

# Invariant density and time asymptotics for collisionless kinetic equations with partly diffuse boundary operators <sup>☆</sup>

B. Lods <sup>a</sup>, M. Mokhtar-Kharroubi <sup>b,\*</sup>, R. Rudnicki <sup>c</sup>

<sup>a</sup> *Università degli Studi di Torino & Collegio Carlo Alberto, Department of Economics and Statistics, Corso Unione Sovietica, 218/bis, 10134 Torino, Italy*

<sup>b</sup> *Université de Bourgogne Franche-Comté, Laboratoire de Mathématiques, CNRS UMR 6623, 16, route de Gray, 25030 Besançon Cedex, France*

<sup>c</sup> *Institute of Mathematics, Polish Academy of Sciences, Bankowa 14, 40-007 Katowice, Poland*

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## Abstract

This paper deals with collisionless transport equations in bounded open domains  $\Omega \subset \mathbb{R}^d$  ( $d \geq 2$ ) with  $\mathcal{C}^1$  boundary  $\partial\Omega$ , orthogonally invariant velocity measure  $\mathbf{m}(dv)$  with support  $V \subset \mathbb{R}^d$  and stochastic partly diffuse boundary operators  $H$  relating the outgoing and incoming fluxes. Under very general conditions, such equations are governed by stochastic  $C_0$ -semigroups  $(U_H(t))_{t \geq 0}$  on  $L^1(\Omega \times V, dx \otimes \mathbf{m}(dv))$ . We give a general criterion of irreducibility of  $(U_H(t))_{t \geq 0}$  and we show that, under very natural assumptions, if an invariant density exists then  $(U_H(t))_{t \geq 0}$  converges strongly (not simply in Cesàro means) to its ergodic projection. We show also that if no invariant density exists then  $(U_H(t))_{t \geq 0}$  is *sweeping* in the sense that, for any density  $\varphi$ , the total mass of  $U_H(t)\varphi$  concentrates near suitable sets of zero measure as  $t \rightarrow +\infty$ . We show also a general weak compactness theorem of interest for the existence of invariant densities. This theorem is based on several results on smoothness and transversality of the dynamical flow associated to  $(U_H(t))_{t \geq 0}$ .

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\* Corresponding author.

E-mail addresses: [bertrand.lods@unito.it](mailto:bertrand.lods@unito.it) (B. Lods), [mustapha.mokhtar-kharroubi@univ-fcomte.fr](mailto:mustapha.mokhtar-kharroubi@univ-fcomte.fr) (M. Mokhtar-Kharroubi), [rudnicki@us.edu.pl](mailto:rudnicki@us.edu.pl) (R. Rudnicki).

## 1. Introduction

Kinetic transport equations in bounded geometry is an important field of investigation which can be traced back to the seminal work [9] where absorbing boundary conditions have been considered. For more general boundary conditions, relating the incoming and outgoing fluxes at the boundary of the physical domain, the well-posedness of associated transport equations with general force terms – including Vlasov-like equations – have been considered in [10,7,8] while a thorough analysis of the free transport equation with abstract boundary conditions on general domains have been performed in [43] (see also [21, Appendix of §2, p. 249]). Notice that, for a nonlinear and collisional kinetic equation such as Boltzmann equation, taking into account general boundary conditions induces notoriously additional difficulties; we just mention here the works [14,25] (dealing with close-to-equilibrium solutions) and [33] (for renormalized solutions) and the references therein.

We aim to emphasize right now that, even though the present contribution is dealing with collisionless kinetic equations, we hope that the tools developed in the paper will be of some interest in nonlinear kinetic theory with general partly diffuse boundary conditions, especially in the study of the regularity up to the boundary for both the linearized and nonlinear Boltzmann equation as in [18,27].

The object of this paper is to build a general theory of time asymptotics ( $t \rightarrow \infty$ ) for multi-dimensional collisionless kinetic semigroups with partly diffuse boundary operators. Our construction is twofold:

- (1) On the one hand, we continue previous functional analytic works [5,6,34,43] on substochastic semigroups governing collisionless transport equations with conservative boundary operators in  $L^1$ -spaces and combine them to recent developments on the asymptotics of stochastic partially integral semigroups in  $L^1$ -spaces motivated by piecewise deterministic processes [41].
- (2) On the other hand, we investigate the problem of the existence of invariant densities for collisionless transport equations. Such existence theory depends heavily on our understanding of compactness properties induced by the diffuse parts of the boundary operators. These compactness properties rely on the fine knowledge of smoothness and transversality properties of the dynamical flow induced by the semigroup.

More precisely, we consider transport equations of the form

$$\partial_t \psi(x, v, t) + v \cdot \nabla_x \psi(x, v, t) = 0, \quad (x, v) \in \Omega \times V, \quad t \geq 0 \quad (1.1a)$$

with initial data

$$\psi(x, v, 0) = \psi_0(x, v), \quad (x, v) \in \Omega \times V, \quad (1.1b)$$

under abstract (conservative) boundary conditions

$$\psi|_{\Gamma_-} = H(\psi|_{\Gamma_+}), \quad (1.1c)$$

where

$$\Gamma_{\pm} = \{(x, v) \in \partial\Omega \times V; \pm v \cdot n(x) > 0\}$$

( $n(x)$  being the outward unit normal at  $v \in \partial\Omega$ , see Fig. 1) and  $H$  is a linear boundary operator relating the outgoing and incoming fluxes  $\psi|_{\Gamma_+}$  and  $\psi|_{\Gamma_-}$  and is *bounded* on the trace spaces

$$L^1_{\pm} = L^1(\Gamma_{\pm}; |v \cdot n(x)| \pi(dx) \otimes m(dv)) = L^1(\Gamma_{\pm}, d\mu_{\pm}(x, v))$$

where  $\pi$  denotes the Lebesgue surface measure on  $\partial\Omega$ . We will focus our attention to the case of *nonnegative* and *conservative* boundary conditions, i.e.

$$H\psi \geq 0 \quad \text{and} \quad \|H\psi\|_{L^1_-} = \|\psi\|_{L^1_+}, \quad \text{for any nonnegative } \psi \in L^1_+. \quad (1.2)$$

Here

$$\Omega \subset \mathbb{R}^d \quad (d \geq 2) \text{ is an open subset with } \mathcal{C}^1 \text{ boundary } \partial\Omega$$

and our analysis takes place in the functional space

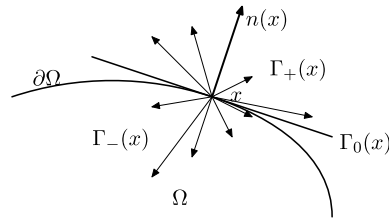


Fig. 1.  $x \in \partial\Omega$ ;  $\Gamma_0(x)$  – the tangent space to  $\partial\Omega$  at  $x$ ;  $\Gamma_+(x)$  – outward velocities;  $\Gamma_-(x)$  – inward velocities.

$$X = L^1(\Omega \times V; dx \otimes \mathbf{m}(dv))$$

where  $V \subset \mathbb{R}^d$  is the support of a nonnegative Borel measure  $\mathbf{m}$  which is orthogonally invariant (i.e. invariant under the action of the orthogonal group of matrices in  $\mathbb{R}^d$ ). Such a measure covers the Lebesgue measure on  $\mathbb{R}^d$ , the surface Lebesgue measure on spheres (one speed or multi-group models) or even combinations of them.

Very precise one-dimensional results corresponding to slab geometry have been obtained in [35]. Their extension to multi-dimensional geometries ( $d \geq 2$ ) is far from being elementary and is completely open. It is the main concern of the present work to provide such a generalization.

Let

$$W = \left\{ \varphi \in X; v \cdot \nabla_x \varphi \in X, \varphi|_{\Gamma_{\pm}} \in L^1(\Gamma_{\pm}) \right\}$$

where  $v \cdot \nabla_x \varphi$  is meant in a distributional sense, (see Section 2 below for a reminder of the trace theory) and let

$$T_H : \mathcal{D}(T_H) \subset X \rightarrow X$$

be defined by

$$T_H \varphi = -v \cdot \nabla_x \varphi, \quad \mathcal{D}(T_H) = \{ \varphi \in W; \varphi|_{\Gamma_-} = H(\varphi|_{\Gamma_+}) \}.$$

In contrast to the one-dimensional case [35], in general,  $T_H$  need *not* be a generator. However, there exists a unique extension

$$A \supset T_H$$

which generates a positive contraction  $C_0$ -semigroup  $(U_H(t))_{t \geq 0}$ , see [5,34,43]. Notice that  $(U_H(t))_{t \geq 0}$  need *not* be stochastic, i.e. mass-preserving on the positive cone  $X_+$  of  $X$ . Actually  $(U_H(t))_{t \geq 0}$  is stochastic if and only if

$$A = \overline{T_H} \tag{1.3}$$

and different characterizations of this property are also available [5,34]. A general sufficient condition for  $(U_H(t))_{t \geq 0}$  to be stochastic is given in Proposition 4.1 below.

Let us briefly describe the main contributions of this paper. We restrict ourselves to the stochastic case (1.3). A very important role is played here by the irreducibility of  $(U_H(t))_{t \geq 0}$  (see Definition 4.3 below). When  $T_H$  is not a generator, it is not possible to handle easily its closure  $A = \overline{T_H}$ . Despite this fact, the resolvent of  $A$  is given by an “explicit” series converging strongly, see (2.1) below. By exploiting this series one can derive a very general sufficient criterion of irreducibility of  $(U_H(t))_{t \geq 0}$  in terms of properties of the stochastic boundary operator  $H$ , see Proposition 4.6 below. It is well known (see [22]) that if the kernel of the generator of an irreducible stochastic  $C_0$ -semigroup is *not trivial* (and consequently one-dimensional) then the semigroup is *ergodic* and converges strongly in *Cesàrò means* to its one-dimensional (positive) ergodic projection (as  $t \rightarrow +\infty$ ). Thus the existence of an invariant density of  $(U_H(t))_{t \geq 0}$  is a *cornerstone* of this construction and is a fundamental problem for the understanding of the long-time behaviour of (1.1).

We mainly consider (local in space) stochastic boundary operators  $H : L^1_+ \rightarrow L^1_-$  which are (locally in space) convex combinations of reflection and diffuse operators of the form

$$\begin{aligned} H\varphi(x, v) &= \alpha(x) R\varphi(x, v) + (1 - \alpha(x)) K\varphi(x, v) \\ &= \alpha(x) \varphi(x, \mathcal{V}(x, v)) + (1 - \alpha(x)) \int_{\Gamma_+(x)} k(x, v, v') \varphi(x, v') \mu_x(dv') \end{aligned} \tag{1.4}$$

where

$$\Gamma_- \ni (x, v) \longmapsto (x, \mathcal{V}(x, v)) \in \Gamma_+$$

is a general  $\mu$ -preserving reflection law,  $\mu_x(dv) = |v \cdot n(x)| m(dv)$  and

$$\int_{\Gamma_-(x)} k(x, v, v') \mu_x(dv) = 1, \quad (x, v') \in \Gamma_+$$

where  $\alpha : x \in \partial\Omega \longmapsto \alpha(x) \in [0, 1]$  is a measurable function.

Regarding the long-time behaviour of the solution to (1.1), when

$$\operatorname{ess\,sup}_{x \in \partial\Omega} \alpha(x) < 1, \tag{1.5}$$

we show under quite general assumptions on the kernel  $k(x, v, v')$  that  $(U_H(t))_{t \geq 0}$  is *partially integral* (i.e. for each  $t > 0$ ,  $U_H(t)$  dominates a non trivial integral operator). It follows that if  $(U_H(t))_{t \geq 0}$  has an invariant density  $\Psi_H$  then  $(U_H(t))_{t \geq 0}$  is *asymptotically stable*, i.e.

$$\lim_{t \rightarrow +\infty} \|U_H(t)f - \Psi_H\| = 0$$

for any density  $f$ ; see Theorem 7.5 for a precise statement. This result provides us with a much more precise result than the mere Cesàro convergence given by the general theory. Converse results are also given; indeed we show that if  $(U_H(t))_{t \geq 0}$  has *no invariant density* then  $(U_H(t))_{t \geq 0}$  is *sweeping* with respect to suitable sets. In a more precise way, the total mass of any trajectory of (1.1)

$$t \geq 0 \longmapsto U_H(t)\psi_0$$

concentrates for large time  $t \rightarrow \infty$  near small (or large) velocities or near the boundary  $\partial\Omega \times V$ , see Theorem 8.3 for a precise statement. Such asymptotics follow from general results on partially integral stochastic semigroups [38–40] which we recall in Appendix B of the paper. These general theorems on asymptotic stability or sweeping of stochastic collisionless kinetic semigroups  $(U_H(t))_{t \geq 0}$  (and also some related results) are the first object of this paper. Our second object is to deal with the *existence* of an invariant density for stochastic collisionless kinetic semigroups  $(U_H(t))_{t \geq 0}$ . As far as we know, the existence of an invariant density is known *only* for the classical Maxwell diffuse model (see Example 6.4 below) for which it is known that  $(U_H(t))_{t \geq 0}$  is asymptotically stable [3].

Thus our second object is to provide an existence theory of invariant density for such kinetic models. We show first, for general stochastic boundary operators  $H$ , that 0 is an eigenvalue of  $T_H$  associated to a nonnegative eigenfunction if and only if there exists a nonnegative solution  $\varphi \in L_+^1$  to the eigenvalue problem

$$M_0 H \varphi = \varphi, \tag{1.6}$$

which satisfies the additional condition

$$\int_{\Gamma_+} \varphi(x, v) |v|^{-1} d\mu_+(x, v) < +\infty \tag{1.7}$$

where

$$M_0 : L_-^1 \rightarrow L_+^1$$

is the *stochastic* operator defined by

$$(M_0 \varphi)(x, v) = \varphi(x - \tau_-(x, v)v, v); \quad (x, v) \in \Gamma_+, \quad \varphi \in L_-^1$$

where  $\tau_-(x, v)$  is the exit time function (see the definition in Section 2 below).

To study the existence of an invariant density, we introduce the sub-class of *regular* partly diffuse boundary operators such that the diffuse part is “*weakly compact with respect to velocities*” (see Definition 3.5 below) which enjoys nice approximation properties. The part of the paper concerned with the existence of an invariant density is very involved and is based on a series of highly technical results culminating in a key spectral result

$$r_{\text{ess}}(\mathbf{M}_0\mathbf{H}) < 1 \quad (1.8)$$

(see Theorem 5.6) where  $r_{\text{ess}}$  refers to the essential spectral radius. Inequality (1.8) is shown to be true under some smallness assumption on the oscillations of the diffuse parameter  $\beta(\cdot) = 1 - \alpha(\cdot)$

$$\left( \|1 - \beta(\cdot)\|_{L^\infty(\partial\Omega)} + \|\beta(\cdot)\|_{L^\infty(\partial\Omega)} \right)^2 - \|\beta(\cdot)\|_{L^\infty(\partial\Omega)}^2 < 1 \quad (1.9)$$

(see Theorem 5.6). However, we believe such an assumption to be purely technical, (see Remark 5.8). It is then straightforward to check that the spectral problem (1.6) has a solution under (1.8). If the corresponding eigenfunction satisfies the additional condition (1.7) then  $(U_{\mathbf{H}}(t))_{t \geq 0}$  is asymptotically stable. If *not* we show a more precise sweeping behaviour: the total mass of any trajectory  $t \geq 0 \mapsto U_{\mathbf{H}}(t)\psi_0$  of (1.1) concentrates near the zero velocity as  $t \rightarrow +\infty$ , see Theorem 8.5.

The above spectral inequality (1.8) is a consequence of a key weak compactness theorem namely: for any regular diffuse operator  $\mathbf{K}$

$$\mathbf{K}\mathbf{M}_0\mathbf{K} : L_+^1 \rightarrow L_-^1 \quad \text{is weakly compact.}$$

The proof of this important result (Theorem 5.1), using the Dunford-Pettis criterion, is highly technical and is given in numerous steps. Roughly speaking, the main difficulty lies in the fact that  $\mathbf{K}$  induces compactness only in the velocity variables and several iterations and changes of variables are necessary to produce the missing compactness in the space variable  $x \in \partial\Omega$ . Such changes of variables are non trivial and have to be carefully justified. To do this, we take advantage of the stochastic character of the various operators involved and we show (see Corollary A.7) *smoothness properties* of the ballistic flow

$$\xi : (x, v) \in \Gamma_+ \mapsto \xi(x, v) = (x - \tau_-(x, v)v, v).$$

and its inverse

$$\xi^{-1} : (x, \omega) \in \Gamma_- \mapsto \xi^{-1}(x, \omega) = (x + \tau_+(x, \omega)\omega, \omega) \in \Gamma_+.$$

We prove in particular the following property: for any  $x \in \partial\Omega$ , there exists a set  $S(x) \subset \mathbb{S}^{d-1}$  of zero surface Lebesgue measure such that the differential of the mapping

$$\omega \in \Gamma_-(x) \setminus S(x) \mapsto x + \tau_+(x, \omega)\omega$$

has maximal rank (see Proposition A.8). These non trivial smoothness and transversality results involve intrinsic tools from differential geometry and are postponed in Appendix A for the simplicity of reading but we wish to point out that our analysis of the flow induced by  $(U_{\mathbf{H}}(t))_{t \geq 0}$  is new (even if results similar to some of ours appear e.g. in [25], see Remark A.6) and has its own interest independently of the main motivation of this paper.

As far as we know, most of our results are new and appear here for the first time. Finally, we note that the assumption that  $\partial\Omega$  is of class  $\mathcal{C}^1$  plays a role only for the results on smoothness and transversality of the flow stated in Appendix A; it is likely that the results stated there remain valid for  $\partial\Omega$  which is only *piecewise of class  $\mathcal{C}^1$* .

As a by-product of our contribution, we obtain for instance the following typical statement:

**Theorem 1.1.** *Let  $\mathbf{H} \in \mathcal{B}(L_+^1, L_-^1)$  be a stochastic boundary operator of the form (1.4) satisfying some technical assumptions (see Assumptions 6.1 and 4.4 for details). Then,  $(\mathbf{T}_{\mathbf{H}}, \mathcal{D}(\mathbf{T}_{\mathbf{H}}))$  is the generator of a stochastic  $C_0$ -semigroup  $(U_{\mathbf{H}}(t))_{t \geq 0}$  which is irreducible and has a unique invariant density  $\Psi_{\mathbf{H}} \in \mathcal{D}(\mathbf{T}_{\mathbf{H}})$  with*

$$\Psi_{\mathbf{H}}(x, v) > 0 \quad \text{for a. e. } (x, v) \in \Omega \times \mathbb{R}^d, \quad \|\Psi_{\mathbf{H}}\|_X = 1$$

and  $\text{Ker}(\mathbf{T}_{\mathbf{H}}) = \text{Span}(\Psi_{\mathbf{H}})$ . Moreover, if the kernel  $k(x, v, v')$  of the diffuse part of  $\mathbf{H}$  satisfies some suitable positivity assumptions (see (7.4) for a precise statement), then

$$\lim_{t \rightarrow \infty} \|U_{\mathbf{H}}(t)f - \varrho_f \Psi_{\mathbf{H}}\|_X = 0$$

for any  $f \in X$  where  $\varrho_f = \int_{\Omega \times V} f(x, v) dx m(dv)$ .

**Remark 1.2.** All the assumptions of Theorem 1.1 are satisfied by the most common boundary operator used in kinetic theory given by

$$\begin{aligned} \mathcal{H}f(x, v) = & (1 - \beta(x))f(x, v - 2(v \cdot n(x))n(x)) \\ & + \beta(x)\mathcal{M}_{\theta(x)}(v) \int_{v' \cdot n(x) > 0} f(x, v')|v' \cdot n(x)|dv' \end{aligned}$$

where  $\beta(\cdot)$  satisfies the above condition (1.9) and  $\mathcal{M}_{\theta(x)}$  is a given normalized Maxwellian distribution with variable temperature  $\theta(x)$ :

$$\begin{aligned} \mathcal{M}_{\theta(x)}(v) = & \frac{\gamma^{-1}(x)}{(2\pi\theta(x))^{d/2}} \exp\left(-\frac{|v|^2}{2\theta(x)}\right), \\ \text{and } \gamma(x) = & \int_{u \cdot n(x) < 0} \mathcal{M}_{\theta(x)}(u)|u \cdot n(x)|du, \quad \forall x \in \partial\Omega \end{aligned}$$

under the additional assumption that  $\theta(x) \geq \theta_0 > 0$  for any  $x \in \partial\Omega$ . Notice that, in this case, the invariant density  $\Psi_H$  is *not* given by  $\mathcal{M}_{\theta(x)}$  unless  $\theta(\cdot)$  is constant.

We end this Introduction by mentioning recent works [11,12] dealing with rates of convergence to equilibrium, in the spirit of [1,26], while a quantitative tauberian approach to these questions is given in [29] extending the results of [36] devoted to slab geometry. Moreover, we hope also to take advantage of the tools developed here to revisit some important works (see e.g. [20,23] and references therein) on stochastic billiards [31].

The paper is organized as follows: in Section 2, we introduce the mathematical framework and notations used in the rest of the paper and establish several properties of the various operators involved in our subsequent analysis. In Section 3 we introduce and analyse the general class of boundary operators we investigate in the rest of the paper. Section 4 is devoted to general criteria for the ergodic convergence of the semigroup  $(U_H(t))_{t \geq 0}$  (see Theorem 4.7) which is related to the study of the eigenvalue problem (1.6) as well as the irreducibility property of  $(U_H(t))_{t \geq 0}$ . In Section 5 we establish the main technical result of the paper (Theorem 5.1) as well of its consequence on the essential radius (1.8), see Theorem 5.6. Section 6 is devoted to the main existence result for an invariant density, Theorem 6.7. The question of the asymptotic stability of  $(U_H(t))_{t \geq 0}$  is then discussed in Section 7 while the *sweeping properties* of  $(U_H(t))_{t \geq 0}$ , when no invariant density exists, are given in Section 8. As already mentioned, the paper ends with two Appendices. A first one, Appendix A contains all the technical results regarding the smoothness and transversality of the ballistic flow while Appendix B recall several important results about partially integral semigroup and sweeping properties used in Section 7 and 8.

## 2. Mathematical setting and useful formulae

### 2.1. Functional setting

We introduce the partial Sobolev space

$$W_1 = \{\psi \in X; v \cdot \nabla_x \psi \in X\}.$$

It is known [16,17,21] that any  $\psi \in W_1$  admits traces  $\psi|_{\Gamma_{\pm}}$  on  $\Gamma_{\pm}$  such that

$$\psi|_{\Gamma_{\pm}} \in L^1_{\text{loc}}(\Gamma_{\pm}; d\mu_{\pm}(x, v))$$

where

$$d\mu_{\pm}(x, v) = |v \cdot n(x)|\pi(dx) \otimes m(dv),$$

denotes the “natural” measure on  $\Gamma_{\pm}$ . Notice that, since  $d\mu_+$  and  $d\mu_-$  share the same expression, we will often simply denote them by

$$d\mu(x, v) = |v \cdot n(x)|\pi(dx) \otimes m(dv),$$

the fact that it acts on  $\Gamma_-$  or  $\Gamma_+$  being clear from the context. Note that

$$\partial\Omega \times V := \Gamma_- \cup \Gamma_+ \cup \Gamma_0,$$

where

$$\Gamma_0 := \{(x, v) \in \partial\Omega \times V; v \cdot n(x) = 0\}.$$

We introduce the space

$$W = \left\{ \psi \in W_1; \psi|_{\Gamma_{\pm}} \in L^1_{\pm} \right\}.$$

One can show [16, 17] that  $W = \left\{ \psi \in W_1; \psi|_{\Gamma_+} \in L^1_+ \right\} = \left\{ \psi \in W_1; \psi|_{\Gamma_-} \in L^1_- \right\}$ . Then, the trace operators  $B^{\pm}$ :

$$\begin{cases} B^{\pm}: & W_1 \subset X \rightarrow L^1_{\text{loc}}(\Gamma_{\pm}; d\mu_{\pm}) \\ & \psi \mapsto B^{\pm}\psi = \psi|_{\Gamma_{\pm}}, \end{cases}$$

are such that  $B^{\pm}(W) \subseteq L^1_{\pm}$ . Let us define the maximal transport operator  $T_{\max}$  as follows:

$$\begin{cases} T_{\max}: & \mathcal{D}(T_{\max}) \subset X \rightarrow X \\ & \psi \mapsto T_{\max}\psi(x, v) = -v \cdot \nabla_x \psi(x, v), \end{cases}$$

with domain  $\mathcal{D}(T_{\max}) = W_1$ . Now, for any bounded boundary operator  $H \in \mathcal{B}(L^1_+, L^1_-)$ , define  $T_H$  as

$$T_H\varphi = T_{\max}\varphi \quad \text{for any } \varphi \in \mathcal{D}(T_H),$$

where

$$\mathcal{D}(T_H) = \{\psi \in W; \psi|_{\Gamma_-} = H(\psi|_{\Gamma_+})\}.$$

In particular, the transport operator with absorbing conditions (i.e. corresponding to  $H = 0$ ) will be denoted by  $T_0$ . We recall here that there exists a *unique* minimal extension  $(A, \mathcal{D}(A))$  of  $(T_H, \mathcal{D}(T_H))$  which generates a nonnegative  $C_0$ -semigroup  $(U_H(t))_{t \geq 0}$  in  $X$ . We note that  $\mathcal{D}(A) \subset W_1$  and  $A\varphi = -v \cdot \nabla_x \varphi = T_{\max}\varphi$  for any  $\varphi \in \mathcal{D}(A)$  but the traces  $B^{\pm}\varphi$  need not to belong to  $L^1(\Gamma_{\pm}, d\mu_{\pm})$ . The resolvent of  $A$  is given by

$$\mathcal{R}(\lambda, A)f = R_{\lambda}f + \sum_{n=0}^{\infty} \Xi_{\lambda} H (M_{\lambda} H)^n G_{\lambda}f, \quad \forall f \in X, \lambda > 0 \quad (2.1)$$

where the series is *strongly* converging in  $X$ . See Section 2.3 for precise definitions of the various operators involved in (2.1). Moreover,  $(U_H(t))_{t \geq 0}$  is a *stochastic*  $C_0$ -semigroup, i.e.

$$\|U_H(t)f\|_X = \|f\|_X \quad \forall f \in X_+; t \geq 0$$

if and only if

$$A = \overline{T_H}.$$

Actually, under suitable assumptions on  $H$  (see Proposition 4.1),  $A = \overline{T_H}$  so that  $(U_H(t))_{t \geq 0}$  is stochastic.

## 2.2. Exit time and integration formula

Let us now introduce the *exit time* of particles in  $\Omega$  (with the notations of [7]), defined as:

**Definition 2.1.** For any  $(x, v) \in \overline{\Omega} \times V$ , define

$$t_{\pm}(x, v) = \inf\{s > 0; x \pm sv \notin \Omega\}.$$

To avoid confusion, we will set  $\tau_{\pm}(x, v) := t_{\pm}(x, v)$  if  $(x, v) \in \partial\Omega \times V$ .

With the notations of [25],  $t_-$  is the *backward exit time*  $t_b$ . From a heuristic viewpoint,  $t_-(x, v)$  is the time needed by a particle having the position  $x \in \Omega$  and the velocity  $-v \in V$  to reach the boundary  $\partial\Omega$ . One can prove [43, Lemma 1.5] that  $t_{\pm}(\cdot, \cdot)$  is measurable on  $\Omega \times V$ . Moreover  $\tau_{\pm}(x, v) = 0$  for any  $(x, v) \in \Gamma_{\pm}$  whereas  $\tau_{\mp}(x, v) > 0$  on  $\Gamma_{\pm}$ . It holds

$$(x, v) \in \Gamma_{\pm} \iff \exists y \in \Omega \quad \text{with} \quad t_{\pm}(y, v) < \infty \quad \text{and} \quad x = y \pm t_{\pm}(y, v)v.$$

In that case,  $\tau_{\mp}(x, v) = t_{\pm}(y, v)$ . Notice also that,

$$t_{\pm}(x, v)|v| = t_{\pm}(x, \omega), \quad \forall (x, v) \in \overline{\Omega} \times V, \quad v \neq 0, \quad \omega = |v|^{-1}v \in \mathbb{S}^{d-1}. \quad (2.2)$$

We have the following integration formulae from [7].

**Proposition 2.2.** *For any  $h \in X$ , it holds*

$$\int_{\Omega \times V} h(x, v) dx \otimes m(dv) = \int_{\Gamma_{\pm}} d\mu_{\pm}(z, v) \int_0^{\tau_{\mp}(z, v)} h(z \mp sv, v) ds, \quad (2.3)$$

and for any  $\psi \in L^1(\Gamma_-, d\mu_-)$ ,

$$\int_{\Gamma_-} \psi(z, v) d\mu_-(z, v) = \int_{\Gamma_+} \psi(x - \tau_-(x, v)v, v) d\mu_+(x, v). \quad (2.4)$$

**Remark 2.3.** Notice that with the notations introduced in [7],

$$\Gamma_{\pm\infty} = \{(x, v) \in \Gamma_{\pm}; \tau_{\mp}(x, v) = \infty\} = \{(x, v) \in \Gamma_{\pm}; v = 0\}$$

so that  $\mu_{\pm}(\Gamma_{\pm\infty}) = 0$ . This explains why the above integration formulae do not involve the sets  $\Gamma_{\pm\infty}$ . Moreover, because  $\mu_-(\Gamma_0) = \mu_+(\Gamma_0) = 0$ , we can extend the above identity (2.4) as follows: for any  $\psi \in L^1(\Gamma_- \cup \Gamma_0, d\mu_-)$  it holds

$$\int_{\Gamma_- \cup \Gamma_0} \psi(z, v) d\mu_-(z, v) = \int_{\Gamma_+ \cup \Gamma_0} \psi(x - \tau_-(x, v)v, v) d\mu_+(x, v). \quad (2.5)$$

### 2.3. About the resolvent of $T_H$

For any  $\lambda \in \mathbb{C}$  such that  $\operatorname{Re} \lambda > 0$ , define

$$\begin{cases} M_{\lambda} : & L_-^1 \longrightarrow L_+^1 \\ & u \longmapsto M_{\lambda} u(x, v) = u(x - \tau_-(x, v)v, v) e^{-\lambda \tau_-(x, v)}, \quad (x, v) \in \Gamma_+; \\ \Xi_{\lambda} : & L_-^1 \longrightarrow X \\ & u \longmapsto \Xi_{\lambda} u(x, v) = u(x - t_-(x, v)v, v) e^{-\lambda t_-(x, v)} \mathbf{1}_{\{t_-(x, v) < \infty\}}, \quad (x, v) \in \Omega \times V; \\ G_{\lambda} : & X \longrightarrow L_+^1 \\ & \varphi \longmapsto G_{\lambda} \varphi(x, v) = \int_0^{\tau_-(x, v)} \varphi(x - sv, v) e^{-\lambda s} ds, \quad (x, v) \in \Gamma_+; \end{cases}$$

and

$$\begin{cases} R_{\lambda} : & X \longrightarrow X \\ & \varphi \longmapsto R_{\lambda} \varphi(x, v) = \int_0^{t_-(x, v)} \varphi(x - tv, v) e^{-\lambda t} dt, \quad (x, v) \in \Omega \times V; \end{cases}$$

where  $\mathbf{1}_E$  denotes the characteristic function of the measurable set  $E$ . All these operators are bounded on their respective spaces. More precisely, for any  $\operatorname{Re} \lambda > 0$

$$\begin{aligned}\|\mathbf{M}_\lambda\| &\leq 1, & \|\Xi_\lambda\| &\leq (\operatorname{Re} \lambda)^{-1}, \\ \|\mathbf{G}_\lambda\| &\leq 1, & \|\mathbf{R}_\lambda\| &\leq (\operatorname{Re} \lambda)^{-1}.\end{aligned}$$

The interest of these operators is related to the resolution of the boundary value problem:

$$\begin{cases} (\lambda - \mathbf{T}_{\max})f = g, \\ \mathbf{B}^- f = u, \end{cases} \quad (2.6)$$

where  $\lambda > 0$ ,  $g \in X$  and  $u$  is a given function over  $\Gamma_-$ . Such a boundary value problem, with  $u \in L_-^1$  can be uniquely solved (see [7])

**Theorem 2.4.** *Given  $\lambda > 0$ ,  $u \in L_-^1$  and  $g \in X$ , the function*

$$f = \mathbf{R}_\lambda g + \Xi_\lambda u$$

*is the **unique** solution  $f \in \mathcal{D}(\mathbf{T}_{\max})$  of the boundary value problem (2.6).*

**Remark 2.5.** Notice that  $\Xi_\lambda$  is a lifting operator which, to a given  $u \in L_-^1$ , associates a function  $f = \Xi_\lambda u \in \mathcal{D}(\mathbf{T}_{\max})$  whose trace on  $\Gamma_-$  is exactly  $u$ . More precisely,

$$\mathbf{T}_{\max} \Xi_\lambda u = \lambda \Xi_\lambda u, \quad \mathbf{B}^- \Xi_\lambda u = u, \quad \mathbf{B}^+ \Xi_\lambda u = \mathbf{M}_\lambda u, \quad \forall u \in L_-^1. \quad (2.7)$$

We can complement the above result with the following whose proof can be extracted from [8, Theorem 4.2]:

**Proposition 2.6.** *If  $r_\sigma(\mathbf{M}_\lambda \mathbf{H}) < 1$  ( $\lambda > 0$ ), then  $A = \mathbf{T}_\mathbf{H}$  and*

$$\mathcal{R}(\lambda, \mathbf{T}_\mathbf{H}) = \mathbf{R}_\lambda + \Xi_\lambda \mathbf{H} \mathcal{R}(1, \mathbf{M}_\lambda \mathbf{H}) \mathbf{G}_\lambda$$

*where the series converges in  $\mathcal{B}(X)$ .*

#### 2.4. Some auxiliary operators

For  $\lambda = 0$ , we can extend the definition of these operators in an obvious way but not all the resulting operators are bounded in their respective spaces. However, we see from the above integration formula (2.4), that

$$\mathbf{M}_0 \in \mathcal{B}(L_-^1, L_+^1) \quad \text{with} \quad \|\mathbf{M}_0 u\|_{L_+^1} = \|u\|_{L_-^1}, \quad \forall u \in L_-^1.$$

In the same way, one deduces from (2.3) that for any nonnegative  $\varphi \in X$ :

$$\int_{\Gamma_+} \mathbf{G}_0 \varphi(x, v) d\mu_+(x, v) = \int_{\Gamma_+} d\mu_+(x, v) \int_0^{\tau_-(x, v)} \varphi(x - sv, v) ds = \int_{\Omega \times V} \varphi(x, v) dx \otimes \mathbf{m}(dv) \quad (2.8)$$

which proves that

$$\mathbf{G}_0 \in \mathcal{B}(X, L_+^1) \quad \text{with} \quad \|\mathbf{G}_0 \varphi\|_{L_+^1} = \|\varphi\|_X, \quad \forall \varphi \in X.$$

To be able to provide a rigorous definition of the operators  $\Xi_0$  and  $\mathbf{R}_0$  we need the following

**Definition 2.7.** Introduce the function spaces

$$\mathbb{Y}_1^\pm = L^1(\Gamma_\pm, |v|^{-1} d\mu_\pm)$$

with its associated  $L^1$ -norm  $\|\cdot\|_{\mathbb{Y}_1^\pm}$  and

$$\mathbb{X}_\tau = L^1(\Omega \times V, t_+(x, v) dx \otimes \mathbf{m}(dv))$$

with the associated  $L^1$ -norm  $\|\cdot\|_\tau$ .

The interest of the above boundary spaces lies in the following:

**Lemma 2.8.** *For any  $u \in \mathbb{Y}_1^-$  one has  $\Xi_0 u \in X$  with*

$$\|\Xi_0 u\|_X = \int_{\Gamma_-} u(x, v) \tau_+(x, v) d\mu_+(x, v) \leq D \|u\|_{\mathbb{Y}_1^-}, \quad \forall u \in \mathbb{Y}_1^- \quad (2.9)$$

where  $D$  is the diameter of  $\Omega$ ,  $D = \sup_{x, y \in \partial\Omega} |x - y|$ . Moreover, if  $u \in \mathbb{Y}_1^-$  then  $\mathbf{M}_0 u \in \mathbb{Y}_1^+$  with

$$\|\mathbf{M}_0 u\|_{\mathbb{Y}_1^+} = \|u\|_{\mathbb{Y}_1^-}. \quad (2.10)$$

If  $f \in \mathbb{X}_\tau$  then  $\mathbf{G}_0 f \in \mathbb{Y}_1^+$  and  $\mathbf{R}_0 f \in \mathcal{D}(\mathbf{T}_0) \subset X$  and  $\mathbf{T}_0 \mathbf{R}_0 f = -f$ .

**Proof.** From (2.3), for nonnegative  $u \in \mathbb{Y}_1^-$ :

$$\begin{aligned} & \int_{\Omega \times V} \Xi_0 u(x, v) dx \otimes m(dv) \\ &= \int_{\Gamma_+} d\mu_+(z, v) \int_0^{\tau_-(z, v)} u(z - sv - t_-(z - sv, v)v, v) \mathbf{1}_{\{t_-(z - sv, v) < \infty\}} ds \\ &= \int_{\Gamma_+} u(z - \tau_-(z, v)v, v) \tau_-(z, v) d\mu_+(z, v) \end{aligned}$$

which, using now (2.4) yields (2.9). If now  $u \in \mathbb{Y}_1^-$ , then

$$\int_{\Gamma_+} \mathbf{M}_0 u(x, v) |v|^{-1} d\mu_+(x, v) = \int_{\Gamma_+} u(x - \tau_-(x, v)v, v) |v|^{-1} d\mu_+(x, v)$$

and we deduce from (2.4) that

$$\int_{\Gamma_+} \mathbf{M}_0 u(x, v) |v|^{-1} d\mu_+(x, v) = \int_{\Gamma_-} u(z, v) |v|^{-1} d\mu_-(z, v)$$

which is (2.10). If now  $f \in \mathbb{X}_\tau$  is nonnegative, one has directly from (2.3) that

$$\int_{\Gamma_+} |v|^{-1} \mathbf{G}_0 f(x, v) d\mu_+(x, v) = \| |v|^{-1} f \|_X \leq D \|f\|_\tau < \infty$$

which proves that  $\mathbf{G}_0 f \in \mathbb{Y}_1^+$ . Moreover, using (2.3),

$$\begin{aligned} \int_{\Omega \times V} \mathbf{R}_0 f(x, v) dx \otimes m(dv) &= \int_{\Gamma_+} d\mu_+(z, v) \int_0^{\tau_-(z, v)} [\mathbf{R}_0 f](z - sv, v) ds \\ &= \int_{\Gamma_+} d\mu_+(z, v) \int_0^{\tau_-(z, v)} ds \int_0^{t_-(z - sv, v)} f(z - sv - tv, v) dt \end{aligned}$$

and, since  $t_-(z - sv, v) = \tau_-(z, v) - s$  for all  $(z, v) \in \Gamma_+$  and all  $0 < s < \tau_-(z, v)$  we get

$$\begin{aligned} \int_{\Omega \times V} R_0 f(x, v) dx \otimes m(dv) &= \int_{\Gamma_+} d\mu_+(z, v) \int_0^{\tau_-(z, v)} ds \int_s^{\tau_-(z, v)} f(z - tv, v) dt \\ &= \int_{\Gamma_+} d\mu_+(z, v) \int_0^{\tau_-(z, v)} t f(z - tv, v) dt \end{aligned}$$

Now, since  $t_+(z - tv, v) = t$  for any  $(z, v) \in \Gamma_+$ , the above reads

$$\int_{\Omega \times V} R_0 f(x, v) dx \otimes m(dv) = \int_{\Gamma_+} d\mu_+(z, v) \int_0^{\tau_-(z, v)} t_+(z - tv, v) f(z - tv, v) dt$$

and, using again (2.3), one gets

$$\int_{\Omega \times V} R_0 f(x, v) dx \otimes m(dv) = \int_{\Omega \times V} t_+(x, v) f(x, v) dx \otimes m(dv).$$

This proves that  $R_0 f \in X$ . Now, it is easy to see that actually  $g = R_0 f$  satisfies  $T_{\max} g = -f$  and  $B^- g = 0$ , i.e.  $g \in \mathcal{D}(T_0)$  with  $T_0 g = -f$ .  $\square$

**Remark 2.9.** Notice that, for any nonnegative  $u \in L^1_+$ ,

$$\int_{\Gamma_+} M_0 u(x, v) \tau_-(x, v) d\mu_+(x, v) = \int_{\Gamma_+} u(x - \tau_-(x, v)v, v) \tau_-(x, v) d\mu_+(x, v)$$

and, since  $\tau_+(x - \tau_-(x, v)v, v) = \tau_-(x, v)$  for any  $(x, v) \in \Gamma_-$ , we deduce from (2.4) that

$$\int_{\Gamma_+} M_0 u(x, v) \tau_-(x, v) d\mu_+(x, v) = \int_{\Gamma_-} u(z, v) \tau_+(z, v) d\mu_-(z, v)$$

This shows that, in (2.10), we can replace  $\mathbb{Y}_1^\pm$  with  $L^1(\Gamma_\pm, \tau_\mp(x, v) d\mu_\pm(x, v))$ . In the same way, one sees that, for  $g \in \mathbb{X}_\tau$  it holds  $\|G_0 g\|_{L^1(\Gamma_-, \tau_+ d\mu_-)} = \|g\|_\tau$ .

One has the following result:

**Proposition 2.10.** *Let  $g \in \mathbb{X}_\tau$  be given and  $u \in L^1(\Gamma_-, \tau_+(x, v) d\mu_-)$ . The boundary value problem*

$$\begin{cases} -T_{\max} f = g \\ B^- f = u \end{cases} \quad (2.11)$$

*admits a unique solution  $f \in X$  given by  $f = R_0 g + \Xi_0 u$ .*

**Proof.** Let  $u \in L^1(\Gamma_-, \tau_+(x, v) d\mu_-)$  and  $g \in \mathbb{X}_\tau$ , since  $\Xi_0 u \in \mathcal{D}(T_{\max})$  with  $T_{\max} \Xi_0 u = 0$ ,  $B^+ \Xi_0 u = M_0 u$  and  $B^- \Xi_0 u = u$  one sees that  $f \in \mathcal{D}(T_{\max})$  with

$$T_{\max} f = T_{\max} R_0 g + T_{\max} \Xi_0 u = T_0 R_0 g = -g$$

while  $B^- f = B^- R_0 g + B^- \Xi_0 u = B^- \Xi_0 u = u$ . This shows that  $f = R_0 g + \Xi_0 u$  is a solution to (2.11). To prove the uniqueness, it suffices to assume that  $g = u = 0$  but then (2.11) reads  $T_0 f = 0$  which admits the unique solution  $f = 0$ .  $\square$

### 3. General stochastic partly diffuse boundary conditions

Let us explicit here the general class of boundary conditions we aim to deal with. Typical boundary operators arising in the kinetic theory of gases are *local with respect to*  $x \in \partial\Omega$ . In order to exploit this local nature of the boundary conditions, we introduce the following notations. For any  $x \in \partial\Omega$ , we define

$$\Gamma_{\pm}(x) = \{v \in V ; \pm v \cdot n(x) > 0\}, \quad \Gamma_0(x) = \{v \in V ; v \cdot n(x) = 0\}$$

and we define the measure  $\mu_x(dv)$  on  $\Gamma_{\pm}(x)$  given by

$$\mu_x(dv) = |v \cdot n(x)| m(dv).$$

This allows to define the  $L^1$ -space  $L^1(\Gamma_{\pm}(x), d\mu_x)$  in an obvious way. We shall denote the  $L^1(\Gamma_{\pm}(x), \mu_x)$  norm by  $\|\cdot\|_{L^1(\Gamma_{\pm}(x))}$ . Since, for any  $\varphi \in L^1(\Gamma_{\pm}, \mu_{\pm})$  one has

$$\|\varphi\|_{L^1_{\pm}} = \int_{\partial\Omega} \left[ \int_{\Gamma_{\pm}(x)} |\varphi(x, v)| \mu_x(dv) \right] \pi(dx) = \int_{\partial\Omega} \|\varphi(x, \cdot)\|_{L^1(\Gamma_{\pm}(x))} \pi(dx)$$

we can identify isometrically any  $\varphi \in L^1_{\pm}$  to the *field*

$$x \in \partial\Omega \mapsto \varphi(x, \cdot) \in L^1(\Gamma_{\pm}(x)). \quad (3.1)$$

#### 3.1. Reflection boundary operators

We begin with the following definition of pure reflection boundary conditions (see [43, Definition 6.1, p. 104]):

**Definition 3.1.** One says that  $R \in \mathcal{B}(L^1_+, L^1_-)$  is a pure reflection boundary operator if

$$R(\varphi)(x, v) = \varphi(x, \mathcal{V}(x, v)) \quad \forall (x, v) \in \Gamma_-, \varphi \in L^1_+$$

where  $\mathcal{V} : x \in \partial\Omega \mapsto \mathcal{V}(x, \cdot)$  is a field of bijective bi-measurable and  $\mu_x$ -preserving mappings

$$\mathcal{V}(x, \cdot) : \Gamma_-(x) \cup \Gamma_0(x) \rightarrow \Gamma_+(x) \cup \Gamma_0(x)$$

such that

- i)  $|\mathcal{V}(x, v)| = |v|$  for any  $(x, v) \in \Gamma_-$ .
- ii) If  $(x, v) \in \Gamma_0$  then  $(x, \mathcal{V}(x, v)) \in \Gamma_0$ , i.e.  $\mathcal{V}(x, \cdot)$  maps  $\Gamma_0(x)$  in  $\Gamma_0(x)$ .
- iii) The mapping

$$(x, v) \in \Gamma_- \mapsto (x, \mathcal{V}(x, v)) \in \Gamma_+$$

is a  $C^1$  diffeomorphism.

**Remark 3.2.** This last regularity property on  $\mathcal{V}$  may require additional regularity of  $\partial\Omega$  as seen in Example 3.3.

Note that

$$[R\varphi](x, v) = [R(x)\varphi(x, \cdot)](v), \quad \forall \varphi \in L^1_+, (x, v) \in \Gamma_-$$

where we identify (isometrically)  $\varphi \in L^1_+$  to the integrable field (3.1) and

$$x \in \partial\Omega \mapsto R(x) \in \mathcal{B}(L^1(\Gamma_+(x), \Gamma_-(x)))$$

is the field of operators defined by

$$R(x)\psi(v) = \psi(\mathcal{V}(x, v)), \quad \psi \in L^1(\Gamma_+(x)), \quad v \in \Gamma_-(x).$$

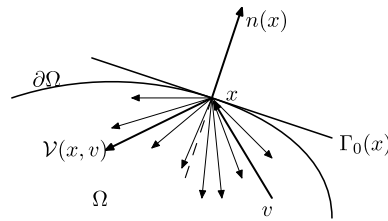


Fig. 2. Regular and diffuse reflection:  $v$  – an outward vector,  $\mathcal{V}(x, v)$  – the specular reflection, thin vectors – diffuse reflection.

It holds

$$\|\mathbf{R}(x)\psi\|_{L^1(\Gamma_-(x))} = \|\psi\|_{L^1(\Gamma_+(x))}, \quad \forall x \in \partial\Omega, \quad \psi \in L^1(\Gamma_+(x)),$$

therefore,  $\mathbf{R} \in \mathcal{B}(L_+^1, L_-^1)$  is stochastic since

$$\|\mathbf{R}\varphi\|_{L_-^1} = \int_{\partial\Omega} \|\mathbf{R}(x)\varphi(x, \cdot)\|_{L^1(\Gamma_-(x))} \pi(dx) = \int_{\partial\Omega} \|\varphi(x, \cdot)\|_{L^1(\Gamma_+(x))} \pi(dx) = \|\varphi\|_{L_+^1}.$$

Notice that this last identity is equivalent to the property that the mapping

$$(x, v) \in \Gamma_- \longmapsto (x, \mathcal{V}(x, v)) \in \Gamma_+$$

is  $\mu$ -preserving.

**Example 3.3.** In practical situations, the most frequently used pure reflection conditions are

- (a) the *specular reflection boundary conditions* (see Fig. 2) which corresponds to the case in which  $V$  and  $m$  are invariant under the orthogonal group and

$$\mathcal{V}(x, v) = v - 2(v \cdot n(x))n(x) \quad (x, v) \in \Gamma_-.$$

Notice that, for  $\mathcal{V}$  to be a  $\mathcal{C}^1$  diffeomorphism, we need  $\partial\Omega$  to be of class  $\mathcal{C}^2$ .

- (b) The *bounce-back reflection conditions* for which  $\mathcal{V}(x, v) = -v$ ,  $(x, v) \in \Gamma_-$ .

### 3.2. Diffuse boundary operators

We introduce the following definition

**Definition 3.4.** One says that  $\mathbf{K} \in \mathcal{B}(L_+^1, L_-^1)$  is a *stochastic diffuse boundary operator* if

$$\mathbf{K}\psi(x, v) = \int_{\Gamma_+(x)} k(x, v, v')\psi(x, v')\mu_x(dv'), \quad (x, v) \in \Gamma_-, \quad \psi \in L_+^1 \quad (3.2)$$

where the kernel  $k(x, v, v')$  induces a field of nonnegative measurable functions

$$x \in \partial\Omega \mapsto k(x, \cdot, \cdot)$$

where

$$k(x, \cdot, \cdot) : \Gamma_-(x) \times \Gamma_+(x) \rightarrow \mathbb{R}^+$$

is such that

$$\int_{\Gamma_-(x)} k(x, v, v')\mu_x(dv) = 1, \quad \forall (x, v') \in \Gamma_+.$$

As we did for reflection operators, we identify  $K \in \mathcal{B}(L_+^1, L_-^1)$  to a field of integral operators

$$x \in \partial\Omega \mapsto K(x) \in \mathcal{B}(L^1(\Gamma_+(x), \Gamma_-(x)))$$

by the formula

$$[K\psi](x, v) = [K(x)\psi(x, \cdot)](v)$$

where, for any  $x \in \partial\Omega$

$$K(x) : \psi \in L^1(\Gamma_+(x)) \mapsto [K(x)\psi](v) = \int_{\Gamma_+(x)} k(x, v, v')\psi(v')\mu_x(dv') \in L^1(\Gamma_-(x)).$$

Note that  $K(x) : L^1(\Gamma_+(x)) \rightarrow L^1(\Gamma_-(x))$  is stochastic for any  $x \in \partial\Omega$  and therefore so is  $K \in \mathcal{B}(L_+^1, L_-^1)$ , i.e.

$$\|K\psi\|_{L_-^1} = \|\psi\|_{L_+^1} \quad \forall \psi \in L_+^1.$$

We introduce now a useful class of diffuse boundary operators. Before giving the formal definition, let us recall that, if  $K \in \mathcal{B}(L_+^1, L_-^1)$  given by (3.2) is such that

$$K(x) \in \mathcal{B}(L^1(\Gamma_+(x)), L^1(\Gamma_-(x))) \quad \text{is weakly compact for any } x \in \partial\Omega \quad (3.3)$$

then, according to the Dunford-Pettis criterion (see [13, Theorem 4.30, p. 115 & Exercise 4.36, p. 129]), for any  $x \in \partial\Omega$  and any  $\varepsilon > 0$ , there is  $\delta > 0$  such that

$$\sup_{v' \in \Gamma_+(x)} \int_A k(x, v, v')\mu_x(dv) < \varepsilon \quad \forall A \subset \Gamma_-(x) \text{ such that } \mu_x(A) < \delta$$

and

$$\lim_{m \rightarrow \infty} \sup_{v' \in \Gamma_+(x)} \int_{\Gamma_-(x) \setminus A_m} k(x, v, v')\mu_x(dv) = 0$$

for any sequence  $(A_m)_m \subset \Gamma_-(x)$  with  $A_m \subset A_{m+1}$ ,  $\mu_x(A_m) < \infty$  and  $\cup_m A_m = \Gamma_-(x)$ . In particular, for any  $x \in \partial\Omega$ ,

$$\lim_{m \rightarrow \infty} \sup_{v' \in \Gamma_+(x)} \int_{\{v \in \Gamma_-(x); |v| \geq m\}} k(x, v, v')\mu_x(dv) = 0.$$

Moreover, since

$$\begin{aligned} 1 &= \int_{\Gamma_-(x)} k(x, v, v')\mu_x(dv) \geq \int_{\{v \in \Gamma_-(x); k(x, v, v') \geq m\}} k(x, v, v')\mu_x(dv) \\ &\geq m \mu_x(\{v \in \Gamma_-(x); k(x, v, v') \geq m\}), \quad \forall m \in \mathbb{N}, \quad (x, v') \in \Gamma_+, \end{aligned}$$

we have

$$\lim_{m \rightarrow \infty} \sup_{v' \in \Gamma_+(x)} \mu_x(\{v \in \Gamma_-(x); k(x, v, v') \geq m\}) = 0.$$

In other words, for any  $x \in \partial\Omega$ , the following holds

$$\lim_{m \rightarrow \infty} \sup_{v' \in \Gamma_+(x)} \int_{S_m(x, v')} k(x, v, v')\mu_x(dv) = 0 \quad (3.4)$$

where, for any  $m \in \mathbb{N}$  and any  $(x, v') \in \Gamma_+$

$$S_m(x, v') = \{v \in \Gamma_-(x); |v| \geq m\} \cup \{v \in \Gamma_-(x); k(x, v, v') \geq m\}.$$

We introduce then the following class of diffuse boundary operators:

**Definition 3.5.** We say that a diffuse boundary operator  $K \in \mathcal{B}(L_+^1, L_-^1)$  is *regular* if the family of operators

$$K(x) \in \mathcal{B}(L^1(\Gamma_+(x)), L^1(\Gamma_-(x))), \quad x \in \partial\Omega$$

is collectively weakly compact in the sense that (3.3) holds true for any  $x \in \partial\Omega$  and the convergence in (3.4) is *uniform* with respect to  $x \in \partial\Omega$ .

**Remark 3.6.** A diffuse boundary operator  $K$  is regular for instance whenever there exists  $h : V \rightarrow \mathbb{R}^+$  such that  $\int_V h(v) |v| \mathbf{m}(dv) < +\infty$  and

$$k(x, v, v') \leq h(v) \quad \forall x \in \partial\Omega, \quad v' \in \Gamma_+(x), \quad v \in \Gamma_-(x).$$

In particular, the classical Maxwell boundary operator (see Example 6.4 below) is a regular diffuse boundary operator.

We have then the following approximation result.

**Lemma 3.7.** Assume that  $K \in \mathcal{B}(L_+^1, L_-^1)$  is a regular diffuse boundary operator in the sense of the above definition. Then, there exists a sequence  $(K_m)_m \subset \mathcal{B}(L_+^1, L_-^1)$  such that

- (1)  $0 \leq K_m \leq K$  for any  $m \in \mathbb{N}$ ;
- (2)  $\lim_{m \rightarrow \infty} \|K - K_m\|_{\mathcal{B}(L_+^1, L_-^1)} = 0$ ;
- (3) For any  $m \in \mathbb{N}$  and any nonnegative  $f \in L_+^1$  it holds

$$K_m f(x, v) \leq \psi_m(v) \int_{\Gamma_+(x)} f(x, v') |v' \cdot n(x)| \mathbf{m}(dv'), \quad (x, v) \in \Gamma_- \quad (3.5)$$

with  $\psi_m = m \mathbf{1}_{B_m}$  where  $B_m = \{v \in \mathbb{R}^d; |v| \leq m\}$ .

**Proof.** Let  $k(x, v, v')$  be the kernel associated to  $K$  through (3.2). Introduce then  $k_m(x, v, v') = \inf\{k(x, v, v'); m \mathbf{1}_{B_m}(v)\}$  for any  $m \in \mathbb{N}$ , where  $B_m$  is the ball of  $\mathbb{R}^d$  centered in 0 and with radius  $m$ , and set

$$K_m \varphi(x, v) = \int_{\Gamma_+(x)} k_m(x, v, v') \varphi(x, v') |v' \cdot n(x)| \mathbf{m}(dv'), \quad \varphi \in L_+^1, \quad (x, v) \in \Gamma_-.$$

Clearly,  $K_m \in \mathcal{B}(L_+^1, L_-^1)$  is a diffuse boundary operator with  $0 \leq K_m \leq K$  and (3.5) holds. Moreover, for any  $x \in \partial\Omega$  and any  $\varphi \in L^1(\Gamma_+(x))$ , it is easy to check that

$$\begin{aligned} \|K(x)\varphi - K_m(x)\varphi\|_{L^1(\Gamma_-(x))} &\leq \|\varphi\|_{L^1(\Gamma_+(x))} \\ &\quad \times \sup_{v' \in \Gamma_+(x)} \int_{\{v \in \Gamma_-(x); k(x, v, v') \geq m \mathbf{1}_{B_m}(v)\}} k(x, v, v') \mu_x(dv), \end{aligned}$$

i.e.

$$\begin{aligned} \|K(x) - K_m(x)\|_{\mathcal{B}(L^1(\Gamma_+(x)), L^1(\Gamma_-(x)))} &\leq \sup_{v' \in \Gamma_+(x)} \int_{\{v \in \Gamma_-(x); k(x, v, v') \geq m \mathbf{1}_{B_m}(v)\}} k(x, v, v') \mu_x(dv) \\ &\leq \sup_{v' \in \Gamma_+(x)} \int_{S_m(x, v')} k(x, v, v') \mu_x(dv). \end{aligned}$$

One sees then that

$$\|K - K_m\|_{\mathcal{B}(L_+^1, L_-^1)} = \sup_{x \in \partial\Omega} \|K(x) - K_m(x)\|_{\mathcal{B}(L^1(\Gamma_+(x)), L^1(\Gamma_-(x)))}$$

goes to zero as  $m \rightarrow \infty$  since the convergence in (3.4) is uniform with respect to  $x \in \partial\Omega$ .  $\square$

We complement the above result with a different kind of approximation which will turn useful in Section 8:

**Lemma 3.8.** *Let  $K$  be a regular stochastic diffuse boundary operator with kernel  $k(x, v, v')$ . Let*

$$\beta_n(x, v') = \int_{\Gamma_-(x) \cap \{|v| > \frac{1}{n}\}} k(x, v, v') \mu_x(dv), \quad (x, v') \in \Gamma_+$$

and

$$k_n(x, v, v') = \frac{k(x, v, v')}{\beta_n(x, v')} \mathbf{1}_{\{|v| > \frac{1}{n}\}}, \quad x \in \partial\Omega, v' \in \Gamma_+(x), v \in \Gamma_-(x)$$

Then, denoting by  $K_n$  the regular stochastic diffuse boundary operator with kernel  $k_n$ , it holds

- (i)  $\lim_{n \rightarrow +\infty} \beta_n(x, v') = 1$  uniformly in  $(x, v') \in \Gamma_+$ .
- (ii)  $\lim_{n \rightarrow \infty} \|K_n - K\|_{\mathcal{B}(L^1_+, L^1_-)} = 0$ .

**Proof.** (i) For any  $x \in \partial\Omega$ , set  $A_n(x) = \{v \in \Gamma_-(x); |v| < \frac{1}{n}\}$ . One has

$$\mu_x(A_n(x)) = \int_{\Gamma_-(x) \cap \{|v| < n^{-1}\}} |v \cdot n(x)| m(dv) \leq \frac{m(B_1)}{n}$$

where  $B_1$  is the unit ball of  $\mathbb{R}^d$ . Thus,

$$\lim_{n \rightarrow \infty} \sup_{x \in \partial\Omega} \mu_x(A_n(x)) = 0. \quad (3.6)$$

Now, since  $K$  is regular, (3.4) holds uniformly with respect to  $x \in \partial\Omega$ , i.e. for any  $\varepsilon > 0$ , there is  $m \in \mathbb{N}$  large enough so that

$$\sup_{(x, v') \in \Gamma_+} \int_{S_m(x, v')} k(x, v, v') \mu_x(dv) < \varepsilon.$$

Then, for any  $n \in \mathbb{N}$  and any  $(x, v') \in \Gamma_+$ ,

$$\begin{aligned} \int_{A_n(x)} k(x, v, v') \mu_x(dv) &= \int_{A_n(x) \cap S_m(x, v')} k(x, v, v') \mu_x(dv) \\ &\quad + \int_{A_n(x) \setminus S_m(x, v')} k(x, v, v') \mu_x(dv) \leq \varepsilon + m \mu_x(A_n(x)) \end{aligned}$$

since  $k(x, v, v') \leq m$  on  $\Gamma_-(x) \setminus S_m(x, v')$ . Using then (3.6) we get

$$\lim_{n \rightarrow \infty} \sup_{(x, v') \in \Gamma_+} \int_{\Gamma_-(x) \cap \{|v| < n^{-1}\}} k(x, v, v') \mu_x(dv) = 0 \quad (3.7)$$

which shows (i) since  $\int_{\Gamma_-(x)} k(x, v, v') \mu_x(dv) = 1$  for any  $x \in \partial\Omega$ .

(ii) Set  $\widehat{K}_n$  the boundary operator with kernel  $\mathbf{1}_{\{|v| > n^{-1}\}} k(x, v, v')$ . One checks easily that

$$\|K - \widehat{K}_n\|_{\mathcal{B}(L^1_+, L^1_-)} \leq \sup_{(x, v') \in \Gamma_+} \int_{\Gamma_-(x) \cap \{|v| < n^{-1}\}} k(x, v, v') \mu_x(dv)$$

so that  $\lim_n \|K - \widehat{K}_n\|_{\mathcal{B}(L^1_+, L^1_-)} = 0$  from (3.7). Since moreover

$$\|K_n - \widehat{K}_n\|_{\mathcal{B}(L^1_+, L^1_-)} \leq \sup_{(x, v') \in \Gamma_+} |1 - \beta_n(x, v')| \|K\|_{\mathcal{B}(L^1_+, L^1_-)}$$

which goes to zero from point (i), we get the desired result.  $\square$

### 3.3. Stochastic partly diffuse boundary operators

We introduce now the general class of boundary operator we aim at investigating.

**Definition 3.9.** We shall say that a boundary operator  $H \in \mathcal{B}(L_+^1, L_-^1)$  is *stochastic partly diffuse* if it writes

$$H\psi(x, v) = \alpha(x) R\psi(x, v) + (1 - \alpha(x)) K\psi(x, v), \quad (x, v) \in \Gamma_-, \psi \in L_+^1 \quad (3.8)$$

where  $\alpha(\cdot) : \partial\Omega \rightarrow [0, 1]$  is measurable,  $R$  is a reflection operator, and  $K \in \mathcal{B}(L_+^1, L_-^1)$  is a *stochastic diffuse boundary operator* given by (3.2).

If the diffuse part  $K$  is regular we say that  $H$  is a *regular stochastic partly diffuse boundary operator*.

**Remark 3.10.** Notice that, being a convex combination of stochastic operators, a stochastic partly diffuse operator  $H$  is stochastic.

## 4. General results for abstract stochastic boundary operators

We begin with the following which is a direct consequence of [34, Theorem 21]:

**Proposition 4.1.** Let  $H \in \mathcal{B}(L_+^1, L_-^1)$  be a stochastic boundary operator. Let there exist  $\varphi \in L_+^1$  such that  $\varphi > 0$   $\mu$ -a.e. and  $\varphi \geq M_0 H\varphi$ . Then,  $A = \overline{T_H}$  and  $(U_H(t))_{t \geq 0}$  is a stochastic  $C_0$ -semigroup, where we recall that  $(A, \mathcal{D}(A))$  is the generator of  $(U_H(t))_{t \geq 0}$ .

We give a general result about the spectrum of  $T_H$ :

**Proposition 4.2.** Let  $H \in \mathcal{B}(L_+^1, L_-^1)$  be a stochastic boundary operator. Then, there is a nonnegative  $\Psi \in \mathcal{D}(T_H)$  with  $T_H\Psi = 0$  if and only if 1 is an eigenvalue of  $M_0 H$  associated to a nonnegative eigenfunction  $\varphi \in \mathbb{Y}_1^+$  such that  $H\varphi \in \mathbb{Y}_1^-$ .

**Proof.** Assume first that there exists  $\varphi \in \mathbb{Y}_1^+$  such that

$$M_0 H\varphi = \varphi, \quad \text{and} \quad H\varphi \in \mathbb{Y}_1^-.$$

Then, as already seen (see (2.9)),  $\Xi_0 H\varphi \in X$ . Let  $\Psi = \Xi_0 H\varphi$ . One has  $T_{\max}\Psi = 0$ ,  $B^-\Psi = B^-\Xi_0 H\varphi = H\varphi$ , i.e.  $B^-\Psi = HB^+\Psi$ . This means that  $\Psi \in \mathcal{D}(T_H)$  with  $T_H\Psi = 0$ .

Assume now that  $0 \in \mathfrak{S}_p(T_H)$  is associated to a nonnegative eigenfunction  $\Psi \in \mathcal{D}(T_H)$ . Let  $\varphi = B^+\Psi$  and  $u = B^-\Psi$ . It holds  $u = H\varphi$  and, solving the boundary value problem (2.11) (see Proposition 2.10) yields  $\Psi = \Xi_0 u$ . It is easy to check then that  $\varphi = M_0 H\varphi$ . Since  $u \neq 0 \implies \varphi \neq 0$ , we get  $1 \in \mathfrak{S}_p(M_0 H)$ .  $\square$

We recall the definition of irreducible operators or semigroups in  $L^1$ -spaces and refer to [2] for more details.

**Definition 4.3.** Let  $(E, \Sigma, m)$  be a given  $\sigma$ -finite measure space. Let  $B \in \mathcal{B}(L^1(E, \Sigma, m))$  be given. Then, we say that

- i)  $B$  is *positive* and write  $B \geq 0$ , if  $B$  leaves invariant the cone of nonnegative functions of  $L^1(E, \Sigma, m)$  i.e. for any  $h \in L^1(E, \Sigma, m)$ ,

$$h(s) \geq 0 \quad \text{for } m\text{-a. e. } s \in E \implies Bh(s) \geq 0 \quad \text{for } m\text{-a. e. } s \in E.$$

- ii)  $B$  is *positivity improving* if for any  $h \in L^1(E, \Sigma, m)$  non identically zero

$$h(s) \geq 0 \quad \text{for } m\text{-a. e. } s \in E \implies Bh(s) > 0 \quad \text{for } m\text{-a. e. } s \in E.$$

iii)  $B$  is *irreducible* if for any non trivial and nonnegative  $h \in L^1(E, \Sigma, m)$  and any non trivial nonnegative  $g \in L^\infty(E, \Sigma, m)$ , there exists  $n \in \mathbb{N}$  such that

$$\langle B^n h, g \rangle_{1,\infty} > 0$$

where  $\langle \cdot, \cdot \rangle_{1,\infty}$  is the duality bracket between  $L^1(E, \Sigma, m)$  and  $L^\infty(E, \Sigma, m)$ .

iv) A positive  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  on  $L^1(E, \Sigma, m)$  with generator  $(G, \mathcal{D}(G))$  is irreducible if, for any non trivial nonnegative  $h \in L^1(E, \Sigma, m)$  and any non trivial nonnegative  $g \in L^\infty(E, \Sigma, m)$ , there exists  $t \geq 0$  such that  $\langle S(t)h, g \rangle_{1,\infty} > 0$ . This property is equivalent to the fact that  $\mathcal{R}(\lambda, G)$  is positivity improving for  $\lambda > 0$  large enough.

We introduce the following:

**Assumption 4.4.**  $H \in \mathcal{B}(L_+^1, L_-^1)$  is a stochastic operator such that  $M_0 H \in \mathcal{B}(L_+^1)$  is irreducible and there exists  $\psi_0 \in L_+^1$  such that  $H\psi_0 > 0$   $\mu$ -a.e. on  $\Gamma_-$ .

**Remark 4.5.** If  $H$  is stochastic partly diffuse operator of the form (3.8) and if  $M_0(1 - \alpha)K$  is irreducible then so is  $M_0 H$ . This is the case for instance if  $\|\alpha\|_\infty < 1$  and if, for  $\pi$ -a.e.  $x \in \partial\Omega$

$$k(x, v, v') > 0 \quad \text{for } \mu_x\text{-a.e. } v \in \Gamma_-(x), v' \in \Gamma_+(x). \quad (4.1)$$

In addition, if  $\alpha(x) > 0$   $\pi$ -a.e.  $x \in \partial\Omega$ , the second condition  $H\psi_0 > 0$   $\mu$ -a.e. is satisfied by any  $\psi_0 > 0$   $\mu$ -a.e. Otherwise, we assume that, for any  $x \in \partial\Omega$  such that  $\alpha(x) = 0$  there is a subset  $\gamma_+(x) \subset \Gamma_+(x)$  of positive  $\mu_x$ -measure and such that  $k(x, v, v') > 0$  for  $\mu_x$ -a.e.  $v \in \Gamma_+(x), v' \in \gamma_+(x)$ . Then, again the second condition is satisfied by any  $\psi_0 > 0$   $\mu$ -a.e. In particular, the second assumption in Assumptions 4.4 is always satisfied under (4.1).

One has the following (see also [34, Remark 20]):

**Proposition 4.6.** Let  $H \in \mathcal{B}(L_+^1, L_-^1)$  be a stochastic operator and let Assumptions 4.4 be satisfied. Then, the  $C_0$ -semigroup  $(U_H(t))_{t \geq 0}$  is irreducible.

**Proof.** Let  $h \in X$  and  $g \in L^\infty(\Omega \times V, dx \otimes m(dv))$  be nonnegative and non trivial. Denoting the duality parity between  $X$  and its dual simply by  $\langle \cdot, \cdot \rangle$ , we have for any  $\lambda > 0$

$$\langle \mathcal{R}(\lambda, A)h, g \rangle = \langle R_\lambda h, g \rangle + \sum_{k=0}^{\infty} \langle \Xi_\lambda H (M_\lambda H)^k G_\lambda g, g \rangle = \langle R_\lambda h, g \rangle + \sum_{k=0}^{\infty} \langle (M_\lambda H)^k G_\lambda h, H^* \Xi_\lambda^* g \rangle$$

where  $H^*$  and  $\Xi_\lambda^*$  denote the dual operator of  $H$  and  $\Xi_\lambda$  respectively. Notice that  $g_\star = \Xi_\lambda^* g$  is nonnegative and nontrivial and the same holds for  $H^* \Xi_\lambda^* = H^* g_\star$  since, under Assumption 4.4

$$\langle \psi_0, H^* g_\star \rangle = \langle H\psi_0, g_\star \rangle > 0.$$

Now, since the irreducibility of  $M_0 H$  is equivalent to that of  $M_\lambda H$  for any  $\lambda > 0$ , we deduce that  $\langle \mathcal{R}(\lambda, A)h, g \rangle > 0$  for any nonnegative and nontrivial  $g \in L^\infty(\Omega \times V, dx \otimes m(dv))$  which proves that  $\mathcal{R}(\lambda, A)h$  is positive a. e. on  $\Omega \times V$  and  $\mathcal{R}(\lambda, A)$  is positivity improving.  $\square$

The main result of this section is then the following:

**Theorem 4.7.** Let  $H \in \mathcal{B}(L_+^1, L_-^1)$  be a stochastic boundary operator and let Assumptions 4.4 be satisfied. Assume there exists  $\Psi_H \in \mathcal{D}(\Gamma_H)$  (with unit norm) such that

$$\Gamma_H \Psi_H = 0 \quad \text{and} \quad \Psi_H \geq 0 \quad \text{a.e. on } \Omega \times V.$$

Then,  $\overline{\Gamma_H}$  generates an irreducible and stochastic  $C_0$ -semigroup  $(U_H(t))_{t \geq 0}$  on  $X$  and  $\Psi_H$  is the unique invariant density of  $(U_H(t))_{t \geq 0}$ . Moreover,  $(U_H(t))_{t \geq 0}$  is ergodic with ergodic projection

$$\mathbb{P}f = \varrho_f \Psi_H, \quad \text{where} \quad \varrho_f = \int_{\Omega \times \mathbb{R}^d} f(x, v) dx \otimes m(dv), \quad f \in X,$$

i.e.

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t U_H(s) f ds - \varrho_f \Psi_H \right\|_X = 0, \quad \forall f \in X \quad (4.2)$$

and  $X = \text{Ker}(\overline{T_H}) \oplus \overline{\text{Range}(\overline{T_H})}$ .

**Proof.** According to Proposition 4.2, there exists  $\varphi \in L_+^1$  such that  $\varphi \geq 0$   $\mu$ -a.e. and  $\varphi = M_0 H \varphi$ . Since  $M_0 H$  is irreducible,  $\varphi > 0$   $\mu$ -a.e. and we deduce from Proposition 4.1 that  $A = \overline{T_H}$ . Moreover, Lemma 4.6 ensures that  $(U_H(t))_{t \geq 0}$  is irreducible. Since  $\text{Ker}(\overline{T_H}) \neq 0$ , the ergodicity of  $(U_H(t))_{t \geq 0}$  follows from [22, Theorem 7.3, p. 174 and Theorem 5.1 p. 123].  $\square$

## 5. Weak compactness result and existence of an invariant density

### 5.1. Weak compactness result

We prove here the main compactness result of the paper. The proof of the result is based on a series of important geometrical results regarding regularity and transversality of the ballistic flow

$$\xi : (x, v) \in \Gamma_+ \mapsto \xi(x, v) = (x - \tau_-(x, v)v, v) \in \partial\Omega \times V.$$

Such technical results have been postponed to Appendix A for the clarity of the reading and will be used repeatedly in the proof of the following.

**Theorem 5.1.** Let  $K \in \mathcal{B}(L_+^1, L_-^1)$  be regular diffuse boundary operator. Then, one has

$$KM_0 K \in \mathcal{B}(L_+^1, L_-^1)$$

is weakly compact.

**Proof.** Let  $k, \ell \geq 1$  be fixed. Let  $(K_m)_m \subset \mathcal{B}(L_+^1, L_-^1)$  be the sequence of approximation obtained from Lemma 3.7, which is such that  $\lim_m \|K - K_m\|_{\mathcal{B}(L_+^1, L_-^1)} = 0$ . It is then enough to prove the weak-compactness of  $K_n M_0 K_m$  for any  $n, m \in \mathbb{N}$ . Still using the notations of Lemma 3.7, introduce

$$\overline{K}_m f(x, v) = \psi_m(v) \int_{v' \cdot n(x) > 0} f(x, v') \mu_x(dv'), \quad f \in L_+^1, \quad (x, v) \in \Gamma_-, \quad m \in \mathbb{N}$$

where  $\psi_m(v) = m \mathbf{1}_{B_m}(v)$ . Given  $n, m \in \mathbb{N}$ , using (3.5) and a domination argument, the weak-compactness of  $\overline{K}_n M_0 \overline{K}_m$  would imply the result. To avoid too heavy notations, and setting for instance  $F(v) = \max(\psi_n, \psi_m)$ , it suffices to prove that  $K M_0 K$  is weakly-compact for

$$K\varphi(x, v) = F(v) \int_{\Gamma_+(x)} \varphi(x, v') \mu_x(dv'). \quad (5.1)$$

Since  $F$  is compactly supported and bounded we can assume without loss of generality that

$$F = \mathbf{1}_{B_1} \quad (5.2)$$

which amounts to consider only velocities  $|v| \leq 1$ . Recall that, thanks to Corollary A.7,

$$\xi^{-1} : (x, \omega) \in \widehat{\Gamma}_- \mapsto \xi^{-1}(x, \omega) = (x + \tau_+(x, \omega)\omega, \omega) \in \Gamma_+$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\widehat{\Gamma}_-$  onto its image. Denoting here for simplicity

$$\Gamma_-^{(0)} = \Gamma_- \setminus \widehat{\Gamma}_-, \quad \Gamma_+^{(0)} = \xi^{-1}(\Gamma_-^{(0)})$$

one has  $\mu_-(\Gamma_-^{(0)}) = \mu_+(\Gamma_+^{(0)}) = 0$  (see for instance (A.16)) and we may make the identification

$$L_+^1 = L^1(\Gamma_+^{(0)}, d\mu), \quad L_-^1 = L^1(\Gamma_-^{(0)}, d\mu) \quad (5.3)$$

so that we only have to prove the weak-compactness of

$$\mathbf{KM}_0\mathbf{K} : L^1(\Gamma_+^{(0)}, d\mu) \rightarrow L^1(\Gamma_-^{(0)}, d\mu).$$

Notice that, by (5.1) and (5.2), the range of  $\mathbf{KM}_0\mathbf{K}$  can be rather considered as  $L^1(\Gamma_-^{(0)}, d\tilde{\mu})$  where

$$d\tilde{\mu}(x, v) = F(v)d\mu(x, v) = \mathbf{1}_{B_1}(v)|v \cdot n(x)|\pi(dx) \otimes m(dv) \quad (5.4)$$

is nothing but the restriction of  $\mu_-$  to  $\partial\Omega \times B_1$ . In particular,  $\tilde{\mu}$  is a *finite* measure. From a simple consequence of the Dunford-Pettis criterion (see [15, Corollary 4.7.21, p. 288]), we need to prove that, for *any* nonincreasing sequence of measurable subsets  $(\mathcal{A}_j)_j \subset \Gamma_-^{(0)}$  with  $\bigcap_j \mathcal{A}_j = \emptyset$ , it holds

$$\lim_{j \rightarrow \infty} \sup_{\|\varphi\|_{L_+^1} \leq 1} \int_{\mathcal{A}_j} |\mathbf{KM}_0\mathbf{K}\varphi(x, v)| d\mu(x, v) = 0. \quad (5.5)$$

Since  $\mathbf{K}$  and  $\mathbf{M}_0$  are nonnegative operators, it suffices of course to consider nonnegative  $\varphi \in L_+^1$  in (5.5). Let us fix a sequence  $(\mathcal{A}_j)_j \subset \Gamma_-^{(0)}$  with  $\bigcap_j \mathcal{A}_j = \emptyset$  and consider a nonnegative  $\varphi \in L_+^1$ . We set

$$I_j(\varphi) = \int_{\mathcal{A}_j} \mathbf{KM}_0\mathbf{K}\varphi(x, v) d\mu(x, v).$$

Introduce then the  $\mu$ -preserving change of variables

$$(x, v) \in \Gamma_+ \mapsto (y, w) = \xi(x, v) = (\xi_s(x, v), v) = (x - \tau_-(x, v)v, v) \in \Gamma_-,$$

and denote the position component of the inverse of  $\xi$  by  $\mathbf{Y}$ , i.e. given  $(y, w) \in \Gamma_-$ ,

$$\mathbf{Y}(y, w) = \xi_s^{-1}(y, w) \in \partial\Omega \quad \text{and} \quad \xi^{-1}(y, w) = (\mathbf{Y}(y, w), w) \in \Gamma_+.$$

Notice that here and everywhere in the text,  $\xi_s^{-1}$  denotes the position component of the inverse  $\xi^{-1}$ , i.e.  $\xi_s^{-1} = (\xi^{-1})_s$ . Defining, for any  $y \in \partial\Omega$ ,

$$B_j(y) = \int_{\Gamma_-(y)} G_j(\mathbf{Y}(y, w))F(w)\mu_y(dw), \quad G_j(x) = \int_{\Gamma_-(x)} \mathbf{1}_{\mathcal{A}_j}(x, v)F(v)\mu_x(dv),$$

one can show (see the Lemma 5.5 hereafter) that

$$I_j(\varphi) = \int_{\Gamma_+} B_j(y)\varphi(y, w_1)d\mu(y, w_1).$$

It is clear then that (5.5) will hold true if

$$\lim_{j \rightarrow \infty} \sup_{y \in \partial\Omega} B_j(y) = 0. \quad (5.6)$$

The proof of this property is given in the next Lemma yielding the desired weak-compactness.  $\square$

**Lemma 5.2.** *With the notations of the proof of Theorem 5.1, given  $y \in \partial\Omega$ , introduce*

$$B_j(y) = \int_{\Gamma_-(y)} G_j(\mathbf{Y}(y, w))F(w)\mu_y(dw).$$

*Then,  $\lim_{j \rightarrow \infty} \sup_{y \in \partial\Omega} B_j(y) = 0$ .*

The proof of the above will use the following polar decomposition theorem (see [43, Lemma 6.13, p. 113]):

**Lemma 5.3.** *If  $\mathbf{m}$  is a orthogonally invariant Borel measure with support  $V \subset \mathbb{R}^d$ , introduce  $\mathbf{m}_0$  as the image of the measure  $\mathbf{m}$  under the transformation  $v \in \mathbb{R}^d \mapsto |v| \in [0, \infty)$ , i.e.  $\mathbf{m}_0(I) = \mathbf{m}(\{v \in \mathbb{R}^d; |v| \in I\})$  for any Borel subset  $I \subset \mathbb{R}^+$ . Then, for any  $\psi \in L^1(\mathbb{R}^d, \mathbf{m})$  it holds*

$$\int_{\mathbb{R}^d} \psi(v) \mathbf{m}(dv) = \frac{1}{|\mathbb{S}^{d-1}|} \int_0^\infty \mathbf{m}_0(d\rho) \int_{\mathbb{S}^{d-1}} \psi(\rho \omega) \sigma(d\omega)$$

where  $d\sigma$  denotes the Lebesgue measure on  $\mathbb{S}^{d-1}$  with total mass  $|\mathbb{S}^{d-1}|$ .

**Remark 5.4.** We shall use in the proof of Lemma 5.2 that, with the notations of Proposition A.8, for any  $y \in \partial\Omega$  we can construct an orthonormal basis  $\{e_1(y), \dots, e_d(y)\}$  of  $\mathbb{R}^d$  depending continuously on  $y \in \partial\Omega$  with

$$e_d(y) = -n(y)$$

and such that, in this basis, any  $\varpi \in \mathbb{S}^{d-1}$  can be written as

$$\varpi = \sum_{i=1}^d \omega_i e_i(y)$$

where  $\omega = \omega(\theta) = (\omega_1, \dots, \omega_d)$  is given by (A.14) in terms of the polar coordinates

$$\theta = (\theta_1, \dots, \theta_{d-1}) \in U = [0, 2\pi] \times [0, \pi]^{d-3} \times \left[0, \frac{\pi}{2}\right].$$

In this case,  $\omega$  is independent of  $y \in \partial\Omega$ . We also recall that, according to Remark A.9, for any  $\varepsilon > 0$ , one can define  $S_\varepsilon(y)$  as those  $\varpi \in \mathbb{S}^{d-1}$  for which

$$\theta \in U_\varepsilon := \left\{ (\theta_1, \dots, \theta_{d-1}) \in U; \sin^{d-2} \theta_{d-1} \sin^{d-3} \theta_{d-2} \dots \sin \theta_2 \leq \varepsilon \right\}.$$

and prove that  $\lim_{\varepsilon \rightarrow 0^+} \sup_{y \in \partial\Omega} \sigma(S_\varepsilon(y)) = 0$ . See Remark A.9 for more details.

**Proof of Lemma 5.2.** Recall that the identification (5.3) is in force and  $(\mathcal{A}_j)_j \subset \Gamma_-^{(0)}$  is non-increasing with  $\bigcap_j \mathcal{A}_j = \emptyset$ . Introducing the polar coordinates  $w = \rho \varpi$ , with  $\rho > 0$  and  $\varpi \in \mathbb{S}^{d-1}$  and using Lemma 5.3 (recall that  $\mu_x(dv) = |v \cdot n(x)| \mathbf{m}(dv)$ ) we get

$$B_j(y) = |\mathbb{S}^{d-1}|^{-1} \int_0^1 \rho \mathbf{m}_0(d\rho) \int_{\Gamma_-(y)} G_j(\mathbf{Y}(y, \rho \varpi)) |\varpi \cdot n(y)| \sigma(d\varpi).$$

Since  $\{0\}$  is not an atom for the measure  $\rho \mathbf{m}_0(d\rho)$ , according to the dominated convergence theorem, it is enough to prove that, for any given  $\rho \in (0, 1)$ ,

$$\lim_{j \rightarrow \infty} \sup_{y \in \partial\Omega} \int_{\Gamma_-(y)} G_j(\mathbf{Y}(y, \rho \varpi)) |\varpi \cdot n(y)| \sigma(d\varpi) = 0 \quad (5.7)$$

and, since  $\rho > 0$ , there is no loss of generality in proving the result only for  $\rho = 1$ . Notice that, for any  $\varepsilon > 0$  and  $y \in \partial\Omega$ ,

$$\begin{aligned} \int_{\Gamma_-(y)} G_j(\mathbf{Y}(y, \varpi)) |\varpi \cdot n(y)| \sigma(d\varpi) &= \int_{\Gamma_-(y) \cap S_\varepsilon(y)} G_j(\mathbf{Y}(y, \varpi)) |\varpi \cdot n(y)| \sigma(d\varpi) \\ &\quad + \int_{\Gamma_-(y) \setminus S_\varepsilon(y)} G_j(\mathbf{Y}(y, \varpi)) |\varpi \cdot n(y)| \sigma(d\varpi) \end{aligned}$$

where  $S_\varepsilon(y)$  has been introduced in the above Remark 5.4. Clearly, since  $\|G_j\|_\infty \leq \int_{\mathbb{R}^d} F(u)|u|\mathbf{m}(du)$ , there is  $C > 0$  independent of  $j$  such that

$$\sup_{y \in \partial\Omega} \int_{\Gamma_-(y) \cap S_\varepsilon(y)} G_j(\mathbf{Y}(y, \varpi)) |\varpi \cdot n(y)| \sigma(d\varpi) \leq C \sup_{y \in \partial\Omega} \sigma(S_\varepsilon(y))$$

which goes to 0 as  $\varepsilon \rightarrow 0$  according to Remark A.9. Therefore, to show (5.7), we only have to prove that, for any  $\varepsilon > 0$ ,

$$\lim_{j \rightarrow \infty} \sup_{y \in \partial\Omega} \int_{\Gamma_-(y) \setminus S_\varepsilon(y)} G_j(\mathbf{Y}(y, \varpi)) \sigma(d\varpi) = 0. \quad (5.8)$$

Recall that, from Proposition A.8, for any  $y \in \partial\Omega$ , the mapping

$$\varpi \in \widehat{\Gamma}_-(y) \setminus S(y) \mapsto \mathbf{Y}(y, \varpi) \in \partial\Omega \quad \text{is of class } \mathcal{C}^1 \text{ with differential of rank } d-1 \quad (5.9)$$

with moreover  $\sigma(S(y)) = 0$ . On the other hand, with the notations and parametrization used in Proposition A.8 and recalled in Remark 5.4, the mapping  $(\theta, y) \in U \times \partial\Omega \mapsto \varpi = \varpi(\theta, y)$  is continuous while

$$\theta \in U \mapsto \varpi = \varpi(\theta, y)$$

is of class  $\mathcal{C}^1$  with a continuous derivative  $(\theta, y) \in A \times \partial\Omega \mapsto \partial_\theta \varpi(\theta, y)$ . Since the mapping  $\theta \in U_\varepsilon \mapsto \varpi(\theta, y)$  is a  $\mathcal{C}^1$  parametrization of  $S_\varepsilon(y)$ , by virtue of (5.9) we have that, for any  $y \in \partial\Omega$ ,

$$\theta \in U \setminus U_\varepsilon \mapsto \mathbf{Y}(y, \varpi(\theta, y)) \in \partial\Omega$$

is a regular parametrization of

$$\mathcal{E}_y := \{\mathbf{Y}(y, \varpi); \varpi \in \Gamma_-(y) \setminus S_\varepsilon(y)\} \subset \partial\Omega.$$

Then, according to [42, Lemma 5.2.11 & Theorem 5.2.16, pp. 128–131], the Lebesgue surface measure  $\pi_{\mathcal{E}_y}(dY)$  on  $\mathcal{E}_y$  is given by

$$J_{\mathbf{Y}}(y, \varpi) d\theta_1 \dots d\theta_{d-1} = J_{\mathbf{Y}}(y, \varpi) d\theta$$

where

$$J_{\mathbf{Y}}(y, \varpi) = \left[ \det \left( (\partial_{\theta_i} \mathbf{Y}(y, \varpi), \partial_{\theta_\ell} \mathbf{Y}(y, \varpi))_{1 \leq i, \ell \leq d-1} \right) \right]^{1/2} > 0$$

on  $U \setminus U_\varepsilon$ . Since the mapping

$$(\theta, y) \in U \times \partial\Omega \mapsto \partial_{\theta_i} \mathbf{Y}(y, \varpi(\theta, y)) = d_\varpi \mathbf{Y}(y, \varpi(\theta, y)) \partial_{\theta_i} \varpi(\theta, y)$$

is continuous for any  $i = 1, \dots, d-1$ , then so is the mapping

$$(\theta, y) \in U \times \partial\Omega \mapsto J_{\mathbf{Y}}(y, \varpi(\theta, y))$$

and there exists  $C_\varepsilon > 0$  such that

$$J_{\mathbf{Y}}(y, \varpi(\theta, y)) \geq C_\varepsilon > 0, \quad \forall (\theta, y) \in (U \setminus U_\varepsilon) \times \partial\Omega.$$

Hence, for any  $y \in \partial\Omega$

$$\begin{aligned} & \int_{\Gamma_-(y) \setminus S_\varepsilon(y)} G_j(\mathbf{Y}(y, \varpi)) \sigma(d\varpi) \\ & \leq \frac{1}{C_\varepsilon} \int_{U \setminus U_\varepsilon} G_j(\mathbf{Y}(y, \varpi(\theta, y))) J_{\mathbf{Y}}(y, \varpi(\theta, y)) d\theta \\ & = \frac{1}{C_\varepsilon} \int_{\mathcal{E}_y} G_j(Y) \pi_{\mathcal{E}_y}(dY). \end{aligned}$$

Clearly, recalling the definition of  $G_j$  – and because the measures  $\pi$  and  $\pi_{\mathcal{E}_y}$  coincide on  $\mathcal{E}_y$  – we get

$$\begin{aligned} \int_{\mathcal{E}_y} G_j(Y) \pi_{\mathcal{E}_y}(dY) &= \int_{\mathcal{E}_y} \left( \int_{\Gamma_-(Y)} F(v) \mathbf{1}_{\mathcal{A}_j}(Y, v) \mu_Y(dv) \right) \pi_{\mathcal{E}_y}(dY) \\ &\leq \int_{\partial\Omega} \left( \int_{\Gamma_-(Y)} F(v) \mathbf{1}_{\mathcal{A}_j}(Y, v) \mu_Y(dv) \right) \pi(dY) = \tilde{\mu}(\mathcal{A}_j), \end{aligned}$$

where  $\tilde{\mu}$  is given by (5.4). Thus,

$$\sup_{y \in \partial\Omega} \int_{\Gamma_-(y) \setminus S_\varepsilon(y)} G_j(\mathbf{Y}(y, \varpi)) \sigma(d\varpi) \leq \frac{\tilde{\mu}(\mathcal{A}_j)}{C_\varepsilon} \quad \forall j \in \mathbb{N}.$$

Since  $(\mathcal{A}_j)_j$  is non-increasing with  $\bigcap_j \mathcal{A}_j = \emptyset$  and  $\tilde{\mu}$  is a finite measure, we have  $\lim_j \tilde{\mu}(\mathcal{A}_j) = 0$  which implies (5.8) and proves the Lemma.  $\square$

**Lemma 5.5.** *With the notations of the proof of Theorem 5.1, it holds, for any  $j \in \mathbb{N}$ ,*

$$I_j(\varphi) = \int_{\Gamma_+} B_j(y) \varphi(y, w_1) d\mu(y, w_1) \quad (5.10)$$

where, for any  $y \in \partial\Omega$

$$B_j(y) = \int_{\Gamma_-(y)} G_j(\mathbf{Y}(y, w)) F(w) \mu_y(dw), \quad G_j(x) = \int_{\Gamma_-(x)} \mathbf{1}_{\mathcal{A}_j}(x, v) F(v) \mu_x(dv).$$

**Proof.** We use the notations of the above proof. In particular, we assume  $K$  to be given by (5.1). Notice that, for any nonnegative  $\psi \in L^1_+$ ,

$$\begin{aligned} \int_{\mathcal{A}_j} K M_0 \psi(x, v) d\mu_-(x, v) &= \int_{\mathcal{A}_j} F(v) \left( \int_{\Gamma_+(x)} M_0 \psi(x, v') \mu_x(dv') \right) d\mu_-(x, v) \\ &=: \int_{\mathcal{A}_j} F(v) \Psi_0(x) d\mu_-(x, v) \end{aligned}$$

where one has

$$\Psi_0(x) = \int_{\Gamma_+(x)} M_0 \psi(x, v') \mu_x(dv') = \int_{\Gamma_+(x)} \psi(\xi(x, v')) \mu_x(dv'), \quad x \in \partial\Omega.$$

Simple use of Fubini's theorem yields

$$\int_{\mathcal{A}_j} F(v) \Psi_0(x) d\mu_-(x, v) = \int_{\partial\Omega} G_j(x) \Psi_0(x) \pi(dx) = \int_{\Gamma_+} G_j(x) \psi(\xi(x, v')) d\mu_+(x, v').$$

Introduce then the  $\mu$ -preserving change of variables

$$(x, v') \in \Gamma_+ \mapsto (y, w) = \xi(x, v') \in \Gamma_- \quad (5.11)$$

and recalling that  $x = \mathbf{Y}(y, \omega)$  we have,

$$\int_{\mathcal{A}_j} \mathbf{K} \mathbf{M}_0 \psi(x, v) d\mu_-(x, v) = \int_{\Gamma_-} G_j(\mathbf{Y}(y, w)) \psi(y, w) d\mu_-(y, w).$$

Applying this with  $\psi = \mathbf{K}\varphi$  for some nonnegative  $\varphi \in L^1_+$ , we get

$$\begin{aligned} \int_{\mathcal{A}_j} \mathbf{K} \mathbf{M}_0 \mathbf{K} \varphi(x, v) d\mu_-(x, v) &= \int_{\Gamma_-} G_j(\mathbf{Y}(y, w)) \mathbf{K} \varphi(y, w) d\mu_-(y, w) \\ &= \int_{\Gamma_-} G_j(\mathbf{Y}(y, w)) F(w) d\mu_-(y, w) \int_{\Gamma_+(y)} \varphi(y, w_1) \mu_y(dw_1) \end{aligned}$$

which is the desired result.  $\square$

## 5.2. About the essential spectral radius of $\mathbf{M}_0 \mathbf{H}$

We are ready to show:

**Theorem 5.6.** Let  $\mathbf{H} \in \mathcal{B}(L^1_+, L^1_-)$  be a stochastic regular partly diffuse boundary operator given by (3.8) and denote for simplicity  $\beta(x) = 1 - \alpha(x)$  for  $\pi$ -a. e.  $x \in \partial\Omega$  and  $\beta_\infty := \text{ess sup}_{x \in \partial\Omega} \beta(x)$ . If

$$\text{ess inf}_{x \in \partial\Omega} \beta(x) > 1 + \beta_\infty - \sqrt{1 + \beta_\infty^2} \quad (5.12)$$

then  $r_{\text{ess}}(\mathbf{M}_0 \mathbf{H}) < 1$ .

**Proof.** For notation simplicity, we simply denote by  $\alpha \mathbf{R}$  the operator  $\alpha(\cdot) \mathbf{R}$  and by  $\beta \mathbf{K}$  the operator  $\beta(\cdot) \mathbf{K}$ . Note first that  $(\mathbf{M}_0 \beta \mathbf{K})^2 \leq (\mathbf{M}_0 \mathbf{K})^2$  and Theorem 5.1 imply that  $[\mathbf{M}_0 \beta \mathbf{K}]^2$  is weakly compact. We recall that in  $L^1$  spaces, the ideal of strictly singular operators coincides with the ideal of weakly compact operators [37]. Since

$$[\mathbf{M}_0 \mathbf{H}]^2 = [\mathbf{M}_0 \alpha \mathbf{R} + \mathbf{M}_0 \beta \mathbf{K}]^2 = [\mathbf{M}_0 \beta \mathbf{K}]^2 + [\mathbf{M}_0 \alpha \mathbf{R}]^2 + \mathbf{M}_0 \alpha \mathbf{R} \mathbf{M}_0 \beta \mathbf{K} + \mathbf{M}_0 \beta \mathbf{K} \mathbf{M}_0 \alpha \mathbf{R}$$

then the stability of essential spectra by strictly singular perturbations [28, Proposition 2.c.10, p. 79] shows that  $[\mathbf{M}_0 \mathbf{H}]^2$  and  $[\mathbf{M}_0 \alpha \mathbf{R}]^2 + \mathbf{M}_0 \alpha \mathbf{R} \mathbf{M}_0 \beta \mathbf{K} + \mathbf{M}_0 \beta \mathbf{K} \mathbf{M}_0 \alpha \mathbf{R}$  share the same essential spectrum. In particular,

$$\begin{aligned} r_{\text{ess}}([\mathbf{M}_0 \mathbf{H}]^2) &= r_{\text{ess}}([\mathbf{M}_0 \alpha \mathbf{R}]^2 + \mathbf{M}_0 \alpha \mathbf{R} \mathbf{M}_0 \beta \mathbf{K} + \mathbf{M}_0 \beta \mathbf{K} \mathbf{M}_0 \alpha \mathbf{R}) \\ &\leq \left\| [\mathbf{M}_0 \alpha \mathbf{R}]^2 + \mathbf{M}_0 \alpha \mathbf{R} \mathbf{M}_0 \beta \mathbf{K} + \mathbf{M}_0 \beta \mathbf{K} \mathbf{M}_0 \alpha \mathbf{R} \right\|_{\mathcal{B}(L^1_+)} \\ &\leq \|\alpha(\cdot)\|_{L^\infty(\partial\Omega)}^2 + 2\|\alpha(\cdot)\|_{L^\infty(\partial\Omega)} \|\beta(\cdot)\|_{L^\infty(\partial\Omega)} \\ &= (\|\alpha(\cdot)\|_{L^\infty(\partial\Omega)} + \|\beta(\cdot)\|_{L^\infty(\partial\Omega)})^2 - \|\beta(\cdot)\|_{L^\infty(\partial\Omega)}^2 \end{aligned}$$

Since  $\alpha(\cdot) = 1 - \beta(\cdot)$ , this means that

$$r_{\text{ess}}([\mathbf{M}_0 \mathbf{H}]^2) \leq (1 + \text{osc}(\beta))^2 - \|\beta(\cdot)\|_{L^\infty(\partial\Omega)}^2$$

where  $\text{osc}(\beta) = \text{ess sup}_{x \in \partial\Omega} \beta(x) - \text{ess inf}_{x \in \partial\Omega} \beta(x)$  is the oscillation of  $\beta(\cdot)$ . Finally, the condition  $(1 + \text{osc}(\beta))^2 - \|\beta(\cdot)\|_{L^\infty(\partial\Omega)}^2 < 1$  amounts to

$$(\text{osc}(\beta))^2 + 2\text{osc}(\beta) < \beta_\infty^2$$

which is equivalent to (5.12). This ends the proof since  $r_{\text{ess}}([\mathbf{M}_0 \mathbf{H}]^2) = (r_{\text{ess}}(\mathbf{M}_0 \mathbf{H}))^2$  by the spectral mapping theorem.  $\square$

**Remark 5.7.** We can view (5.12) as

$$\text{osc}(\beta) < \sqrt{1 + \beta_\infty^2} - 1$$

which expresses a smallness of the oscillation  $\text{osc}(\beta)$  relatively to  $\text{esssup}_{x \in \partial\Omega} \beta(x)$ . Notice that this condition is always satisfied if  $\beta > 0$  is a constant.

**Remark 5.8.** We strongly believe that the assumption (5.12) is purely technical and we conjecture the above result to be true with the sole assumption that  $\text{essinf}_{x \in \partial\Omega} \beta(x) > 0$ , i.e. when the diffuse reflection is active everywhere on  $\partial\Omega$ .

In a previous version of the paper [30], we erroneously established the inequality  $r_{\text{ess}}(\mathbf{M}_0\mathbf{H}) < 1$  from the stability of the essential spectral radius, proving that

$$r_{\text{ess}}(\mathbf{M}_0\mathbf{H}) = r_{\text{ess}}(\mathbf{M}_0\alpha\mathbf{R})$$

without any condition on the oscillation of  $\beta(\cdot)$ . The key point to establish such a stability result was the following (erroneous) property: for any integers  $k, \ell \geq 1$ , of the operators

$$\mathbf{K}(\mathbf{M}_0\mathbf{R})^k \mathbf{M}_0 \mathbf{K}(\mathbf{M}_0\mathbf{R})^\ell \mathbf{M}_0 \mathbf{K} : L_+^1 \rightarrow L_-^1 \quad \text{is weakly-compact.} \quad (5.13)$$

As pointed out by an anonymous referee, the proof of such a result contained a gap and the result cannot be true whenever  $\mathbf{R}$  is associated to bounce-back boundary conditions (see Example A.11 for details). We however conjecture that the above operators are indeed weakly-compact for any  $k, \ell \geq 1$  whenever  $\mathbf{R}$  is associated to *specular boundary reflection*, i.e.

$$\mathbf{R}\varphi(x, v) = \varphi(x, v - 2(v \cdot n(x))n(x)), \quad \varphi \in L_+^1, \quad (x, v) \in \Gamma_-.$$

More generally, it would be interesting to characterize the domains  $\Omega$  and the class of reflection boundary operators  $\mathbf{R}$  for which the above (5.13) holds true for any  $k, \ell \geq 1$ .

**Remark 5.9.** We point out here that the conclusion in Theorem 5.6 applies to any stochastic operator  $\mathbf{R}$  and not only to reflection boundary operators.

## 6. Kinetic semigroup for regular partly diffuse boundary operators

We introduce the following set of Assumptions:

**Assumption 6.1.** The *regular* stochastic partly diffuse boundary operator

$$\mathbf{H} = \alpha(\cdot)\mathbf{R} + (1 - \alpha(\cdot))\mathbf{K}$$

is such that

**A1)**  $\text{Range}(\mathbf{K}) \subset \mathbb{Y}_1^-$ ;

**A2)**  $\mathbf{R}(\mathbb{Y}_1^+) \subset \mathbb{Y}_1^-$ ;

**A3)** Inequality (5.12) is satisfied.

**Remark 6.2.** In the above set of Assumptions, it is possible to replace  $\mathbb{Y}_1^\pm$  with  $L^1(\Gamma_\pm, \tau_\mp d\mu_\pm)$ . However, in this case, Assumption **A2)** is not necessarily satisfied for practical examples of boundary conditions (see Example 6.5).

**Remark 6.3.** The above Assumption **A3)** can be replaced with

$$\mathbf{A3'}) \quad r_{\text{ess}}(\mathbf{M}_0\mathbf{H}) < 1 \quad \text{and} \quad \text{esssup}_{x \in \partial\Omega} \alpha(x) < 1.$$

Notice that, as seen from Theorem 5.6, **A3)  $\implies$  A3')**. In the rest of the analysis, this is only **A3')** that will be used.

**Example 6.4.** Consider the classical Maxwell diffuse boundary condition for which

$$[\mathbf{H}f](x, v) = \frac{\mathcal{M}(v)}{\gamma(x)} \int_{v' \cdot n(x) > 0} f(x, v') |v' \cdot n(x)| dv', \quad \forall (x, v) \in \Gamma_-$$

with

$$\mathcal{M}(v) = \frac{1}{(2\pi\theta)^{d/2}} \exp\left(-\frac{|v|^2}{2\theta}\right), \quad \text{and} \quad \gamma(x) = \int_{u \cdot n(x) < 0} \mathcal{M}(u) |u \cdot n(x)| du, \quad \forall x \in \partial\Omega$$

for some  $\theta > 0$ . Notice that, actually,  $\gamma$  is independent of  $x$  and

$$\gamma(x) = \gamma_d := C_d \int_{\mathbb{R}^d} |v| \mathcal{M}(v) dv, \quad \forall x \in \partial\Omega$$

for some universal constant  $C_d = \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|} \int_0^1 t(1-t^2)^{\frac{d-3}{2}} dt$ . One has then  $\mathbf{H}(L_+^1) \subset \mathbb{Y}_1^-$ . Indeed, for  $f \in L_+^1$  nonnegative, one has

$$\begin{aligned} \|\mathbf{H}f\|_{\mathbb{Y}_1^-} &= \frac{1}{\gamma_d} \int_{\partial\Omega} \pi(dx) \int_{v \cdot n(x) < 0} |v|^{-1} \mathcal{M}(v) |v \cdot n(x)| dv \\ &\quad \times \int_{v' \cdot n(x) > 0} f(x, v') |v' \cdot n(x)| dv' \\ &\leq \frac{1}{\gamma_d} \int_{\partial\Omega} \pi(dx) \int_{v' \cdot n(x) > 0} f(x, v') |v' \cdot n(x)| dv' = \frac{1}{2\gamma_d} \int_{\Gamma_+} f d\mu \end{aligned}$$

where we used that  $\int_{\mathbb{R}^d} \mathcal{M}(v) dv = 1$  for the first inequality. This shows that  $\text{Range } \mathbf{H} \subset \mathbb{Y}_1^-$ . This result extends easily to the case in which the temperature  $\theta$  depends on  $x \in \partial\Omega$ , i.e.  $\theta = \theta(x)$  with  $\theta(x) \geq \theta_0 > 0$  for any  $x \in \partial\Omega$ .

**Example 6.5.** Recalling that both  $V$  and the measure  $\mathbf{m}$  are invariant under the orthogonal group let us consider the pure reflection boundary operator

$$\mathbf{R}\varphi(x, v) = \varphi(x, v - 2(v \cdot n(x))n(x)), \quad (x, v) \in \Gamma_-, \quad \varphi \in L_+^1.$$

and let  $\varphi \in \mathbb{Y}_1^+$ . Then, with the change of variables  $w = v - 2(v \cdot n(x))n(x)$  such that  $|w| = |v|$  and  $v = w - 2(w \cdot n(x))n(x)$  (which preserves the measure  $d\mu_{\pm}$ ) we get

$$\begin{aligned} \|\mathbf{R}\varphi\|_{\mathbb{Y}_1^-} &= \int_{\Gamma_-} \varphi(x, v - 2(v \cdot n(x))n(x)) |v|^{-1} |v \cdot n(x)| \mathbf{m}(dv) \pi(dx) \\ &= \int_{\Gamma_+} \varphi(x, w) |w|^{-1} |w \cdot n(x)| \pi(dx) \mathbf{m}(dw), \end{aligned}$$

i.e.  $\mathbf{R}(\mathbb{Y}_1^+) \subset \mathbb{Y}_1^-$ .

Notice that, in this example, in full generality, we can not replace  $\mathbb{Y}_1^-$  with  $L^1(\Gamma_-, \tau_+ d\mu_-)$ . Actually, requiring that

$$\mathbf{R}(L^1(\Gamma_+, \tau_- d\mu_+)) \subset L^1(\Gamma_-, \tau_+ d\mu_-)$$

amounts to assume that there exists  $c > 0$  such that  $\tau_+(x, \mathcal{V}(x, v)) \leq c\tau_-(x, v)$  for any  $(x, v) \in \Gamma_+$  which is a geometrical condition not satisfied if  $\Omega$  is not strictly convex.

A key point is that, under Assumptions 6.1, the following holds:

**Lemma 6.6.** Assume  $\mathbf{H} \in \mathcal{B}(L_+^1, L_-^1)$  satisfies Assumptions 6.1. Then, for any  $\varphi \in L_+^1$

$$\psi = \mathcal{R}(1, \mathbf{M}_0(\alpha \mathbf{R})) \mathbf{M}_0((1 - \alpha) \mathbf{K}) \varphi \in \mathbb{Y}_1^+$$

so that  $\mathbf{H}\psi \in \mathbb{Y}_1^-$ .

**Proof.** Notice that, since  $\sup_{x \in \partial\Omega} \alpha(x) = \alpha_0 < 1$ , one has  $\|M_0(\alpha R)\|_{\mathcal{B}(L^1_+)} \leq \alpha_0 < 1$  and  $\mathcal{R}(1, M_0(\alpha R))$  is well-defined. From Assumption 6.1 A1),  $(1 - \alpha(\cdot))K\varphi \in \mathbb{Y}_1^-$ . Then, from (2.10),  $g = M_0(1 - \alpha)K\varphi \in \mathbb{Y}_1^+$ . From Assumption 6.1 A2),  $\alpha Rg \in \mathbb{Y}_1^-$  and, from (2.10),  $M_0\alpha Rg \in \mathbb{Y}_1^+$ . More precisely,  $\|M_0\alpha R\|_{\mathcal{B}(\mathbb{Y}_1^+)} \leq \alpha_0 < 1$  so that

$$\psi = \sum_{n=0}^{\infty} (M_0(\alpha R))^n g \in \mathbb{Y}_1^+.$$

Now, it is clear that  $H\psi \in \mathbb{Y}_1^-$  since  $H(\mathbb{Y}_1^+) \subset \mathbb{Y}_1^-$  (notice that  $K$  maps any function in  $\mathbb{Y}_1^-$  while  $R(\mathbb{Y}_1^+) \subset \mathbb{Y}_1^-$ ).  $\square$

We can now state our main existence and uniqueness result about invariant density:

**Theorem 6.7.** *Let  $H \in \mathcal{B}(L^1_+, L^1_-)$  be a regular stochastic partly diffuse boundary operator and let Assumptions 6.1 and 4.4 be satisfied. Then,  $(T_H, \mathcal{D}(T_H))$  is the generator of a stochastic  $C_0$ -semigroup  $(U_H(t))_{t \geq 0}$ . Moreover,  $(U_H(t))_{t \geq 0}$  is irreducible and has a unique invariant density  $\Psi_H \in \mathcal{D}(T_H)$  with*

$$\Psi_H(x, v) > 0 \quad \text{for a. e. } (x, v) \in \Omega \times \mathbb{R}^d, \quad \|\Psi_H\|_X = 1$$

and  $\text{Ker}(T_H) = \text{Span}(\Psi_H)$ . Moreover,  $(U_H(t))_{t \geq 0}$  is ergodic, Eq. (4.2) holds and  $X = \text{Ker}(T_H) \oplus \overline{\text{Range}(T_H)}$ .

**Proof.** We begin with proving that, under Assumptions 6.1,  $1 \in \mathfrak{S}_p(M_0H)$ . Indeed, being both  $M_0$  and  $H$  stochastic, the spectral radius of  $M_0H$  is  $r_\sigma(M_0H) = 1 \in \mathfrak{S}(M_0H)$ . According to Theorem 5.6, one has

$$r_{\text{ess}}(M_0H) < 1 = r_\sigma(M_0H).$$

As well-known (see [32, Theorem 2.1]), this implies that 1 is an isolated eigenvalue of  $M_0H$ . Moreover, being  $M_0H$  irreducible, we deduce from [32, Theorem 2.2] the uniqueness and the strict positivity (almost everywhere) of a nonnegative eigenfunction  $\varphi$ .

Let us consider now  $\lambda \in \mathbb{C}$  with  $\text{Re}\lambda > 0$ . Considering the modulus operator  $|M_\lambda H|$  (see [19] for a precise definition) one has

$$|M_\lambda H| \leq M_0H \quad \text{and} \quad |M_\lambda H| \neq M_0H.$$

In particular, from [32, Theorem 4.3],  $r_\sigma(|M_\lambda H|) < r_\sigma(M_0H) = 1$ . Since moreover,  $r_\sigma(M_\lambda H) \leq r_\sigma(|M_\lambda H|)$  according to [19, Theorem 1], this proves that  $r_\sigma(M_\lambda H) < 1$ , i.e.  $1 \notin \mathfrak{S}(M_\lambda H)$ . We conclude that  $A = T_H$  thanks to [5, Theorem 4.5]. Let us now show that the eigenfunction  $\varphi$  lies in  $\mathbb{Y}_1^+$ . Being  $M_0H\varphi = \varphi$ , we have  $\varphi = M_0(\alpha R)\varphi + M_0(1 - \alpha)K\varphi$  so that, since  $1 - M_0(\alpha R)$  is invertible,

$$\varphi = \mathcal{R}(1, M_0(\alpha R))M_0((1 - \alpha)K)\varphi.$$

From Lemma 6.6, we get that  $\varphi \in \mathbb{Y}_1^+$ . We deduce then from Proposition 4.2 that there exists  $\Psi_H \in \mathcal{D}(T_H)$  nonnegative and such that  $T_H\Psi_H = 0$ . We conclude with Theorem 4.7.  $\square$

**Remark 6.8.** The fact that  $T_H$  is the generator of  $(U_H(t))_{t \geq 0}$  does not depend on A1) and A2) in Assumptions 6.1.

## 7. Asymptotic stability of collisionless kinetic semigroups

The object of this section is to complement Theorem 4.7 and Theorem 6.7 where a convergence in *Cesarò means* of  $(U_H(t))_{t \geq 0}$  to its ergodic projection is given. Indeed, under a quite weak additional assumption on the kernel of  $K$  we will show that  $(U_H(t))_{t \geq 0}$  is asymptotically stable, i.e.  $U_H(t)f$  converges in norm as  $t \rightarrow +\infty$ . In particular

$$\lim_{t \rightarrow \infty} \|U_H(t)f - \Psi_H\|_X = 0 \tag{7.1}$$

for any density  $f \in X$ , i.e. any nonnegative  $f$  with  $\|f\|_X = 1$ . For the sake of simplicity, we restrict ourselves to the case in which  $m(dv) = dv$  is the Lebesgue measure over

$$V = \left\{ v \in \mathbb{R}^d ; m \leq |v| \leq M \right\}$$

where  $0 \leq m < M \leq \infty$ , although the surface Lebesgue measure on the unit sphere can also be dealt with, see Remark 8.2 below.

In order to prove asymptotic stability of  $(U_H(t))_{t \geq 0}$  we first describe the movement of particles as a piecewise deterministic Markov process. Then we explain how the stochastic semigroup  $(U_H(t))_{t \geq 0}$  can be defined by this process and finally we prove the asymptotic stability of this semigroup.

### 7.1. Piecewise deterministic Markov process

Consider the following stochastic process which describes the movement of particles. A particle is moving in the space  $\overline{\Omega}$  with a constant velocity and when it strikes the boundary  $\partial\Omega$  a new direction is chosen randomly from the directions that point back into the interior of  $\overline{\Omega}$  and the motion continues. We recall that if  $x \in \partial\Omega$  and  $v' \in \Gamma_+(x)$  then the distribution of velocity  $v$  after reflection is given by a probability measure  $P(x, v', \cdot)$  defined on Borel subsets  $B$  of  $\Gamma_-(x)$  by

$$P(x, v', B) = \alpha(x) \delta_{\mathcal{V}(x, v')}(B) + (1 - \alpha(x)) \int_B h(x, v, v') \mathbf{m}(dv).$$

where  $\mathcal{V}(x, v)$  is the regular reflection law. From the Assumptions 6.1 A3) it follows that there exists  $\varepsilon_0 = 1 - \sup_{x \in \partial\Omega} \alpha(x) > 0$  such that  $1 - \alpha(x) \geq \varepsilon_0$  for all  $x \in \partial\Omega$ . This implies that

$$P(x, v', dv) \geq \varepsilon_0 h(x, v', v) \mathbf{m}(dv). \quad (7.2)$$

Let a particle starts at time  $t = 0$  from some point  $x \in \Omega$  with some initial velocity  $v \in V \setminus \{0\}$  or from  $x \in \partial\Omega$  with velocity  $v \in \Gamma_-(x)$ . Let  $x(t)$  be the position and  $v(t)$  be the velocity of the particle at time  $t$  and let  $t_1 = t_+(x, v)$ . Then  $x(t) = x + vt$  and  $v(t) = v$  for  $t \in [0, t_1)$ . Let  $0 < t_1 < t_2 < \dots$  be a sequence of times when a particle hits the boundary  $\partial\Omega$ . Then

$$\text{Prob}(v(t_n) \in B \mid x(t_n^-) = x, v(t_n^-) = v') = P(x, v', B)$$

for every Borel subset  $B$  of  $\Gamma_+(x)$ , where  $x(t_n^-)$  and  $v(t_n^-)$  are the left-hand side limits of  $x(t)$  and  $v(t)$ , respectively, at the point  $t_n$ . Moreover

$$x(t) = x(t_n) + v(t_n)(t - t_n), \quad v(t) = v(t_n) \quad \text{for } t \in [t_n, t_{n+1}),$$

$x(t_n) = x(t_n^-)$  and  $t_{n+1} = t_n + t_+(x(t_n), v(t_n))$  for  $n \geq 1$ .

It is easy to observe that

$$\xi(t) = (x(t), v(t)), \quad t \geq 0,$$

defines a piecewise deterministic Markov process [41] with values in the space

$$\mathcal{E} = (\Omega \times V) \cup \Gamma_-.$$

The process  $\{\xi(t)\}_{t \geq 0}$  has càdlàg sample paths, i.e., they are right-continuous with left limits. Let  $\mathcal{P}(t, x, v, B)$  be the transition probability function for this process, i.e.

$$\mathcal{P}(t, x, v, B) = \text{Prob}(\xi(t) \in B \mid \xi(0) = (x, v)),$$

where  $B$  are Borel subsets of  $\mathcal{E}$ . The semigroup  $(U_H(t))_{t \geq 0}$  can be uniquely determined by the transition probability function  $\mathcal{P}(t, x, v, B)$  because the following relation holds

$$\int_B U_H(t) f(y, w) dy \otimes \mathbf{m}(dw) = \int_{\Omega \times V} \mathcal{P}(t, x, v, B) f(x, v) dx \otimes \mathbf{m}(dv)$$

for all  $f \in X$ , Borel subsets  $B$  of  $\Omega \times V$  and  $t \geq 0$ .

**Remark 7.1.** It should be noted that we do not assume here that  $\Omega$  is a strictly convex set and it can happen that at some boundary points  $x$  some outward or inward vectors belong to the tangent space  $\Gamma_0(x)$ . In such cases trajectories can be tangent to the boundary  $\partial\Omega$ , especially in the case when we consider the specular reflection (see Fig. 3). But there is no need to consider such pathological trajectories because the set  $\Gamma_0$  has zero measure for  $\mu$  and does not play any role in the definition of the boundary operator  $H$ .



Fig. 3. Examples of pathological trajectories in the case of the specular reflection  $v'$  – outward vector,  $v$  – inward vector.

## 7.2. Asymptotic stability

Now we check that the semigroup  $(U_H(t))_{t \geq 0}$  is partially integral (see Appendix B.1 for precise definition), i.e. that for some  $t > 0$  there exists an integrable function  $q: \Omega \times V \times \Omega \times V \rightarrow [0, \infty)$ ,  $q \not\equiv 0$ , such that

$$\mathcal{P}(t, x, v, B) \geq \int_B q(x, v, y, w) dy \otimes m(dw). \quad (7.3)$$

In order to prove this property we need a rather weak assumption concerning function  $h(x, v, v')$ .

**Definition 7.2.** Let  $H \in \mathcal{B}(L_+^1, L_-^1)$  be a stochastic partly diffuse boundary operator of the form (3.8). We say that the boundary operator  $H$  is *weakly locally diffuse* (WLD) if for each point  $x \in \partial\Omega$  and  $v'_0 \in \Gamma_+(x)$  there exists a  $v_0 \in \Gamma_-(x)$  and  $\delta > 0$  such that

$$h(x, v, v') > 0 \quad \text{for } \mu_x\text{-a. e. } v \in \Gamma_-(x) \cap B(v_0, \delta), \quad v' \in \Gamma_+(x) \cap B(v'_0, \delta). \quad (7.4)$$

If we replace condition (7.4) by a stronger one:

$$h(x, v, v') \geq \delta \quad \text{for all } v \in \Gamma_-(x) \cap B(v_0, \delta), \quad v' \in \Gamma_+(x) \cap B(v'_0, \delta). \quad (7.5)$$

then the boundary operator  $H$  will be called *strongly locally diffuse* (SLD).

**Lemma 7.3.** Assume that the operator  $H$  is weakly locally diffuse and satisfies Assumptions 4.4 and 6.1. Then the semigroup  $(U_H(t))_{t \geq 0}$  is partially integral.

**Proof.** Let  $(x, v) \in \mathcal{E}_0 = \Omega \times (V \setminus \{0\})$  be the initial position and velocity of a particle. At time  $t_1 = t_+(x, v)$  it hits the point  $x_1 = x + t_+(x, v)v$  on the boundary  $\partial\Omega$ . Then we choose a new velocity  $\bar{v}_1 \in \Gamma_-(x_1)$  and at time  $t_1 + t_+(x_1, \bar{v}_1)$  the particle hits the boundary for the second time at the point  $\bar{x}_2 = x_1 + t_+(x_1, \bar{v}_1)\bar{v}_1$ . We choose a new velocity  $\bar{v}_2 \in \Gamma_-(x_2)$ . Let  $t > 0$  satisfy inequalities

$$t_1 + t_+(x_1, \bar{v}_1) < t < t_1 + t_+(x_1, \bar{v}_1) + t_+(x_2, \bar{v}_2) \quad (7.6)$$

and let  $\tau = t - t_1$ . We will find an neighbourhood  $U$  of  $(\bar{v}_1, \bar{v}_2) \in V$  such that for  $(v_1, v_2) \in U$  we have  $v_1 \in \Gamma_-(x_1)$ ,  $v_2 \in \Gamma_-(x_1 + t_+(x_1, v_1)v_1)$  and (7.6) is satisfied for  $(v_1, v_2) \in U$ . Then

$$\begin{aligned} x(t) &= x_2 + (\tau - t_+(x_1, v_1))v_2 \\ &= x_1 + t_+(x_1, v_1)v_1 + (\tau - t_+(x_1, v_1))v_2 \end{aligned}$$

and  $v(t) = v_2$  is the position at time  $t$  of the particle if it starts from  $x$  with velocity  $v$ , and after hitting the boundary we choose velocities  $v_1$  and  $v_2$  (see Fig. 4). We define the function  $F: U \rightarrow V$  by

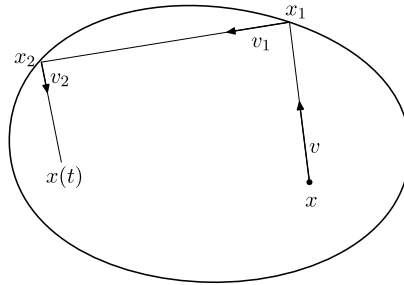
$$F(v_1, v_2) = (x_1 + t_+(x_1, v_1)v_1 + (\tau - t_+(x_1, v_1))v_2, v_2) = (F_1(v_1, v_2), F_2(v_1, v_2)).$$

Now we check that if

$$\nabla_{\bar{v}_1} t_+(x_1, \bar{v}_1) \cdot \bar{v}_2 \neq t_+(x_1, \bar{v}_1) + \nabla_{\bar{v}_1} t_+(x_1, \bar{v}_1) \cdot \bar{v}_1 \quad (7.7)$$

then the function  $F$  is a local diffeomorphism in some neighbourhood of  $(\bar{v}_1, \bar{v}_2)$ . Indeed, let us denote the Jacobian matrix of  $F$  by  $\mathcal{J}_F = \left( \frac{\partial F}{\partial v_1}, \frac{\partial F}{\partial v_2} \right)$ . One checks easily that

$$\det \mathcal{J}_F(v_1, v_2) = \det \left( \frac{\partial F_1}{\partial v_1} \right) = \det \left( \frac{\partial}{\partial v_1} (t_+(x_1, v_1)(v_1 - v_2)) \right) =: \det M$$

Fig. 4. Position  $x(t)$  of the particle after two collisions with the boundary.

where the matrix  $M$  is given by

$$M = c \operatorname{Id} + A, \quad A = a \otimes u = a^t u,$$

where  $a$  is the vector  $a = \partial_{v_1} t_+(x_1, v_1)$ ,  $u = v_1 - v_2$  and  $c = t_+(x_1, v_1)$ . We check then that<sup>1</sup>

$$\det M = c^{d-1} (c + a \cdot u) = t_+^d(x_1, v_1) + t_+^{d-1}(x_1, v_1) \nabla_{v_1} t_+(x_1, v_1) \cdot (v_1 - v_2).$$

Consequently, if (7.7) holds then the Jacobian matrix  $\mathcal{J}_F(\bar{v}_1, \bar{v}_2)$  is non singular and  $F$  is a diffeomorphism in some neighbourhood of  $(\bar{v}_1, \bar{v}_2)$ . Observe that condition (7.7) does not hold only on a  $(2d - 1)$ -dimensional differentiable manifold in  $\mathbb{R}^{2d}$  and we can change equality in (7.7) to inequality after a small perturbation of the vector  $\bar{v}_2$ . We have

$$\begin{aligned} \mathcal{P}(t, x, v, B) &= \operatorname{Prob}((x(t), v(t)) \in B) \\ &\geq \operatorname{Prob}((v_1, v_2) \in U : F(v_1, v_2) \in B) \\ &\geq \int_{F^{-1}(B)} \varepsilon_0^2 k(x_1, v_1, v) k(x_1 + v_1 t_+(x_1, v_1), v_2, v_1) \mathbf{m}(dv_1) \mathbf{m}(dv_2). \end{aligned}$$

Since  $x_1 = x + v t_+(x, v)$  we can define

$$\kappa_{(x,v)}(v_1, v_2) := \varepsilon_0^2 k(x_1, v_1, v) k(x_1 + v_1 t_+(x_1, v_1), v_2, v_1).$$

Since  $k$  satisfies WLD, for each  $(x_0, v_0) \in \mathcal{E}_0$  there exist  $\delta' > 0$  and  $\bar{v}_1 \in \Gamma_-(x_1)$ ,  $\bar{v}_2 \in \Gamma_-(x_1 + t_+(x_1, \bar{v}_1)\bar{v}_1)$  such that  $\kappa_{(x,v)}(v_1, v_2) > 0$  a. e. for  $(x, v) \in B((x_0, v_0), \delta')$  and  $(v_1, v_2) \in B((\bar{v}_1, \bar{v}_2), \delta')$ . Without loss of generality we can assume that condition (7.7) holds and  $F$  is a diffeomorphism from  $U_0 = B((\bar{v}_1, \bar{v}_2), \delta')$  onto  $F(U_0)$ . Then

$$\begin{aligned} \mathcal{P}(t, x, v, B) &\geq \int_{F^{-1}(B)} \kappa_{(x,v)}(v_1, v_2) \mathbf{m}(dv_1) \mathbf{m}(dv_2) \\ &= \int_{B \cap F(U_0)} \kappa_{(x,v)}(F^{-1}(y, w)) |\det \mathcal{J}_{F^{-1}}(y, w)| dy \otimes \mathbf{m}(dw) \end{aligned}$$

where  $\mathcal{J}_{F^{-1}}(y, w) = \left( \frac{\partial F^{-1}}{\partial y}, \frac{\partial F^{-1}}{\partial w} \right)$  is the Jacobian matrix of  $F^{-1}$ . From the last inequality it follows that (7.3) holds for

$$q(x, v, y, w) = \mathbf{1}_{F(U_0)}(y, w) \kappa_{(x,v)}(F^{-1}(y, w)) |\det \mathcal{J}_{F^{-1}}(y, w)|$$

and the semigroup  $(U_H(t))_{t \geq 0}$  is partially integral.  $\square$

<sup>1</sup> Indeed, let  $p(z)$  be the characteristic polynomial of  $A$ ,  $p(z) = \det(A - z \operatorname{Id})$ . Since the rank of  $A$  is less or equal to 1,  $z = 0$  is a root of  $p$  with multiplicity at least  $d - 1$  while the trace  $\operatorname{tr}(A) = u \cdot a$  should also be a root of  $p$ . Therefore,

$$p(z) = (-1)^d z^{d-1} (z - u \cdot a)$$

and, taking  $z = -c$  gives the result.

**Remark 7.4.** It is very likely that an analytical proof based upon the Dyson-Phillips-like representation of the semigroup  $(U_H(t))_{t \geq 0}$  obtained in [4,6] may replace the adopted probabilistic proof. Such a proof seems more involved than the probabilistic one given here and we did not investigate further on this point.

Combining Theorem 4.7, Theorem B.2 and Lemma 7.3 we obtain:

**Theorem 7.5.** *Let the assumptions of Theorem 4.7 be satisfied. Assume moreover that  $H$  is weakly locally diffuse, then the semigroup  $(U_H(t))_{t \geq 0}$  is asymptotically stable.*

**Remark 7.6.** In particular, under the conditions of Theorem 6.7, the semigroup  $(U_H(t))_{t \geq 0}$  is asymptotically stable.

## 8. Sweeping properties of collisionless kinetic semigroups

The asymptotic stability of the semigroup  $(V_H(t))_{t \geq 0}$  is strictly connected with the existence of an invariant density which was assumed in Theorem 4.7 and proved in Theorem 6.7. We investigate here the behaviour of  $(U_H(t))_{t \geq 0}$  when this semigroup has no invariant density. A crucial role is played by sweeping property (see Appendix B.2). We first establish the following which complements Lemma 7.3:

**Lemma 8.1.** *If  $H$  is strongly locally diffuse (in the sense of Definition 7.2), then, defining  $\mathcal{E}_0 = \Omega \times (V \setminus \{0\})$ , for every  $(x_0, v_0) \in \mathcal{E}_0$  there exist  $\varepsilon > 0$ ,  $t > 0$ , and a measurable function  $\eta_0: \mathcal{E}_0 \rightarrow [0, \infty)$  such that*

$$\int_{\mathcal{E}_0} \eta(x, v) dx \otimes m(dv) > 0$$

and

$$q(x, v, y, w) \geq \eta(y, w) \mathbf{1}_{B_0(\varepsilon)}(x, v) \quad \text{for } (y, w) \in \Omega \times V, \quad (8.1)$$

where  $B_0(\varepsilon)$  is the open ball in  $\mathcal{E}_0$  centered at  $(x_0, v_0)$  with radius  $\varepsilon$ .

**Proof.** The proof uses the notations introduced in the proof of Lemma 7.3. Recall that  $(U_H(t))_{t \geq 0}$  is partially integral with

$$q(x, v, y, w) = \mathbf{1}_{F(U_0)}(y, w) \kappa_{(x, v)}(F^{-1}(y, w)) |\det \mathcal{J}_{F^{-1}}(y, w)|.$$

If the operator  $H$  is strongly locally diffuse then there exist  $\delta' > 0$  and  $\bar{v}_1 \in \Gamma_-(x_1)$ ,  $\bar{v}_2 \in \Gamma_-(x_1 + t_+(x_1, \bar{v}_1)\bar{v}_1)$  such that  $\kappa_{(x, v)}(v_1, v_2) \geq \delta'$  for all  $(x, v) \in B((x_0, v_0), \delta')$  and  $(v_1, v_2) \in B((\bar{v}_1, \bar{v}_2), \delta')$ . Now setting

$$\eta(y, w) = \mathbf{1}_{F(U_0)}(y, w) |\det \mathcal{J}_{F^{-1}}(y, w)|$$

we check that  $\eta$  satisfies the desired properties.  $\square$

**Remark 8.2.** We note that Lemma 8.1 and Lemma 7.3 are also true when  $m(dv)$  is a surface Lebesgue measure on a sphere but the proofs are slightly more technical. Indeed, instead of two reflections at the boundary (see Fig. 4) we need one more reflection to achieve the property that the semigroup is partially integral.

According to Theorem B.4 and the previous Lemma, we have:

**Theorem 8.3.** *Let us assume that  $(U_H(t))_{t \geq 0}$  is stochastic and has no invariant density. If the boundary operator  $H$  is strongly local diffuse then  $(U_H(t))_{t \geq 0}$  is sweeping from all compact subsets of  $\mathcal{E}_0$ .*

**Proof.** Since  $((\Omega \times V) \setminus \mathcal{E}_0)$  is of zero measure for the measure  $dx \otimes m(dv)$ , we can assume that the semigroup  $(U_H(t))_{t \geq 0}$  is defined on the space  $L^1(\mathcal{E}_0, \mathcal{B}(\mathcal{E}_0), dx \otimes m(dv))$ . Then, on this space, Lemma 8.1 exactly means that  $(U_H(t))_{t \geq 0}$  satisfies property (K) of Theorem B.4.  $\square$

**Remark 8.4.** For any  $\varepsilon > 0$  and  $M > \varepsilon$  we define the set

$$F_{\varepsilon, M} = \{(x, v) \in \Omega \times V : \varepsilon \leq |v| \leq M, \text{dist}(x, \partial\Omega) \geq \varepsilon\},$$

where  $\text{dist}(x, \partial\Omega) = \inf\{|x - y| : y \in \partial\Omega\}$ . Since the set  $F_{\varepsilon, M}$  is compact in the space  $\mathcal{E}_0$ , for every  $f \in X$  we have

$$\lim_{t \rightarrow \infty} \int_{F_{\varepsilon, M}} U_H(t) f(x, v) dx \otimes m(dv) = 0. \quad (8.2)$$

This result has the following probabilistic interpretation. If the semigroup  $(U_H(t))_{t \geq 0}$  has no invariant density, the velocity of almost all particles converges to 0 or to  $\infty$  or particles get close to the boundary  $\partial\Omega$  when time goes to infinity.

We complement Theorem 8.3 by a more precise sweeping result:

**Theorem 8.5.** Assume that  $H \in \mathcal{B}(L_+^1, L_-^1)$  is a regular stochastic partly diffuse operator given by (3.8) and satisfying Assumptions 4.4. Assume moreover that  $H$  is weakly locally diffuse (WLD),  $\sup_{x \in \partial\Omega} \alpha(x) < 1$  and

$$\mu_+ \left( \left\{ (x, v') \in \Gamma_+; \int_{\Gamma_-(x)} k(x, v, v') \tau_+(x, v) \mu_x(dv) = +\infty \right\} \right) > 0. \quad (8.3)$$

Then

$$\lim_{t \rightarrow \infty} \int_{\Omega \times V} \mathbf{1}_{\{|v| \geq \varepsilon\}} U_H(t) f(x, v) dx \otimes m(dv) = 0, \quad \forall \varepsilon > 0, \quad \forall f \in X. \quad (8.4)$$

**Proof.** Note first that  $T_H$  is the generator of  $(U_H(t))_{t \geq 0}$  (see Remark 6.8). By virtue of Theorem B.5, the proof simply consists in showing that  $(U_H(t))_{t \geq 0}$  has no invariant density and in constructing a function  $\Psi = \Psi(x, v)$  such that

$$0 < \Psi(x, v) < \infty \quad \text{a. e. on } \Omega \times V, \quad \int_{\Omega \times V} \Psi(x, v) dx \otimes m(dv) = +\infty, \quad (8.5)$$

$$\int_{\Omega \times V} \mathbf{1}_{\{|v| \geq \varepsilon\}} \Psi(x, v) dx \otimes m(dv) < +\infty \quad (\varepsilon > 0)$$

and

$$U_H(t) \Psi \leq \Psi, \quad \forall t \geq 0. \quad (8.6)$$

The proof will be given in several steps. First of all, according to Theorem 5.6, there exists  $\varphi \in L_+^1$  such that

$$M_0 H \varphi = \varphi, \quad \|\varphi\|_{L_+^1} = 1. \quad (8.7)$$

Since  $M_0 K$  is irreducible then this  $\varphi$  is unique.

- *First step:* The function  $\Psi = \Xi_0 H\varphi$  satisfies (8.5). Indeed, one first notices that

$$\begin{aligned} & \int_{\Gamma_-} [K\varphi](x, v) \tau_+(x, v) d\mu_-(x, v) \\ &= \int_{\partial\Omega} d\pi(x) \int_{\Gamma_-(x)} \left( \int_{\Gamma_+(x)} k(x, v, v') \varphi(x, v') \mu_x(dv') \right) \tau_+(x, v) \mu_x(dv) \\ &= \int_{\partial\Omega} d\pi(x) \int_{\Gamma_+(x)} \left( \int_{\Gamma_-(x)} k(x, v, v') \tau_+(x, v) \mu_x(dv) \right) \varphi(x, v') \mu_x(dv') \\ &= \int_{\Gamma_+} \left( \int_{\Gamma_-(x)} k(x, v, v') \tau_+(x, v) \mu_x(dv) \right) \varphi(x, v') d\mu_+(x, v'). \end{aligned}$$

Therefore, under assumption (8.3) and because  $\varphi(x, v') > 0$  a.e. on  $\Gamma_+$ , we have

$$\int_{\Gamma_-} [K\varphi](x, v) \tau_+(x, v) d\mu_-(x, v) = +\infty.$$

Using Lemma 2.8 – identity (2.9) – and with  $\varepsilon_0 = 1 - \sup_{x \in \partial\Omega} \alpha(x) > 0$

$$\begin{aligned} \int_{\Omega \times V} \Psi(x, v) dx \otimes m(dv) &= \int_{\Gamma_-} [H\varphi](x, v) \tau_+(x, v) d\mu_-(x, v) \\ &\geq \varepsilon_0 \int_{\Gamma_-} [K\varphi](x, v) \tau_+(x, v) d\mu_-(x, v) = +\infty. \end{aligned}$$

Hence,

$$\int_{\Gamma_-} [H\varphi](x, v) |v|^{-1} d\mu_-(x, v) = \infty$$

since  $\tau_-(x, v) \leq |v|^{-1} D$  (where we recall that  $D$  is the diameter of  $\Omega$ ). Thus,  $H\varphi \notin \mathbb{Y}_1^-$  and 0 is not an eigenvalue of  $T_H$  (associated to a nonnegative eigenvalue) according to Proposition 4.2. Since  $T_H$  is the generator of  $(U_H(t))_{t \geq 0}$ , this means that  $(U_H(t))_{t \geq 0}$  has no invariant density. Moreover,

$$\begin{aligned} \int_{\Omega \times V} \mathbf{1}_{\{|v| \geq \varepsilon\}} \Psi(x, v) dx \otimes m(dv) &= \int_{\Gamma_-} \mathbf{1}_{\{|v| \geq \varepsilon\}} [H\varphi](x, v) \tau_+(x, v) d\mu_-(x, v) \\ &\leq \frac{D}{\varepsilon} \|H\varphi\|_{L_-^1} = \frac{D}{\varepsilon} \|\varphi\|_{L_+^1}. \end{aligned}$$

Using that  $M_0 H\varphi = \varphi$ , one has  $\varphi(x, v) = [H\varphi](x - \tau_-(x, v)v, v)$ , for any  $(x, v) \in \Gamma_+$  and, from the irreducibility of  $M_0 H$ , we get  $0 < \varphi(x, v) < +\infty$  a.e. on  $\Gamma_+$  which in turns implies that

$$0 < \Psi(x, v) = \Xi_0 H\varphi(x, v) = [H\varphi](x - t_-(x, v)v, v) < +\infty \quad \text{a.e. on } \Omega \times V.$$

This proves that  $\Psi$  satisfies (8.5).

In order to prove that  $\Psi$  satisfies also (8.6) we shall resort to Lemma 3.8 and for any  $n \in \mathbb{N}$ , introduce the regular diffuse operator given by  $H_n = \alpha R + (1 - \alpha)K_n$  with  $K_n$  is defined as in Lemma 3.8. As before, for any  $n \in \mathbb{N}$ , there exists  $\varphi_n \in L_+^1$  such that

$$M_0 H_n \varphi_n = \varphi_n, \quad \|\varphi_n\|_{L_+^1} = 1.$$

- *Second step:*  $\lim_n \|\varphi_n - \varphi\|_{L_+^1} = 0$ .

To prove this, we notice that

$$M_0 H \varphi_n = g_n + \varphi_n, \quad g_n = M_0 (H - H_n) \varphi_n \quad (8.8)$$

with  $\|g_n\|_{L_+^1} \leq \|H - H_n\|_{\mathcal{B}(L_+^1)}$  since  $\|\varphi_n\|_{L_+^1} = 1$ . In particular,

$$\lim_{n \rightarrow \infty} \|g_n\|_{L_+^1} = 0.$$

Now, denote by  $P$  the spectral projection associated to the (simple) eigenvalue 1 of  $M_0 H$  one has

$$M_0 H = M_0 H P + M_0 H (I - P)$$

with  $M_0 H P$  compact (since  $P$  is of finite rank) and  $1 \notin \mathcal{S}(M_0 H (I - P))$ . One can then write (8.8) as  $\varphi_n - M_0 H (I - P) \varphi_n = g_n + M_0 H P \varphi_n$  or equivalently,

$$\varphi_n = \mathcal{R}(1, M_0 H (I - P)) g_n + \mathcal{K}_0 \varphi_n$$

with  $\mathcal{K}_0 = \mathcal{R}(1, M_0 H (I - P)) M_0 H P$  compact. The sequence  $(\mathcal{K}_0 \varphi_n)_n$  is then relatively compact in  $L_+^1$  and, since

$$\lim_n \|\mathcal{R}(1, M_0 H (I - P)) g_n\|_{L_+^1} = 0,$$

the sequence  $(\varphi_n)_n$  is also relatively compact. There exists then a subsequence, still denoted  $(\varphi_n)_n$ , which converges in  $L_+^1$  to some  $\psi \in L_+^1$  with unit norm. One clearly has then

$$\psi = \mathcal{K}_0 \psi = \mathcal{R}(1, M_0 H (I - P)) M_0 H P \psi$$

i.e.  $\psi - M_0 H (I - P) \psi = M_0 H P \psi$  or equivalently

$$\psi = M_0 H \psi.$$

We deduce from this that  $\psi = \varphi$  by uniqueness. This shows finally that the whole sequence  $(\varphi_n)_n$  converges to  $\varphi$  in  $L_+^1$ .

- *Third step:* Introducing the semigroup  $(V_{H_n}(t))_{t \geq 0}$  associated to the boundary operator  $H_n$   $n \in \mathbb{N}$ , it holds

$$\lim_n \|V_{H_n}(t) f - U_H(t) f\|_X = 0, \quad \forall t \geq 0, \quad f \in X. \quad (8.9)$$

Indeed, for any  $n \in \mathbb{N}$  the resolvent of the generator  $T_{H_n}$  is given by

$$\mathcal{R}(\lambda, T_{H_n}) = \Xi_\lambda H_n (\lambda - M_\lambda H_n)^{-1} G_\lambda + R_\lambda$$

and it is easy to check, using again Lemma 3.8 and Eq. (2.1) that

$$\lim_n \|\mathcal{R}(\lambda, T_{H_n}) f - \mathcal{R}(\lambda, A) f\|_X = 0, \quad \forall \lambda > 0, \quad f \in X$$

where we recall that  $(A, \mathcal{D}(A))$  is the generator of  $(U_H(t))_{t \geq 0}$ . We deduce the second step from the Trotter-Kato approximation Theorem [22, Theorem 3.19, p. 83].

- *Fourth step.* Introduce then  $\Psi_n = M_0 H_n \varphi_n$ . According to Theorem 6.7,

$$V_{H_n}(t) \Psi_n = \Psi_n, \quad \forall n \in \mathbb{N}, t \geq 0.$$

On the other hand, since  $\lim_n \|H_n \varphi_n - H \varphi\|_{L_-^1} = 0$ , we have, for any  $\varepsilon > 0$ ,

$$\lim_n \|\mathbf{1}_{\{|v| > \varepsilon\}} H_n \varphi_n - \mathbf{1}_{\{|v| > \varepsilon\}} H \varphi\|_{L_-^1} = 0$$

and also

$$\lim_n \|\Xi_0(\mathbf{1}_{\{|v| > \varepsilon\}} H_n \varphi_n) - \Xi_0(\mathbf{1}_{\{|v| > \varepsilon\}} H \varphi)\|_{L_+^1} = 0$$

or equivalently

$$\lim_n \|\mathbf{1}_{\{|v| > \varepsilon\}} \Psi_n - \mathbf{1}_{\{|v| > \varepsilon\}} \Psi\|_{L_+^1} = 0.$$

Let then

$$\Psi_n^\varepsilon = \mathbf{1}_{\{|v|>\varepsilon\}} \Psi_n, \quad \Psi^\varepsilon = \mathbf{1}_{\{|v|>\varepsilon\}} \Psi, \quad n \in \mathbb{N}, \varepsilon > 0,$$

we note that

$$V_{H_n}(t) \Psi_n^\varepsilon \leq \Psi_n \quad \text{and} \quad \mathbf{1}_{\{|v|>\varepsilon\}} V_{H_n}(t) \Psi_n^\varepsilon \leq \Psi_n^\varepsilon$$

for any  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ ,  $t > 0$ . Using the *Third step*, we can pass to the limit in norm in this inequality as  $n \rightarrow +\infty$  and get

$$\mathbf{1}_{\{|v|>\varepsilon\}} U_H(t) \Psi^\varepsilon \leq \Psi^\varepsilon \leq \Psi.$$

Letting  $\varepsilon \rightarrow 0$ , the monotone convergence theorem yields to  $U_H(t) \Psi \leq \Psi$ , i.e.  $\Psi$  satisfies (8.6) and the proof is concluded.  $\square$

## Declaration of competing interest

We declare that there is no competing interest.

## Appendix A. About the ballistic flow

We establish in this appendix several important properties of the so-called *ballistic flow*

$$\xi : (x, v) \in \Gamma_+ \mapsto \xi(x, v) = (x - \tau_-(x, v)v, v) \in \partial\Omega \times V$$

which are fundamental for the proof of our main weak compactness result Theorem 5.1. For the clarity of exposition, we postponed these results in an Appendix but strongly believe that the results stated here have their own mathematical interest. In this Appendix, we will use the following notations: for any element  $z = (x, v)$  of the extended phase space  $\overline{\Omega} \times V$ , we will call  $x$  the space component of  $z$  and  $v$  the velocity component of  $z$ , writing  $x = z_s$  and  $v = z_v$ .

With the notations of [25],  $\xi = x_b$ . Notice that, as already observed in [25, 43], in non convex domain this deterministic flow does not avoid the grazing set  $\Gamma_0$ , i.e. in full generality

$$\xi(x, v) \in \Gamma_- \cup \Gamma_0$$

and – as far as the regularity of  $\xi$  is concerned – the set  $\{(x, v) \in \Gamma_+; \xi(x, v) \in \Gamma_0\}$  will be particularly relevant. Notice though that

$$\xi \in \Gamma_+ \cup \Gamma_0 \rightarrow \Gamma_- \cup \Gamma_0$$

is invertible with inverse

$$\xi^{-1} : (x, v) \in \Gamma_- \cup \Gamma_0 \mapsto \xi^{-1}(x, v) = (x + \tau_-(x, v)v, v) \in \Gamma_+ \cup \Gamma_0.$$

Moreover, according to (2.5) with  $\psi = \mathbf{1}_{\Gamma_0}$  we see that

$$\int_{\Gamma_+ \cup \Gamma_0} \mathbf{1}_{\Gamma_0}(\xi(x, v)) \mu_+(dx, dv) = \mu_-(\Gamma_0) = 0$$

which proves that

$$\mu_+((\{(x, v) \in \Gamma_+ \cup \Gamma_0; \xi(x, v) \in \Gamma_0\})) = 0. \quad (\text{A.1})$$

We introduce the following where we focus on velocity which are unit vectors (this is no loss of generality by virtue of (2.2))

**Definition A.1.** Let

$$\widehat{\Gamma}_\pm = \left\{ (x, \omega) \in \partial\Omega \times \mathbb{S}^{d-1}; (x, \omega) \in \Gamma_\pm \quad \text{and} \quad \xi(x, \omega) \in \Gamma_\mp \right\}$$

and introduce, for any  $x \in \partial\Omega$  the section

$$\widehat{\Gamma}_\pm(x) = \left\{ \omega \in \mathbb{S}^{d-1}; (x, \omega) \in \widehat{\Gamma}_\pm \right\}.$$

### A.1. Regularity of the travel time

The main result of this section is the following:

**Theorem A.2.** *The set  $\widehat{\Gamma}_{\pm}$  are open subsets of  $\Gamma_{\pm}$  and*

$$\tau_{\mp} : (x, \omega) \in \widehat{\Gamma}_{\pm} \mapsto \tau_{\mp}(x, \omega) \in \mathbb{R}^+$$

*is of class  $\mathcal{C}^1$ .*

We will split the proof of the above in a series of Lemma – dealing with  $\tau_-$  but all the results have their counterpart for  $\tau_+$ :

**Lemma A.3.** *The set  $\widehat{\Gamma}_+$  is an open subset of  $\Gamma_+$  and  $\tau_-$  is continuous on  $\widehat{\Gamma}_+$ .*

**Proof.** Let us fix  $x_0 \in \partial\Omega$  and  $\omega_0 \in \widehat{\Gamma}_+(x_0)$ , i.e.  $\omega_0 \cdot n(x_0) > 0$  and  $\omega_0 \cdot n(\xi_0) < 0$  where  $\xi_0 = x_0 - \tau_-(x_0, \omega_0)\omega_0$  is the space component of  $\xi(x_0, \omega_0)$ . For simplicity, set

$$\tau_0 = \tau_-(x_0, \omega_0).$$

Let  $(x_n, v_n) \subset \Gamma_+$  be a given sequence such that  $\lim_n (x_n, v_n) = (x_0, \omega_0)$ . In particular, we can assume that  $|v_n| \neq 0$  for any  $n \in \mathbb{N}$ . Set then  $\omega_n = |v_n|^{-1} v_n \in \mathbb{S}^{d-1}$  for any  $n \in \mathbb{N}$  and

$$y_n = x_n - \tau_-(x_n, \omega_n)\omega_n \in \partial\Omega \quad \forall n \in \mathbb{N}.$$

Taking a subsequence if necessary (recall that  $\partial\Omega$  is compact), we may assume that  $(y_n)_n$  converges to some  $y_0 \in \partial\Omega$ . Then, since  $\tau_-(x_n, \omega_n) = |y_n - x_n|$  we get that

$$\lim_n \tau_-(x_n, v_n) = \lim_n \tau_-(x_n, \omega_n)|v_n|^{-1} = \lim_n \tau_-(x_n, \omega_n) = |y_0 - x_0| =: \tau_1$$

and, consequently, letting  $n$  goes to infinity in the definition of  $y_n$  yields

$$y_0 = x_0 - \tau_1 \omega_0 \in \partial\Omega.$$

This in particular shows that  $\tau_0 \leq \tau_1$ . To prove that  $\tau_1 = \tau_0$ , let us argue by contradiction and assume that  $\tau_1 > \tau_0$ . Since both  $y_0 = x_0 - \tau_1 \omega_0$  and  $\xi_0 = x_0 - \tau_0 \omega_0$  belong to  $\partial\Omega$  and since  $\omega_0 \cdot n(\xi_0) < 0$ , the set

$$\left\{ t \in (\tau_0, \tau_1); x_0 - t\omega_0 \notin \overline{\Omega} \right\}$$

is open and not empty. Therefore, there exists  $\delta > 0$  such that  $\tau_1 > \delta + \tau_0$  and

$$x_0 - t\omega_0 \notin \overline{\Omega} \quad \forall t \in (\tau_0, \delta + \tau_0). \quad (\text{A.2})$$

Notice that  $x_n - t\omega_n \in \Omega$  for all  $t \in (0, \tau_-(x_n, \omega_n))$  and any  $n \in \mathbb{N}$ . Since  $\lim_n \tau_-(x_n, \omega_n) = \tau_1$ , we get that, for  $n \in \mathbb{N}$  large enough,

$$x_n - t\omega_n \in \Omega \quad \forall t \in (0, \tau_0 + \delta).$$

Letting then  $n$  goes to infinity, we obtain  $x_0 - t\omega_0 \in \overline{\Omega}$  for any  $t \in (0, \tau_0 + \delta)$  which contradicts (A.2). Therefore,  $\tau_1 = \tau_0$  which proves the continuity of  $\tau_-$  on  $\widehat{\Gamma}_+$ . Let us now show that  $\widehat{\Gamma}_+$  is open. We keep the previous notations, fixing  $(x_0, \omega_0) \in \widehat{\Gamma}_+$ . Let us assume that there exists a sequence  $(x_n, \omega_n) \subset \Gamma_+$  such that  $\lim_n (x_n, \omega_n) = (x_0, \omega_0)$  where  $\omega_n \in \mathbb{S}^{d-1}$  for any  $n \in \mathbb{N}$  but  $(x_n, \omega_n) \notin \widehat{\Gamma}_+$ . This means that

$$\omega_n \cdot n(x_n - \tau_-(x_n, \omega_n)\omega_n) = 0, \quad \forall n \in \mathbb{N}.$$

From the previous part of the proof, we know that  $\lim_n x_n - \tau_-(x_n, \omega_n)\omega_n = x_0 - \tau_-(x_0, \omega_0)\omega_0$  and, since  $n(\cdot)$  is continuous, we get

$$\omega_0 \cdot n(x_0 - \tau_-(x_0, \omega_0)\omega_0) = 0,$$

which contradicts the assumption that  $(x_0, \omega_0) \in \widehat{\Gamma}_+$ . Therefore, no sequence with the above properties can exist and  $\widehat{\Gamma}_+$  is open.  $\square$

**Lemma A.4.** For any  $x \in \partial\Omega$ , the mapping

$$\omega \in \widehat{\Gamma}_+(x) \longmapsto \tau_-(x, \omega) \in \mathbb{R}^+$$

is differentiable and

$$\nabla_\omega \tau_- : (x, \omega) \in \widehat{\Gamma}_+ \longmapsto \tau_-(x, \omega) \in \mathbb{R}^+ \quad (\text{A.3})$$

is continuous.

**Proof.** As before, let us fix  $x_0 \in \partial\Omega$  and  $\omega_0 \in \widehat{\Gamma}_+(x_0)$ . Since the normal vector  $n(\cdot)$  is continuous on  $\partial\Omega$ , we deduce from Lemma A.3 that there exists a radius  $r > 0$  such that

$$\omega \cdot n(x) > 0 \quad \text{and} \quad \omega \cdot n(\xi_s(x, \omega)) < 0 \quad \forall (x, \omega) \in \mathbb{U}(x_0) \times \mathbb{V}(\omega_0), \quad (\text{A.4})$$

where  $\mathbb{U}(x_0) = \mathbb{B}(x_0, r) \cap \partial\Omega$  is an open neighbourhood of  $x_0$  and  $\mathbb{V}(\omega_0) = \mathbb{B}(\omega_0, r) \cap \mathbb{S}^{d-1}$  is an open neighbourhood of  $\omega_0$ . The continuity of  $\tau_-$  implies that there exists  $t_0 > 0$  such that

$$\tau_-(x, \omega) \geq t_0 > 0 \quad \forall (x, \omega) \in \mathbb{U}(x_0) \times \mathbb{V}(\omega_0). \quad (\text{A.5})$$

Since the mapping  $\xi : (x, \omega) \in \widehat{\Gamma}_+ \mapsto \xi(x, \omega) = (x - \tau_-(x, \omega)\omega, \omega) \in \Gamma_-$  is continuous, invertible with inverse

$$\xi^{-1} : (y, \omega) \in \Gamma_- \mapsto \xi^{-1}(y, \omega) = (y + \tau_+(y, \omega)\omega, \omega) \in \widehat{\Gamma}_+$$

and since  $\xi(\widehat{\Gamma}_+) \subset \widehat{\Gamma}_-$ , one has  $\xi^{-1}$  continuous. In particular

$$\{(x - \tau_-(x, \omega)\omega, \omega), (x, \omega) \in \mathbb{U}(x_0) \times \mathbb{V}(\omega_0)\}$$

is an open neighbourhood of  $\xi(x_0, \omega_0)$  and

$$\mathbb{W}(z_0) := \{x - \tau_-(x, \omega)\omega, (x, \omega) \in \mathbb{U}(x_0) \times \mathbb{V}(\omega_0)\}$$

is an open neighbourhood of  $z_0 = x_0 - \tau_-(x_0, \omega_0)\omega_0 = \xi_s(x_0, \omega_0) \in \partial\Omega$ . Since  $\partial\Omega$  is of class  $\mathcal{C}^1$  then (up to choosing a smaller neighbourhood  $\mathbb{W}(z_0)$  if necessary), there exists a  $\mathcal{C}^1$  bijective mapping

$$\Psi : y \in (-1, 1)^{d-1} \longmapsto \Psi(y) \in \mathbb{W}(z_0)$$

with  $\Psi(0) = z_0$  and such that the range of the differential  $d\Psi(y)$  has dimension  $d - 1$  for any  $y \in (-1, 1)^{d-1}$ . We introduce open pieces of  $\partial\Omega$  indexed by  $x \in \partial\Omega$

$$\mathcal{S}_x = \{x - \tau_-(x, \omega)\omega, \omega \in \mathbb{V}(\omega_0)\} \subset \mathbb{W}(z_0).$$

Define then

$$\mathcal{O}_x = \Psi^{-1}(\mathcal{S}_x), \quad x \in \mathbb{U}(x_0),$$

one sees that, for any  $x \in \partial\Omega$ , the mapping  $\Psi : y \in \mathcal{O}_x \mapsto \Psi(y) \in \mathcal{S}_x$  is a parametrization of  $\mathcal{S}_x$ . Namely, given  $(x, \omega) \in \mathbb{U}(x_0) \times \mathbb{V}(\omega_0)$ , there is a unique  $y \in \mathcal{O}_x$  such that  $x - \tau_-(x, \omega)\omega = \Psi(y)$ . Thus,

$$\tau_-(x, \omega) = |x - \Psi(y)| \quad \text{and} \quad \omega = \frac{x - \Psi(y)}{|x - \Psi(y)|}. \quad (\text{A.6})$$

In particular  $x_0 - \tau_-(x_0, \omega_0)\omega_0 = z_0 = \Psi(0)$ , and  $\omega_0 = \frac{x_0 - \Psi(0)}{|x_0 - \Psi(0)|}$ . Introduce the  $\mathcal{C}^1$  mapping

$$H : (x, y) \in \mathbb{U}(x_0) \times (-1, 1)^{d-1} \mapsto H(x, y) = \frac{x - \Psi(y)}{|x - \Psi(y)|} \in \mathbb{V}(\omega_0) \quad (\text{A.7})$$

and, for any  $z \in \mathbb{R}^d \setminus \{0\}$ , let  $\mathbb{P}_z$  denote the orthogonal projection on the hyperplane orthogonal to  $z$ ,

$$\mathbb{P}_z h = h_z^\perp := h - \langle h, \bar{z} \rangle \bar{z}, \quad \bar{z} = \frac{z}{|z|} \in \mathbb{S}^{d-1}, h \in \mathbb{R}^d.$$

Notice that  $\mathbb{P}_z = \mathbb{P}_{\bar{z}}$  for any  $z \in \mathbb{R}^d \setminus \{0\}$ . Because the differential of the mapping  $z \in \mathbb{R}^d \setminus \{0\} \mapsto \frac{z}{|z|}$  is given by

$$h \in \mathbb{R}^d \mapsto -|z|^{-1} \mathbb{P}_z h, \quad h \in \mathbb{R}^d$$

it follows that the differential  $d_y H(x, y)$  of  $H$  is given by

$$d_y H(x, y) : h \in \mathbb{R}^d \mapsto -\frac{\mathbb{P}_{x-\Psi(y)}(d\Psi(y)h)}{|x-\Psi(y)|} = -\frac{\mathbb{P}_\omega(d\Psi(y)h)}{|x-\Psi(y)|}, \quad (\text{A.8})$$

where  $\omega = H(x, y) = |x - \Psi(y)|^{-1} (x - \Psi(y))$ . Note that the differential  $d_y H(x, y)$  depends *continuously* on  $(x, y) \in \mathbb{U}(x_0) \times (-1, 1)^{d-1}$ . Let us assume for a while that

$$\text{Rank}(d_y H(x_0, 0)) = d - 1. \quad (\text{A.9})$$

Then, the dimension of the range of  $d_y H(x, y)$  remains of dimension  $d - 1$  for  $x$  close enough to 0. Recalling that  $\omega_0 = |x_0 - \Psi(0)|^{-1} (x_0 - \Psi(0))$ , we deduce from the *local inverse function theorem* that, in some open neighbourhood  $\mathbb{U}'(x_0) \times (-\delta, \delta)^{d-1}$  of  $(x_0, 0)$  and a neighbourhood  $\mathbb{V}'(\omega_0)$  of  $\omega_0$  such that the equation

$$\omega = H(x, y) \quad (x, y) \in \mathbb{U}'(x_0) \times (-\delta, \delta)^{d-1}$$

is solved uniquely as

$$y = G(x, \omega), \quad (x, \omega) \in \mathbb{U}'(x_0) \times \mathbb{V}'(\omega_0)$$

where  $G(x, \cdot)$  is a  $\mathcal{C}^1$  mapping on a neighbourhood  $\mathbb{V}'(\omega_0)$  of  $\omega_0$  and the mapping  $(x, \omega) \mapsto G(x, \omega)$  is continuous on  $\mathbb{U}'(x_0) \times \mathbb{V}'(\omega_0)$ . It follows that, for  $x \in \mathbb{U}'(x_0)$  the mapping

$$\omega \in \mathbb{V}'(\omega_0) \mapsto \tau_-(x, \omega) = |x - \Psi(y)| = |x - \Psi(G(x, \omega))|$$

is differentiable with differential  $d_\omega \tau_-(x, \omega)$  given by

$$d_\omega \tau_-(x, \omega) : h \in \mathbb{R}^d \mapsto -\frac{\langle x - \Psi(G(x, \omega)), d\Psi(G(x, \omega)) d_\omega G(x, \omega) h \rangle}{|x - \Psi(G(x, \omega))|}. \quad (\text{A.10})$$

Since  $d_\omega G(x, \omega) = (d_y H(x, G(x, \omega)))^{-1}$  and the mapping  $(x, \omega) \mapsto G(x, \omega)$  is continuous then so is

$$(x, \omega) \in \mathbb{U}'(x_0) \times \mathbb{V}'(\omega_0) \mapsto d_\omega \tau_-(x, \omega)$$

which proves the Lemma under assumption (A.9). It only remains to prove (A.9). Notice that

$$\text{Range}(d\Psi(0)) = \left\{ d\Psi(0)h; h \in \mathbb{R}^{d-1} \right\}$$

is the  $(d - 1)$ -dimensional tangent space of  $\partial\Omega$  at  $z_0 = \Psi(0)$  with  $x_0 - \Psi(0) = \tau_-(x_0, \omega_0)\omega_0$  and

$$\left\{ \mathbb{P}_{x_0-\Psi(0)}(d\Psi(0)h); h \in \mathbb{R}^{d-1} \right\} = \mathbb{P}_{\omega_0}(\text{Range}(d\Psi(0)))$$

is the orthogonal projection of  $\text{Range}(d\Psi(0))$  on the orthogonal hyperplane to  $\omega_0$ . One sees that

$$\omega_0 \cdot n(\Psi(0)) < 0 \implies \omega_0 \notin \text{Range}(d\Psi(0))$$

and consequently  $\mathbb{P}_{\omega_0}(\text{Range}(d\Psi(0)))$  coincides with the orthogonal hyperplane to  $\omega_0$ . In particular, it has dimension  $d - 1$  which is exactly (A.9).  $\square$

**Lemma A.5.** *For any  $\omega \in \mathbb{S}^{d-1}$ , the mapping*

$$x \in \widehat{\Gamma}_+(\omega) \mapsto \tau_-(x, \omega) \in \mathbb{R}^+$$

*is differentiable and*

$$\nabla_x \tau_- : (x, \omega) \in \widehat{\Gamma}_+ \mapsto \tau_-(x, \omega) \in \mathbb{R}^+$$

*is continuous.*

**Proof.** As in the previous Lemma, we fix  $x_0 \in \partial\Omega$  and  $\omega_0 \in \widehat{\Gamma}_+(x_0)$  and consider an open neighbourhood  $\mathbb{U}(x_0) \times \mathbb{V}(\omega_0)$  of  $(x_0, \omega_0)$  on which (A.4) and (A.5) hold. For any  $\omega \in \mathbb{V}(\omega_0)$ , define

$$\mathcal{S}_\omega = \{x - \tau_-(x, \omega)\omega; x \in \mathbb{U}(x_0)\}$$

and, with the notation of the previous Lemma,  $\mathcal{S}_\omega \subset \mathbb{W}(z_0)$  for any  $\omega \in \mathbb{V}(\omega_0)$  where  $\mathbb{W}(z_0)$  is the image of the  $\mathcal{C}^1$  function

$$\Psi : y \in (-1, 1)^{d-1} \mapsto \Psi(y) \in \mathbb{W}(z_0)$$

with  $\Psi(0) = z_0$  and  $\text{Rank}(d\Psi(y)) = d - 1$  for any  $y \in (-1, 1)^{d-1}$ . This allows to introduce, as in the previous Lemma,  $\mathcal{O}_\omega = \Psi^{-1}(\mathcal{S}_\omega)$  and  $\mathcal{S}_\omega$  is parametrized by  $\Psi$  (defined now on  $\mathcal{O}_\omega$ ), i.e. given  $(x, \omega) \in \mathbb{U}(x_0) \times \mathbb{V}(\omega_0)$ , there is a unique  $y \in \mathcal{O}_\omega$  such that  $x - \tau_-(x, \omega)\omega = \Psi(y)$  and (A.6) and (A.7) still hold. We have seen in the proof of Lemma A.4 that  $H$  is  $\mathcal{C}^1$  with differential  $d_y H(x, y)$  given by (A.8) and depending continuously on  $(x, y) \in \mathbb{U}(x_0) \times (-1, 1)^{d-1}$ . In particular, as seen earlier, at  $(x, y) = (x_0, 0)$  the differential  $d_y H$  is given by

$$d_y H(x_0, 0) : h \in \mathbb{R}^{d-1} \mapsto -\frac{\mathbb{P}_{\omega_0}(d\Psi(0)h)}{|x_0 - \Psi(0)|} = -\frac{\mathbb{P}_{\omega_0}(d\Psi(0)h)}{\tau_-(x_0, \omega_0)}$$

and has  $(d - 1)$ -dimensional range. As before, from the *implicit function theorem*, there is a neighbourhood  $(-\delta, \delta)^{d-1} \times \mathbb{U}''(x_0)$  of  $(0, x_0)$  and a neighbourhood  $\mathbb{V}''(\omega_0)$  of  $\omega_0$  on which the equation  $\omega = |x - \Psi(y)|^{-1}(x - \Psi(y))$  with  $(y, x) \in (-\delta, \delta)^{d-1} \times \mathbb{U}''(x_0)$  is solved uniquely as

$$y = \widehat{G}(\omega, x), \quad (\omega, x) \in \mathbb{V}''(\omega_0) \times \mathbb{U}''(x_0)$$

where  $\widehat{G}(\omega, \cdot)$  is a  $\mathcal{C}^1$  mapping and  $(\omega, x) \in \mathbb{V}''(\omega_0) \times \mathbb{U}''(x_0) \mapsto \widehat{G}(\omega, x)$  is continuous. It follows that the mapping  $x \in \mathbb{U}''(x_0) \mapsto \tau_-(x, \omega) = |\Psi(\widehat{G}(\omega, x)) - x|$  is differentiable for any  $\omega \in \mathbb{V}''(\omega_0)$  with differential given by

$$d_x \tau_-(x, \omega) : h \in \mathcal{T}_x \mapsto -\frac{\langle \Psi(\widehat{G}(\omega, x)) - x, d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h - h \rangle}{|\Psi(\widehat{G}(\omega, x)) - x|} \quad (\text{A.11})$$

where  $\mathcal{T}_x$  is the tangent space of  $\partial\Omega$  at  $x \in \partial\Omega$ . Let us prove now the continuity of  $d_x \tau_-(\cdot, \cdot)$ . Because  $\omega + \frac{\Psi(\widehat{G}(\omega, x)) - x}{|\Psi(\widehat{G}(\omega, x)) - x|} = 0$ , differentiating with respect to  $x$  along the direction  $h$  tangent at  $\partial\Omega$  at  $x$  yields

$$\frac{\mathbb{P}_{\Psi(\widehat{G}(\omega, x)) - x}(d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h - h)}{|x - \Psi(\widehat{G}(\omega, x))|} = 0$$

i.e.

$$\mathbb{P}_\omega(d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h) = \mathbb{P}_\omega h, \quad \forall h \in \mathcal{T}_x$$

where we used that  $\omega$  is the unit vector in the direction of  $(\Psi(\widehat{G}(\omega, x)) - x)$ . This implies that

$$d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h = \mathbb{P}_\omega h + \langle d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h, \omega \rangle \omega. \quad (\text{A.12})$$

Since  $d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h$  is a tangent vector to  $\partial\Omega$  at  $\Psi(\widehat{G}(\omega, x)) = \xi_s(x, \omega) = x - \tau_-(x, \omega)\omega$ , taking the inner product of the above identity with the normal unit vector  $n(\xi_s(x, \omega))$  yields

$$\langle d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h, \omega \rangle \langle \omega, n(\xi_s(x, \omega)) \rangle = -\langle \mathbb{P}_\omega h, n(\xi_s(x, \omega)) \rangle.$$

Inserting this into (A.12) and since  $\omega \cdot n(\xi_s(x, \omega)) \neq 0$  we get

$$d\Psi(\widehat{G}(\omega, x))d_x \widehat{G}(\omega, x)h = \mathbb{P}_\omega h - \frac{\langle \mathbb{P}_\omega h, n(\xi_s(x, \omega)) \rangle}{\langle \omega, n(\xi_s(x, \omega)) \rangle} \omega$$

which, plugged into (A.11), yields

$$\begin{aligned} d_x \tau_-(x, \omega)h &= -\langle \omega, \mathbb{P}_\omega h - \frac{\langle \mathbb{P}_\omega h, n(\xi_s(x, \omega)) \rangle}{\langle \omega, n(\xi_s(x, \omega)) \rangle} \omega - h \rangle \\ &= \langle h; \omega \rangle - \frac{\langle \mathbb{P}_\omega h; n(\xi_s(x, \omega)) \rangle}{\langle \omega; n(\xi_s(x, \omega)) \rangle} \end{aligned}$$

i.e.

$$d_x \tau_-(x, \omega)h = \frac{\langle h, n(\xi_s(x, \omega)) \rangle}{\langle \omega, n(\xi_s(x, \omega)) \rangle} \quad (\text{A.13})$$

This gives directly the continuity of the mapping  $(x, \omega) \mapsto d_x \tau_-(x, \omega)$  since  $\xi_s$  is continuous on  $\widehat{\Gamma}_+$ .  $\square$

**Remark A.6.** Notice that (A.13) allows to recover the expression

$$\nabla_x \tau_-(x, \omega) = \frac{1}{\omega \cdot n(\xi_s(x, \omega))} n(\xi_s(x, \omega)), \quad (x, \omega) \in \widehat{\Gamma}_+$$

which was obtained in [25, Lemma 3] for some special structure of  $\Omega$ . Moreover, using (A.10) and using the range of the differential of  $\Psi(G(x, \omega))$  is the orthogonal of  $\text{Span}(n(G(x, \omega)))$  we can prove

$$\nabla_\omega \tau_-(x, \omega) = \frac{\tau_-(x, \omega)}{\omega \cdot n(\xi_s(x, \omega))} n(\xi_s(x, \omega)), \quad (x, \omega) \in \widehat{\Gamma}_+$$

which, again, is a result obtained in a special case in [25, Lemma 3].

**Proof of Theorem A.2.** The above three Lemmas give directly the proof of Theorem A.2 for  $\tau_-$  and  $\widehat{\Gamma}_+$ . The proof for  $\tau_+$  and  $\widehat{\Gamma}_-$  is done similarly.  $\square$

An immediate consequence of Theorem A.2 is the following regularity of the ballistic flow:

**Corollary A.7.** *The ballistic flow:*

$$\xi : (x, \omega) \in \widehat{\Gamma}_+ \mapsto \xi(x, \omega) = (x - \tau_-(x, \omega)\omega, \omega) \in \Gamma_-$$

is a  $C^1$  diffeomorphism from  $\widehat{\Gamma}_+$  onto its image and

$$\xi^{-1} : (x, \omega) \in \widehat{\Gamma}_- \mapsto \xi^{-1}(x, \omega) = (x + \tau_+(x, \omega)\omega, \omega) \in \Gamma_+$$

is a  $C^1$  diffeomorphism from  $\widehat{\Gamma}_-$  onto its image.

## A.2. Further non degeneracy results

We introduce a local polar parametrization of the boundary which will turn useful later on. Here and everywhere in the text,  $\xi_s^{-1}$  denotes the position component of the inverse  $\xi$ , i.e.  $\xi_s^{-1} = (\xi^{-1})_s$ <sup>2</sup>:

**Proposition A.8.** *For any  $x \in \partial\Omega$ , there is a closed subset  $S(x) \subset \widehat{\Gamma}_-(x)$  with zero surface Lebesgue measure  $d\sigma$  and such that the mapping*

$$\xi_s^{-1}(x, \cdot) : \omega \in \widehat{\Gamma}_-(x) \setminus S(x) \mapsto x + \tau_+(x, \omega)\omega \in \partial\Omega$$

has a differential of rank  $d - 1$ .

**Proof.** For any  $x \in \partial\Omega$ , we choose an orthonormal basis  $\{e_1(x), \dots, e_{d-1}(x), e_d(x)\}$  – depending continuously on  $x \in \partial\Omega$  – where

$$e_d(x) = -n(x).$$

Let us write the components of  $\omega \in \widehat{\Gamma}_-(x)$  in this basis using polar coordinates

<sup>2</sup> This should not be confused with the inverse of the position component  $(\xi_s)^{-1}$ .

$$\left\{ \begin{array}{l} \omega_1 = \sin \theta_{d-1} \dots \sin \theta_3 \sin \theta_2 \sin \theta_1, \\ \omega_2 = \sin \theta_{d-1} \dots \sin \theta_3 \sin \theta_2 \cos \theta_1, \\ \omega_3 = \sin \theta_{d-1} \dots \sin \theta_3 \cos \theta_2, \\ \vdots \\ \omega_{d-1} = \sin \theta_{d-1} \cos \theta_{d-2}, \\ \omega_d = \cos \theta_{d-1} \end{array} \right. \quad (\text{A.14})$$

with  $\theta_1 \in [0, 2\pi]$  and  $\theta_2, \dots, \theta_{d-1} \in [0, \pi]$ . Notice that the assumption  $\omega \in \widehat{\Gamma}_-(x)$  actually implies that  $\theta_{d-1} \in [0, \pi/2]$ . We will write  $\theta = (\theta_1, \dots, \theta_{d-2}, \theta_{d-1})$  and  $U := [0, 2\pi] \times [0, \pi]^{d-3} \times [0, \frac{\pi}{2}]$ , so that  $\omega \in \widehat{\Gamma}_-(x) \implies \theta \in U$ . Notice that the set  $U$  is independent of  $x \in \partial\Omega$ . Within this frame and with the above set of coordinates, we write

$$T(x, \theta_1, \dots, \theta_{d-2}, \theta_{d-1}) := \tau_+(x, \omega) \quad \text{and} \\ G(x, \theta_1, \dots, \theta_{d-2}, \theta_{d-1}) = x + T(x, \theta_1, \dots, \theta_{d-2}, \theta_{d-1})\omega, \quad \forall \omega \in \widehat{\Gamma}_-(x).$$

From Theorem A.2, for any  $x \in \partial\Omega$ , the mapping  $\theta \in U \mapsto T(x, \theta)$  is of class  $C^1$ . It is then clear that the set of  $\omega \in \mathbb{S}^{d-1}$  given by (A.14) such that the vectors

$$(\partial_{\theta_j} G(x, \theta))_{j=1, \dots, d-1}$$

are linearly independent coincide with the set at which the differential of the mapping  $\omega \in \widehat{\Gamma}_-(x) \mapsto x + \tau_+(x, \omega)\omega \in \partial\Omega$  is of full rank  $d - 1$ .

One has

$$\partial_{\theta_j} G(x, \theta) = \partial_{\theta_j} T(x, \theta)\omega + T(x, \theta)\partial_{\theta_j}\omega$$

and, because  $|G(x, \theta) - x| = T(x, \theta)$ , it is easy to check that  $\partial_{\theta_j} T(x, \theta) = \omega \cdot \partial_{\theta_j} G(x, \theta)$ , i.e.

$$\partial_{\theta_j} G(x, \theta) - (\partial_{\theta_j} G(x, \theta) \cdot \omega)\omega = T(x, \theta)\partial_{\theta_j}\omega.$$

Notice that  $\partial_{\theta_j} G(x, \theta) - (\partial_{\theta_j} G(x, \theta) \cdot \omega)\omega$  is nothing but the projection of  $\partial_{\theta_j} G(x, \theta)$  on the hyperplane  $\omega^\perp$ . Recalling that  $T(x, \theta) > 0$ , we see that  $(\partial_{\theta_j} G(x, \theta) - (\partial_{\theta_j} G(x, \theta) \cdot \omega)\omega)_{j=1, \dots, d-1}$  are independent if and only if  $(\partial_{\theta_j} \omega)_{j=1, \dots, d-1}$  are independent, or equivalently, if the Gram matrix  $\mathcal{J}_\theta(\omega) = (\partial_{\theta_i} \omega, \partial_{\theta_j} \omega)_{i,j}$  is not singular. It is well known that

$$\det(\mathcal{J}_\theta(\omega)) = \sin^{d-2} \theta_{d-1} \sin^{d-3} \theta_{d-2} \dots \sin \theta_2. \quad (\text{A.15})$$

In other words, if  $\det(\mathcal{J}_\theta(\omega)) \neq 0$ , then  $(\partial_{\theta_j} G(x, \theta) - (\partial_{\theta_j} G(x, \theta) \cdot \omega)\omega)_{j=1, \dots, d-1}$  are independent and one deduces easily that then  $(\partial_{\theta_j} G(x, \theta))_{j=1, \dots, d-1}$  are also independent. We define then

$$S(x) = \{\omega \in \widehat{\Gamma}_-(x); \det(\mathcal{J}_\theta(\omega)) = 0\}$$

the mapping  $\omega \in \widehat{\Gamma}_- \setminus S(x) \mapsto x + \tau_+(x, \omega)\omega$  has a differential of rank  $d - 1$ . It is clear that  $S(x)$  is closed. Let us now prove that indeed  $S(x)$  has a zero surface Lebesgue measure. Using then (A.15), we get that

$$\omega \in S(x) \quad \text{if and only if} \quad \theta_j \in \{0, \pi\} \text{ for some } j = 2, \dots, d-1.$$

The condition  $\theta_{d-1} \in \{0, \pi\}$  only means  $\theta_{d-1} = 0$  (recall that  $\theta_{d-1} \leq \pi/2$ ) which means that  $\omega = (0, \dots, 0, 1)$ . Then, for  $d \geq 3$ , the condition  $\theta_{d-2} \in \{0, \pi\}$  means that

$$\omega = (0, \dots, 0, \pm \sin \theta_{d-1}, \cos \theta_{d-1}),$$

i.e.  $\omega$  belongs to some (half) unit circle of  $\mathbb{S}^{d-1}$ . More generally, the condition  $\theta_{d-j} \in \{0, \pi\}$  for  $2 \leq j \leq d-2$  describes a unit  $(j-1)$ -dimensional (half)-spheres of  $\mathbb{S}^{d-1}$ . This means that  $S(x)$  can be written as  $S(x) = \bigcup_{j=2}^{d-2} C_j$  where  $C_j$  is a closed set with positive  $(d-j)$ -Lebesgue measure<sup>3</sup> ( $2 \leq j \leq d-2$ ). Therefore,  $\sigma(S(x)) = 0$  (where we recall that  $d\sigma$  is the Lebesgue surface measure over  $\mathbb{S}^{d-1}$ ) and the conclusion follows.  $\square$

<sup>3</sup> Namely  $C_j = \{\omega \in \widehat{\Gamma}_+(x); \theta_{d-j} = 0 \text{ or } \theta_{d-j} = \pi\}$ .

**Remark A.9.** For any  $x \in \partial\Omega$  and any  $\varepsilon > 0$ , introduce the set  $S_\varepsilon(x)$  of all  $\omega \in \widehat{\Gamma}_-(x) \subset \mathbb{S}^{d-1}$  whose polar coordinates  $\theta = (\theta_1, \dots, \theta_{d-1})$  in the basis  $\{e_1(x), \dots, e_d(x)\}$  are such that

$$\det(\mathcal{J}_\theta(\omega)) \leq \varepsilon$$

where we recall that the determinant  $\det(\mathcal{J}_\theta(\omega))$  is given by (A.15) with  $\omega = \omega(\theta)$  given by (A.14). Notice that  $\det(\mathcal{J}_\theta(\omega))$  is actually *independent* of  $x \in \partial\Omega$ . Then, the surface Lebesgue measure  $\sigma(S_\varepsilon(x))$  of  $S_\varepsilon(x)$  is given by

$$\int_{\mathbb{S}^{d-1}} \mathbf{1}_{S_\varepsilon(x)}(\omega) \sigma(d\omega) = \int_{U_\varepsilon} \det(\mathcal{J}_\theta(\omega)) d\theta \leq \varepsilon \pi^{d-2}$$

where  $U_\varepsilon = \{\theta \in U ; \sin^{d-2} \theta_{d-1} \sin^{d-3} \theta_{d-2} \dots \sin \theta_2 \leq \varepsilon\}$ . Therefore

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{x \in \partial\Omega} \sigma(S_\varepsilon(x)) = 0.$$

We have then the following result (which is not used in the core of the paper but has its own interest):

**Proposition A.10.** Assume that, for  $\pi$ -a. e.  $x \in \partial\Omega$ ,  $\mathcal{V}(x, \cdot) : \Gamma_-(x) \rightarrow \Gamma_+(x)$  is a field of measurable mappings associated to a pure reflection boundary operator as in Definition 3.1 and let

$$\left\{ \begin{array}{l} \mathcal{U} : (x, v) \in \Gamma_+ \cup \Gamma_0 \mapsto \mathcal{U}(x, v) = (x - \tau_-(x, v)v, \mathcal{V}(x - \tau_-(x, v)v, v)) \\ \quad = (\xi_s(x, v), \mathcal{V}(\xi(x, v))) \in \Gamma_+ \cup \Gamma_0. \end{array} \right.$$

For any  $k \in \mathbb{N}$  there exists a subset  $\gamma_k \subset \Gamma_-$  such that:

- (1)  $\gamma_k$  is a closed subset of  $\Gamma_-$  with  $\mu(\gamma_k) = 0$ .
- (2)  $\mathcal{U}^{-k} \circ \xi^{-1}(\Gamma_- \setminus \gamma_k)$  is an open subset of  $\Gamma_+$  and

$$\mathcal{U}^{-k} \circ \xi^{-1} : \Gamma_- \setminus \gamma_k \rightarrow \Gamma_+$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\Gamma_- \setminus \gamma_k$  onto its image.

**Proof.** We first notice that, thanks to Corollary A.7,  $\mathcal{U} : \widehat{\Gamma}_+ \rightarrow \Gamma_+$  is a  $\mathcal{C}^1$  diffeomorphism from  $\widehat{\Gamma}_+$  onto its image  $\mathcal{U}(\widehat{\Gamma}_+)$  which is an open set of  $\Gamma_+$ . Let us introduce

$$\Delta := \Gamma_+ \setminus \widehat{\Gamma}_+ = \{(x, v) \in \Gamma_+ \cup \Gamma_0 ; \xi(x, v) \in \Gamma_0\}$$

As already noticed in (A.1),

$$\mu(\Delta) = 0, \tag{A.16}$$

i.e.  $\Delta$  is a closed set of  $\Gamma_+$  of zero  $d\mu$ -measure. Since

$$\mathcal{U}(\Gamma_+ \cup \Gamma_0) = \mathcal{U}(\Gamma_0) \cup \mathcal{U}(\widehat{\Gamma}_+) \cup \mathcal{U}(\Delta)$$

we have that  $\mu(\mathcal{U}(\widehat{\Gamma}_+)) = \mu(\mathcal{U}(\Gamma_+ \cup \Gamma_0)) = \mu(\Gamma_+ \cup \Gamma_0)$  since  $\mathcal{U}$  is  $\mu$ -preserving as a composition of the  $\mu$ -preserving mapping  $\xi$  and  $\mathcal{V}$ . Therefore

$$\mu(\Gamma_+ \setminus \mathcal{U}(\widehat{\Gamma}_+)) = 0.$$

Introduce then  $\Gamma_+^{(1)} = \widehat{\Gamma}_+$ ,

$$\Gamma_+^{(2)} = \left\{ (x, \omega) \in \Gamma_+^{(1)} ; \mathcal{U}(x, \omega) \in \widehat{\Gamma}_+ \right\}$$

and

$$\Lambda_1 = \Gamma_+^{(1)} \setminus \Gamma_+^{(2)}.$$

One has  $\Lambda_1 = \mathcal{U}^{-1}(\Delta)$  is a closed subset of  $\widehat{\Gamma}_+$  with  $\mu(\Lambda_1) = 0$ . Moreover,

$$\mathcal{U}^2 : \Gamma_+^{(2)} \rightarrow \Gamma_+$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\Gamma_+^{(2)}$  onto its image. Since  $\mathcal{U}^2$  is  $\mu$ -preserving, writing the disjoint unions

$$\Gamma_+^{(1)} = \Gamma_+^{(2)} \cup \Lambda_1, \quad \mathcal{U}^2(\Gamma_+^{(1)}) = \mathcal{U}^2(\Gamma_+^{(2)}) \cup \mathcal{U}^2(\Lambda_1)$$

we see that  $\mu(\mathcal{U}^2(\Gamma_+^{(2)})) = \mu(\mathcal{U}^2(\Gamma_+^{(1)})) = \mu(\mathcal{U}(\Gamma_+^{(1)}))$  and

$$\mu(\Gamma_+ \setminus \mathcal{U}^2(\Gamma_+^{(2)})) = 0.$$

By induction, assuming that there is  $\Gamma_+^{(k-1)} \subset \widehat{\Gamma}_+$  such that

$$\mathcal{U}^{k-1} : \Gamma_+^{(k-1)} \rightarrow \Gamma_+$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\Gamma_+^{(k-1)}$  onto its image  $\mathcal{U}^{k-1}(\Gamma_+^{(k-1)})$  which is of full  $\mu$ -measure, i.e.

$$\mu(\Gamma_+ \setminus \mathcal{U}^{k-1}(\Gamma_+^{(k-1)})) = 0,$$

then define

$$\Gamma_+^{(k)} = \{(x, \omega) \in \Gamma_+^{(k-1)} ; \mathcal{U}^{k-1}(x, \omega) \in \widehat{\Gamma}_+\} \quad (\text{A.17})$$

so that

$$\Lambda_{k-1} = \Gamma_+^{(k)} \setminus \Gamma_+^{(k-1)} = (\mathcal{U}^{k-1})^{-1}(\Delta)$$

is a closed subset of  $\Gamma_+$  with  $\mu(\Lambda_{k-1}) = 0$  while

$$\mathcal{U}^k : \Gamma_+^{(k)} \rightarrow \Gamma_+$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\Gamma_+^{(k)}$  onto its image  $\mathcal{U}^k(\Gamma_+^{(k)})$ . As before, writing the disjoint unions

$$\Gamma_+^{(k-1)} = \Gamma_+^{(k)} \cup \Lambda_{k-1}, \quad \mathcal{U}^k(\Gamma_+^{(k-1)}) = \mathcal{U}^k(\Gamma_+^{(k)}) \cup \mathcal{U}^k(\Lambda_{k-1})$$

we see that  $\mu(\mathcal{U}^k(\Gamma_+^{(k)})) = \mu(\mathcal{U}^k(\Gamma_+^{(k-1)})) = \mu(\mathcal{U}^{k-1}(\Gamma_+^{(k-1)}))$  so that

$$\mu(\Gamma_+ \setminus \mathcal{U}^k(\Gamma_+^{(k)})) = 0.$$

On the other hand, according to Corollary A.7,

$$\xi : \widehat{\Gamma}_+ \rightarrow \Gamma_-$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\widehat{\Gamma}_+$  onto its image. Using the definition (A.17) for  $k+1$  we see that

$$\Lambda_k = \Gamma_+^{(k)} \setminus \Gamma_+^{(k+1)} = (\mathcal{U}^k)^{-1}(\Delta)$$

is a closed subset of  $\Gamma_+$  with  $\mu(\Lambda_k) = 0$  and

$$\xi \circ \mathcal{U}^k : \Gamma_+^{(k+1)} \rightarrow \Gamma_-$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\Gamma_+^{(k+1)}$  onto its image  $(\xi \circ \mathcal{U}^k)(\Gamma_+^{(k+1)})$ . Arguing as before and writing

$$\Gamma_+^{(k)} = \Gamma_+^{(k+1)} \cup \Lambda_k, \quad (\xi \circ \mathcal{U}^k)(\Gamma_+^{(k)}) = (\xi \circ \mathcal{U}^k)(\Gamma_+^{(k+1)}) \cup (\xi \circ \mathcal{U}^k)(\Lambda_k)$$

and since  $\xi \circ \mathcal{U}^k$  is  $\mu$ -preserving, we have that

$$\mu((\xi \circ \mathcal{U}^k)(\Gamma_+^{(k+1)})) = \mu((\xi \circ \mathcal{U}^k)(\Gamma_+^{(k)})) = \mu(\mathcal{U}^k(\Gamma_+^{(k)}))$$

where we used that  $\xi$  is also  $\mu$ -preserving. Therefore

$$\mu\left(\Gamma_- \setminus \left(\left(\xi \circ \mathcal{U}^k\right)\left(\Gamma_+^{(k+1)}\right)\right)\right) = 0.$$

Since  $(\xi \circ \mathcal{U}^k)(\Gamma_+^{(k+1)})$  is an open subset of  $\Gamma_-$  then setting

$$\gamma_k := \Gamma_- \setminus \left(\left(\xi \circ \mathcal{U}^k\right)\left(\Gamma_+^{(k+1)}\right)\right)$$

one sees that  $\gamma_k$  is a closed subset of  $\Gamma_-$  with  $\mu(\gamma_k) = 0$  and

$$\mathcal{U}^{-k} \circ \xi^{-1} : \Gamma_- \setminus \gamma_k \rightarrow \Gamma_+^{(k+1)}$$

is a  $\mathcal{C}^1$  diffeomorphism from  $\Gamma_- \setminus \gamma_k$  onto its image.  $\square$

In a previous version of the paper [30, Lemma A.11], we claimed that, with the above notations, for any  $x \in \partial\Omega$  and any  $k \in \mathbb{N}$ , introducing

$$\mathcal{O}_k(x) = \left\{ \omega \in \widehat{\Gamma}_-(x) ; \xi^{-1}(x, \omega) \in \mathcal{U}^k(\Gamma_+^{(k+1)}) \right\}$$

where  $\Delta_{k+1}$  is defined thanks to (A.17), it holds that, for any  $\omega \in \mathcal{O}_k(x) \setminus S(x)$ , the differential

$$d_\omega \left[ \mathcal{U}^{-k} \circ \xi^{-1} \right]_s(x, \omega)$$

has rank  $(d-1)$ . As pointed out by an anonymous referee, this result is not true. We provide here a simple counterexample for bounce-back boundary conditions:

**Example A.11.** Assume that  $\mathcal{V}$  is associated to bounce-back boundary conditions (see Example 3.3), i.e.

$$\mathcal{V}(x, v) = -v \quad \forall (x, v) \in \Gamma_-.$$

Then, one checks easily that

$$\xi \circ \mathcal{U}^k(x, v) = \begin{cases} (x, -v) & \text{if } k \text{ is odd} \\ \xi(x, v) = (x - \tau_-(x, v)v, v) & \text{if } k \text{ is even} \end{cases}$$

which results easily in

$$\left[ \mathcal{U}^{-k} \circ \xi^{-1} \right]_s(x, \omega) = \begin{cases} x & \text{if } k \text{ is odd} \\ \xi_s^{-1}(x, \omega) = x + \tau_+(x, \omega)\omega & \text{if } k \text{ is even} \end{cases}.$$

In particular, the rank of the differential is zero for odd  $k$  while it is  $d-1$  for even  $k$  provided  $\omega \in \widehat{\Gamma}_+(x) \setminus S(x)$  (see Proposition A.8). We aim also to point out that, with this choice of the boundary condition and for a diffuse boundary operator  $K$  of the form

$$K\varphi(x, v) = F(v) \int_{\Gamma_+(x)} \varphi(x, v') \mu_x(dv')$$

as considered in the proof of Theorem 5.1, one has, for any  $k, \ell$  odd,

$$K(M_0 R)^k M_0 K(M_0 R)^\ell M_0 K\varphi(x, v) = \tilde{F}(x) K\varphi(x, v), \quad \varphi \in L_+^1, \quad (x, v) \in \Gamma_-$$

where  $\tilde{F}(x) = \int_{\Gamma_+(x)} F(-v') \mu_x(dv')$ ,  $x \in \partial\Omega$ . In particular, one sees that such an operator is not weakly compact (see Remark 5.8).

## Appendix B. Reminders on partially integral stochastic semigroups

We collect here several results on partially integral stochastic semigroups in  $L^1(E, \Sigma, m)$  where  $(E, \Sigma, m)$  is a given  $\sigma$ -finite measure space.

### B.1. Partially integral stochastic semigroup

We begin with the following definition

**Definition B.1.** A stochastic semigroup  $\{P(t)\}_{t \geq 0}$  on the space  $L^1(E, \Sigma, m)$  is called *partially integral* if there exists a measurable function  $k: (0, \infty) \times E \times E \rightarrow [0, \infty)$ , called a *kernel*, such that for every  $y \in E$  all nonnegative  $f \in L^1(E, \Sigma, m)$  we have

$$P(t)f(y) \geq \int_E k(t, x, y) f(x) m(dx) \quad (\text{B.1})$$

and

$$\int_E \int_E k(t, x, y) m(dx) \otimes m(dy) > 0$$

for some  $t > 0$ .

We have then the following (see [38])

**Theorem B.2.** Let  $\{P(t)\}_{t \geq 0}$  be a partially integral stochastic semigroup. Assume that the semigroup  $\{P(t)\}_{t \geq 0}$  has a unique invariant probability density  $f_*$ . If  $f_* > 0$  a.e., then the semigroup  $\{P(t)\}_{t \geq 0}$  is asymptotically stable.

Let  $\mathcal{P}(t, y, B)$  be a probability transition function for the semigroup  $\{P(t)\}_{t \geq 0}$ , i.e.

$$\int_B P(t)f(y) m(dy) = \int_E \mathcal{P}(t, x, B) f(x) m(dx)$$

for all  $f \in L^1(E, \Sigma, m)$ ,  $B \in \Sigma$  and  $t \geq 0$ . Then inequality (B.1) can be rewritten as

$$\mathcal{P}(t, x, dy) \geq k(t, x, y) m(dy).$$

### B.2. Sweeping property

We define now the sweeping property for stochastic semigroups:

**Definition B.3.** A stochastic semigroup  $\{P(t)\}_{t \geq 0}$  on the space  $L^1(E, \Sigma, m)$  is sweeping from a set  $A$  if

$$\lim_{t \rightarrow \infty} \int_A P(t)f(x) m(dx) = 0$$

for each density  $f$ .

If moreover  $(E, \rho)$  is a metric space and  $\Sigma = \mathcal{B}(E)$  is the  $\sigma$ -algebra of Borel subsets of  $E$ , a partially integral semigroup  $\{P(t)\}_{t \geq 0}$  with kernel  $k(t, \xi, z)$  is said to satisfy the property (K) on  $E$  if the following holds:

(K) for every  $\xi_0 \in E$  there exist  $\varepsilon > 0$ ,  $t > 0$ , and a measurable function  $\eta: E \rightarrow [0, \infty)$  such that

$$\int_E \eta(\xi) m(d\xi) > 0$$

and

$$k(t, \xi, z) \geq \eta(z) \mathbf{1}_{B(\xi_0, \varepsilon)}(\xi), \quad (\xi, z) \in E \times E,$$

where  $B(\xi_0, \varepsilon) = \{\xi \in E: \rho(\xi, \xi_0) < \varepsilon\}$ .

We have the following which is a simple consequence of a general result concerning asymptotic decomposition of stochastic semigroups (see [39, Corollary 2]):

**Theorem B.4.** *Let  $\{P(t)\}_{t \geq 0}$  be a stochastic semigroup on  $L^1(E, \Sigma, m)$ , where  $E$  is a separable metric space,  $\Sigma = \mathcal{B}(E)$ , and  $m$  is a  $\sigma$ -finite measure on  $(E, \Sigma)$ . Assume that  $\{P(t)\}_{t \geq 0}$  has the property (K) and has no invariant density. Then  $\{P(t)\}_{t \geq 0}$  is sweeping from all compact sets.*

### B.3. Foguel alternative

If a stochastic semigroup has no invariant density but we are able to find a subinvariant function  $f_* > 0$ , then we can precisely point out all sets having the sweeping property [40]. We start with some general description.

Let a stochastic semigroup  $\{P(t)\}_{t \geq 0}$  be given and assume that this semigroup is partially integral. If the kernel  $k(t, x, y)$  satisfies

$$\int_E \int_0^\infty k(t, x, y) dt m(dx) > 0 \quad y - \text{a.e.},$$

then  $\{P(t)\}_{t \geq 0}$  is called a *pre-Harris semigroup*. In particular, if a semigroup is partially integral and irreducible then it is pre-Harris semigroup. The following condition plays a crucial role in studying sweeping.

**(KT):** There exists a measurable function  $f_*$  such that:  $0 < f_* < \infty$  a.e.,  $P(t)f_* \leq f_*$  for  $t \geq 0$ ,  $f_* \notin L^1$  and  $\int_A f_*(x) m(dx) < \infty$ .

In **(KT)** we have written  $P(t)f_*$  for a non-integrable function. We can use such notation because any substochastic operator  $P$  may be extended beyond the space  $L^1$  (see [24] Chap. I). If  $f$  is an arbitrary non-negative measurable function, then we define  $Pf$  as a pointwise limit of the sequence  $Pf_n$ , where  $(f_n)$  is any monotonic sequence of non-negative functions from  $L^1$  pointwise convergent to  $f$  almost everywhere.

**Theorem B.5** ([40], Corollary 3). *Let  $\{P(t)\}_{t \geq 0}$  be a pre-Harris stochastic semigroup which has no invariant density. Assume that the semigroup  $\{P(t)\}_{t \geq 0}$  and a set  $A \in \Sigma$  satisfy condition (KT). Then the semigroup  $\{P(t)\}_{t \geq 0}$  is sweeping with respect to  $A$ .*

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