

In particular, if $E \in \mathfrak{h}_\delta(\Sigma)$ then its boundary is a compact $C^{1+s+\alpha}$-hypersurface.

We begin with a standard calculation.

Lemma 3.1. *Let $E \subset \mathbb{R}^{n+1}$ be a smooth bounded set, let Φ_τ be a family of diffeomorphisms such that $\Phi_0(x) = x$, denote the velocity field by $X(x) = \frac{d}{d\tau}\big|_{\tau=0}\Phi_\tau(x)$ and suppose $X \in C^{1+s+\alpha}(\Sigma)$. Then it holds*

$$-\frac{d}{d\tau}\Big|_{\tau=0} H^s_{\Phi_\tau(E)}(\Phi_\tau(x)) = 2\int_{\partial E} \frac{(X(y) - X(x)) \cdot \nu_E(y)}{|y-x|^{n+1+s}}\, d\mathcal{H}^n_y.$$

Proof. Let us denote $E_\tau = \Phi_\tau(E)$ and $k_\varepsilon(z) = (|z|^2 + \varepsilon)^{-\frac{n+s+1}{2}}$. It is enough to show that

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_{E_\tau} k_\varepsilon(\Phi_\tau(x) - y)dy = \int_{\partial E} k_\varepsilon(x-y)(X(y) - X(x)) \cdot \nu_E(y)\, d\mathcal{H}^n_y.$$

Indeed, by repeating the same calculations for the second term in (1.2) and letting $\varepsilon \to 0$ yields the result. We split the above term as

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_{E_\tau} k_\varepsilon(\Phi_\tau(x) - y)dy = \underbrace{\frac{d}{d\tau}\Big|_{\tau=0} \int_{E_\tau} k_\varepsilon(x-y)\, dy}_{=I} + \underbrace{\frac{d}{d\tau}\Big|_{\tau=0} \int_{E} k_\varepsilon(\Phi_\tau(x) - y)\, dy}_{=II}.$$

Let us denote the Jacobian of Φ_τ by $J_{\Phi_\tau,\mathbb{R}^{n+1}}$. Since $\frac{d}{d\tau}\big|_{\tau=0} J_{\Phi_\tau,\mathbb{R}^{n+1}} = \operatorname{div}_{\mathbb{R}^{n+1}} X$, we may write the first term by change of variables as

$$\begin{aligned}
I &= \frac{d}{d\tau}\Big|_{\tau=0} \int_E k_\varepsilon(\Phi_\tau(y) - x)\, J_{\Phi_\tau,\mathbb{R}^{n+1}}(y)\, dy \\
&= \int_E \big(k_\varepsilon(y-x)\,(\operatorname{div}_{\mathbb{R}^{n+1}} X)(y) + D_y k_\varepsilon(y-x) \cdot X(y)\big)\, dy \\
&= \int_E \operatorname{div}_{\mathbb{R}^{n+1}}(k_\varepsilon(y-x)\, X(y))\, dy \\
&= \int_{\partial E} k_\varepsilon(y-x)\,(X(y) \cdot \nu_E(y))\, d\mathcal{H}^n_y.
\end{aligned}$$

By symmetry it holds $D_x k_\varepsilon(x-y) = -D_y k_\varepsilon(x-y)$. Therefore we have for the second term

$$
\begin{aligned}
II &= \int_E (D_x k_\varepsilon(x-y)\cdot X(x))\,dy \\
&= -\int_E \mathrm{div}_{\mathbb{R}^{n+1}}(k_\varepsilon(y-x)\,X(x))\,dy \\
&= -\int_{\partial E} k_\varepsilon(y-x)\,(X(x)\cdot \nu_E(y))\,d\mathcal{H}^n_y. \quad \square
\end{aligned}
$$

We may use Lemma 3.1 to write the fractional mean curvature H^s_E over the reference surface Σ. Let $E \in \mathfrak{h}_\delta(\Sigma)$ with $\partial E = \{x + h(x)\nu(x) : x \in \Sigma\}$. We define the sets $E_{t'}$ as $\partial E_{t'} := \{x + t'h(x)\nu(x) : x \in \Sigma\}$, with $t' \in [0,1]$, and family of diffeomorphisms $\Phi_{t'h} : \Sigma \to \partial E_{t'}$ as

$$
\Phi_{t'h}(x) = x + t'h(x)\nu(x).
$$

Then for $x \in \Sigma$ we have

$$
-H^s_E(x + h(x)\nu(x)) = -\int_0^1 \frac{d}{dt'} H^s_{\Phi_{t'h}(\Sigma)}(\Phi_{t'h}(x))\,dt' - H^s_\Sigma(x). \tag{3.5}
$$

We denote the tangential Jacobian of $\Phi_{t'h}$ by $J_{\Phi_{t'h}}$ (see [1] for details) and define $\Phi_\tau(x) := \Phi_{(t'+\tau)h}(\Phi^{-1}_{t'h}(x))$. Note that $\Phi_\tau : \partial E_{t'} \to \partial E_{t'+\tau}$ is a diffeomorphism and

$$
\frac{d}{d\tau}\Big|_{\tau=0} \Phi_\tau(x) = h(\Phi^{-1}_{t'h}(x))\nu(\Phi^{-1}_{t'h}(x)) \qquad \text{for } x \in \partial E_{t'}.
$$

We apply Lemma 3.1 and change of variables to deduce

$$
\begin{aligned}
&-\frac{d}{dt'} H^s_{\Phi_{t'h}(\Sigma)}(\Phi_{t'h}(x)) \\
&\quad = 2\int_{\partial E_{t'}} \frac{1}{|y-x|^{n+1+s}} \Big(h(\Phi^{-1}_{t'h}(y))\nu(\Phi^{-1}_{t'h}(y)) - h(\Phi^{-1}_{t'h}(x))\nu(\Phi^{-1}_{t'h}(x))\Big)\cdot \nu_{E_{t'}}(y)\,d\mathcal{H}^n_y \\
&\quad = 2\int_\Sigma \frac{1}{|\Phi_{t'h}(x) - \Phi_{t'h}(y)|^{n+1+s}} (h(y)\nu(y) - h(x)\nu(x))\cdot \nu_{E_{t'}}(\Phi_{t'h}(y)) J_{\Phi_{t'h}}(y)\,d\mathcal{H}^n_y \\
&\quad = 2\int_\Sigma \frac{(h(y)-h(x))}{|\Phi_{t'h}(x) - \Phi_{t'h}(y)|^{n+1+s}} (\nu(y)\cdot \nu_{E_{t'}}(\Phi_{t'h}(y)) J_{\Phi_{t'h}}(y))\,d\mathcal{H}^n_y \\
&\qquad + 2\left(\int_\Sigma \frac{(\nu(y)-\nu(x))\cdot \nu_{E_{t'}}(\Phi_{t'h}(y))}{|\Phi_{t'h}(x) - \Phi_{t'h}(y)|^{n+1+s}} J_{\Phi_{t'h}}(y)\,d\mathcal{H}^n_y\right) h(x).
\end{aligned}
$$

We may write the normal $\nu_{E_{t'}}$ (see [26, Section 1.5]) as

$$
\nu_{E'_t}(\Phi_{t'h}(y)) = \frac{1}{J_{\Phi_{t'h}}}\big((1 + Q_1(y, t'h, t'\nabla h))\nu(y) + Q_2(y, t'h, t'\nabla h)\big), \tag{3.6}
$$

where Q_i are smooth functions which depend on the second fundamental form of Σ and $Q_i(y,0,0) = 0$, for $i = 1,2$ for all $y \in \Sigma$. Moreover, Q_2 takes values on the tangent space, i.e., $Q_2(y,\cdot,\cdot)\cdot\nu(y) = 0$. We may thus write

$$
\begin{aligned}
&2\int_\Sigma \frac{(h(y)-h(x))}{|\Phi_{t'h}(x)-\Phi_{t'h}(y)|^{n+1+s}}\,(\nu(y)\cdot\nu_{E_{t'}}(\Phi_{t'h}(y))J_{\Phi_{t'h}}(y))\,d\mathcal{H}^n_y\\
&\quad=2\int_\Sigma \frac{(h(y)-h(x))}{|\Phi_{t'h}(x)-\Phi_{t'h}(y)|^{n+1+s}}\,(1+Q_1(y,t'h,t'\nabla h))\,d\mathcal{H}^n_y\\
&\quad=2\Delta^{\frac{s+1}{2}}h(x)\\
&\qquad+2\int_\Sigma (h(y)-h(x))\left(\frac{1+Q_1(y,t'h,t'\nabla h)}{|\Phi_{t'h}(x)-\Phi_{t'h}(y)|^{n+1+s}}-\frac{1}{|y-x|^{n+1+s}}\right)d\mathcal{H}^n_y.
\end{aligned}
$$

We write $2(\nu(y)-\nu(x))\cdot\nu(y)=|\nu(y)-\nu(x)|^2$ and obtain by (3.6)

$$
\begin{aligned}
&2\int_\Sigma \frac{(\nu(y)-\nu(x))\cdot\nu_{E_{t'}}(\Phi_{t'h}(y))}{|\Phi_{t'h}(x)-\Phi_{t'h}(y)|^{n+1+s}}\,J_{\Phi_{t'h}}(y)\,d\mathcal{H}^n_y\\
&\quad=\int_\Sigma \frac{|\nu(y)-\nu(x)|^2}{|y-x|^{n+1+s}}\,d\mathcal{H}^n_y\\
&\qquad+\int_\Sigma |\nu(y)-\nu(x)|^2\left(\frac{1+Q_1(y,t'h,t'\nabla h)}{|\Phi_{t'h}(x)-\Phi_{t'h}(y)|^{n+1+s}}-\frac{1}{|y-x|^{n+1+s}}\right)d\mathcal{H}^n_y\\
&\qquad+2\int_\Sigma \frac{(\nu(y)-\nu(x))\cdot Q_2(y,t'h,t'\nabla h)}{|\Phi_{t'h}(x)-\Phi_{t'h}(y)|^{n+1+s}}\,d\mathcal{H}^n_y.
\end{aligned}
$$

To shorten the notation we denote the kernel $K_u:\Sigma\times\Sigma\to[0,\infty]$ generated by $u\in C^{1+s+\alpha}(\Sigma)$ as

$$
K_u(y,x):=\frac{1}{|y-x+u(y)\nu(y)-u(x)\nu(x)|^{n+1+s}}. \tag{3.7}
$$

Recall that $Q_1(y,0,0)=0$. We may thus write

$$
\frac{1+Q_1(y,t'h,t'\nabla h)}{|\Phi_{t'h}(x)-\Phi_{t'h}(y)|^{n+1+s}}-\frac{1}{|y-x|^{n+1+s}}=\int_0^{t'}\frac{d}{d\xi}\big((1+Q_1(y,\xi h,\xi\nabla h))K_{\xi h}(y,x)\big)d\xi.
$$

We may finally write the fractional mean curvature of $E\in\mathfrak{h}_\delta(\Sigma)$ by recalling the linear operator $L[\cdot]$ in (2.5), by (3.5) and by the previous calculations

$$
-H^s_E(x+h(x)\nu(x))=L[h](x)-H^s_\Sigma(x)+R_{1,h}(x)+R_{2,h}(x)\,h(x). \tag{3.8}
$$

The remainder terms $R_{1,u}$ and $R_{2,u}$ are defined for a generic function $u\in C^{1+s+\alpha}(\Sigma)$ with $\|u\|_{C^{1+s+\alpha}(\Sigma)}\le\delta$ as

$$
R_{1,u}(x):=2\int_0^1\int_0^{t'}\int_\Sigma (u(y)-u(x))\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y d\xi\,dt' \tag{3.9}
$$

and

$$
\begin{aligned}
R_{2,u}(x):=&\int_0^1\int_0^{t'}\int_\Sigma |\nu(y)-\nu(x)|^2\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y d\xi\,dt'\\
&+2\int_0^1\int_\Sigma (\nu(y)-\nu(x))\cdot Q_2(y,t'u,t'\nabla u)\,K_{t'u}(y,x)\,d\mathcal{H}^n_y dt',
\end{aligned} \tag{3.10}
$$

where the kernel K_u is defined in (3.7), and Q_1, Q_2 are smooth functions which satisfy $Q_1(y,0,0) = Q_2(y,0,0) = 0$ for all $y \in \Sigma$.

In order to write the flow (1.1) as an equation we recall from [26] that the normal velocity of the flow $(E_t)_t$ given by $E_t = \Phi_t(E)$, where $\Phi_t(x) = x + h(x,t)\nu(x)$ on Σ, is

$$V_t = \big(\nu_{E_t}(\Phi_t(x)) \cdot \nu(x)\big)\partial_t h.$$

By choosing in $t' = 1$ in (3.6) we have

$$\nu_{E_t}(\Phi_t(x)) \cdot \nu(x) = \frac{1}{J_{\Phi_t}}(1 + Q_1(y,h,\nabla h))$$

and the Jacobian can be written as $J_{\Phi_t} = 1 + Q_3(y,h,\nabla h)$. Here Q_1 and Q_3 are smooth functions with $Q_1(x,0,0) = Q_3(x,0,0) = 0$ for all $x \in \Sigma$. Thus when $E_t \in \mathfrak{h}_\delta(\Sigma)$ for small enough δ we may write

$$V_t = \big(\nu_{E_t}(\Phi_t(x)) \cdot \nu(x)\big)\partial_t h = \frac{\partial_t h}{1 + Q(y, h(\cdot,t), \nabla h(\cdot,t))} \tag{3.11}$$

where Q is a smooth function with $Q(x,0,0) = 0$ for all $x \in \Sigma$. We may finally write the equation for h by combining (3.8) and (3.11). We state this in the following proposition.

Proposition 3.2. *Assume that the flow $(E_t)_{t\in(0,T]}$, with $E_t \in \mathfrak{h}_\delta(\Sigma)$ for $t \in (0,T]$, is a classical solution of (1.1) starting from E_0 with $\partial E_0 = \{x + h_0(x)\nu(x) : x \in \Sigma\}$ and assume δ is small. Then the function $h \in C(\Sigma \times [0,T]) \cap C^\infty(\Sigma \times (0,T])$ with $\partial E_t = \{x + h(x,t)\nu(x) : x \in \Sigma\}$ is a solution of the equation*

$$\partial_t h = (1 + Q(x,h,\nabla h))\Big(L[h] - H^s_\Sigma(x) + R_{1,h(\cdot,t)}(x) + R_{2,h(\cdot,t)}(x)\,h(x,t)\Big) \qquad \text{on } \Sigma \times (0,T] \tag{3.12}$$

with $h(x,0) = h_0$. Here L is the linear operator defined in (2.5) and H^s_Σ is the fractional mean curvature of the reference surface Σ. The remainder terms $R_{1,h(\cdot,t)}$ and $R_{2,h(\cdot,t)}$ are defined in (3.9) and (3.10) respectively and Q is a smooth function with $Q(x,0,0) = 0$ for all $x \in \Sigma$.

Conversely, if $h \in C(\Sigma \times [0,T]) \cap C^\infty(\Sigma \times (0,T])$ is a solution of (3.12) with

$$h(x,0) = h_0 \quad \text{and} \quad \sup_{0<t<T} \|h(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le \delta,$$

then $\partial E_t = \{x + h(x,t)\nu(x) : x \in \Sigma\}$ defines a family of sets which is a solution of (1.1) starting from E_0.

4. Regularity estimates for the nonlocal operators

In this section we study the spatial regularity issues related to the equation (3.12) and, in particular, the remainder terms $R_{1,u}$ and $R_{2,u}$ defined in (3.9) and (3.10). As we mentioned in the previous section, our goal is to prove that if $\|u\|_{C^{1+s+\alpha}}$ is small then $R_{1,u}$ and $R_{2,u}$ are small in the C^α-sense, which then implies that we may regard the equation (3.12) as a linear equation with a small perturbation. We study also the higher order regularity of $R_{1,u}$ and $R_{2,u}$ in order to prove that the solution of (3.12) becomes instantaneously smooth. The complicated structure of $R_{1,u}$ and $R_{2,u}$ makes this section challenging.

Throughout this section K denotes a generic kernel, if not otherwise mentioned, while K_u is the kernel defined in (3.7). Next we define the class of kernels which we will use throughout the section.

Definition 4.1. Let $\kappa > 0$ and $K : \Sigma \times \Sigma \to \mathbb{R} \cup \{\pm\infty\}$. We say that $K \in \mathcal{S}_\kappa$ if the following three conditions hold:

(i) K is continuous at every $y, x \in \Sigma$, $x \neq y$, and it holds

$$|K(y,x)| \le \frac{\kappa}{|y-x|^{n+1+s}}.$$

(ii) The function $x \mapsto K(y,x)$ is differentiable at every $x, y \in \Sigma$, $x \neq y$, and

$$|\nabla_x K(y,x)| \le \frac{\kappa}{|y-x|^{n+2+s}}.$$

(iii) The function

$$\psi(x) := \int_\Sigma (y-x)\, K(y,x)\, d\mathcal{H}^n_y$$

is Hölder continuous with $\|\psi\|_{C^\alpha(\Sigma)} \le \kappa$.

Remark 4.2. Throughout the paper we assume that Σ is a compact hypersurface, but Definition 4.1 can be extended to the case $\Sigma = \mathbb{R}^n$. For instance the autonomous kernel $|y-x|^{-n-1-s}$ in $\mathbb{R}^n$ trivially satisfies the conditions (i)-(iii) for $\kappa = n+1+s$.

The first two conditions in Definition 4.1 state that the kernel K behaves similarly as the model case $|y-x|^{-n-1-s}$, while the third condition is somewhat more involved. Indeed, it is not trivial to prove the condition (iii) for K_u defined in (3.7), since Σ is not flat and thus there is no cancellation due to symmetry as in the case $\Sigma = \mathbb{R}^n$. (We will prove this in Lemma 4.5.) However, it is important first to notice that there are cases when we do not need the condition (iii) to prove Hölder continuity estimates. This is stated in the following useful auxiliary lemma.

Lemma 4.3. *Assume that $K : \Sigma \times \Sigma \to \mathbb{R} \cup \{\pm\infty\}$ satisfies the conditions (i) and (ii) in Definition 4.1 with constant $\kappa > 0$. Moreover, assume that $F \in C(\Sigma \times \Sigma)$ satisfies the following:*

(1) For all $x, y \in \Sigma$ it holds

$$|F(y,x)| \le \kappa_0 |y-x|^{1+s+\alpha}.$$

(2) For all $x, y, z \in \Sigma$ with $|y-x| \ge 2|z-x|$ it holds

$$|F(y,z) - F(y,x)| \le \kappa_0 |z-x|^{s+\alpha}\, |y-x|.$$

Then the function

$$\psi(x) = \int_\Sigma F(y,x) K(y,x)\, d\mathcal{H}^n_y$$

is Hölder continuous and $\|\psi\|_{C^\alpha(\Sigma)} \le C\kappa_0\kappa$.

Proof. By the condition (i) in Definition 4.1 and by the assumption (1) we immediately obtain $\|\psi\|_{C^0(\Sigma)} \le C\kappa_0\kappa$, because the function $y \mapsto F(y,x)K(y,x)$ is integrable over Σ for every x. To show the Hölder continuity we may assume that $0 \in \Sigma$ and we need to show $|\psi(z) - \psi(0)| \le C\kappa_0\kappa|z|^\alpha$ for $z \in \Sigma$ close to 0. We divide the set Σ into $\Sigma_- = \Sigma \cap \{|y| \le 2|z|\}$ and $\Sigma_+ = \Sigma \cap \{|y| > 2|z|\}$. For all $y \in \Sigma_-$ it holds by the condition (i) in Definition 4.1 and by the assumption (1) that

$$\int_{\Sigma_-} |F(y,z)||K(y,z)|\, d\mathcal{H}^n_y \le \kappa_0\kappa \int_{\Sigma_-} \frac{1}{|y-z|^{n-\alpha}}\, d\mathcal{H}^n_y \le C\kappa_0\kappa \int_0^{3|z|} \rho^{\alpha-1}\, d\rho \le C\kappa_0\kappa|z|^\alpha.$$

Similarly it holds

$$\int_{\Sigma_-} |F(y,0)||K(y,0)|\, d\mathcal{H}^n_y \le C\kappa_0\kappa|z|^\alpha.$$

On the other hand it follows from the condition (ii) in Definition 4.1 that for all $y \in \Sigma_+$, i.e., $|y| \ge 2|z|$, it holds

$$|K(y,z) - K(y,0)| \le C\kappa \frac{|z|}{|y|^{n+2+s}} \le C\kappa \frac{|z|^{s+\alpha}}{|y|^{n+1+2s+\alpha}}.$$

This together with the condition (i) and with the assumptions (1) and (2) yield

$$\begin{aligned}
&\int_{\Sigma_+} |F(y,z)K(y,z) - F(y,0)K(y,0)|\, d\mathcal{H}^n_y \\
&\qquad \le \int_{\Sigma_+} |F(y,z) - F(y,0)||K(y,z)| + |F(y,0)||K(y,z) - K(y,0)|\, d\mathcal{H}^n_y \\
&\qquad \le C\kappa_0\kappa |z|^{s+\alpha} \int_{\Sigma_-} \frac{1}{|y|^{n+s}}\, d\mathcal{H}^n_y \\
&\qquad \le C\kappa_0\kappa |z|^{s+\alpha} \int_{2|z|}^{\infty} \rho^{-1-s}\, d\rho \le C\kappa_1\kappa_2 |z|^{\alpha}.
\end{aligned}$$

These imply $|\psi(z) - \psi(0)| \le C\kappa_1\kappa_2|z|^{\alpha}$. $\square$

We proceed by stating first the crucial regularity estimates we need repeatedly in the paper, and then prove that $K_u \in \mathcal{S}_\kappa$ (see Definition 4.1) for bounded κ.

Lemma 4.4. *Let $K \in \mathcal{S}_\kappa$ (see Definition 4.1) and assume $v_1 \in C^{1+s+\alpha}(\Sigma)$, $v_2 \in C^{s+\alpha}(\Sigma)$ and $v_3 \in C^{\alpha}(\Sigma)$. Then the function*

$$\psi(x) = \int_{\Sigma} (v_1(y) - v_1(x))v_2(y)v_3(x)\, K(y,x)\, d\mathcal{H}^n_y$$

is Hölder continuous and

$$\|\psi\|_{C^\alpha(\Sigma)} \le C\kappa \|v_1\|_{C^{1+s+\alpha}(\Sigma)} \|v_2\|_{C^{s+\alpha}(\Sigma)} \|v_3\|_{C^\alpha(\Sigma)}.$$

Proof. We write ψ as

$$\psi(x) = v_3(x) \underbrace{\int_{\Sigma} (v_1(y) - v_1(x))(v_2(y) - v_2(x))\, K(y,x)\, d\mathcal{H}^n_y}_{=:\psi_1(x)} + v_2(x)v_3(x) \underbrace{\int_{\Sigma} (v_1(y) - v_1(x))\, K(y,x)\, d\mathcal{H}^n_y}_{=:\psi_2(x)}.$$

Note that $\|\psi\|_{C^\alpha(\Sigma)} \le \|v_3\|_{C^\alpha(\Sigma)}\|\psi_1\|_{C^\alpha(\Sigma)} + \|v_3\|_{C^\alpha(\Sigma)}\|v_2\|_{C^\alpha(\Sigma)}\|\psi_2\|_{C^\alpha(\Sigma)}$. Therefore it is enough to estimate $\|\psi_1\|_{C^\alpha(\Sigma)}$ and $\|\psi_2\|_{C^\alpha(\Sigma)}$. We define

$$F_1(y,x) := (v_1(y) - v_1(x))(v_2(y) - v_2(x)).$$

It is straightforward to check that F_1 satisfies the assumptions of Lemma 4.3 with $\kappa_0 \le C\, \|v_1\|_{C^1(\Sigma)}\|v_2\|_{C^{s+\alpha}(\Sigma)}$. Therefore Lemma 4.3 yields $\|\psi_1\|_{C^\alpha(\Sigma)} \le C\kappa\, \|v_1\|_{C^1(\Sigma)}\|v_2\|_{C^{s+\alpha}(\Sigma)}$. We need thus to show that

$$\|\psi_2\|_{C^\alpha} \le C\kappa \|v_1\|_{C^{1+s+\alpha}(\Sigma)}. \tag{4.1}$$

To this aim we write ψ_2 as

$$\psi_2(x) = \underbrace{\int_{\Sigma} \big(v_1(y) - v_1(x) - D_\tau v_1(x)\cdot(y-x)\big)\, K(y,x)\, d\mathcal{H}^n_y}_{=:\psi_3(x)} + D_\tau v_1(x) \cdot \int_{\Sigma} (y-x)\, K(y,x)\, d\mathcal{H}^n_y.$$

It follows immediately from the condition (iii) in Definition 4.1 that the second term on the right-hand-side is Hölder-continuous with C^α-norm bounded by $C\kappa\|v_1\|_{C^{1+\alpha}(\Sigma)}$. We need thus to prove the Hölder-continuity of ψ_3. We notice that for every $x, y \in \Sigma$ it holds

$$|v_1(y) - v_1(x) - D_\tau v_1(x)\cdot(y-x)| \le \|v_1\|_{C^{1+s+\alpha}} |y-x|^{1+s+\alpha}.$$

Therefore the function

$$F_2(y,x) := v_1(y) - v_1(x) - D_\tau v_1(x)\cdot(y-x) \tag{4.2}$$

satisfies the assumption (1) of Lemma 4.3 with $\kappa_0 \le \|v_1\|_{C^{1+s+\alpha}(\Sigma)}$. Moreover for every $x, y, z \in \Sigma$ with $|y-x| \ge 2|x-z|$ it holds

$$\begin{aligned}\big|F_2(y,z) - F_2(y,x)\big| &= \big|\big(v_1(y) - v_1(z) - D_\tau v_1(z)\cdot(y-z)\big) - \big(v_1(y) - v_1(x) - D_\tau v_1(x)\cdot(y-x)\big)\big| \\ &= \big|\big(v_1(x) - v_1(z) - D_\tau v_1(z)\cdot(x-z)\big) + (D_\tau v_1(x) - D_\tau v_1(z))\cdot(y-x)\big| \\ &\le \|v_1\|_{C^{1+s+\alpha}}|z-x|^{1+s+\alpha} + \|v_1\|_{C^{1+s+\alpha}}|z-x|^{s+\alpha}|y-x| \\ &\le 2\|v_1\|_{C^{1+s+\alpha}}|z-x|^{s+\alpha}|y-x|.\end{aligned}$$

Therefore F_2 satisfies the assumption (2) of Lemma 4.3 with $\kappa_0 \le 2\|v_1\|_{C^{1+s+\alpha}(\Sigma)}$, and we conclude by Lemma 4.3 that

$$\|\psi_3\|_{C^\alpha} \le C\kappa\|v_1\|_{C^{1+s+\alpha}(\Sigma)}. \quad \square$$

Let us now prove that the kernel K_u defined in (3.7) belongs to the class $\mathcal{S}_\kappa$ (see Definition 4.1) for bounded κ, when the norm $\|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$ is small. Recall that this is a reasonable assumption by (3.2). We denote

$$\Phi_u(x) := x + u(x)\nu(x)$$

and write K_u defined in (3.7) as

$$K_u(y,x) = \frac{1}{|\Phi_u(y) - \Phi_u(x)|^{n+1+s}}.$$

We study also the linearization of K_u, which means that for a given $w \in C^{1+s+\alpha}(\Sigma)$ we consider

$$\frac{d}{d\xi}\Big|_{\xi=0} K_{u+\xi w}(y,x) = -\frac{n+1+s}{|\Phi_u(y) - \Phi_u(x)|^{n+3+s}}\big(\Phi_u(y) - \Phi_u(x)\big)\cdot\big(w(y)\nu(y) - w(x)\nu(x)\big). \tag{4.3}$$

Lemma 4.5. *Assume that $u \in C^{1+s+\alpha}(\Sigma)$ is such that $\|u\|_{C^{1+s+\alpha}(\Sigma)} + \|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ and $w \in C^{1+s+\alpha}(\Sigma)$. Then the following hold.*

(a) When δ is small enough the kernel K_u defined in (3.7) belongs to the class $\mathcal{S}_{\kappa_1}$, with $\kappa_1 \le C$.
(b) When δ is small enough the kernel $\frac{d}{d\xi}\Big|_{\xi=0} K_{u+\xi w}$ belongs to the class $\mathcal{S}_{\kappa_2}$, with

$$\kappa_2 \le C\|w\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}.$$

Proof. Claim (a): We denote $\Phi_u(x) := x + u(x)\nu(x)$ and recall that

$$K_u(y,x) = \frac{1}{|\Phi_u(y) - \Phi_u(x)|^{n+1+s}}.$$

It follows from the assumption $\|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ that

$$\frac{1}{2}|y-x| \le |\Phi_u(y) - \Phi_u(x)| \le 2|y-x|, \tag{4.4}$$

when δ is small. Therefore it is clear that K_u satisfies the conditions (i) and (ii) in Definition 4.1 of $\mathcal{S}_{\kappa_1}$ with $\kappa_1 \le C$.

The condition (iii) in Definition 4.1 is technically more involved to verify. We note that in principle we should regularize the kernel K_u for the forthcoming calculations as in the proof of Lemma 3.1, but we ignore this since it can be done with obvious changes. Recall that we need to show that the function

$$\psi(x) = \int_\Sigma (y-x)K_u(y,x)\,d\mathcal{H}^n_y \tag{4.5}$$

is Hölder continuous. The idea is to use integration parts in order to write ψ as a nonsingular integral. To be more precise, we prove the following equality

$$\big(I+\tilde{Q}(x,D_\tau(u\,\nu_\Sigma))\big)\overbrace{\int_\Sigma (y-x)K_u(y,x)\,d\mathcal{H}^n_y}^{=\psi(x)}=\int_\Sigma F_{\Phi_u}(y,x)\,K_u(y,x)d\mathcal{H}^n_y, \tag{4.6}$$

where $\tilde{Q}$ is a smooth function with $\tilde{Q}(x,0)=0$ for all $x\in\Sigma$, and

$$\begin{aligned} F_{\Phi_u}(y,x)=&-\frac{H_\Sigma(y)\,\nu(y)}{(n-1+s)}|\Phi_u(y)-\Phi_u(x)|^2+\big((y-x)\cdot\nu(x)\big)\,\nu(x)\\ &-(D_\tau\Phi_u(y)-D_\tau\Phi_u(x))^T(\Phi_u(y)-\Phi_u(x))\\ &-D_\tau\Phi_u(x)^T\big(\Phi_u(y)-\Phi_u(x)-D_\tau\Phi_u(x)(y-x)\big). \end{aligned} \tag{4.7}$$

Recall that we already know that K_u satisfies the conditions (i) and (ii) in Definition 4.1. The idea is then to show that F_{Φ_u} defined in (4.7) satisfies the assumptions of Lemma 4.3, which then implies that the RHS of (4.6) defines a Hölder continuous function.

In order to show (4.6) we shorten the notation by $\Phi(x)=\Phi_u(x)$ and notice that the tangential gradient of $y\mapsto|\Phi(y)-\Phi(x)|^{-n+1-s}$ is

$$\begin{aligned} D_{\tau(y)}|\Phi(y)-\Phi(x)|^{-n+1-s}&=-(n-1+s)\frac{D_\tau\Phi(y)^T(\Phi(y)-\Phi(x))}{|\Phi(y)-\Phi(x)|^{n+1+s}}\\ &=-(n-1+s)\frac{D_\tau\Phi(x)^T(\Phi(y)-\Phi(x))}{|\Phi(y)-\Phi(x)|^{n+1+s}}\\ &\quad-(n-1+s)\frac{(D_\tau\Phi(y)-D_\tau\Phi(x))^T(\Phi(y)-\Phi(x))}{|\Phi(y)-\Phi(x)|^{n+1+s}}. \end{aligned}$$

By the divergence theorem it holds

$$\int_\Sigma D_{\tau(y)}|\Phi(y)-\Phi(x)|^{-n+1-s}\,d\mathcal{H}^n_y=\int_\Sigma\frac{H_\Sigma(y)\,\nu(y)}{|\Phi(y)-\Phi(x)|^{n-1+s}}\,d\mathcal{H}^n_y.$$

Therefore the two previous equalities yield

$$\begin{aligned} D_\tau\Phi(x)^T\int_\Sigma\frac{\Phi(y)-\Phi(x)}{|\Phi(y)-\Phi(x)|^{n+1+s}}\,d\mathcal{H}^n_y=&-\frac{1}{(n-1+s)}\int_\Sigma\frac{H_\Sigma(y)\,\nu(y)}{|\Phi(y)-\Phi(x)|^{n-1+s}}\,d\mathcal{H}^n_y\\ &-\int_\Sigma\frac{(D_\tau\Phi(y)-D_\tau\Phi(x))^T(\Phi(y)-\Phi(x))}{|\Phi(y)-\Phi(x)|^{n+1+s}}\,d\mathcal{H}^n_y. \end{aligned} \tag{4.8}$$

We write the term on the left-hand-side as

$$\begin{aligned} D_\tau\Phi(x)^T\int_\Sigma\frac{\Phi(y)-\Phi(x)}{|\Phi(y)-\Phi(x)|^{n+1+s}}\,d\mathcal{H}^n_y=&(D_\tau\Phi(x)^T\,D_\tau\Phi(x))\int_\Sigma\frac{y-x}{|\Phi(y)-\Phi(x)|^{n+1+s}}\,d\mathcal{H}^n_y\\ &+D_\tau\Phi(x)^T\int_\Sigma\frac{\Phi(y)-\Phi(x)-D_\tau\Phi(x)(y-x)}{|\Phi(y)-\Phi(x)|^{n+1+s}}\,d\mathcal{H}^n_y. \end{aligned} \tag{4.9}$$

The equality (4.6) then follows from (4.8), (4.9) and from the fact that

$$D_\tau\Phi(x)^T\,D_\tau\Phi(x)=I-\nu(x)\otimes\nu(x)+\tilde{Q}(x,D_\tau(u\,\nu_\Sigma)),$$

where $\tilde{Q}(x,0)=0$ for all $x\in\Sigma$.

When $\|u\,\nu_\Sigma\|_{C^1}\le\delta$, with δ small enough, the matrix $I+\tilde{Q}(x,D_\tau(u\,\nu_\Sigma))$ is invertible. Therefore in order to show that

$$\psi(x) = \int_\Sigma (y-x) K_u(y,x)\, d\mathcal{H}^n_y$$

is Hölder continuous it is enough to show that the RHS in (4.6) is Hölder continuous. As we mentioned, we will do this by showing that F_{Φ_u} defined in (4.7) satisfies the assumptions of Lemma 4.3 with $\kappa_0 \le C$.

First, by using (4.4) and $\|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ it is straightforward to check that

$$F_1(y,x) = -\frac{1}{(n-1+s)} H_\Sigma(y)\,\nu(y)|\Phi_u(y)-\Phi_u(x)|^2 - (D_\tau\Phi_u(y) - D_\tau\Phi_u(x))^T(\Phi_u(y)-\Phi_u(x))$$

satisfies the assumptions of Lemma 4.3 with $\kappa_0 \le C$. Moreover we have that

$$|\Phi_u(y) - \Phi_u(x) - D_\tau\Phi_u(x)(y-x)| \le C|y-x|^{1+s+\alpha}.$$

Therefore, arguing as with (4.2) we deduce that

$$F_2(y,x) = -D_\tau\Phi_u(x)^T\big(\Phi_u(y)-\Phi_u(x) - D_\tau\Phi_u(x)(y-x)\big)$$

also satisfies the assumptions of Lemma 4.3 with $\kappa_0 \le C$.

We need yet to treat the term

$$F_3(y,x) = \big((y-x)\cdot\nu(x)\big)\,\nu(x).$$

Since Σ is uniformly $C^{1,1}$-regular hypersurface, there is a constant C such that $|(y-x)\cdot\nu(x)| \le C|y-x|^2$ for every $x, y \in \Sigma$. Therefore F_3 satisfies the assumption (1) of Lemma 4.3. The assumption (2) is straightforward to verify but we do this for the reader's convenience. We have

$$\begin{aligned}
|F_3(y,z) - F_3(y,x)| &= \big|(\nu(z)\otimes\nu(z))(y-z) - (\nu(x)\otimes\nu(x))(y-x)\big| \\
&\le \big|(\nu(z)\otimes\nu(z) - \nu(x)\otimes\nu(x))(y-z)\big| + \big|(\nu(x)\otimes\nu(x))((y-z)-(y-x))\big| \\
&\le \big|(\nu(z)\otimes\nu(z) - \nu(x)\otimes\nu(x))\big||y-z| + \big|\nu(x)\cdot(x-z)\big| \\
&\le C|z-x||y-z| + C|z-x|^2.
\end{aligned}$$

Note that $|y-x| \ge 2|z-x|$ implies $|y-z| \le 2|y-x|$. Therefore F_3 satisfies the assumption (2) of Lemma 4.3 and the RHS of (4.6) is Hölder continuous. This proves the claim (a).

Claim (b):

We denote $\partial_w K_u = \frac{d}{d\xi}\Big|_{\xi=0} K_{u+\xi w}$ for short. Note that from (4.3) and (4.4) it follows

$$|\partial_w K_u(y,x)| \le \frac{C\|w\,\nu_\Sigma\|_{C^1(\Sigma)}}{|y-x|^{n+1+s}}$$

for every $x, y \in \Sigma, x \ne y$. Therefore $\partial_w K_u$ satisfies the condition (i) in Definition 4.1 with $\kappa \le C\|w\,\nu_\Sigma\|_{C^1(\Sigma)}$. It is straightforward to check that $\partial_w K_u$ satisfies also the condition (ii) in Definition 4.1 with $\kappa \le C\|w\|_{C^1(\Sigma)}$.

We need thus to verify the last condition, i.e., we show that

$$\tilde\psi(x) = \int_\Sigma (y-x)\,\partial_w K_u\, d\mathcal{H}^n_y \tag{4.10}$$

satisfies $\|\tilde\psi\|_{C^\alpha(\Sigma)} \le C\|w\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$. To this aim we recall that by (4.6) for small ξ it holds

$$\big(I + \tilde Q(x, D_\tau(u\,\nu_\Sigma + \xi w\,\nu_\Sigma))\big)\int_\Sigma (y-x) K_{u+\xi w}(y,x)\, d\mathcal{H}^n_y = \int_\Sigma F_{\Phi_{u+\xi w}}(y,x)\, K_{u+\xi w}(y,x) d\mathcal{H}^n_y, \tag{4.11}$$

where $F_{\Phi_{u+\xi w}}$ is defined in (4.7).

Let us denote the RHS of (4.11) by

$$\varphi_\xi(x) := \int_\Sigma F_{\Phi_{u+\xi w}}(y,x)\, K_{u+\xi w}(y,x) d\mathcal{H}^n_y.$$

By differentiating we have

$$\frac{\partial}{\partial\xi}\Big|_{\xi=0}\varphi_\xi(x) = \underbrace{\int_\Sigma \frac{\partial}{\partial\xi}\Big|_{\xi=0} F_{\Phi_{u+\xi w}}(y,x)\, K_u(y,x)\, d\mathcal{H}^n_y}_{=:\varphi_{\xi,1}} + \underbrace{\int_\Sigma F_{\Phi_u}(y,x)\, \partial_w K_u(y,x)\, d\mathcal{H}^n_y}_{=:\varphi_{\xi,2}}.$$

Recall that $\partial_w K_u$ satisfies the conditions (i) and (ii) in Definition 4.1 with $\kappa \le C\|w\,\nu_\Sigma\|_{C^1(\Sigma)}$ and we already proved that F_{Φ_u} satisfies the assumptions (1) and (2) of Lemma 4.3 with $\kappa_0 \le C$. Lemma 4.3 then implies that

$$\|\varphi_{\xi,2}\|_{C^\alpha(\Sigma)} \le C\|w\,\nu_\Sigma\|_{C^1(\Sigma)}.$$

To treat $\varphi_{\xi,1}$ we recall that we already proved that $K_u \in \mathcal{S}_\kappa$ for $\kappa \le C$. We simplify the expression (4.7) by writing it as

$$\begin{aligned} F_{\Phi_{u+\xi w}}(y,x) = &-\frac{H_\Sigma(y)\,\nu(y)}{(n-1+s)}|\Phi_{u+\xi w}(y) - \Phi_{u+\xi w}(x)|^2 + \big((y-x)\cdot\nu(x)\big)\,\nu(x) \\ &- D_\tau\Phi_{u+\xi w}(y)^T(\Phi_{u+\xi w}(y) - \Phi_{u+\xi w}(x)) + D_\tau\Phi_{u+\xi w}(x)^T D_\tau\Phi_{u+\xi w}(x)(y-x). \end{aligned}$$

We denote $\Phi'_u(x) = \frac{\partial}{\partial\xi}\big|_{\xi=0}\Phi_{u+\xi w}(x) = w(x)\nu(x)$, differentiate the above equality and obtain

$$\begin{aligned} \frac{\partial}{\partial\xi}\Big|_{\xi=0} F_{\Phi_{u+\xi w}}(y,x) = &-\frac{2H_\Sigma(y)\,\nu(y)}{(n-1+s)}(\Phi_u(y) - \Phi_u(x))\cdot(\Phi'_u(y) - \Phi'_u(x)) \\ &- D_\tau\Phi'_u(y)^T(\Phi_u(y) - \Phi_u(x)) - D_\tau\Phi_u(y)^T(\Phi'_u(y) - \Phi'_u(x)) \\ &+ \big(D_\tau\Phi'_u(x)^T D_\tau\Phi_u(x) + D_\tau\Phi_u(x)^T D_\tau\Phi'_u(x)\big)(y-x). \end{aligned}$$

Since $\|\Phi'_u\|_{C^{1+s+\alpha}(\Sigma)}, \|D_\tau\Phi'_u\|_{C^{s+\alpha}(\Sigma)} \le C\|w\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$ we may use Lemma 4.4 to deduce

$$\|\varphi_{\xi,1}\|_{C^{1+s+\alpha}(\Sigma)} \le C\|w\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}.$$

The Hölder continuity of $\tilde\psi$, defined in (4.10), then follows from (4.11) and from the fact that

$$\|\frac{d}{d\xi}\Big|_{\xi=0}\tilde Q(x, D_\tau(u\,\nu_\Sigma + \xi w\,\nu_\Sigma))\|_{C^\alpha(\Sigma)} \le \|w\,\nu_\Sigma\|_{C^{1+\alpha}(\Sigma)}.$$

Hence we have

$$\|\tilde\psi\|_{C^\alpha(\Sigma)} \le C\|w\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}.$$

This proves the claim (b). □

Remark 4.6. It is clear that the results of Lemma 4.4 and Lemma 4.5 hold also in the case $\Sigma = \mathbb{R}^n$ if we assume that the functions v_1, v_2, v_3, u and w are in the corresponding Hölder spaces globally, i.e., $v_i \in C^{1+s+\alpha}(\mathbb{R}^n)$ for $i = 1, 2, 3$ and $\|u\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le \delta$.

We may use the previous results to prove that when $\|u\|_{C^{1+s+\alpha}}$ is small then the remainder terms $R_{1,u}$ and $R_{2,u}$ defined in (3.9) and (3.10) are small. This is stated more precisely in the following proposition. Recall that we may ignore the dependence on C^1-norm of ν_Σ, but we do however need to keep track on the dependence on higher norm of ν_Σ for later purpose.

Proposition 4.7. *Assume $u \in C^{1+s+\alpha}(\Sigma)$ is such that $\|u\|_{C^{1+s+\alpha}(\Sigma)} + \|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$, and let $R_{1,u}$ and $R_{2,u}$ be the functions defined in (3.9) and in (3.10) respectively. Then for $\delta > 0$ small enough it holds*

$$\|R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta\,\|u\|_{C^{1+s+\alpha}(\Sigma)} \qquad \textit{and} \qquad \|R_{2,u}\|_{C^\alpha(\Sigma)} \le C\delta\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}.$$

Proof. Estimate for $R_{1,u}$: Recall that

$$R_{1,u}(x) = 2\int_0^1\int_0^{t'}\int_\Sigma (u(y)-u(x))\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y d\xi dt'.$$

For later purpose we prove a slightly more general claim. Assume that u is as in the assumption, $v \in C^{1+s+\alpha}(\Sigma)$ and define

$$\varphi(x) := \int_\Sigma (v(y)-v(x))\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y. \tag{4.12}$$

We claim that it holds

$$\|\varphi\|_{C^\alpha(\Sigma)} \le C\delta\,\|v\|_{C^{1+s+\alpha}} \tag{4.13}$$

for all $\xi \in [0,1]$. The estimate for $R_{1,u}$ then follows from (4.13) by choosing $v = u$.

In order to prove (4.13) we write (4.12) as

$$\begin{aligned}\varphi(x) = &\int_\Sigma (v(y)-v(x))\big(\frac{d}{d\xi}Q_1(y,\xi u,\xi\nabla u)\big)\,K_{\xi u}(y,x)\,d\mathcal{H}^n_y\\ &+\int_\Sigma (v(y)-v(x))\,(1+Q_1(y,\xi u,\xi\nabla u))\big(\frac{d}{d\xi}K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y.\end{aligned}$$

When δ is small Lemma 4.5 yields $K_{\xi u} \in \mathcal{S}_{\kappa_1}$ with $\kappa_1 \le C$ and $\frac{d}{d\xi}K_{\xi u} \in \mathcal{S}_{\kappa_2}$ with $\kappa_2 \le C\|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\delta$ for all $\xi \in [0,1]$. We note that the functions Q_1 and Q_2 in (3.6) depend on the second fundamental form of Σ such that $\|Q_i(x,u,\nabla u)\|_{C^\beta(\Sigma)} \le C(\|u\|_{C^{1+\beta}(\Sigma)} + \|u\nu_\Sigma\|_{C^{1+\beta}(\Sigma)})$ for $\beta \in (0,1)$. Therefore it holds

$$\|1+Q_1(y,\xi u,\xi\nabla u)\|_{C^{s+\alpha}(\Sigma)} \le C \tag{4.14}$$

and we also have

$$\|\frac{d}{d\xi}Q_1(y,\xi u,\xi\nabla u)\|_{C^{s+\alpha}(\Sigma)} \le C(\|u\|_{C^{1+s+\alpha}(\Sigma)} + \|u\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}) \le C\delta \tag{4.15}$$

for all $\xi \in [0,1]$. Hence, the estimate (4.13) follows from Lemma 4.4.

Estimate for $R_{2,u}$: Recall that

$$\begin{aligned}R_{2,u}(x) := &\int_0^1\int_0^{t'}\int_\Sigma |\nu(y)-\nu(x)|^2\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y d\xi dt'\\ &+2\int_0^1\int_\Sigma (\nu(y)-\nu(x))\cdot Q_2(y,t'u,t'\nabla u)\,K_{t'u}(y,x)\,d\mathcal{H}^n_y dt'.\end{aligned}$$

We use $|\nu(y)-\nu(x)|^2 = -2\nu(x)\cdot(\nu(y)-\nu(x))$ and write the first term as

$$\begin{aligned}\psi_1(x) &= \int_\Sigma |\nu(y)-\nu(x)|^2\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y\\ &= -2\nu(x)\cdot\int_\Sigma (\nu(y)-\nu(x))\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y\end{aligned}$$

The function inside the integral is of type (4.12) with $v = \nu$ and therefore (4.13) implies

$$\|\psi_1\|_{C^\alpha(\Sigma)} \le C\delta\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}.$$

Let us fix $t' \in [0,1]$. We need yet to prove that the function

$$\psi_2(x) = \int_\Sigma (\nu(y) - \nu(x)) \cdot Q_2(y, t'u, t'\nabla u)\, K_{t'u}(y,x)\, d\mathcal{H}^n_y$$

satisfies $\|\psi_2\|_{C^\alpha(\Sigma)} \le C\delta$. To this aim we recall that we may estimate

$$\|Q_2(x, t'u, t'\nabla u)\|_{C^{\alpha+s}(\Sigma)} \le C(\|u\|_{C^{1+s+\alpha}(\Sigma)} + \|u\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}) \le C\delta.$$

Since $K_{t'u} \in \mathcal{S}_{\kappa_1}$ with $\kappa_1 \le C$, Lemma 4.4 implies $\|\psi_2\|_{C^\alpha(\Sigma)} \le C\delta\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$. □

We need similar estimate as Proposition 4.7 for the linearization of the remainder terms $R_{1,u}$ and $R_{2,u}$.

Proposition 4.8. *Assume $u, w \in C^{1+s+\alpha}(\Sigma)$ and $\|u\|_{C^{1+s+\alpha}(\Sigma)} + \|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$. Let $R_{1,u+\xi w}$ and $R_{2,u+\xi w}$ be functions defined in (3.9) and in (3.10) respectively and define*

$$\partial_w R_{1,u}(x) = \frac{d}{d\eta}\Big|_{\eta=0} R_{1,u+\eta w}(x) \qquad \text{and} \qquad \partial_w R_{2,u}(x) = \frac{d}{d\eta}\Big|_{\eta=0} R_{2,u+\eta w}(x).$$

Then for δ small it holds

$$\|\partial_w R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}$$

and

$$\|\partial_w R_{2,u}\|_{C^\alpha(\Sigma)} \le C\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}.$$

Here C is a uniform constant while C_Σ depends on $\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$.

Proof. This time we prove the claim only for $\partial_w R_{1,u}$ since the argument for $\partial_w R_{2,u}$ is similar. We differentiate $R_{1,u+\eta w}$, defined in (3.9), and obtain

$$\begin{aligned}
\partial_w R_{1,u}(x) &= \frac{d}{d\eta}\Big|_{\eta=0} R_{1,u+\eta w}(x) \\
&= 2\int_0^1\int_0^{t'}\int_\Sigma (w(y)-w(x))\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\, d\mathcal{H}^n_y d\xi\, dt' \\
&\quad + 2\int_0^1\int_\Sigma (u(y)-u(x))\frac{d}{d\eta}\Big|_{\eta=0} Q_1(y, t'(u+\eta w), t'\nabla(u+\eta w))\, K_{t'u}(y,x)\, d\mathcal{H}^n_y dt' \\
&\quad + 2\int_0^1\int_\Sigma (u(y)-u(x))(1+Q_1(y,t'u,t'\nabla u))\left(\frac{d}{d\eta}\Big|_{\eta=0} K_{t'u+t'\eta w}(y,x)\right) d\mathcal{H}^n_y dt'
\end{aligned} \tag{4.16}$$

Note that the first term is of type (4.12) with $v = w$ and therefore (4.13) implies that it is Hölder continuous and its C^α-norm is bounded by $C\delta\|w\|_{C^{1+s+\alpha}(\Sigma)}$. Concerning the two last terms in (4.16), note first that Lemma 4.5 (b) yields $\frac{d}{d\eta}\Big|_{\eta=0} K_{t'u+t'\eta w} \in \mathcal{S}_{\kappa_2}$ with $\kappa_2 \le C\|w\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$. By the interpolation inequality in Lemma 2.1 we may estimate

$$\|w\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}.$$

Moreover, we have by (4.14)

$$\|1 + Q_1(y, t'u, t'\nabla u)\|_{C^{s+\alpha}(\Sigma)} \le C$$

and as in (4.15) we have

$$\|\frac{d}{d\eta}\Big|_{\eta=0} Q_1(y,t'(u+\eta w),t'\nabla(u+\eta w))\|_{C^{s+\alpha}(\Sigma)} \le C(\|w\|_{C^{1+s+\alpha}(\Sigma)} + \|w\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)})$$
$$\le C\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}.$$

Thus we deduce by Lemma 4.4 that the two last terms in (4.16) are Hölder continuous with C^α-norms bounded by $C\delta\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}$. Hence, we have

$$\|\partial_w R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}. \quad \square$$

At the end of the section we study how to control the higher order norms of $R_{1,u}$ and $R_{2,u}$ in order to differentiate the equation (3.12) with respect to x. Moreover, even if the fractional Laplacian is linear it is not obvious how to bound its higher order covariant derivatives. Before that we remark on how we write the derivative of the function

$$\psi(x) = \int_\Sigma G(y,x)\,d\mathcal{H}^n_y$$

with respect to a vector field $X \in \mathcal{T}(\Sigma)$. First it holds

$$\nabla_X\psi(x) = \int_\Sigma \nabla_{X(x)}G(y,x)\,d\mathcal{H}^n_y,$$

where $\nabla_{X(x)}G(y,x)$ denotes the derivative of $G(y,\cdot)$ with respect to X. On the other hand it holds

$$\mathrm{div}_y(G(y,x)X(y)) = \nabla_{X(y)}G(y,x) + G(y,x)\,\mathrm{div}(X)(y).$$

Therefore we have by the divergence theorem that

$$\nabla_X\psi(x) = \int_\Sigma (\nabla_{X(y)} + \nabla_{X(x)})G(y,x)\,d\mathcal{H}^n_y + \int_\Sigma G(y,x)\,\mathrm{div}(X)(y)\,d\mathcal{H}^n_y, \tag{4.17}$$

where

$$(\nabla_{X(y)} + \nabla_{X(x)})G(y,x) = \nabla_{X(y)}G(y,x) + \nabla_{X(x)}G(y,x).$$

The following proposition gives us a formula to commute differentiation and the fractional Laplacian. In the following, and in the rest of the paper, C_k denotes a constant which depends on k and on Σ in an unspecified way while C denotes a uniform constant.

Proposition 4.9. *Let $X_1,\dots,X_k \in \mathcal{T}(\Sigma)$ be vector fields such that $\|X_i\|_{C^{k+2}(\Sigma)} \le 1$ for $i = 1,\dots,k$ and assume $u \in C^\infty(\Sigma)$. Then*

$$\nabla_{X_k}\cdots\nabla_{X_1}\big(\Delta^{\frac{1+s}{2}}u\big) = \Delta^{\frac{1+s}{2}}\big(\nabla_{X_k}\cdots\nabla_{X_1}u\big) + \partial^{k+s}u,$$

where $\partial^{k+s}u$ denotes a function which satisfies $\|\partial^{k+s}u\|_{C^\alpha(\Sigma)} \le C_k\|u\|_{C^{k+s+\alpha}(\Sigma)}$. Moreover, it holds

$$\|\Delta^{\frac{1+s}{2}}u\|_{C^{k+\alpha}(\Sigma)} \le C_k\|u\|_{C^{k+1+s+\alpha}(\Sigma)}$$

for every $k \in \mathbb{N}$.

Proof. Let us denote $K(y,x) = |y-x|^{-n-1-s}$ and let $X_1,\dots,X_k \in \mathcal{T}(\Sigma)$ be as in the assumption. We define $\partial_1 K(y,x) := \nabla_{X_1(y)}K(y,x) + \nabla_{X_1(x)}K(y,x)$ and $\partial_j K(y,x)$, for $2 \le j \le k$, recursively as

$$\partial_j K(y,x) := \nabla_{X_j(y)}\partial_{j-1}K(y,x) + \nabla_{X_j(x)}\partial_{j-1}K(y,x).$$

We begin by claiming that $\partial_k K(y,x)$ satisfies the conditions (i)-(iii) in Definition 4.1 with $\kappa \le C_k$, i.e., $\partial_k K(y,x) \in \mathcal{S}_{C_k}$. Note that the constant does not depend on the chosen vector fields $X_1,\dots,X_k$ once they satisfy $\|X_i\|_{C^{k+2}(\Sigma)} \le 1$.

It is straightforward to check that $\partial_k K(y,x)$ satisfies the conditions (i) and (ii) in the Definition 4.1 with $\kappa \le C_k$ for some C_k. We need thus to prove the condition (iii). We prove this by induction and fix $X \in \mathcal{T}(\Sigma)$ such that $\|X\|_{C^3(\Sigma)} \le 1$. We need first to show that the function

$$\psi_1(x) = \int_\Sigma (y-x)\partial_1 K(y,x)\, d\mathcal{H}^n_y \tag{4.18}$$

is α-Hölder continuous.

The formula (4.6) in the case $u = 0$ reads as

$$\int_\Sigma (y-x)K(y,x)\, d\mathcal{H}^n_y = \int_\Sigma F(y,x)K(y,x)\, d\mathcal{H}^n_y, \tag{4.19}$$

where

$$F(y,x) = -\frac{H_\Sigma(y)\,\nu(y)}{(n-1+s)}|y-x|^2 + \big((y-x)\cdot\nu(y)\big)\,\nu(y).$$

We differentiate (4.19) with respect to $X(x)$ and obtain by (4.17)

$$\int_\Sigma (y-x)\partial_1 K(y,x)\, d\mathcal{H}^n_y = \underbrace{\int_\Sigma F(y,x)\partial_1 K(y,x)\, d\mathcal{H}^n_y}_{=:\tilde\psi_1} + \underbrace{\int_\Sigma F_1(y,x)K(y,x)\, d\mathcal{H}^n_y}_{=:\varphi_1},$$

where

$$F_1(y,x) = (\nabla_{X(y)} + \nabla_{X(x)})(F(y,x) - (y-x)) + \operatorname{div} X(y)(F(y,x) - (y-x)).$$

First, recall that in the proof of Lemma 4.5 we already verified that F satisfies the assumptions of Lemma 4.3. Since $\partial_1 K(y,x)$ satisfies the conditions (i) and (ii) in Definition 4.1 with $\kappa \le C_1$, Lemma 4.3 yields $\|\tilde\psi_1\|_{C^\alpha(\Sigma)} \le C_1$. Second, we may write F_1 as

$$F_1(y,x) = \sum_{j=1}^{N_1} (v_{1,i}(y) - v_{1,i}(x))v_{2,i}(y)v_{3,i}(x),$$

where $v_{j,i}$ are such that $\|v_{j,i}\|_{C^2(\Sigma)} \le C_1$. Moreover, by Lemma 4.5 it holds $K(y,x) \in \mathcal{S}_\kappa$ with $\kappa \le C$ and we may thus use Lemma 4.4 to conclude that $\|\varphi_1\|_{C^\alpha(\Sigma)} \le C_1$. Hence

$$\|\psi\|_{C^\alpha(\Sigma)} \le C_1$$

and therefore $\partial_1 K(y,x) \in \mathcal{S}_\kappa$ with $\kappa \le C_1$.

We argue by induction and assume that $\partial_{k-1} K(y,x) \in \mathcal{S}_\kappa$ with $\kappa \le C_{k-1}$. Note that this holds for any vector fields $X_1, \dots, X_{k-1}$ with $\|X_i\|_{C^{k+1}(\Sigma)} \le 1$. Let us fix $X_1, \dots, X_k$ as in the assumption. We differentiate (4.19) with respect to $X_1, \dots, X_k$ and obtain by (4.17)

$$\int_\Sigma (y-x)\partial_k K(y,x)\, d\mathcal{H}^n_y = \underbrace{\int_\Sigma F(y,x)\partial_k K(y,x)\, d\mathcal{H}^n_y}_{=:\tilde\psi_k} + \underbrace{\sum_{i=0}^{k-1}\int_\Sigma \tilde F_i(y,x)\partial_i K(y,x)\, d\mathcal{H}^n_y}_{=:\varphi_k}.$$

Here $\tilde F_i$ can be written as

$$\tilde F_i(y,x) = \sum_{j=1}^{N_k} (v_{1,i}(y) - v_{1,i}(x))v_{2,i}(y)v_{3,i}(x),$$

where $v_{j,i}$ are such that $\|v_{j,i}\|_{C^2(\Sigma)} \le C_k$. Again we recall that F satisfies the assumptions of Lemma 4.3 and $\partial_k K(y,x)$ satisfies the conditions (i) and (ii) in Definition 4.1 for $\kappa \le C_k$. Lemma 4.3 then yields $\|\tilde\psi_k\|_{C^\alpha(\Sigma)} \le C_k$. To prove the Hölder continuity of φ_k we recall that by induction assumption $\partial_j K(y,x) \in \mathcal{S}_\kappa$ with $\kappa \le C_{k-1}$ for every $j \le k-1$. We may thus use Lemma 4.4 to deduce $\|\varphi_k\|_{C^\alpha(\Sigma)} \le C_k$. Therefore we conclude that

$$\psi_k(x) = \int_\Sigma (y-x)\partial_k K(y,x)\, d\mathcal{H}^n_y$$

is Hölder continuous with $\|\psi_k\|_{C^\alpha(\Sigma)} \le C_k$ and thus $\partial_k K(y,x)$ satisfies the condition (iii) in Definition 4.1 with $\kappa \le C_k$.

We prove the claim by first choosing $\tilde{X}_1, \dots, \tilde{X}_l$, with $1 \le l \le k$, such that $\|\tilde{X}_i\|_{C^{k+2}(\Sigma)} \le 1$ for $1 \le i \le l$. Recall that the function $\tilde{X}_l \cdots \tilde{X}_1 u$ is defined recursively as $\tilde{X}_1 u = \nabla_{\tilde{X}_1} u$ and $\tilde{X}_{j+1}\tilde{X}_j \cdots \tilde{X}_1 u = \nabla_{\tilde{X}_{j+1}}(\tilde{X}_j \cdots \tilde{X}_1 u)$ for $j \ge 1$. We apply (4.17) l times for $\tilde{X}_1, \dots, \tilde{X}_l$ and obtain

$$\begin{aligned}\tilde{X}_l \cdots \tilde{X}_1 \Delta^{\frac{1+s}{2}} u(x) &= 2\int_\Sigma (\tilde{X}_l \cdots \tilde{X}_1 u(y) - \tilde{X}_l \cdots \tilde{X}_1 u(x)) K(y,x)\, d\mathcal{H}^n_y \\ &\quad + 2\int_\Sigma (u(y)-u(x))\partial_l K(y,x)\, d\mathcal{H}^n_y + \sum_{j=0}^{l-1}\int_\Sigma (\partial^{l-1-j}u(y) - \partial^{l-1-j}u(x)) v_j(y)\partial_j K(y,x)\, d\mathcal{H}^n_y,\end{aligned}$$

where $\partial^{l-1-j}u$ denotes a function which satisfies $\|\partial^{l-1-j}u\|_{C^{1+s+\alpha}(\Sigma)} \le C_{l-1}\|u\|_{C^{l-j+s+\alpha}(\Sigma)}$ and v_j are such that $\|v_j\|_{C^2(\Sigma)} \le C_l$. By using Lemma 4.4 and $\partial_j K(y,x) \in \mathcal{S}_\kappa$ with $\kappa \le C_j$ we deduce

$$\tilde{X}_l \cdots \tilde{X}_1 (\Delta^{\frac{1+s}{2}} u) = \Delta^{\frac{1+s}{2}}(\tilde{X}_l \cdots \tilde{X}_1 u) + \partial^{l+s} u, \tag{4.20}$$

where $\partial^{l+s}u$ denotes a function which satisfies $\|\partial^{l+s}u\|_{C^\alpha(\Sigma)} \le C_l\|u\|_{C^{l+s+\alpha}(\Sigma)}$. Hence, we deduce by (2.4), Lemma 4.4 and (4.20) that

$$\|\Delta^{\frac{1+s}{2}} u\|_{C^{l+\alpha}(\Sigma)} \le C_l\big(\|u\|_{C^{l+1+s+\alpha}(\Sigma)} + \|\Delta^{\frac{1+s}{2}} u\|_{C^{l-1+\alpha}(\Sigma)}\big)$$

for every $l \le k$. Note that Lemma 4.4 implies $\|\Delta^{\frac{1+s}{2}} u\|_{C^\alpha(\Sigma)} \le C\|u\|_{C^{1+s+\alpha}(\Sigma)}$. Therefore by iterating the above inequality for $l = 1, \dots, k$ we obtain

$$\|\Delta^{\frac{1+s}{2}} u\|_{C^{k+\alpha}(\Sigma)} \le C_k\|u\|_{C^{k+1+s+\alpha}(\Sigma)}. \tag{4.21}$$

This implies the second statement.

Let $X_1, \dots, X_k$ be as in the assumption. We deduce from (4.20) that

$$X_k \cdots X_1(\Delta^{\frac{1+s}{2}} u) = \Delta^{\frac{1+s}{2}}(X_k \cdots X_1 u) + \partial^{k+s} u,$$

where $\partial^{k+s}u$ denotes a function which satisfies $\|\partial^{k+s}u\|_{C^\alpha(\Sigma)} \le C_k\|u\|_{C^{k+s+\alpha}(\Sigma)}$. By (2.3) we have

$$\nabla_{X_k} \cdots \nabla_{X_1}(\Delta^{\frac{1+s}{2}} u) = \Delta^{\frac{1+s}{2}}(\nabla_{X_k} \cdots \nabla_{X_1} u) + \partial^{k-1}(\Delta^{\frac{1+s}{2}} u) + \partial^{k+s} u,$$

where $\partial^{k-1}(\Delta^{\frac{1+s}{2}} u)$ denotes a function which satisfies

$$\|\partial^{k-1}(\Delta^{\frac{1+s}{2}} u)\|_{C^\alpha(\Sigma)} \le C_k\|\Delta^{\frac{1+s}{2}} u\|_{C^{k-1+\alpha}(\Sigma)}.$$

The estimate (4.21) applied to $(k-1)$ yields

$$\|\partial^{k-1}(\Delta^{\frac{1+s}{2}} u)\|_{C^\alpha(\Sigma)} \le C_k\|u\|_{C^{k+s+\alpha}(\Sigma)}$$

and the claim follows. □

Similar result holds for the remainder terms R_1 and R_2.

Lemma 4.10. *Assume $u \in C^\infty(\Sigma)$ with $\|u\|_{C^{1+s+\alpha}(\Sigma)} + \|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ and let $R_{1,u}$ and $R_{2,u}$ be the functions defined in (3.9) and in (3.10) respectively. Let $X_1, \dots, X_k \in \mathcal{T}(\Sigma)$ be vector fields with $\|X_i\|_{C^{k+2}(\Sigma)} \le 1$ for $i = 1, \dots, k$. There is a constant C_k, which depends on k and Σ, such that for $\delta > 0$ small enough it holds*

$$\|\nabla_{X_k} \cdots \nabla_{X_1} R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta\|\nabla_{X_k} \cdots \nabla_{X_1} u\|_{C^{1+s+\alpha}(\Sigma)} + C_k(1 + \|u\|^k_{C^{k+s+\alpha}(\Sigma)})$$

and

$$\|\nabla_{X_k}\cdots\nabla_{X_1}R_{2,u}\|_{C^\alpha(\Sigma)}\le C\|\nu\|_{C^{1+s+\alpha}(\Sigma)}\|\nabla_{X_k}\cdots\nabla_{X_1}u\|_{C^{1+s+\alpha}(\Sigma)}+C_k(1+\|u\|^k_{C^{k+s+\alpha}(\Sigma)}).$$

In particular, it holds

$$\|R_{1,u}\|_{C^{k+\alpha}(\Sigma)}+\|R_{2,u}\|_{C^{k+\alpha}(\Sigma)}\le C_k(1+\|u\|^k_{C^{k+1+s+\alpha}(\Sigma)})$$

for every $k\in\mathbb{N}$.

Proof. Since the proof is similar to the proof of Proposition 4.9 we only sketch it. In addition we only prove the claim for $R_{1,u}$ as the estimate for $R_{2,u}$ follows from a similar argument. Let $u\in C^\infty(\Sigma)$ be as in the assumption and let $K_u(y,x)$ be as defined in (3.7). As in the proof of the previous proposition we define $\partial_j K_u(y,x)$, for $1\le j\le k$, by $\partial_1 K_u(y,x)=\nabla_{X_1(y)}K_u(y,x)+\nabla_{X_1(x)}K_u(y,x)$ and for $j\ge 2$ recursively as

$$\partial_j K_u(y,x):=\nabla_{X_j(y)}\partial_{j-1}K_u(y,x)+\nabla_{X_j(x)}\partial_{j-1}K_u(y,x).$$

We claim that $\partial_k K_u(y,x)$ satisfies the conditions (i)-(iii) in Definition 4.1 with

$$\kappa_k\le C\|X_k\cdots X_1u\|_{C^{1+s+\alpha}(\Sigma)}+C_k(1+\|u\|^k_{C^{k+s+\alpha}(\Sigma)}) \tag{4.22}$$

for some C_k and C, where the latter is independent of k. Moreover, the constants in (4.22) do not depend on the chosen vector fields $X_1,\dots,X_k$. The argument for (4.22) is similar to the one in the beginning of Proposition 4.9 and thus we omit it.

To prove the claim we recall the definition of $R_{1,u}$ in (3.9). As in Proposition 4.9 we first choose $\tilde X_1,\dots,\tilde X_l$, with $1\le l\le k$, such that $\|\tilde X_i\|_{C^{l+2}(\Sigma)}\le 1$ for $1\le i\le l$. We apply (4.17) l times for $\tilde X_1,\dots,\tilde X_l$ and obtain

$$\begin{aligned}
&\tilde X_l\cdots\tilde X_1R_{1,u}(x)\\
&=2\int_0^1\int_0^{t'}\int_\Sigma(\tilde X_l\cdots\tilde X_1u(y)-\tilde X_l\cdots\tilde X_1u(x))\frac{d}{d\xi}\big((1+Q_1(y,\xi u,\xi\nabla u))K_{\xi u}(y,x)\big)\,d\mathcal{H}^n_y d\xi dt'\\
&+2\int_0^1\int_\Sigma(u(y)-u(x))(\tilde X_l\cdots\tilde X_1\,Q_1(y,t'u,t'\nabla u))K_{t'u}(y,x)\,d\mathcal{H}^n_y dt'\\
&+2\int_0^1\int_\Sigma(u(y)-u(x))\big((1+Q_1(y,t'u,t'\nabla u))\partial_l K_{t'u}(y,x)-\partial_l K(y,x)\big)\,d\mathcal{H}^n_y dt'\\
&+\sum_{\substack{i,j,m\le l-1\\ i+j+m=l}}\int_0^1\int_\Sigma(\partial^i u(y)-\partial^i u(x))\,\partial^j Q_1(y,t'u,t'\nabla u)\,v_j(y)\,\big(\partial_m K_{t'u}(y,x)-\partial_m K(y,x)\big)\,d\mathcal{H}^n_y dt'\\
&=\varphi_1+\varphi_2+\varphi_3+f
\end{aligned}\tag{4.23}$$

where $\partial^j w$ denotes a function which satisfies $\|\partial^j w\|_{C^{1+s+\alpha}(\Sigma)}\le C\|w\|_{C^{j+1+s+\alpha}(\Sigma)}$ and v_j are such that $\|v_j\|_{C^2(\Sigma)}\le C_l$. Here φ_1 denotes the fuction on the first row in (4.23), φ_2 the function on the second row etc. The function φ_1 is of type (4.12), with $v=\tilde X_l\cdots\tilde X_1u$ and therefore (4.13) implies

$$\|\varphi_1\|_{C^\alpha(\Sigma)}\le C\delta\|\tilde X_l\cdots\tilde X_1u\|_{C^{1+s+\alpha}(\Sigma)}.$$

On the other hand Lemma 4.4, the assumption $\|u\|_{C^{1+s+\alpha}(\Sigma)}\le\delta$, (2.1) and (2.3) imply

$$\begin{aligned}
\|\varphi_2\|_{C^\alpha(\Sigma)}&\le C\|u\|_{C^{1+s+\alpha}(\Sigma)}\|\tilde X_l\cdots\tilde X_1Q_1(x,t'u,t'\nabla u)\|_{C^{s+\alpha}(\Sigma)}\\
&\le C\delta\|\tilde X_l\cdots\tilde X_1u\|_{C^{1+s+\alpha}(\Sigma)}+C_l(1+\|u\|^l_{C^{l+s+\alpha}(\Sigma)}).
\end{aligned}$$

Since $\partial_l K_{t'u}(y,x)$ belongs to the class $\mathcal{S}_{\kappa_l}$, with κ_l given by (4.22), we conclude by Lemma 4.4 that

$$\|\varphi_3\|_{C^\alpha(\Sigma)}\le C\delta\|\tilde X_l\cdots\tilde X_1u\|_{C^{1+s+\alpha}(\Sigma)}+C_l(1+\|u\|^l_{C^{l+s+\alpha}(\Sigma)}).$$

Similarly we deduce that $\|f\|_{C^\alpha(\Sigma)} \le C_l(1+\|u\|^l_{C^{l+s+\alpha}(\Sigma)})$. Combining the previous inequalities with (4.23) yields

$$\|\tilde{X}_l \cdots \tilde{X}_1 R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta \|\tilde{X}_l \cdots \tilde{X}_1 u\|_{C^{1+s+\alpha}(\Sigma)} + C_l(1+\|u\|^l_{C^{l+s+\alpha}(\Sigma)}) \tag{4.24}$$

We deduce by (2.4) and (4.24) that

$$\|R_{1,u}\|_{C^{l+\alpha}(\Sigma)} \le C_l\big(1+\|u\|^l_{C^{l+1+s+\alpha}(\Sigma)} + \|R_{1,u}\|_{C^{l-1+\alpha}(\Sigma)}\big)$$

for every $l \le k$. Recall that by Proposition 4.7 we have $\|R_{1,u}u\|_{C^\alpha(\Sigma)} \le C\delta\|u\|_{C^{1+s+\alpha}(\Sigma)}$. Therefore by using the above inequality k times for $l = 1, \dots, k$ we obtain

$$\|R_{1,u}\|_{C^{k+\alpha}(\Sigma)} \le C_k(1+\|u\|^k_{C^{k+1+s+\alpha}(\Sigma)}). \tag{4.25}$$

This proves the second claim.

Let $X_1, \dots, X_k$ be as in the assumption. We deduce from (4.24) that

$$\|X_k \cdots X_1 R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta \|X_k \cdots X_1 u\|_{C^{1+s+\alpha}(\Sigma)} + C_k(1+\|u\|^k_{C^{k+s+\alpha}(\Sigma)}).$$

Then (2.3) implies

$$\|\nabla_{X_k} \cdots \nabla_{X_1} R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta \|\nabla_{X_k} \cdots \nabla_{X_1} u\|_{C^{1+s+\alpha}(\Sigma)} + C_k(1+\|u\|^k_{C^{k+s+\alpha}(\Sigma)} + \|R_{1,u}\|_{C^{k-1+\alpha}(\Sigma)}).$$

The claim follows from (4.25) with $(k-1)$. □

5. Proof of the Main Theorem

We will first prove the main theorem for the flow (1.1) and explain at the end of the section how the proof can be applied to deal with the volume preserving case (1.3). By Proposition 3.2 we need to prove that the equation (3.12) has a unique solution $h \in C(\Sigma \times [0,T]) \cap C^\infty(\Sigma \times (0,T])$ with $h(x,0) = h_0(x)$ for $x \in \Sigma$.

Suppose that δ is small such that the results in Section 4 hold. Recall that by the discussion at the beginning of Section 3 we may choose Σ in such a way that we have

$$\|h_0\|_{C^0(\Sigma)} < \varepsilon/2, \quad \|h_0\|_{C^{1+s+\alpha}(\Sigma)} < (2C)^{-1}\delta \quad \text{and} \quad \|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\varepsilon^{-s-\alpha}, \tag{5.1}$$

where $\varepsilon \in (0,\delta)$ and C will be chosen later. Here is the statement of the main theorem for (1.1).

Theorem 5.1 ***(Main Theorem)***. *Let $0 < \alpha < (1-s)/2$. Assume $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface and $h_0 : \Sigma \to \mathbb{R}$ is such that (5.1) holds. For δ and ε small enough, there is $T \in (0,1)$, depending on δ and ε, such that the equation (3.12) has a unique classical solution $h \in C(\Sigma \times [0,T]) \cap C^\infty(\Sigma \times (0,T])$ with initial value $h(x,0) = h_0(x)$ for all $x \in \Sigma$. Moreover, it holds*

$$\sup_{0<t<T} \|h(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$$

and for every $k \in \mathbb{N}$ there is a constant Λ_k such that

$$\sup_{0<t<T} \big(t^{k!}\|h(\cdot,t)\|_{C^k(\Sigma)}\big) \le \Lambda_k.$$

Note that Theorem 5.1 implies that the solution of (3.12) exists as long as its $C^{1+s+\alpha}$-norm stays small. This means that the fractional mean curvature flow has a smooth solution as long as it stays $C^{1+s+\alpha}$-close to the initial set. We also remark that the exponent $k!$ in the final statement is not optimal and we expect the optimal exponent to be linear in k. However, the most important consequence of the last inequality is that it quantifies the smoothness of $h(\cdot,t)$ for every $t \in (0,T]$.

Proof. Step 1: (*Set-up and basic estimates.*)

Let us write the equation (3.12) as

$$\partial_t h = L[h] + P(x,h,\nabla h) - H^s_\Sigma(x), \tag{5.2}$$

where the remainder term is defined for a generic function $u \in C^\infty(\Sigma)$ as

$$P(x,u,\nabla u) = Q(x,u,\nabla u)\big(L[u] - H^s_\Sigma(x)\big) + (1 + Q(x,u,\nabla u))\big(R_{1,u}(x) + R_{2,u}(x)\,u\big). \tag{5.3}$$

Recall that Q is a smooth function with $Q(x,0,0) = 0$ for all $x \in \Sigma$, and $R_{1,u}$ and $R_{2,u}$ are defined in (3.9) and (3.10) respectively.

Let us first fix a small $\delta > 0$ for which the results in Section 4 hold. Let us assume that

$$\|u\|_{C^{1+s+\alpha}(\Sigma)} \le \delta \quad \text{and} \quad \|u\|_{C^0(\Sigma)} \le \varepsilon$$

and prove that this implies

$$\|P(x,u,\nabla u)\|_{C^\alpha(\Sigma)} \le C\delta^2 + C_\delta \varepsilon, \tag{5.4}$$

when ε is small enough. Here C_δ depends on δ.

First, we have by the assumption (5.1) that $\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\varepsilon^{-s-\alpha}$. Therefore $\|u\|_{C^0(\Sigma)} \le \varepsilon$ and (3.1) applied to u imply

$$\|u\|_{C^\alpha(\Sigma)}\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\varepsilon^{1-s-2\alpha} \le \delta \tag{5.5}$$

when ε is small. In particular, these imply $\|u\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\delta$. It follows from Proposition 4.7 that

$$\|R_{1,u}\|_{C^\alpha(\Sigma)} \le C\delta\,\|u\|_{C^{1+s+\alpha}(\Sigma)} \qquad \text{and} \qquad \|R_{2,u}\|_{C^\alpha(\Sigma)} \le C\delta\|\nu\|_{C^{1+s+\alpha}(\Sigma)}.$$

The latter inequality and (5.5) yield

$$\|R_{2,u}\,u\|_{C^\alpha(\Sigma)} \le C\delta^2.$$

Similarly it follows from Lemma 4.4 that $\|L[u]\|_{C^\alpha(\Sigma)} \le C\delta$. Moreover, we may estimate as with (4.14) that

$$\|Q(x,u,\nabla u)\|_{C^\alpha(\Sigma)} \le C(\|u\|_{C^{1+\alpha}(\Sigma)} + \|u\nu_\Sigma\|_{C^{1+\alpha}(\Sigma)}). \tag{5.6}$$

By the interpolation inequality in Lemma 2.1 we estimate

$$\|u\|_{C^{1+\alpha}(\Sigma)} + \|u\nu_\Sigma\|_{C^{1+\alpha}(\Sigma)} \le \delta(\|u\|_{C^{1+s+\alpha}(\Sigma)} + \|u\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}) + C_\delta\|u\|_{C^0(\Sigma)}.$$

Hence, we have (5.4) by (5.5) and by the fact that $\|H^s_\Sigma\|_{C^\alpha(\Sigma)}$ is uniformly bounded for $\alpha < \frac{1-s}{2}$.

We will also 'linearize' the equation (5.3). To this aim we prove that if $v_1, v_2 \in C^{1+s+\alpha}(\Sigma)$ are such that $\|v_i\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ and $\|v_i\|_{C^0(\Sigma)} \le \varepsilon$, for $i = 1,2$, then it holds

$$\|P(x,v_2,\nabla v_2) - P(x,v_1,\nabla v_1)\|_{C^\alpha(\Sigma)} \le C\delta\|v_2 - v_1\|_{C^{1+s+\alpha}(\Sigma)} + C_{\Sigma,\delta}\|v_2 - v_1\|_{C^0(\Sigma)}, \tag{5.7}$$

when ε is small enough. Here $C_{\Sigma,\delta}$ depends on δ and on $\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}$.

Indeed, we denote $w = v_2 - v_1$ and write

$$P(x,v_2,\nabla v_2) - P(x,v_1,\nabla v_1) = \int_0^1 \frac{d}{d\xi} P\big(x, v_1 + \xi w, \nabla(v_1 + \xi w)\big)\,d\xi.$$

Denote further $v_\xi = v_1 + \xi w$ and recall that (5.5) holds also for v_ξ. In particular, it holds $\|v_\xi\,\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \le C\delta$. By recalling the definition of P in (5.3) we obtain by differentiating

$$\begin{aligned}
&\frac{d}{d\xi} P(x, v_1 + \xi w, \nabla(v_1 + \xi w)) \\
&\quad = \Big(\frac{d}{d\xi} Q(x, v_\xi, \nabla v_\xi)\Big)\big(L[v_\xi] - H^s_\Sigma(x) + R_{1,v_\xi}(x) + R_{2,v_\xi}(x)\,v_\xi\big) \\
&\qquad + Q(x, v_\xi, \nabla v_\xi)\,L[w] + (1 + Q(x, v_\xi, \nabla v_\xi))\left(\frac{d}{d\xi} R_{1,v_\xi} + \frac{d}{d\xi} R_{2,v_\xi}\,v_\xi + R_{2,v_\xi}\,w\right)
\end{aligned}$$

It follows from Proposition 4.7 that

$$\|R_{1,v_\xi}\|_{C^\alpha(\Sigma)} \le C\delta^2 \qquad \text{and} \qquad \|R_{2,v_\xi}\|_{C^\alpha(\Sigma)} \le C\delta\|\nu\|_{C^{1+s+\alpha}(\Sigma)}$$

for all $\xi \in (0,1)$. Moreover, we have by Proposition 4.8 that

$$\|\frac{d}{d\xi}R_{1,v_\xi}\|_{C^\alpha(\Sigma)} \le C\delta\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}$$

and

$$\|\frac{d}{d\xi}R_{2,v_\xi}\|_{C^\alpha(\Sigma)} \le C\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}.$$

Note that the latter inequality and (5.5) yield

$$\|\frac{d}{d\xi}R_{2,v_\xi}\, v_\xi\|_{C^\alpha(\Sigma)} \le C\delta\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}.$$

Lemma 4.4 implies

$$\|L[w]\|_{C^\alpha(\Sigma)} \le C\|w\|_{C^{1+s+\alpha}(\Sigma)}$$

and

$$\|L[v_\xi]\|_{C^\alpha(\Sigma)} \le C\|v_\xi\|_{C^{1+s+\alpha}(\Sigma)} \le C\delta.$$

Finally we have by (5.5) and (5.6) that

$$\|Q(x, v_\xi, \nabla v_\xi)\|_{C^\alpha(\Sigma)} \le C(\|v_\xi\|_{C^{1+\alpha}(\Sigma)} + \|v_\xi\nu_\Sigma\|_{C^{1+\alpha}(\Sigma)}) \le C\delta$$

and since Q is smooth we have

$$\|\frac{d}{d\xi}Q(x, v_\xi, \nabla v_\xi)\|_{C^\alpha(\Sigma)} \le C_\Sigma\|w\|_{C^{1+\alpha}(\Sigma)}.$$

By combining the previous estimates we obtain

$$\begin{aligned}
&\|P(x, v_2, \nabla v_2) - P(x, v_1, \nabla v_1)\|_{C^\alpha(\Sigma)} \\
&\qquad \le \int_0^1 \|\frac{d}{d\xi}P\big(x, v_1 + \xi w, \nabla v_1 + \xi\nabla w\big)\|_{C^\alpha(\Sigma)}\, d\xi \\
&\qquad \le C\delta\|w\|_{C^{1+s+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^{1+\alpha}(\Sigma)} + C_\Sigma\|w\|_{C^0(\Sigma)}.
\end{aligned}$$

The inequality (5.7) then follows from the interpolation inequality in Lemma 2.1.

Step 2: (*Existence and Uniqueness of the strong solution.*)

We define X as the space of function $u \in C(\Sigma \times [0,T])$ such that $u \in X$ if

$$\sup_{0<t<T}\|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le \delta, \qquad \sup_{0<t<T}\|u(\cdot,t)\|_{C^0(\Sigma)} \le \varepsilon, \qquad \sup_{0<t<T}\|\partial_t u(\cdot,t)\|_{C^\alpha(\Sigma)} \le C$$

and $h(x,0) = h_0(x)$ for all $x \in \Sigma$. We choose $\delta > 0$ so small that the results in Section 4 hold and $\varepsilon > 0$ even smaller if necessary and assume that h_0 satisfies (5.1). Finally C is a large constant which we choose later.

We define a map $\mathcal{L} : X \to X$ such that for a given $h \in X$ the value $\mathcal{L}[h] := u$ is the solution of the following linear equation with a forcing term

$$\begin{cases} \partial_t u - \Delta^{\frac{1+s}{2}} u = c_s^2(x)h(\cdot,t) + P(x, h(\cdot,t), \nabla h(\cdot,t)) - H_\Sigma^s(x) \\ u(x,0) = h_0. \end{cases} \tag{5.8}$$

Recall that by definition (2.5) $L[u] = \Delta^{\frac{1+s}{2}}u + c_s^2(x)u$. Therefore a fixed point of $\mathcal{L} : X \to X$ is a strong solution of (5.2). By a strong solution we mean that $h : \Sigma \times [0,T] \to \mathbb{R}$ is Lipschitz continuous in time, $C^{1+s+\alpha}$-regular in space and satisfies the equation (5.2) for almost every $t \in (0,T]$.

Let us first show that $\mathcal{L} : X \to X$ is well defined, i.e., u defined by (5.8) belongs to X. By a standard approximation argument we may assume that h_0 and $h \in X$ are smooth. By Theorem 2.2 the solution u is smooth and it holds

$$\sup_{0<t<T} \|u(\cdot,t)\|_{C^0(\Sigma)} \le \|h_0\|_{C^0(\Sigma)} + T \sup_{0<t<T} \big(C\|h(\cdot,t)\|_{C^0(\Sigma)} + \|P(x,h(\cdot,t),\nabla h(\cdot,t))\|_{C^0(\Sigma)} + \|H^s_\Sigma\|_{C^0(\Sigma)}\big).$$

Since we assume $\sup_{0\le t<T} \|h(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ and $\sup_{0\le t<T} \|h(\cdot,t)\|_{C^0(\Sigma)} \le \varepsilon$ we have by (5.4) that

$$\|P(x,h(\cdot,t),\nabla h(\cdot,t))\|_{C^\alpha(\Sigma)} \le C\delta^2 + C_\delta \varepsilon \tag{5.9}$$

for every $t \in (0,T]$. Therefore since $\|h_0\|_{C^0(\Sigma)} \le \varepsilon/2$ we find

$$\sup_{0<t<T} \|u(\cdot,t)\|_{C^0(\Sigma)} \le \frac{\varepsilon}{2} + T(C\varepsilon + C\delta^2 + C_\delta\varepsilon + C) < \varepsilon$$

when T is small. Hence we have the second condition in the definition of X.

In order to prove the first condition we recall that it holds

$$\|h(\cdot,t)\|_{C^\alpha(\Sigma)} \le \|h(\cdot,t)\|^\alpha_{C^1(\Sigma)} \|h(\cdot,t)\|^{1-\alpha}_{C^0(\Sigma)} \le C\varepsilon^{1-\alpha}$$

for every $t \in (0,T]$. We use again Theorem 2.2, (5.9) and $\|h_0\|_{C^{1+s+\alpha}(\Sigma)} \le \frac{\delta}{2C}$ and find

$$\begin{aligned} &\sup_{0<t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \\ &\le C\|h_0\|_{C^{1+s+\alpha}(\Sigma)} + C \sup_{0<t<T} \big(\|h(\cdot,t)\|_{C^\alpha(\Sigma)} + \|P(x,h(\cdot,t),\nabla h(\cdot,t))\|_{C^\alpha(\Sigma)} + T\|H^s_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}\big) \\ &\le \frac{\delta}{2} + C\varepsilon^{1-\alpha} + C\delta^2 + C_\delta\varepsilon + C\|H^s_\Sigma\|_{C^{1+s+\alpha}(\Sigma)}T, \end{aligned} \tag{5.10}$$

where C is a uniform constant and C_δ depends on δ. By choosing first δ, then ε and finally T small we find

$$\sup_{0<t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} < \delta$$

and the first condition follows.

Finally the bound $\sup_{0\le t<T} \|\partial_t u(\cdot,t)\|_{C^\alpha(\Sigma)} \le C$ follows from the equation (5.8) and from (5.4) as

$$\begin{aligned} &\sup_{0<t<T} \|L[u] + P(x,h(\cdot,t),\nabla h(\cdot,t))\|_{C^\alpha(\Sigma)} \\ &\qquad \le \sup_{0<t<T} C(\|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + \|h(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)}) \le C\delta + C_\delta\varepsilon. \end{aligned}$$

Hence we conclude that $\mathcal{L} : X \to X$ is well defined.

Let us next show that $\mathcal{L} : X \to X$ is a contraction with respect to the following norm

$$\|u\|_X := \sup_{0<t<T} \big(\|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + \Lambda_0\|u(\cdot,t)\|_{C^0(\Sigma)}\big),$$

where Λ_0 is a large constant which will be chosen later. Let us fix $h_1, h_2 \in X$ and denote $u_1 = \mathcal{L}[h_1]$ and $u_2 = \mathcal{L}[h_2]$. The function $v = u_2 - u_1$ is a solution of the equation

$$\partial_t v - \Delta^{\frac{1+s}{2}} v = c_s^2(x)(h_2 - h_1) + P(x,h_2,\nabla h_2) - P(x,h_1,\nabla h_1) \tag{5.11}$$

with $v(x,0) = 0$ for all $x \in \Sigma$.

We denote $w = h_2 - h_1$ and use (5.7) for $v_1(x) = h_1(x,t)$ and $v_2(x) = h_2(x,t)$ to conclude that

$$\begin{aligned} &\|P(x,h_2(\cdot,t),\nabla h_2(\cdot,t)) - P(x,h_1(\cdot,t),\nabla h_1(\cdot,t))\|_{C^\alpha(\Sigma)} \\ &\qquad \le C\delta\|w(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_{\Sigma,\delta}\|w(\cdot,t)\|_{C^0(\Sigma)}. \end{aligned}$$

Therefore the equation (5.11) and Theorem 2.2 yield

$$\sup_{0<t<T} \|v(\cdot,t)\|_{C^0(\Sigma)} \le CT \sup_{0\le t<T} \big(\delta\|w(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_{\Sigma,\delta}\|w(\cdot,t)\|_{C^0(\Sigma)}\big),$$

and

$$\sup_{0<t<T} \|v(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le C \sup_{0\le t<T} \left(\delta\|w(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + \|w(\cdot,t)\|_{C^{\alpha}(\Sigma)} + C_{\Sigma,\delta}\|w(\cdot,t)\|_{C^0(\Sigma)}\right)$$
$$\le C \sup_{0\le t<T} \left(\delta\|w(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_{\Sigma,\delta}\|w(\cdot,t)\|_{C^0(\Sigma)}\right),$$

where the last inequality follows from the interpolation inequality in Lemma 2.1. We choose $\Lambda_0 \ge \delta^{-1}C_{\Sigma,\delta}$ and $T \le \Lambda_0^{-1}$ and have by the two above inequalities

$$\sup_{0<t<T} \left(\|v(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + \Lambda_0\|v(\cdot,t)\|_{C^{\alpha}(\Sigma)}\right)$$
$$\le C\delta \sup_{0<t<T} \left(\|w(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + \Lambda_0\|w(\cdot,t)\|_{C^{\alpha}(\Sigma)}\right).$$

In other words

$$\|u_2 - u_1\|_X = \|v\|_X \le C\delta\|w\|_X \le \frac{1}{2}\|h_2 - h_1\|_X$$

when δ is small. Hence, $\mathcal{L} : X \to X$ is a contraction and by a standard fixed point argument we conclude that the equation (3.12) has a unique strong solution in X.

Step 3: (*Higher order regularity.*)

We prove the last statement of the theorem. In fact, we prove slightly stronger estimate, i.e., for every $k \in \mathbb{N}$ there is Λ_k such that

$$\sup_{0<t<T} \left(t^{k!}\|h(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)}\right) \le \Lambda_k. \tag{5.12}$$

In particular, this implies that $h(\cdot,t)$ is smooth for $t > 0$. One may then use the equation (3.12) to deduce that $h \in C^\infty(\Sigma \times (0,T])$.

Since $T < 1$ we know by Step 2 that (5.12) holds for $k = 0$. We argue by induction and assume that (5.12) holds for $k \in \mathbb{N}$ and prove that it holds also for $k+1$ with some large constant $\Lambda_{k+1} \ge \Lambda_k$. To this aim we fix vector fields $X_1, \dots, X_k \in \mathcal{T}(\Sigma)$ with $\|X_i\|_{C^{k+2}(\Sigma)} \le 1$, $i = 1, \dots, k$, and define the function space $Y_{k+1} \subset X$ such that $u \in Y_{k+1}$ if $u \in X$ (defined in the beginning of Step 2) and

$$\sup_{0<t<T} \left(t^{k!}\|u(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)}\right) \le \Lambda_k \quad \text{and} \quad \sup_{0<t<T} \left(t^{(k+1)!}\|\nabla_{X_k}\cdots\nabla_{X_1}u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)}\right) \le \tilde{\Lambda}_{k+1}, \tag{5.13}$$

where Λ_k is the constant given by the induction assumption and $\tilde{\Lambda}_{k+1}$ is a constant which we will fix later. We note first that Y_{k+1} is non-empty since at least the solution of the heat equation

$$\partial_t u = \Delta u \quad \Sigma \times (0,T] \qquad \text{with} \quad u(x,0) = h_0(x) \ \text{ on } x \in \Sigma,$$

belongs to Y_{k+1} when $\tilde{\Lambda}_{k+1}$ is chosen large enough.

Let $\mathcal{L} : X \to X$ be the map defined by (5.8). The goal is to show that for $h \in Y_{k+1}$ it holds $u = \mathcal{L}(h) \in Y_{k+1}$, i.e., $\mathcal{L}(Y_{k+1}) \subset Y_{k+1}$. Therefore since the solution constructed in Step 2 is unique in X we deduce that the solution belongs also to Y_{k+1}. In other words the solution of (3.12) satisfies

$$\sup_{0<t<T} \left(t^{(k+1)!}\|\nabla_{X_k}\cdots\nabla_{X_1}h(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)}\right) \le \tilde{\Lambda}_{k+1}.$$

Therefore it follows from $T < 1$ and from the fact that (5.12) holds for k that

$$\frac{1}{C_k} \sup_{0<t<T} \left(t^{(k+1)!}\|h(\cdot,t)\|_{C^{k+1+s+\alpha}(\Sigma)}\right) \le \tilde{\Lambda}_{k+1} + C_k\Lambda_k,$$

which proves (5.12). We need thus to prove that u satisfies the second inequality in (5.13).

Let u be the solution of (5.8) where h_0 and h are smooth function such that $h \in Y_{k+1}$, i.e., h satisfies (5.13). We denote

$$u_k := \nabla_{X_k}\cdots\nabla_{X_1}u \qquad \text{and} \qquad h_k := \nabla_{X_k}\cdots\nabla_{X_1}h.$$

We claim that u_k is a solution of the equation

$$\partial_t u_k - \Delta^{\frac{1+s}{2}} u_k = P_k(x,t) \qquad \text{on } \Sigma \times (0,T], \tag{5.14}$$

where the function P_k satisfies

$$\|P_k(\cdot,t)\|_{C^\alpha(\Sigma)} \le C\delta \|h_k(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_{k,\delta}\big(1 + \|u(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)} + \|h(\cdot,t)\|^k_{C^{k+s+\alpha}(\Sigma)}\big) \tag{5.15}$$

for all $t \in (0,T]$. Here $C_{k,\delta}$ is a constant which depends on k, δ and Σ, while C is a uniform constant.

Indeed, we first note that since h is smooth then the equation (5.8) and Theorem 2.2 imply that u is smooth. We may thus differentiate (5.8) and obtain by Proposition 4.9 that

$$\partial_t u_k - \Delta^{\frac{1+s}{2}} u_k = \nabla_{X_k} \cdots \nabla_{X_1}\big(c_s^2(x)h + P(x,h,\nabla h) - H^s_\Sigma(x)\big) + \partial^k u(x,t), \tag{5.16}$$

where $\partial^k u(x,t)$ denotes a function which satisfies $\|\partial^k u(x,t)\|_{C^\alpha(\Sigma)} \le C_k \|u(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)}$ for every $t \in (0,T]$. Recall that the function P is defined in (5.3). We use Leibniz rule and Proposition 4.9 to deduce

$$\begin{aligned}\nabla_{X_k} \cdots \nabla_{X_1} P(x,h,\nabla h) =& \big(\nabla_{X_k} \cdots \nabla_{X_1} Q(x,h,\nabla h)\big)\big(L[h] + R_{1,h}(x) + R_{2,h}(x)\,h - H^s_\Sigma(x)\big)\\ &+ (1 + Q(x,h,\nabla h))\big(\nabla_{X_k} \cdots \nabla_{X_1} R_{1,h} + \nabla_{X_k} \cdots \nabla_{X_1}(R_{2,h}\,h)\big)\\ &+ Q(x,h,\nabla h)L[h_k] + f_k(x,t),\end{aligned} \tag{5.17}$$

where f_k is a function which satisfies

$$\begin{aligned}\|f_k(\cdot,t)\|_{C^\alpha(\Sigma)} \le& C_k \sum_{j=1}^{k-1} \|Q(x,h,\nabla h)\|_{C^{j+\alpha}(\Sigma)} \|L[h] + R_{1,h} + R_{2,h}\,h - H^s_\Sigma\|_{C^{k-j+\alpha}(\Sigma)}\\ &+ C_k(1 + \|h\|_{C^{k+s+\alpha}(\Sigma)}).\end{aligned}$$

We have

$$\|Q(x,h,\nabla h)\|_{C^{j+\alpha}(\Sigma)} \le C_j(1 + \|h\|^j_{C^{j+1+\alpha}(\Sigma)})$$

and Proposition 4.9 yields

$$\|L[h]\|_{C^{j+\alpha}(\Sigma)} \le C_j \|h\|_{C^{j+1+s+\alpha}(\Sigma)}.$$

Moreover by Lemma 4.10 we have

$$\|R_{1,h}\|_{C^{j+\alpha}(\Sigma)} + \|R_{1,h}\|_{C^{j+\alpha}(\Sigma)} \le C_j(1 + \|h\|^j_{C^{j+1+\alpha}(\Sigma)}).$$

Therefore it holds

$$\|f_k(\cdot,t)\|_{C^\alpha(\Sigma)} \le C_k(1 + \|h(\cdot,t)\|^k_{C^{k+\alpha}(\Sigma)}) \tag{5.18}$$

for all $t \in (0,T]$.

We use (2.1) to conclude that

$$\|\nabla_{X_k} \cdots \nabla_{X_1} Q(x,h(\cdot,t),\nabla h(\cdot,t))\|_{C^\alpha(\Sigma)} \le C_k(1 + \|h_k(\cdot,t)\|_{C^{1+\alpha}(\Sigma)} + \|h(\cdot,t)\|^k_{C^{k+s+\alpha}(\Sigma)})$$

for all $t \in (0,T]$. By the interpolation inequality in Lemma 2.1 and by Young's inequality we have

$$\|\nabla_{X_k} \cdots \nabla_{X_1} Q(x,h(\cdot,t),\nabla h(\cdot,t))\|_{C^\alpha(\Sigma)} \le \delta \|h_k(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_{k,\delta}(1 + \|h(\cdot,t)\|^k_{C^{k+s+\alpha}(\Sigma)}).$$

Recall that by (5.5) the assumption $h \in X$ implies $\|\nu_\Sigma\|_{C^{1+s+\alpha}(\Sigma)} \|h(\cdot,t)\|_{C^\alpha(\Sigma)} \le \delta$ for every $t \in (0,T)$ when ε is small. Then Lemma 4.10 yields

$$\begin{aligned}&\|\nabla_{X_k} \cdots \nabla_{X_1} R_{1,h(\cdot,t)}\|_{C^\alpha(\Sigma)} + \|\nabla_{X_k} \cdots \nabla_{X_1}(R_{2,h(\cdot,t)} h(\cdot,t))\|_{C^\alpha(\Sigma)}\\ &\qquad \le C\delta \|h_k(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_k(1 + \|h(\cdot,t)\|^k_{C^{k+s+\alpha}(\Sigma)}).\end{aligned}$$

Finally, by (5.6) and by Lemma 4.4 we have

$$\begin{aligned}\|Q(x,h(\cdot,t),\nabla h(\cdot,t))L[h_k(\cdot,t)]\|_{C^\alpha(\Sigma)} &\le C(\|h(\cdot,t)\|_{C^{1+\alpha}(\Sigma)} + \|h(\cdot,t)\nu_\Sigma\|_{C^{1+\alpha}(\Sigma)})\|h_k(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)}\\ &\le C\delta \|h_k(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)}.\end{aligned}$$

The equation (5.14) and the estimate (5.15) then follows from the previous inequalities, from (5.16) (5.17) and (5.18) and from

$$\|L[h] + R_{1,h}(x) + R_{2,h}(x)\,h - H^s_\Sigma(x)\|_{C^\alpha(\Sigma)} \le C.$$

Let us then prove that $u \in Y_{k+1}$. We define $v(x,t) = t^{(k+1)!}u_k(x,t)$. Since u_k is a solution of (5.14) then v is a solution of

$$\partial_t v - \Delta^{\frac{1+s}{2}} v = t^{(k+1)!}P_k(x,t) + (k+1)!t^{(k+1)!-1}u_k(x,t) \qquad \text{on } \Sigma \times (0,T],$$

with $v(x,0) = 0$. Theorem 2.2 and (5.15) imply (recall that $T < 1$)

$$\begin{aligned}
&\sup_{0<t<T} \|v(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \\
&\quad \le C \sup_{0<t<T} \big(t^{(k+1)!}\|P_k(\cdot,t)\|_{C^\alpha(\Sigma)} + (k+1)!t^{k!}\|u_k(\cdot,t)\|_{C^\alpha(\Sigma)}\big) \\
&\quad \le \sup_{0<t<T} \big(C\delta\, t^{(k+1)!}\|h_k(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_{k,\delta}\big(1 + t^{k!}\|u(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)} + t^{(k+1)!}\|h(\cdot,t)\|^k_{C^{k+s+\alpha}(\Sigma)}\big).
\end{aligned}$$

Recall that we assume $\sup_{0<t<T} t^{k!}\|u(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)} \le \Lambda_k$ and $\sup_{0<t<T} t^{k!}\|h(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)} \le \Lambda_k$. In particular, the latter implies

$$\sup_{0<t<T} t^{(k+1)!}\|h(\cdot,t)\|^k_{C^{k+s+\alpha}(\Sigma)} \le \sup_{0<t<T} \Big(t^{k!}\|h(\cdot,t)\|_{C^{k+s+\alpha}(\Sigma)}\Big)^k \le \Lambda_k^k.$$

Therefore we have

$$\sup_{0<t<T} \|v(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le C\delta \sup_{0<t<T} t^{(k+1)!}\|h_k(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_{k,\delta}(1 + \Lambda_k + \Lambda_k^k).$$

Since we assume that the second inequality in (5.13) holds for $h_k = \nabla_{X_k} \cdots \nabla_{X_1} h$, the above inequality yields

$$\sup_{0<t<T} \|\nabla_{X_k} \cdots \nabla_{X_1} u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le C\delta\tilde{\Lambda}_{k+1} + C_{k,\delta}(1 + \Lambda_k + \Lambda_k^k).$$

Let us first choose δ such that $C\delta \le \frac{1}{4}$ and then $\tilde{\Lambda}_{k+1} = 4C_{k,\delta}(1 + \Lambda_k + \Lambda_k^k)$. This gives us

$$\sup_{0<t<T} \|\nabla_{X_k} \cdots \nabla_{X_1} u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le \frac{1}{2}\tilde{\Lambda}_{k+1}.$$

Therefore u satisfies the second inequality in (5.13) and we conclude that $u \in Y_{k+1}$. □

We conclude this section by showing how to modify the previous proof to obtain a result analogous to Theorem 5.1 for the volume preserving fractional mean curvature flow (1.3). We use the same parametrization as in Section 3 to describe the motion of $E_0 \in \mathfrak{h}_\delta(\Sigma)$ given by (1.3). If $E \in \mathfrak{h}_\delta(\Sigma)$ with $\partial E = \{x + h(x)\nu(x) : x \in \Sigma\}$ then by (3.8) and by change of variables we have

$$-\fint_{\partial E} H^s_E(x)\,d\mathcal{H}^n_x = \left(\int_\Sigma J_\Phi(x)\,d\mathcal{H}^n_x\right)^{-1} \left(\int_\Sigma \big(L[h] - H^s_\Sigma + R_{1,h}(x) + R_{2,h}(x)h\big)J_\Phi(x)\,d\mathcal{H}^n_x\right),$$

where J_Φ denotes the tangential Jacobian of $\Phi(x) = x + h(x)\nu(x)$. As we mentioned in Section 3 the tangential Jacobian can be written as $J_\Phi(x) = 1 + Q_3(x,h,\nabla h)$, where Q_3 is a smooth function such that $Q_3(x,0,0) = 0$ for all $x \in \Sigma$. Recall that $L[h]$ is defined in (2.5) and notice that for $h \in C^{1+s+\alpha}(\Sigma)$ it holds

$$\int_\Sigma \Delta^{\frac{1+s}{2}} h(x)\,d\mathcal{H}^n_x = 0.$$

Let us then define the number

$$\bar{R}_{3,u} := \frac{\int_\Sigma \big(-c_s^2(x)u(x) + H^s_\Sigma(x) - R_{1,u}(x) - R_{2,u}(x)u(x)\big)(1 + Q_3(x,u,\nabla u))\,d\mathcal{H}^n_x}{\int_\Sigma 1 + Q_3(x,u,\nabla u)\,d\mathcal{H}^n_x}. \tag{5.19}$$

That is $\bar{R}_{3,u} = \bar{H}^s_E$.

We obtain immediately the following result which is analogous to Proposition 3.2.

Proposition 5.2. *Assume that $E_0 \in \mathfrak{h}_\delta(\Sigma)$ for δ small. There exists a flow $(E_t)_{t\in(0,T]}$ with $E_t \in \mathfrak{h}_\delta(\Sigma)$, for all $t \in (0,T]$, starting from E_0 which is a classical solution of (1.3) if and only if there exists a classical solution $h \in C(\Sigma \times [0,T]) \cap C^\infty(\Sigma \times (0,T])$ of*

$$\begin{cases} \partial_t h = L[h] + \tilde{P}(x,h,\nabla h) - H^s_\Sigma(x) & \text{on } \Sigma \times (0,T] \\ h(x,0) = h_0(x) & \text{for } x \in \Sigma, \end{cases} \tag{5.20}$$

with $\sup_{0<t<T} \|h(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$. Here $\tilde{P}$ is defined for a generic function $u \in C^\infty(\Sigma)$ as

$$\tilde{P}(x,u,\nabla u) = P(x,u,\nabla u) + (1 + Q(x,u,\nabla u))\bar{R}_{3,u},$$

where $P(x,u,\nabla u)$ is given by (5.3), $\bar{R}_{3,u}$ is defined in (5.19) and Q is a smooth function such that $Q(x,0,0) = 0$ for all $x \in \Sigma$.

Arguing as with (5.4) we deduce that if u is such that $\|u\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ and $\|u\|_{C^0(\Sigma)} \le \varepsilon$ with ε small enough it holds

$$\|\bar{R}_{3,u}\|_{C^\alpha(\Sigma)} = |\bar{R}_{3,u}| \le C\delta^2 + C_\delta \varepsilon + |\bar{H}^s_\Sigma|.$$

Similarly, we argue as with (5.7) and obtain for v_1, v_2 with $\|v_i\|_{C^{1+s+\alpha}(\Sigma)} \le \delta$ and $\|v_i\|_{C^\alpha(\Sigma)} \le \varepsilon$, $i = 1,2$, that

$$\|\bar{R}_{3,v_2} - \bar{R}_{3,v_1}\|_{C^\alpha(\Sigma)} \le C\delta \|v_2 - v_1\|_{C^{1+s+\alpha}(\Sigma)} + C_{\Sigma,\delta}\|v_2 - v_1\|_{C^0(\Sigma)}.$$

Therefore we obtain by (5.4) that

$$\|\tilde{P}(x,u,\nabla u)\|_{C^\alpha(\Sigma)} \le C\delta^2 + C_\delta \|u\|_{C^0(\Sigma)} + |\bar{H}^s_\Sigma| \tag{5.21}$$

and by (5.7) that

$$\|\tilde{P}(x,v_2,\nabla v_2) - \tilde{P}(x,v_1,\nabla v_1)\|_{C^\alpha(\Sigma)} \le C\delta \|v_2 - v_1\|_{C^{1+s+\alpha}(\Sigma)} + C_{\Sigma,\delta}\|v_2 - v_1\|_{C^0(\Sigma)}. \tag{5.22}$$

We may thus use the argument in Step 2 in the proof of Theorem 5.1 to obtain the unique strong solution of (5.20). The smoothness of the strong solution follows immediately from Step 3 since $\bar{R}_{3,h(\cdot,t)}$ does not depend on x. We have thus the following result.

Theorem 5.3. *Let $0 < \alpha(1-s)/2$. Assume $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth compact hypersurface and $h_0 : \Sigma \to \mathbb{R}$ is such that (5.1) holds. For δ and ε small enough, there is $T \in (0,1)$, depending on δ and ε, such that (5.20) has a unique classical solution $h \in C(\Sigma \times [0,T]) \cap C^\infty(\Sigma \times (0,T])$ with*

$$\sup_{0<t<T} \|h(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le \delta.$$

Moreover, for every $k \in \mathbb{N}$ there is a constant Λ_k such that

$$\sup_{0<t<T} \left(t^{k!}\|h(\cdot,t)\|_{C^k(\Sigma)}\right) \le \Lambda_k.$$

Theorem 5.3 together with Proposition 5.2 proves the main theorem for the volume preserving flow (1.3).

Declaration of competing interest

No competing interest.

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Appendix A

Here we give the proof of Theorem 2.2. We first recall the result in the case $\Sigma = \mathbb{R}^n$ and then use a perturbation argument to prove it for a compact and smooth hypersurface. The following result can be found in [27].

Theorem A.1. *Assume that $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is smooth and $|f(x,t)| \le C(1+|x|)^{-n-1-s}$ for all $(x,t) \in \Sigma \times [0,T]$. Assume that u with $\operatorname{supp} u(\cdot,t) \subset B_1$ for all $t \in [0,T]$ is the solution of*

$$\begin{cases} \partial_t u = \Delta^{\frac{s+1}{2}} u + f(x,t) & \text{in } \mathbb{R}^n \times (0,T] \\ u(x,0) = 0. \end{cases}$$

Then it holds

$$\sup_{0<t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C \sup_{0<t<T} \|f(\cdot,t)\|_{C^{\alpha}(\mathbb{R}^n)}.$$

Proof of Theorem 2.2. The existence and uniqueness of the weak solution follows from Galerkin method and the smoothness follows by differentiating the equation with respect to time. Since the argument is standard we omit it and simply refer to [15].

Let us first prove the second inequality. It is clear that we may assume $g = 0$. We write $u = v + w$ where

$$\begin{cases} \partial_t v = \Delta^{\frac{s+1}{2}} v & \text{on } \Sigma \times (0,T] \\ v(x,0) = u_0(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t w = \Delta^{\frac{s+1}{2}} w + f(x,t) & \text{on } \Sigma \times (0,T] \\ w(x,0) = 0. \end{cases}$$

By maximum principle it holds $|v(x,t)| \le |u_0(x)|$ for all $(x,t) \in \Sigma \times (0,T]$. Let us then prove

$$w(x,t) \le T \sup_{x\in\Sigma, t\in(0,T]} |f(x,t)| \qquad \text{for all } (x,t) \in \Sigma \times (0,T].$$

To this aim define $\tilde{w}(x,t) = t^{\varepsilon-1} w(x,t)$ for $\varepsilon > 0$. Then $\tilde{w}$ is continuous on $\Sigma \times [0,T]$, $\tilde{w}(x,0) = 0$ and assume it attains its maximum at $(\hat{x},\hat{t}) \in \Sigma \times (0,T]$. By maximum principle it holds

$$0 \le \partial_t \tilde{w}(\hat{x},\hat{t}) = t^{\varepsilon-1} \partial_t w(\hat{x},\hat{t}) - (1-\varepsilon)\hat{t}^{\varepsilon-2} w(\hat{x},\hat{t})$$

and

$$0 \ge \Delta^{\frac{1+s}{2}} \tilde{w}(\hat{x},\hat{t}) = t^{\varepsilon-1} \Delta^{\frac{1+s}{2}} w(\hat{x},\hat{t}).$$

Then the equation for w implies

$$\hat{t}^{\varepsilon-2} w(\hat{x},\hat{t}) \le \frac{\hat{t}^{\varepsilon-1} |f(\hat{x},\hat{t})|}{1-\varepsilon}.$$

Therefore, because $(\hat{x},\hat{t})$ is the maximum point, it holds for all $(x,t) \in \Sigma \times (0,T]$

$$t^{\varepsilon-1} w(x,t) \le \hat{t}^{\varepsilon-1} w(\hat{x},\hat{t}) \le \frac{T^{\varepsilon}}{1-\varepsilon} \sup_{x\in\Sigma, t\in(0,T]} |f(x,t)|.$$

The estimate follows by letting $\varepsilon \to 0$. By repeating the argument for $-w$ we obtain the second inequality in Theorem 2.2.

Let us prove the first inequality in Theorem 2.2. We may assume that $g = u_0 = 0$, since the general case follows by considering the function $v(x,t) = u(x,t) - tg(x) - u_0(x)$.

Let us fix $x_0 \in \Sigma$ and without loss of generality we may assume that $x_0 = 0$ and $\nu(0) = e_{n+1}$. Since Σ is smooth and uniformly $C^{1,1}$-regular we may write it locally as a graph of a smooth function, i.e., there exists a smooth function $\phi : B_{2r} \subset \mathbb{R}^n \to \mathbb{R}$ such that

$$\Sigma \cap C_r = \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = \phi(x')\},$$

where C_r denotes the cylinder

$$C_r = \{x = (x', x_{n+1}) \in \mathbb{R}^{n+1} : |x'| < r,\ |x_{n+1}| < r\}.$$

Note that the assumption $x_0 = 0$ and $\nu(0) = e_{n+1}$ implies $\phi(0) = 0$, $D\phi(0) = 0$ and

$$\|\phi\|_{C^{1+s+\alpha}(B_r)} < Cr^{1-s-\alpha}. \tag{A.1}$$

Let $\zeta : \mathbb{R}^+ \to \mathbb{R}$ be a smooth cut-off function such that $\zeta(\rho) = 1$ for $\rho \in [0, r/2]$ and $\zeta(\rho) = 0$ for $\rho \geq r$.

Let us denote $\Sigma_r = \Sigma \cap C_r$. The above notation in mind we may write the equation in (2.6) for $x \in \Sigma_r$ as

$$\begin{aligned}\partial_t u(x,t) &= 2\int_{\Sigma} \zeta(|y_{n+1}|)\zeta(|y'|)\frac{u(y,t)-u(x,t)}{|y-x|^{n+1+s}}\,d\mathcal{H}^n_y \\ &\quad + 2\int_{\Sigma}\big(1-\zeta(|y_{n+1}|)\zeta(|y'|)\big)\frac{u(y,t)-u(x,t)}{|y-x|^{n+1+s}}\,d\mathcal{H}^n_y + f(x,t) \\ &= 2\int_{\Sigma_r}\zeta(|y'|)\frac{u(y,t)-u(x,t)}{|y-x|^{n+1+s}}\,d\mathcal{H}^n_y + G_1(x,t) + f(x,t),\end{aligned} \tag{A.2}$$

where

$$G_1(x,t) = 2\int_{\Sigma}\big(1-\zeta(|y_{n+1}|)\zeta(|y'|)\big)\frac{u(y,t)-u(x,t)}{|y-x|^{n+1+s}}\,d\mathcal{H}^n_y.$$

Since the function $1-\zeta(|y_{n+1}|)\zeta(|y'|)$ vanishes on $\Sigma \cap C_{r/2}$ the above integral is non-singular on $\Sigma \cap C_{r/4}$ and we have

$$\sup_{0<t<T} ||\zeta(4|x'|)G_1(x,t)||_{C^\alpha(\Sigma_r)} \leq C \sup_{0<t<T} ||u(\cdot,t)||_{C^\alpha(\Sigma)}. \tag{A.3}$$

We may write every $x \in \Sigma_r$ as $x = (x', \phi(x'))$, where $x' \in B_r \subset \mathbb{R}^n$. We denote, by slight abuse of notations, $u((x', \phi(x')), t) = u(x', t)$ and similarly $G_1(x', t)$, $f(x', t)$ and $g(x')$ for every point $x = (x', \phi(x'))$ on Σ_r. By change of variables we have

$$\int_{\Sigma_r}\zeta(|y'|)\frac{u(y,t)-u(x,t)}{|y-x|^{n+1+s}}\,d\mathcal{H}^n_y = \int_{B_r}\zeta(|y'|)\frac{u(y',t)-u(x',t)}{\big(|x'-y'|^2+(\phi(y')-\phi(x'))^2\big)^{\frac{n+1+s}{2}}}\sqrt{1+|D\phi(y')|^2}\,dy'.$$

We define $Q(z) := \sqrt{1+|z|^2} - 1$ and

$$K_\phi(y', x') := \frac{1}{\big(|x'-y'|^2+(\phi(y')-\phi(x'))^2\big)^{\frac{n+1+s}{2}}}.$$

Note that Q is a smooth function with $Q(0) = 0$ and $K_\phi(y', x')$ agrees with (3.7) when we choose $\Sigma = \mathbb{R}^n \times \{0\} \simeq \mathbb{R}^n$ and $u = \phi$. Note also that by (A.1) ϕ satisfies

$$\|\phi\|_{C^{1+s+\alpha}(B_r)} < \delta \tag{A.4}$$

when r is small enough. Using this notation we may write

$$\int_{\Sigma_r}\zeta(|y'|)\frac{u(y,t)-u(x,t)}{|y-x|^{n+s}}d\mathcal{H}^n_y = \int_{B_r}\zeta(|y'|)\big(u(y',t)-u(x',t)\big)\big(1+Q(D\phi(y'))\big)K_\phi(y',x')\,dy'.$$

Let us define $w(x', t) = \zeta(4|x'|)u(x', t)$ and extend ϕ to $\mathbb{R}^n$ such that (A.4) holds in $\mathbb{R}^n$. Then we have by (A.2) and by the above calculations

$$\begin{aligned}\partial_t w(x',t) &= 2\int_{B_r} \zeta(4|x'|)\zeta(|y'|)\big(u(y',t)-u(x',t)\big)\big(1+Q(D\phi(y'))\big)K_\phi(y',x')\,dy' \\ &\qquad + \zeta(4|x'|)(G_1(x',t)+f(x',t)) \\ &= 2\int_{\mathbb{R}^n} \zeta(4|x'|)\zeta(|y'|)\big(u(y',t)-u(x',t)\big)\big(1+Q(D\phi(y'))\big)K_\phi(y',x')\,dy' \\ &\qquad + \zeta(4|x'|)(G_1(x',t)+f(x',t)).\end{aligned} \tag{A.5}$$

We write

$$\begin{aligned}&\zeta(4|x'|)\zeta(|y'|)\big(u(y',t)-u(x',t)\big) \\ &\quad = \big(w(y',t)-w(x',t)\big)\zeta(|y'|) - \big(\zeta(4|y'|)-\zeta(4|x'|)\big)\zeta(|y'|)u(y',t) \\ &\quad = \big(w(y',t)-w(x',t)\big) - \big(1-\zeta(|y'|)\big)\big(w(y',t)-w(x',t)\big) \\ &\qquad - \big(\zeta(4|y'|)-\zeta(4|x'|)\big)\zeta(|y'|)u(y',t).\end{aligned} \tag{A.6}$$

We recall that $Q(0)=0$ and write

$$\big(1+Q(D\phi(y'))\big)K_\phi(y',x') - \frac{1}{|y'-x'|^{n+1+s}} = \int_0^1 \frac{d}{d\xi}\big((1+Q(\xi D\phi(y')))K_{\xi\phi}(y',x')\big)\,d\xi. \tag{A.7}$$

By combining (A.5), (A.6) and (A.7) we obtain

$$\begin{cases} \partial_t w(x',t) = \Delta^{\frac{s+1}{2}} w(x',t) + F(x',t) - G_2(x,t) - G_3(x,t) + \zeta(4|x'|)(G_1(x',t)+f(x',t)) \\ w(x',0)=0, \end{cases} \tag{A.8}$$

where

$$F(x',t) = 2\int_0^1\int_{\mathbb{R}^n} \big(w(y',t)-w(x',t)\big)\frac{d}{d\xi}\big((1+Q(\xi D\phi(y')))K_{\xi\phi}(y',x')\big)\,dy'd\xi,$$

$$G_2(x,t) = 2\int_{\mathbb{R}^n} \big(1-\zeta(|y'|)\big)\big(w(y',t)-w(x',t)\big)\big(1+Q(D\phi(y'))\big)K_\phi(y',x')\,dy'$$

and

$$G_3(x,t) = 2\int_{\mathbb{R}^n} \big(\zeta(4|y'|)-\zeta(4|x'|)\big)\zeta(|y'|)u(y',t)\big(1+Q(D\phi(y'))\big)K_\phi(y',x')\,dy'.$$

We need to estimate the C^α-norms of F, G_2 and G_3. Note first that trivially $\|w(\cdot,t)\|_{C^\gamma(\mathbb{R}^n)} \le C_\gamma \|u(\cdot,t)\|_{C^\gamma(\Sigma)}$ for all $\gamma \in (0,1)$. Since $(1-\zeta(|y'|))$ vanishes for $|y'| \le r/2$ and $w(y',t)$ vanishes for $|y'| \ge r/4$ we have, similarly as with (A.3), that

$$\sup_{0<t<T} ||G_2(\cdot,t)||_{C^\alpha(\mathbb{R}^n)} \le C \sup_{0<t<T} ||u(\cdot,t)||_{C^\alpha(\Sigma)}. \tag{A.9}$$

Next we recall that by Remark 4.6 and by (A.4), Lemma 4.4 and Lemma 4.5 hold also for $\Sigma = \mathbb{R}^n$ and K_ϕ. Hence, we conclude by Lemma 4.4 that

$$\sup_{0<t<T} ||G_3(\cdot,t)||_{C^\alpha(\mathbb{R}^n)} \le C \sup_{0<t<T} ||u(\cdot,t)||_{C^{s+\alpha}(\Sigma)}. \tag{A.10}$$

Similarly we observe that the term F is of type (4.12) with $v = w(\cdot,t)$ and $u=\phi$. Therefore (4.13) and (A.4) yield

$$\|F(\cdot,t)\|_{C^\alpha(\mathbb{R}^n)} \le C\delta \|w(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \tag{A.11}$$

for every $t \in (0,T]$.

We conclude by (A.3), (A.8), (A.9), (A.10), (A.11) and by Theorem A.1 that

$$\sup_{0<t<T} \|w(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C\delta \sup_{0<t<T} ||w(\cdot,t)||_{C^{1+s+\alpha}(\mathbb{R}^n)} + C \sup_{0<t<T} (||f(\cdot,t)||_{C^\alpha(\Sigma)} + ||u(\cdot,t)||_{C^{s+\alpha}(\Sigma)}).$$

When δ is small we have

$$\sup_{0<t<T} \|w(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)} \le C \sup_{0<t<T} (||f(\cdot,t)||_{C^\alpha(\Sigma)} + ||u(\cdot,t)||_{C^{s+\alpha}(\Sigma)}).$$

Note that $\|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma\cap C_{r/8})} \le C\|w(\cdot,t)\|_{C^{1+s+\alpha}(\mathbb{R}^n)}$ for every $t \in (0,T]$. Therefore since Σ is compact we obtain by standard covering argument

$$\sup_{0<t<T} \|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} \le C \sup_{0<t<T} (||f(\cdot,t)||_{C^\alpha(\Sigma)} + ||u(\cdot,t)||_{C^{s+\alpha}(\Sigma)}). \tag{A.12}$$

By the interpolation inequality in Lemma 2.1 and by the second inequality in Theorem 2.2 (recall that $u_0 = g = 0$) we have for all $t \in (0,T]$

$$\begin{aligned} ||u(\cdot,t)||_{C^{s+\alpha}(\Sigma)} &\le \delta\|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_\delta\|u(\cdot,t)\|_{C^0(\Sigma)} \\ &\le \delta\|u(\cdot,t)\|_{C^{1+s+\alpha}(\Sigma)} + C_\delta T \sup_{0<t<T} ||f(\cdot,t)||_{C^0(\Sigma)}. \end{aligned}$$

The claim then follows from (A.12) by choosing δ small. □

References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
[2] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
[3] B. Barrios, A. Figalli, E. Valdinoci, Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 13 (2014) 609–639.
[4] X. Cabré, M.M. Fall, J. Solá-Morales, T. Weth, Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay, J. Reine Angew. Math. 745 (2018) 253–280.
[5] L. Caffarelli, J.-M. Roquejoffre, O. Savin, Nonlocal minimal surfaces, Commun. Pure Appl. Math. 63 (2010) 1111–1144.
[6] L. Caffarelli, P.E. Souganidis, Convergence of nonlocal threshold dynamics approximations to front propagation, Arch. Ration. Mech. Anal. 195 (2010) 1–23.
[7] A. Cesaroni, S. Dipierro, M. Novaga, E. Valdinoci, Fattening and nonfattening phenomena for planar nonlocal curvature flows, Math. Ann. 375 (2019) 687–736.
[8] A. Chambolle, M. Morini, M. Ponsiglione, A nonlocal mean curvature flow and its semi-implicit time-discrete approximation, SIAM J. Math. Anal. 44 (2012) 4048–4077.
[9] A. Chambolle, M. Morini, M. Ponsiglione, Nonlocal curvature flows, Arch. Ration. Mech. Anal. 218 (2015) 1263–1329.
[10] A. Chambolle, M. Novaga, B. Ruffini, Some results on anisotropic fractional mean curvature flows, Interfaces Free Bound. 19 (2017) 393–415.
[11] E. Cinti, C. Sinestrari, E. Valdinoci, Neckpinch singularities in fractional mean curvature flows, Proc. Am. Math. Soc. 146 (2018) 2637–2646.
[12] E. Cinti, C. Sinestrari, E. Valdinoci, Convex sets evolving by volume preserving fractional mean curvature flows, preprint, 2018.
[13] G. Ciraolo, A. Figalli, F. Maggi, M. Novaga, Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature, J. Reine Angew. Math. 741 (2018) 275–294.
[14] J. Dávila, M. del Pino, J. Wei, Nonlocal s-minimal surfaces and Lawson cones, J. Differ. Geom. 109 (2018) 111–175.
[15] L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.
[16] L.C. Evans, J. Spruck, Motion of level sets by mean curvature. II, Trans. Am. Math. Soc. 330 (1992) 321–332.
[17] A. Figalli, N. Fusco, F. Maggi, V. Millot, M. Morini, Isoperimetry and stability properties of balls with respect to nonlocal energies, Commun. Math. Phys. 336 (2015) 441–507.
[18] R.L. Frank, R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, J. Funct. Anal. 255 (2008) 3407–3430.
[19] N. Fusco, V. Millot, M. Morini, A quantitative isoperimetric inequality for fractional perimeters, J. Funct. Anal. 261 (2011) 697–715.
[20] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 2001.
[21] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Differ. Geom. 20 (1984) 237–266.
[22] G. Huisken, The volume preserving mean curvature flow, J. Reine Angew. Math. 382 (1987) 35–48.
[23] G. Huisken, A. Polden, Geometric evolution equations for hypersurfaces, in: Calculus of Variations and Geometric Evolution Problems, Cetraro, 1996, Springer-Verlag, Berlin, 1999, pp. 45–84.
[24] C. Imbert, Level set approach for fractional mean curvature flows, Interfaces Free Bound. 11 (2009) 153–176.
[25] J.M. Lee, Riemannian Manifolds. An Introduction to Curvature, Graduate Texts in Mathematics, vol. 176, Springer-Verlag, New York, 1997.
[26] C. Mantegazza, Lecture Notes on Mean Curvature Flow, Progress in Mathematics, vol. 290, Birkhäuser/Springer, Basel, 2011.
[27] R. Mikulevicius, H. Pragarauskas, On Hölder solutions of the integro-differential Zakai equation, Stoch. Process. Appl. 119 (2009) 3319–3355.
[28] M. Sáez, E. Valdinoci, On the evolution by fractional mean curvature, Commun. Anal. Geom. 27 (2019).
[29] O. Savin, E. Valdinoci, Regularity of nonlocal minimal cones in dimension 2, Calc. Var. Partial Differ. Equ. 48 (2013) 33–39.