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Convergence to equilibrium for the solution of the full compressible Navier-Stokes equations

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Abstract

We study the convergence to equilibrium for the full compressible Navier-Stokes equations on the torus $\mathbb{T}^3$. Under the conditions that both the density ρ and the temperature θ possess uniform in time positive lower and upper bounds, it is shown that global regular solutions converge to equilibrium with exponential rate. We improve the previous result obtained by Villani in (2009) [28] on two levels: weaker conditions on solutions and faster decay rates.

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1. Introduction

The motion of compressible fluids can be described by the conservation of mass, the conservation of momentum and the conservation of energy:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho,\theta) = \mathrm{div}\mathbb{S}, \\ (\rho E)_t + \mathrm{div}(\rho E u + P u) = \mathrm{div}(\kappa \nabla\theta) + \mathrm{div}(\mathbb{S}u), \end{cases} \tag{1.1}$$

where the unknowns ρ, u and θ are the density, velocity and temperature of the fluids, respectively. $\mathbb{S}$ is the stress tensor given by

$$\mathbb{S} := \mu(\nabla u + \nabla u^\top) + \lambda \mathrm{div} u \mathbb{I}_3,$$

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with $\mathbb{I}_3$ the 3×3 unit matrix. The total energy $E = e + \frac{1}{2}|u|^2$, and $e = c_\nu\theta = \frac{R}{\gamma-1}\theta$ is the internal energy, where $\gamma > 1$ is the adiabatic constant, and $R > 0$ is a generic gas constant. For a perfect gas, the pressure is given by

$$P(\rho, \theta) = (\gamma - 1)\rho e = R\rho\theta. \tag{1.2}$$

The viscous coefficients μ and λ are constants satisfying

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0. \tag{1.3}$$

Moreover, $\kappa > 0$ is the thermal conductivity coefficient. In this paper, we consider the system (1.1) for $(t, x) \in [0, \infty) \times \mathbb{T}^3$ with the volume of $\mathbb{T}^3$ normalized to unity:

$$\left|\mathbb{T}^3\right| = 1. \tag{1.4}$$

System (1.1) is supplemented with initial condition

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0). \tag{1.5}$$

Obviously, $(\rho_s, u_s, \theta_s) := (1, 0, 1)$ is a stationary state of system (1.1). Owing to the conservation of total mass, total momentum, and total energy, we assume that (ρ, u, θ) satisfies

$$\int \rho = 1, \quad \int \rho u = 0, \quad c_\nu \int \rho\theta + \frac{1}{2}\int \rho|u|^2 = c_\nu. \tag{1.6}$$

If the solutions are regular enough, (1.1) is equivalent to the following system

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho(u_t + u \cdot \nabla u) + \nabla P(\rho, \theta) = \mu\Delta u + (\lambda + \mu)\nabla \operatorname{div} u, \\ c_\nu\rho(\theta_t + u \cdot \nabla\theta) - \kappa\Delta\theta + P(\rho, \theta)\operatorname{div} u = \frac{\mu}{2}|\nabla u + \nabla u^\top|^2 + \lambda(\operatorname{div} u)^2. \end{cases} \tag{1.7}$$

There are huge literatures on the well-posedness and large time behavior of solutions to the full compressible Navier-Stokes equations (1.1). To the best of our knowledge, the local existence and uniqueness of classical solutions are first established in [24,26] in the absence of vacuum. For the case when the initial density may vanish in open sets, please refer to [5]. The global classical solutions were first obtained by Matsumura and Nishida [22] for initial data close to a non-vacuum equilibrium in H^3. Later on, Hoff [15] proved the existence and uniqueness of small global weak solutions in suitably low regularity spaces for possibly discontinuous initial data. In the presence of vacuum, Huang and Li [16] recently established global classical and weak solutions with small initial energy. Wen and Zhu [30] obtained global classical solutions with small initial density. In the framework of large data, for specific pressure laws that excludes the perfect gas equation of state, Feireisl constructed the so called variational solutions in [11], where the temperature equation is satisfied only as an inequality. For a very particular form of the viscosity coefficients depending on the density, Bresch and Desjardins [3] obtained the existence of global weak solutions with the aid of the Bresch-Desjardins entropy estimates [1] and the construction scheme of approximate solutions in [2].

For the decay rates of solutions to the full compressible Navier-Stokes equations (1.1), most results are achieved in the near-equilibrium regime. To our best knowledge, the L^2 time-decay rate of classical solutions was established in whole space $\mathbb{R}^3$ first by Matsumura and Nishida [21]. Then Ponce obtained the $L^p (p \geq 2)$ decay rates in [25]. For an exterior domain in $\mathbb{R}^3$, the time decay rate was investigated by Kobayashi and Shibata [19]. When the potential force was taken into account, Duan, Ukai, Yang, and Zhao [7] established the optimal convergence rates in $\mathbb{R}^3$ under the smallness assumptions on initial perturbation and the external force. Moreover, we would like to point out that, for a smooth bounded domain in $\mathbb{R}^3$, Matsumura and Nishida [23] proved that the solution decays exponentially to a unique equilibrium state. All these results mentioned above rely heavily on the analysis of the linearization of (1.1) and the perturbation framework.

Beyond the near-equilibrium regime, there are fewer results on the decay rates of solutions to the compressible Navier-Stokes equations. For the barotropic compressible Navier-Stokes equations on bounded domain, Fang, Zhang and the second author of this work [9] proved that if the density is uniformly bounded from above, the Lions-Feireisl weak solutions [20,10] decay exponentially to equilibrium in L^2 space. In the whole space $\mathbb{R}^3$, the time decay rate of strong solutions to the barotropic compressible Navier-Stokes equations was obtained recently by He, Huang and Wang [13] under the condition that the density is uniformly bounded in C^α for some $0 < \alpha < 1$. For the full compressible Navier-Stokes equations (1.1), as far as we know, the only result was established by Villani. On the multidimensional torus, Villani in [28] showed that if the solution remains smooth and bounded, then it converges to the equilibrium with decay rate $O(t^{-\infty})$.

Motivated by [9,28], the aim of this paper is to investigate the time decay rate of the smooth solution to the full compressible Navier-Stokes equations (1.1) on the torus $\mathbb{T}^3$. We try to weaken the conditions in [28] and get a sharp decay rate at the same time. In fact, we prove that the smooth solution to (1.1) converges exponentially to its associated equilibrium as long as the density and temperature have positive uniform in time lower and upper bounds.

Let us now present the strategy of this paper. Roughly speaking, our proof consists of two parts: (i) decay of the lower order quantities and (ii) uniform in time bounds of all derivatives of the solution. Once we have completed these two parts, we can get the decay of quantities of any order by interpolations. To this end, the first step in this paper is to obtain the decay of the energy. The observation in [9] plays an important role here. We employ the Bogovskii operator to exhibit the damping mechanism of ρ that stems from the pressure. This enables us to construct a Lyapunov functional without resorting to $\nabla\rho$ that was used in [28]. As a result, the exponential decay of $(\rho - 1, u, \theta - 1)$ in L^2 is obtained. Next, the techniques from the blow-up criterion of the system [8,27,29] are helpful for us to obtain the uniform in time bounds of the solution. The effective viscous flux $G := (\lambda + 2\mu)\mathrm{div}u - R(\rho\theta - 1)$ and the vorticity $\mathrm{curl}u$ will be used to improve the uniform in time estimates in the absence of information concerning ∇u. This is because G and $\mathrm{curl}u$ satisfy

$$\Delta G = \mathrm{div}(\rho\dot{u}), \tag{1.8}$$

and

$$\Delta\mathrm{curl}u = \mathrm{curl}(\rho\dot{u}), \tag{1.9}$$

where $\dot{u} := u_t + u \cdot \nabla u$ is the material derivative of u, respectively, and thus they are more regular than ∇u itself. We would like to emphasize that the bounds of the solution in [8,27,29] are allowed to depend on the time. The situation is quite different here, and some new ideas are needed to get the uniform bounds. The main barrier is the uniform bound of ∇u in $L^1_t(L^\infty_x)$, or equivalently the uniform bounds of $\nabla\rho$ in L^p, $p > 3$. To overcome this difficulty, by means of the Littlewood-Paley decomposition on $\mathbb{T}^3$, we derive a logarithmic type Sobolev inequality of the form

$$\|\nabla u\|_{L^\infty} \le C\|\rho - 1\|_{L^\infty} \ln \frac{\|\nabla\rho\|_{L^4} + e}{\epsilon} + \text{good terms}, \tag{1.10}$$

with $\epsilon > 0$ a small constant. Thanks to the exponential decay of the energy and the improved uniform bounds obtained before, we can prove with the aid of the effective viscous flux G that $\|\rho - 1\|_{L^\infty}$ decays exponentially fast as well. Combining this key fact with the inequality (1.10), we obtain the uniform bounds of $\nabla\rho$ (see (5.13)) by virtue of a logarithmic type Gronwall inequality. On this basis, the uniform bounds of all derivatives of the solution are achieved by induction.

Throughout this paper, we denote by $\rho_*, \bar{\rho}$, and $\theta_*, \bar{\theta}$ four positive constants, satisfying $\rho_* < \bar{\rho}$, $\theta_* < \bar{\theta}$. Now let us recall the conditional convergence result for the compressible Navier-Stokes equations (1.1) obtained by Villani.

Theorem 1.1 *([28]). Let (ρ, u, θ) be a C^∞ global solution of (1.1), satisfying the uniform bounds*

$$\begin{cases} \forall k \in \mathbb{N}, \quad \sup_{t\ge 0}\left(\|\rho(t)\|_{C^k} + \|u(t)\|_{C^k} + \|\theta(t)\|_{C^k}\right) < \infty; \\ \forall t \ge 0, \quad \rho(t) \ge \rho_* > 0; \quad \theta(t) \ge \theta_* > 0. \end{cases} \tag{1.11}$$

Then $\|(\rho(t), u(t), \theta(t)) - (1, 0, 1)\|_{C^k} = O(t^{-\infty})$ for all k.

Our main result is stated as follows.

Theorem 1.2. *Let $2\mu > \lambda$, and (ρ, u, θ) be a C^∞ global solution of (1.1), satisfying the uniform bounds*

$$\forall t \ge 0, \quad 0 < \rho_* \le \rho(t) \le \bar{\rho}; \quad 0 < \theta_* \le \theta(t) \le \bar{\theta}. \tag{1.12}$$

Then there exist two positive constants M and a, depending on $\mu, \lambda, R, \gamma, \kappa, \rho_*, \theta_*, \bar{\rho}, \bar{\theta}$ and $\|(\rho_0, u_0, \theta_0)\|_{H^\infty}$, such that

$$\|(\rho(t), u(t), \theta(t)) - (1,0,1)\|_{C^k} \le M e^{-\frac{a}{2k+4}t}, \quad \text{for all} \quad k \in \mathbb{N}. \tag{1.13}$$

Remark 1.3. In Theorem 1.1, the viscosity coefficient λ was taken to be the borderline case, i.e., $\lambda = -\frac{2}{3}\mu$. This, of course, meets our condition on μ and λ.

Remark 1.4. We believe that the restriction $2\mu > \lambda$ can be removed. The reason is that the condition $2\mu > \lambda$ is not necessary in Proposition 3.1, which ensures that the energy becomes very small after a certain period of time t_0. Then starting from t_0, one can use the bootstrap arguments like [16,30].

Remark 1.5. Our method is applicable to the two-dimensional periodic case.

The rest of this paper is organized as follows. In section 2, we introduce some preliminaries (Poincaré type inequalities, property of the Bogovskii operator and Littlewood-Paley decomposition on $\mathbb{T}^3$, etc.) that are needed in our proof. In Section 3, we establish the exponential decay of $(\rho - 1, u, \theta - 1)$ in L^2. In section 4, together with some lower order quantities, we get the exponential decay of $\theta - 1$ in L^∞. Section 5 is devoted to derive the uniform bounds of all derivatives of the solution, in particular, the uniform bounds of ∇u. The proof of Theorem 1.2 is given in section 6. Some elementary inequalities in the Littlewood-Paley theory on $\mathbb{T}^3$ appear in the appendix.

Notation.

(1) $\int f = \int_{\mathbb{T}^3} f dx$, and $\langle v \rangle_f = \int_\Omega f v dx$.
(2) $L^r = L^r(\mathbb{T}^3)$, $W^{k,r} = W^{k,r}(\mathbb{T}^3)$, $H^k = W^{k,2}$ for $r \in [1,\infty]$, $k \in \mathbb{Z}$, and $H^\infty = \cap_{k\ge 0} H^k$.
(3) $\dot{f} = f_t + (u \cdot \nabla) f$ is the material derivative of f, and $\ddot{f} = (\dot{f})^{\cdot}$ denotes the material derivative of $\dot{f}$.
(4) Throughout the paper, C denotes various "harmless" positive constants, which is larger than 1 and may change from line to line. We write $C = C(\delta_1, \delta_2, \cdots, \delta_n)$ to emphasize that C depends on $\delta_1, \delta_2, \cdots, \delta_n$. The notation $A \sim B$ means that $A \le CB$ and $B \le CA$.

2. Priliminaries

As Poincaré type inequalities play fundamental role in this paper, we begin this section by recalling a weighted Poincaré inequality that was established by Desvillettes and Villani [6] to study the convergence to equilibrium for solutions of the Boltzmann equation.

Lemma 2.1. *Let Ω be a bounded connected Lipschitz domain, $\bar{\varrho}$ be a positive constant. There exists a positive constant C, depending on $\bar{\varrho}$ and Ω, such that for any nonnegative function ϱ satisfying*

$$\int_\Omega \varrho dx = 1, \quad \varrho \le \bar{\varrho}, \tag{2.1}$$

and any $v \in H^1(\Omega)$, there holds

$$\int_\Omega \varrho \left(v - \langle v \rangle_\varrho\right)^2 dx \le C \|\nabla v\|_{L^2}^2. \tag{2.2}$$

Proof. For the convenience of readers, we give a short proof here following the strategy of [6]. In fact, using (2.1) and the usual Poincare's inequality, one deduces that

$$
\begin{aligned}
\int_\Omega \varrho \left(v-\langle v\rangle_\varrho\right)^2 dx =&\langle v^2\rangle_\varrho - \left(\langle v\rangle_\varrho\right)^2\\
=&\frac{1}{2}\int_\Omega\int_\Omega \varrho(x)\varrho(y)|v(x)-v(y)|^2 dxdy\\
\le&\frac{1}{2}\bar{\varrho}^2\int_\Omega\int_\Omega |v(x)-v(y)|^2 dxdy\\
=&\bar{\varrho}^2|\Omega|^2\left[\langle v^2\rangle_{\frac{1}{|\Omega|}} - \left(\langle v\rangle_{\frac{1}{|\Omega|}}\right)^2\right]\\
=&\bar{\varrho}^2|\Omega|\int_\Omega \left|v-\langle v\rangle_{\frac{1}{|\Omega|}}\right|^2 dx\\
\le&C\|\nabla v\|_{L^2}^2.
\end{aligned}
\tag{2.3}
$$

This completes the proof of Lemma 2.1. □

The following lemma is also a variant of Poincaré inequality, which will be used to remove the weight function ϱ appearing in Lemma 2.1 without resorting to the lower bound of ϱ. For the proof, please refer to Lemma 3.2 in [11].

Lemma 2.2. *Let Ω be a bounded connected Lipschitz domain in $\mathbb{R}^d$ with $d \ge 2$ and $p > 1$ be a constant. Given positive constants M_0 and E_0, there is a constant $C = C(E_0, M_0)$, such that for any non-negative function ϱ satisfying*

$$
M_0 \le \int_\Omega \varrho dx, \quad \int_\Omega \varrho^p dx \le E_0,
$$

and for any $v \in H^1(\Omega)$, there holds

$$
\|v\|_{L^2}^2 \le C\left[\|\nabla v\|_{L^2}^2 + \left(\int_\Omega \varrho|v|dx\right)^2\right].
$$

Now we recall some properties of the Bogovskii operator [4], which play a key role to indicate the dissipation of $\rho - 1$ in L^2. More details of the Bogovskii operator can be found in [12].

Lemma 2.3. *Let $p, r \in (1, \infty)$. For any $f \in L^p(\mathbb{T}^3)$, with $\int_{\mathbb{T}^3} f dx = 0$, we define*

$$
\mathcal{B}[f] := \nabla u, \tag{2.4}
$$

where u is the unique solution of

$$
\Delta u = f \quad \text{with} \quad \int_{\mathbb{T}^3} u = 0. \tag{2.5}
$$

Then the following properties hold:

- *The function $v = \mathcal{B}[f]$ solves the problem*

$$
\mathrm{div} v = f. \tag{2.6}
$$

- *$\mathcal{B}$ is a bounded linear operator from $L^p(\mathbb{T}^3)$ to $\left\{v \in \left[W^{1,p}(\mathbb{T}^3)\right]^3 : \int_{\mathbb{T}^3} v = 0\right\}$, satisfying*

$$
\|\mathcal{B}[f]\|_{W^{1,p}(\mathbb{T}^3)} \le C\|f\|_{L^p(\mathbb{T}^3)}. \tag{2.7}
$$

- *If a function f can be written in the form $f = \mathrm{div} g$ with $g \in \left[L^r(\mathbb{T}^3)\right]^3$ and $\int_{\mathbb{T}^3} g = 0$, then*

$$\|\mathcal{B}[f]\|_{L^r(\mathbb{T}^3)} \le C\|g\|_{L^r(\mathbb{T}^3)}. \tag{2.8}$$

Next, we briefly introduce the Littlewood-Paley decomposition on $\mathbb{T}^3$, which will be used in the estimates of $\|\nabla\rho\|_{L^4}$. Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function such that $\mathrm{supp}\,\chi \subset (-1, 1)$, $\chi \equiv 1$ on $[0, \frac{1}{2}]$, χ is decreasing on $[0, \infty)$, $\chi(\frac{3}{4}) = \frac{1}{2}$ and $\varphi(x) = \chi(x) - \chi(2x)$. Then $\mathrm{supp}\,\varphi \subset (\frac{1}{4}, 1)$ and $\varphi(x) \ge \frac{1}{2}$ for all $x \in \left[\frac{3}{8}, \frac{3}{4}\right]$. Moreover,

$$\chi(x) + \sum_{q\ge 1} \varphi\left(\frac{x}{2^q}\right) = 1 \quad \text{for all} \quad x \in \mathbb{T}^3. \tag{2.9}$$

If u is periodic, then it has a Fourier series

$$u(x) = \sum_{n\in\mathbb{Z}^3} u_{(n)} e^{in\cdot x}, \quad u_{(n)} \in \mathbb{C}.$$

For $q \ge 0$, we define

$$\Delta_0 u = \sum_{n\in\mathbb{Z}^3} u_{(n)} e^{in\cdot x} \chi(|n|) = u_{(0)} = \int_{\mathbb{T}^3} u dx, \tag{2.10}$$

and

$$\Delta_q u = \sum_{n\in\mathbb{Z}^3} u_{(n)} e^{in\cdot x} \varphi\left(\frac{|n|}{2^q}\right) \quad \text{for all} \quad q \ge 1. \tag{2.11}$$

Then we have the decomposition for u:

$$u = \sum_{q\ge 0} \Delta_q u. \tag{2.12}$$

Some elementary inequalities in the Littlewood-Paley theory on $\mathbb{T}^3$ can be found in the Appendix. For more details about the Littlewood-Paley theory on $\mathbb{T}^3$, please refer to [17]. Finally, let us state the following classical product formula in Sobolev spaces, which will be frequently used in the higher order estimates.

Lemma 2.4 *([18]). Assume $f, g \in H^N(\mathbb{T}^3)$. Then for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $1 \le |\alpha| \le N$, we have*

$$\|\partial^\alpha(fg)\|_{L^2} \le C\left(\|f\|_{L^\infty}\|\nabla^N g\|_{L^2} + \|g\|_{L^\infty}\|\nabla^N f\|_{L^2}\right), \tag{2.13}$$

and

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_{L^2} \le C\left(\|\nabla f\|_{L^\infty}\|\nabla^{N-1} g\|_{L^2} + \|g\|_{L^\infty}\|\nabla^N f\|_{L^2}\right). \tag{2.14}$$

3. L^2 exponential decay estimates

The purpose of this section is to establish the following proposition. We would like to remark that neither the uniform lower bound of the density ρ nor the restriction $2\mu > \lambda$ is needed in this section.

Proposition 3.1. *Let (ρ, u, θ) be a C^∞ global solution of (1.1), satisfying the uniform bounds*

$$\forall t \ge 0, \quad 0 \le \rho(t) \le \bar{\rho}; \quad 0 < \theta_* \le \theta(t) \le \bar{\theta}. \tag{3.1}$$

Then there exist two positive constants $\bar{M}$ and $\bar{a}$, depending on $\mu, \lambda, R, \gamma, \kappa, \theta_*, \bar{\rho}, \bar{\theta}$ and $\|u_0\|_{L^2}$, such that

$$\begin{aligned}\sup_{t\ge 0}&\left(\|(\sqrt{\rho}u)(t)\|_{L^2}^2 + \|(\rho-1)(t)\|_{L^2}^2 + \|\left(\sqrt{\rho}(\theta-1)\right)(t)\|_{L^2}^2\right)\\ &+ \int_0^\infty \left(\|\rho-1\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) dt \le \bar{M},\end{aligned} \tag{3.2}$$

and

$$\|(\sqrt{\rho}u)(t)\|_{L^2}+\|(\rho-1)(t)\|_{L^2}+\|(\sqrt{\rho}(\theta-1))(t)\|_{L^2}\le \bar{M}e^{-\bar{a}t}. \tag{3.3}$$

In order to prove this proposition, we give the following three lemmas. The first one concerns the basic energy estimates.

Lemma 3.2. *Under the conditions of Proposition 3.1, there exist two positive constants $M_1=M_1(\mu,\kappa,\bar{\theta})$ and $C=C(\mu,\kappa,R,\gamma,\theta_*,\bar{\rho},\bar{\theta},\|u_0\|_{L^2})$, such that*

$$\frac{d}{dt}\int\left(\rho\frac{|u|^2}{2}+R(\rho\ln\rho-\rho+1)+c_v\rho(\theta-\ln\theta-1)\right)dx+M_1\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\le 0, \tag{3.4}$$

and

$$\sup_{t\ge 0}\int\left(\rho|u|^2+(\rho\ln\rho-\rho+1)+\rho(\theta-\ln\theta-1)\right)(t)dx+\int_0^\infty\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)dt\le C. \tag{3.5}$$

Proof. Taking the L^2 inner product of $(1.7)_2$ with u, and integrating by parts yields

$$\frac{d}{dt}\int\rho\frac{|u|^2}{2}=-\mu\int|\nabla u|^2-(\lambda+\mu)\int(\text{div}u)^2+R\int\rho\theta\text{div}u. \tag{3.6}$$

Next, taking the inner product of $(1.7)_3$ with $1-\theta^{-1}$, integrating by parts, we are led to

$$\begin{aligned}&\frac{d}{dt}\int c_v\rho(\theta-\ln\theta-1)+\kappa\int\frac{|\nabla\theta|^2}{\theta^2}\\ =&-R\int\rho\theta\text{div}u+R\int\rho\text{div}u-\int\frac{\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\text{div}u)^2}{\theta}+\mu\int|\nabla u|^2+(\lambda+\mu)\int(\text{div}u)^2,\end{aligned} \tag{3.7}$$

where we have used the following equality

$$\frac{1}{2}\int|\nabla u+\nabla u^\top|^2=\int\left(|\nabla u|^2+(\text{div}u)^2\right).$$

Putting (3.6) and (3.7) together, using $(1.1)_1$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int\left(\rho\frac{|u|^2}{2}+c_v\rho(\theta-\ln\theta-1)\right)\\ &+\kappa\int\frac{|\nabla\theta|^2}{\theta^2}+\int\frac{\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\text{div}u)^2}{\theta}\\ =&R\int\rho\text{div}u=-R\int\rho\left((\ln\rho)_t+u\cdot\nabla\ln\rho\right)\\ =&-\frac{d}{dt}\int R\rho\ln\rho=-\frac{d}{dt}\int R\left(\rho\ln\rho-\rho+1\right).\end{aligned} \tag{3.8}$$

Consequently, taking $M_1:=\min\left\{\frac{\kappa}{\bar{\theta}^2},\frac{\mu}{\bar{\theta}}\right\}$, we get (3.4). Then (3.5) follows immediately. The proof of Lemma 3.2 is completed. $\square$

Now we give some Poincaré type inequalities with the aid of (1.6) and (3.5).

Lemma 3.3 *(Poincaré type inequality). Under the conditions of Proposition 3.1, there exists a positive constant C, depending on $\mu,\kappa,R,\gamma,\theta_*,\bar{\rho},\bar{\theta}$, and $\|u_0\|_{L^2}$, such that for all $t\in(0,\infty)$, there hold*

$$\|(\sqrt{\rho}u)(t)\|_{L^2}^2\le C\|\nabla u(t)\|_{L^2}^2, \tag{3.9}$$

$$\|u(t)\|_{L^2}^2 \le C\|\nabla u(t)\|_{L^2}^2, \tag{3.10}$$

and

$$\|\sqrt{\rho}(\theta-1)(t)\|_{L^2}^2 \le C\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2\right). \tag{3.11}$$

Proof. Taking $\varrho = \rho$ in Lemma 2.1, noting that $\langle u\rangle_\rho = 0$ by (1.6), we thus get (3.9). Then (3.10) is a consequence of (3.9) and Lemma 2.2. For (3.11), using first (1.6), we have

$$\langle\theta\rangle_\rho - 1 = -\frac{1}{2c_v}\int \rho|u|^2. \tag{3.12}$$

Then we infer from Lemma 2.1, (3.5) and (3.9) that

$$\begin{aligned}\int \rho|\theta-1|^2 \le& 2\int \rho|\theta-\langle\theta\rangle_\rho|^2 + 2\int \rho|\langle\theta\rangle_\rho - 1|^2\\ \le& C\|\nabla\theta\|_{L^2}^2 + C\int \rho|u|^2\\ \le& C\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right).\end{aligned} \tag{3.13}$$

This, together with Lemma 2.2, gives (3.11). We complete the proof of Lemma 3.3. □

In the following lemma, we construct a new energy functional with the aid of the Bogovskii operator.

Lemma 3.4. *Under the conditions of Proposition 3.1, there exist two positive constants K_1 and M_2, depending on $\mu, \lambda, \kappa, R, \gamma, \theta_*, \bar\rho, \bar\theta$, and $\|u_0\|_{L^2}$, such that*

$$\begin{aligned}&\frac{d}{dt}\int \left\{K_1\left(\rho\frac{|u|^2}{2} + R(1+\rho\ln\rho-\rho) + c_v\rho(\theta-\ln\theta-1)\right) - 2\rho u\cdot\mathcal{B}[\rho-1]\right\}\\ &+ M_2\left(\|\rho-1\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) \le 0.\end{aligned} \tag{3.14}$$

Proof. Taking the L^2 inner product of $(1.1)_2$ with $\mathcal{B}[\rho-1]$ yields

$$\begin{aligned}&-\frac{d}{dt}\int \rho u\cdot\mathcal{B}[\rho-1] + R\int(\rho-1)^2\\ =&\int \rho u\mathcal{B}[\mathrm{div}(\rho u)] - \int \rho u\otimes u : \nabla\mathcal{B}[\rho-1] + \int \mu\nabla u : \nabla\mathcal{B}[\rho-1] + (\lambda+\mu)\mathrm{div}u(\rho-1)\\ &- R\int \rho(\theta-1)(\rho-1)\\ \le& C\|\rho u\|_{L^2}^2 + C\|\rho|u|^2\|_{L^2}\|\nabla\mathcal{B}[\rho-1]\|_{L^2} + \frac{R}{2}\int(\rho-1)^2 + C\|\nabla u\|_{L^2}^2 + C\|\rho(\theta-1)\|_{L^2}^2\\ \le& C\|\nabla u\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + \frac{R}{2}\int(\rho-1)^2,\end{aligned}$$

where we have used Lemma 2.3, the Poincaré type inequalities (3.9), (3.11) and the following estimate

$$\|\rho|u|^2\|_{L^2}\|\nabla\mathcal{B}[\rho-1]\|_{L^2} \le C\|u\|_{L^4}^2\|\rho-1\|_{L^2} \le C\|u\|_{H^1}^2 \le C\|\nabla u\|_{L^2}^2,$$

due to Lemma 2.3 and (3.10). Consequently,

$$-2\frac{d}{dt}\int \rho u\cdot\mathcal{B}[\rho-1] + R\int(\rho-1)^2 \le C_1\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right). \tag{3.15}$$

Multiplying (3.4) by a sufficiently large positive number K_1, and then adding the resulting equation to (3.15), we obtain (3.14), where the constant M_2 can be chosen as $\min\{R, K_1M_1 - C_1\}$. This completes the proof Lemma 3.4. □

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. In view of Lemma 2.3, we have

$$2\left|\int \rho u\cdot \mathcal{B}[\rho-1]\right| \le C\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\rho-1\|_{L^2}^2\right). \tag{3.16}$$

By the Taylor expansion, one deduces that

$$1+\rho\ln\rho-\rho\sim(\rho-1)^2,\quad \text{and}\quad \theta-\ln\theta-1\sim(\theta-1)^2, \tag{3.17}$$

provided (3.1) holds. Therefore, if K_1 is large enough, there holds

$$\begin{aligned}&\int\left\{K_1\left(\rho\frac{|u|^2}{2}+R(1+\rho\ln\rho-\rho)+c_v\rho(\theta-\ln\theta-1)\right)-2\rho u\cdot\mathcal{B}[\rho-1]\right\}\\&\sim\|\sqrt{\rho}u\|_{L^2}^2+\|\rho-1\|_{L^2}^2+\|\sqrt{\rho}(\theta-1)\|_{L^2}^2,\end{aligned} \tag{3.18}$$

which can be bounded by $\|\rho-1\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2$ in view of (3.9) and (3.11). Then (3.2) and (3.3) follow from (3.14) easily. This proof of Proposition 3.1 is completed. □

4. H^1 exponential decay estimates

In this section, we improve the uniform estimates obtained in (3.2), and obtain the exponential decay of $\|(\nabla u,\nabla\theta)\|_{L^2}$ and $\|\theta-1\|_{L^\infty}$. More precisely, we will establish the following proposition.

Proposition 4.1. *Let $2\mu>\lambda$, and (ρ,u,θ) be a C^∞ global solution of (1.1), satisfying the uniform bounds (3.1), and $\inf_{x\in\mathbb{T}^3}\rho_0(x)>0$. Then there exist two positive constants $\tilde{M}$ and $\tilde{a}$, depending on $\mu,\lambda,R,\gamma,\kappa,\theta_*,\bar{\rho},\bar{\theta}$, $\|\rho_0\|_{H^1}$, $\|(u_0,\theta_0)\|_{H^2}$ and $\inf_{x\in\mathbb{T}^3}\rho_0$, such that*

$$\begin{aligned}&\|\sqrt{\rho}u\|_{L^2}+\|\rho-1\|_{L^2}+\|\sqrt{\rho}(\theta-1)\|_{L^2}+\|\rho^{\frac14}u\|_{L^4}^2+\|\nabla u\|_{L^2}+\|\sqrt{\rho}\dot{u}\|_{L^2}+\|\nabla\theta\|_{L^2}\\&+\|\sqrt{\rho}\dot{\theta}\|_{L^2}+\|\theta-1\|_{L^\infty}\le\tilde{M}e^{-\tilde{a}t}.\end{aligned} \tag{4.1}$$

In the following, we establish several lemmas to prove the above proposition. It is just the following lemma that needs the restriction $2\mu>\lambda$.

Lemma 4.2. *Under the conditions of Proposition 4.1, there exist two positive constants M_3 and C, depending on $\mu,\lambda,R,\bar{\rho}$ and $\bar{\theta}$, such that*

$$\frac{d}{dt}\int\rho|u|^4+M_3\|u|\nabla u|\|_{L^2}^2\le C\|\nabla u\|_{L^2}^2. \tag{4.2}$$

Proof. Multiplying $(1.7)_2$ by $4|u|^2u$, and then integrating the resulting equation w.r.t. x, we arrive at

$$\begin{aligned}&\frac{d}{dt}\int\rho|u|^4-4\int(\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u)|u|^2u\\&=4R\int\rho\theta\mathrm{div}(|u|^2u)\le C\int\rho\theta|u|^2|\nabla u|\le\epsilon\,\|u|\nabla u|\|_{L^2}^2+C\int\rho|u|^2.\end{aligned} \tag{4.3}$$

On the other hand, it is not difficult to verify, see [13] for instance, that there exist some positive constant $M_3=M_3(\mu,\lambda)$, such that

$$-4\int(\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u)|u|^2u\ge 2M_3\int|u|^2|\nabla u|^2,\quad\text{provided}\quad 2\mu>\lambda. \tag{4.4}$$

Then using (3.9) again, we get (4.2). This completes the proof of Lemma 4.2. □

The following lemma provides the $L_t^\infty L_x^2$ estimate of ∇u and the $L_t^2 L_x^2$ estimate of $\sqrt{\rho}\dot{u}$ simultaneously. The proof will be given in the spirit of Wen and Zhu [30], and the effective viscous flux G plays an important role.

Lemma 4.3. *Under the conditions of Proposition 4.1, there exist three positive constants K_2, M_4 and C, depending on $\mu, \lambda, R, \gamma, \kappa, \theta_*, \bar{\rho}, \bar{\theta}$, and $\|u_0\|_{H^1}$, such that*

$$\begin{aligned}&\frac{d}{dt}\left\{K_2\int \rho|u|^4+\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2+\frac{R^2}{\lambda+2\mu}\|\rho\theta-1\|_{L^2}^2-2R\int(\rho\theta-1)\mathrm{div}u\right\}\\&+M_4\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|u|\nabla u|\|_{L^2}^2\right)\le C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right).\end{aligned}\tag{4.5}$$

Moreover,

$$\sup_{t\ge0}\left(\|(\rho^{\frac14}u)(t)\|_{L^4}^4+\|\nabla u(t)\|_{L^2}^2\right)+\int_0^\infty\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|u|\nabla u|\|_{L^2}^2\right)dt\le C.\tag{4.6}$$

Proof. Taking the L^2 inner product of $(1.7)_2$ with u_t, integrating by parts, and using

$$\mathrm{div}u=\frac{G+R(\rho\theta-1)}{\lambda+2\mu},\tag{4.7}$$

we find that

$$\begin{aligned}&\frac12\frac{d}{dt}\left(\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2\right)+\int\rho|u_t|^2\\&=-\int\rho u_t\cdot(u\cdot\nabla)u+\frac{d}{dt}\int R(\rho\theta-1)\mathrm{div}u-\frac{R^2}{2(\lambda+2\mu)}\frac{d}{dt}\|\rho\theta-1\|_{L^2}^2-\frac{1}{\lambda+2\mu}\int P_tG.\end{aligned}\tag{4.8}$$

In order to estimate the last term in the above equality, we shall use the following equation of the pressure P:

$$\begin{aligned}P_t+\mathrm{div}(Pu)=&R\rho\dot{\theta}\\=&(\gamma-1)\left\{-P\mathrm{div}u+\kappa\Delta\theta+\lambda(\mathrm{div}u)^2+\mu(\nabla u+\nabla u^\top):\nabla u\right\}.\end{aligned}\tag{4.9}$$

Integrating by parts, we obtain

$$\begin{aligned}&-\frac{1}{\lambda+2\mu}\int P_tG\\&=-\frac{1}{\lambda+2\mu}\int Pu\cdot\nabla G+\frac{\gamma-1}{\lambda+2\mu}\int P\mathrm{div}uG+\frac{(\gamma-1)\kappa}{\lambda+2\mu}\int\nabla\theta\cdot\nabla G\\&+\frac{\gamma-1}{\lambda+2\mu}\int\left(\lambda\mathrm{div}uu\cdot\nabla G+\mu(\nabla u+\nabla u^\top):u\otimes\nabla G+[\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u]\cdot uG\right).\end{aligned}$$

Using $(1.7)_2$ to deal with the last term of the above equality, and integrating by parts once more yields

$$\begin{aligned}&-\frac{1}{\lambda+2\mu}\int P_tG\\=&\frac{\gamma-1}{\lambda+2\mu}\int\rho(u_t+(u\cdot\nabla)u)\cdot uG-\frac{\gamma}{\lambda+2\mu}\int Pu\cdot\nabla G+\frac{(\gamma-1)\kappa}{\lambda+2\mu}\int\nabla\theta\cdot\nabla G\\&+\frac{\gamma-1}{\lambda+2\mu}\int\left(\lambda\mathrm{div}uu\cdot\nabla G+\mu(\nabla u+\nabla u^\top):u\otimes\nabla G\right).\end{aligned}\tag{4.10}$$

Substituting (4.10) into (4.8), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2+\frac{R^2}{\lambda+2\mu}\|\rho\theta-1\|_{L^2}^2-2R\int(\rho\theta-1)\mathrm{div}u\right\}+\int\rho|u_t|^2\\
=&-\int\rho u_t\cdot(u\cdot\nabla)u+\frac{\gamma-1}{\lambda+2\mu}\int\rho(u_t+(u\cdot\nabla)u)\cdot uG-\frac{\gamma R}{\lambda+2\mu}\int\rho\theta u\cdot\nabla G\\
&+\frac{(\gamma-1)\kappa}{\lambda+2\mu}\int\nabla\theta\cdot\nabla G+\frac{\gamma-1}{\lambda+2\mu}\int\lambda\mathrm{div}uu\cdot\nabla G+\mu(\nabla u+\nabla u^\top):u\otimes\nabla G\\
=&\sum_{1\le j\le 5}I_j.
\end{aligned}\tag{4.11}$$

Now we estimate I_j, $1\le j\le 5$. Clearly,

$$|I_1|\le\frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2+C\,\|u|\nabla u|\|_{L^2}^2.\tag{4.12}$$

Recalling that $G=(\lambda+2\mu)\mathrm{div}u-R(\rho\theta-1)$, thanks to (3.9), we have

$$\begin{aligned}
\int\rho|u|^2G^2\le&C\,\|u|\nabla u|\|_{L^2}^2+C\int\rho|u|^2(\rho\theta-1)^2\\
\le&C\,\|u|\nabla u|\|_{L^2}^2+C\int\rho|u|^2\\
\le&C\,\|u|\nabla u|\|_{L^2}^2+C\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{4.13}$$

Then we can bound I_2 as follows

$$\begin{aligned}
|I_2|\le&\frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2+C\,\|u|\nabla u|\|_{L^2}^2+C\int\rho|u|^2G^2\\
\le&\frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2+C\,\|u|\nabla u|\|_{L^2}^2+C\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{4.14}$$

The last three terms can be bounded together:

$$\begin{aligned}
\sum_{3\le j\le 5}|I_j|\le&C\left(\|\sqrt{\rho}u\|_{L^2}+\|\nabla\theta\|_{L^2}+\|u|\nabla u|\|_{L^2}\right)\|\nabla G\|_{L^2}\\
\le&C\left(\|\nabla u\|_{L^2}+\|\nabla\theta\|_{L^2}+\|u|\nabla u|\|_{L^2}\right)\left(\|\sqrt{\rho}u_t\|_{L^2}+\|u|\nabla u|\|_{L^2}\right)\\
\le&\frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2+C\left(\|u|\nabla u|\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right),
\end{aligned}\tag{4.15}$$

where we have used (3.9) and the following elliptic estimate

$$\|\nabla G\|_{L^2}\le C\|\rho(u_t+u\cdot\nabla u)\|_{L^2}\le C\left(\|\sqrt{\rho}u_t\|_{L^2}+\|u|\nabla u|\|_{L^2}\right).\tag{4.16}$$

Accordingly,

$$\begin{aligned}
&\frac{d}{dt}\left\{\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2+\frac{R^2}{\lambda+2\mu}\|\rho\theta-1\|_{L^2}^2-2R\int(\rho\theta-1)\mathrm{div}u\right\}\\
&+\|\sqrt{\rho}u_t\|_{L^2}^2\le C\left(\|u|\nabla u|\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right).
\end{aligned}\tag{4.17}$$

Multiplying (4.2) by a sufficiently large positive constant $K_2=K_2(\mu,\lambda,R,\gamma,\bar\rho,\bar\theta)$, and then adding the resulting inequality to (4.17), we find that (4.5) holds for some positive constant M_4. Since

$$2R\left|\int(\rho\theta-1)\mathrm{div}u\right|\le(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2+\frac{R^2}{\lambda+\mu}\|\rho\theta-1\|_{L^2}^2,\tag{4.18}$$

integrating (4.5) over $[0,t]$, and using (3.2), we get (4.6) immediately. The proof of Lemma 4.3 is completed. □

In the following two lemmas, we improve the uniform estimates of u and θ with the aid of the material derivatives $\dot u$ and $\dot\theta$, respectively. This method was first used by Hoff [14] to construct discontinuous global solutions to the isentropic Navier-Stokes equations. See [15] for the extension to the full Navier-Stokes equations. Many calculations in the proof of these two lemmas are borrowed from [8], we give the details for the sake of completeness. Nevertheless, we would like to emphasize that, different from [8], all the bounds in our proof here are independent of the time t.

Lemma 4.4. *Under the conditions of Proposition 4.1, there exist three positive constants K_3, M_5 and C, depending on $\mu, \lambda, R, \gamma, \kappa, \theta_*, \bar{\rho}, \bar{\theta}$, $\|u_0\|_{H^2}$, $\|(\rho_0, \theta_0)\|_{H^1}$ and $\inf_{x\in\mathbb{T}^3}\rho_0$, such that*

$$\sup_{t\geq 0}\left(\|(\sqrt{\rho}\dot{u})(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^2}^2\right)+\int_0^\infty\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\|\nabla\dot{u}\|_{L^2}^2\right)dt\leq C, \tag{4.19}$$

and

$$\begin{aligned}&\frac{d}{dt}\left\{\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\kappa K_3\|\nabla\theta\|_{L^2}^2-K_3\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(divu)^2\right]\theta\right\}\\&\quad+M_5\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\|\nabla\dot{u}\|_{L^2}^2\right)\\&\leq C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\rho-1\|_{L^2}^2\right).\end{aligned} \tag{4.20}$$

Proof. Taking the material derivative of the momentum equation $(1.7)_2$, using $(1.7)_1$ and $(1.7)_2$, we have

$$\begin{aligned}&\rho\ddot{u}+\nabla P_t+\mathrm{div}(\nabla P\otimes u)\\=&\mu[\Delta u_t+\mathrm{div}(\Delta u\otimes u)]+(\lambda+\mu)[\nabla\mathrm{div}u_t+\mathrm{div}((\nabla\mathrm{div}u)\otimes u)],\end{aligned} \tag{4.21}$$

where $\mathrm{div}(f\otimes u):=\sum\partial_j(fu^j)$. Taking the L^2 inner product of the above equation with $\dot{u}$, using $(1.1)_1$ and (4.9), we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\dot{u}|^2+\mu\int|\nabla\dot{u}|^2+(\lambda+\mu)\int|\mathrm{div}\dot{u}|^2\\=&R\int\rho\dot{\theta}\mathrm{div}\dot{u}-\int P\nabla u:\nabla\dot{u}^\top\\&+\mu\int[(\nabla u\nabla u):\nabla\dot{u}+\nabla u:(\nabla u\nabla\dot{u})-\mathrm{div}u\nabla u:\nabla\dot{u}]\\&+(\lambda+\mu)\int\left[\mathrm{div}u\nabla u:\nabla\dot{u}^\top+\nabla u:\nabla u^\top\mathrm{div}\dot{u}-(\mathrm{div}u)^2\mathrm{div}\dot{u}\right].\end{aligned} \tag{4.22}$$

It follows that

$$\frac{d}{dt}\int\rho|\dot{u}|^2+\mu\|\nabla\dot{u}\|_{L^2}^2\leq C\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^4}^4\right). \tag{4.23}$$

Next, taking the L^2 inner product of $(1.7)_3$ with θ_t, integrating by parts, we are led to

$$\begin{aligned}c_v\int\rho\dot{\theta}^2+\frac{\kappa}{2}\frac{d}{dt}\int|\nabla\theta|^2=&c_v\int\rho\dot{\theta}u\cdot\nabla\theta-R\int\rho\theta\mathrm{div}u\dot{\theta}+R\int\rho\theta\mathrm{div}uu\cdot\nabla\theta\\&+\frac{d}{dt}\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right]\theta\\&-\int\left[\mu(\nabla u+\nabla u^\top):(\nabla u_t+\nabla u_t^\top)+2\lambda\mathrm{div}u\mathrm{div}u_t\right]\theta.\end{aligned} \tag{4.24}$$

Direct calculations show that

$$\begin{aligned}&-2\lambda\int\mathrm{div}u\mathrm{div}u_t\theta\\=&-2\lambda\int\mathrm{div}u\mathrm{div}\dot{u}\theta+2\lambda\int\mathrm{div}u\nabla u:\nabla u^\top\theta-\lambda\int\theta(\mathrm{div}u)^3-\lambda\int u\cdot\nabla\theta(\mathrm{div}u)^2,\end{aligned}$$

and

$$
\begin{aligned}
&-\mu\int(\nabla u+\nabla u^\top):(\nabla u_t+\nabla u_t^\top)\theta\\
=&-2\mu\int\left(\nabla u+\nabla u^\top\right):\nabla\dot{u}\theta+2\mu\int\left(\nabla u+\nabla u^\top\right):(\nabla u\nabla u)\theta\\
&-\mu\int\theta\mathrm{div}u\nabla u:\left(\nabla u+\nabla u^\top\right)-\mu\int u\cdot\nabla\theta\nabla u:\left(\nabla u+\nabla u^\top\right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&c_\nu\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\frac{\kappa}{2}\frac{d}{dt}\|\nabla\theta\|_{L^2}^2\\
=&c_\nu\int\rho\dot{\theta}u\cdot\nabla\theta-R\int\rho\theta\mathrm{div}u\dot{\theta}+R\int\rho\theta\mathrm{div}uu\cdot\nabla\theta\\
&+\frac{d}{dt}\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right]\theta-2\lambda\int\mathrm{div}u\mathrm{div}\dot{u}\theta\\
&+2\lambda\int\mathrm{div}u\nabla u:\nabla u^\top\theta-\lambda\int\theta(\mathrm{div}u)^3-\lambda\int u\cdot\nabla\theta(\mathrm{div}u)^2\\
&-2\mu\int\left(\nabla u+\nabla u^\top\right):\nabla\dot{u}\theta+2\mu\int\left(\nabla u+\nabla u^\top\right):(\nabla u\nabla u)\theta\\
&-\mu\int\theta\mathrm{div}u\nabla u:\left(\nabla u+\nabla u^\top\right)-\mu\int u\cdot\nabla\theta\nabla u:\left(\nabla u+\nabla u^\top\right)\\
\le&\frac{c_\nu}{4}\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+C\int|u|^2|\nabla\theta|^2+C\|\nabla u\|_{L^2}^2+\epsilon\|\nabla\dot{u}\|_{L^2}^2\\
&+C\|\nabla u\|_{L^4}^4+\frac{d}{dt}\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right]\theta.
\end{aligned}
\tag{4.25}
$$

Using (4.6), Poincaré inequality (3.10), and the following elliptic estimate

$$
\begin{aligned}
\|\nabla^2\theta\|_{L^2}\le&C\left(\|\rho\dot{\theta}\|_{L^2}+\|\rho\theta\mathrm{div}u\|_{L^2}+\|\nabla u\|_{L^4}^2\right)\\
\le&C\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}+\|\nabla u\|_{L^2}+\|\nabla u\|_{L^4}^2\right),
\end{aligned}
\tag{4.26}
$$

we have

$$
\begin{aligned}
\int|u|^2|\nabla\theta|^2\le&\|u\|_{L^6}^2\|\nabla\theta\|_{L^3}^2\le C\|u\|_{H^1}^2\|\nabla\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}\\
\le&C\|\nabla\theta\|_{L^2}\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}+\|\nabla u\|_{L^2}+\|\nabla u\|_{L^4}^2\right)\\
\le&\frac{c_\nu}{4}\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^4}^4\right).
\end{aligned}
\tag{4.27}
$$

Substituting (4.27) into (4.25) yields

$$
\begin{aligned}
c_\nu\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\kappa\frac{d}{dt}\|\nabla\theta\|_{L^2}^2\le&C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^4}^4\right)+\epsilon\|\nabla\dot{u}\|_{L^2}^2\\
&+\frac{d}{dt}\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right]\theta.
\end{aligned}
\tag{4.28}
$$

Combining this inequality with (4.23), we find that there exist two positive constants K_3 and M_5, depending on $\mu,\lambda,R,\gamma,\kappa,\theta_*,\bar{\rho},\bar{\theta}$, and $\|u_0\|_{H^1}$, such that

$$
\begin{aligned}
&\frac{d}{dt}\left\{\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\kappa K_3\|\nabla\theta\|_{L^2}^2-K_3\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right]\theta\right\}\\
&+M_5\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\|\nabla\dot{u}\|_{L^2}^2\right)\le C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\nabla u\|_{L^4}^4\right).
\end{aligned}
\tag{4.29}
$$

Noting that

$$\Delta u = \nabla \mathrm{div} u - \mathrm{curlcurl} u, \tag{4.30}$$

by virtue of the elliptic estimates, (4.6) and (3.11), we have

$$\begin{aligned}\|\nabla u\|_{L^4}^4 \le & C\left(\|\mathrm{curl} u\|_{L^4}^4 + \|\mathrm{div} u\|_{L^4}^4\right)\\ \le & C\left(\|\mathrm{curl} u\|_{L^2}\|\nabla \mathrm{curl} u\|_{L^2}^3 + \|G\|_{L^4}^4 + \|\rho\theta - 1\|_{L^4}^4\right)\\ \le & C\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^3 + \|G\|_{L^2}^4 + \|\nabla G\|_{L^2}^4 + \|\theta - 1\|_{H^1}^4 + \|\rho - 1\|_{L^4}^4\right)\\ \le & C\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^4 + \|\nabla u\|_{L^2}^4 + \|\nabla \theta\|_{L^2}^4 + \|\rho - 1\|_{L^2}^2\right),\end{aligned} \tag{4.31}$$

where we have used the following interpolation

$$\|\rho - 1\|_{L^4}^4 \le C\|\rho - 1\|_{L^2}^2\|\rho - 1\|_{L^\infty}^2 \le C\|\rho - 1\|_{L^2}^2.$$

Substituting (4.31) into (4.29), we have

$$\begin{aligned}&\frac{d}{dt}\left\{\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \kappa K_3\|\nabla\theta\|_{L^2}^2 - K_3\int\left[\frac{\mu}{2}|\nabla u + \nabla u^\top|^2 + \lambda(\mathrm{div} u)^2\right]\theta\right\}\\ &+ M_5\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2\right)\\ \le & C\left(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2 + \left(\|\nabla\theta\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2\right)^2\right).\end{aligned} \tag{4.32}$$

Now (3.2) and (4.6) enable us to use Gronwall's inequality to get (4.19). Then (4.20) follows from (4.32) and (4.19) easily. This completes the proof of Lemma 4.4. □

Corollary 4.5. *Under the conditions of Proposition 4.1, there exists a positive constant C, depending on μ, λ, R, γ, κ, θ_*, $\bar{\rho}$, $\bar{\theta}$, $\|u_0\|_{H^2}$, $\|(\rho_0, \theta_0)\|_{H^1}$ and $\inf_{x\in\mathbb{T}^3}\rho_0$, such that*

$$\sup_{t\ge 0}\left(\|u(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^6}\right) + \int_0^\infty \|\nabla^2\theta\|_{L^2}^2 dt \le C. \tag{4.33}$$

Proof. Similar to (4.31), using (4.6) and (4.19), we are led to

$$\begin{aligned}\|\nabla u\|_{L^6} &\le C\left(\|\mathrm{curl} u\|_{L^6} + \|\mathrm{div} u\|_{L^6}\right)\\ &\le C\left(\|\mathrm{curl} u\|_{H^1} + \|G\|_{H^1} + \|\rho\theta - 1\|_{L^6}\right)\\ &\le C\left(1 + \|\nabla u\|_{L^2} + \|\rho\dot{u}\|_{L^2}\right)\\ &\le C.\end{aligned} \tag{4.34}$$

Then by the Sobolev embedding $W^{1,6} \hookrightarrow L^\infty$ and (3.10), we obtain

$$\|u\|_{L^\infty} \le C\left(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^6}\right) \le C.$$

Finally, it follows from (4.26), (4.31), and (4.19) that

$$\int_0^\infty \|\nabla^2\theta\|_{L^2}^2 dt \le \int_0^\infty\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2\right)dt \le C.$$

The proof of Corollary 4.5 is completed. □

The following lemma is devoted to improving the estimates of θ.

Lemma 4.6. *Under the conditions of Proposition 4.1, there exists a positive constant C, depending on $\mu, \lambda, R, \gamma, \kappa$, θ_*, $\bar{\rho}$, $\bar{\theta}$, $\|\rho_0\|_{H^1}$, $\|(u_0, \theta_0)\|_{H^2}$ and $\inf_{x\in\mathbb{T}^3}\rho_0$, such that*

$$\begin{aligned}
&c_\nu \frac{d}{dt}\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + \kappa\|\nabla\dot{\theta}\|_{L^2}^2 \\
&\le C\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + \|\nabla\dot{u}\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2\right),
\end{aligned} \tag{4.35}$$

and

$$\sup_{t\ge 0}\left(\|\nabla^2\theta(t)\|_{L^2}^2 + \|\left(\sqrt{\rho}\dot{\theta}\right)(t)\|_{L^2}^2\right) + \int_0^\infty \|\nabla\dot{\theta}\|_{L^2}^2 dt \le C. \tag{4.36}$$

Proof. Taking the material derivative of $(1.7)_3$, it is not difficult to verify that

$$\begin{aligned}
c_\nu\rho\ddot{\theta} - \kappa(\Delta\theta)^{\cdot} =& (c_\nu - R)\rho\dot{\theta}\mathrm{div}u + R\rho\theta\left[(\mathrm{div}u)^2 + \nabla u : \nabla u^\top\right] \\
& - R\rho\theta\mathrm{div}\dot{u} + \left[\mu(\nabla u + \nabla u^\top) : (\nabla\dot{u} + \nabla\dot{u}^\top) + 2\lambda\mathrm{div}u\mathrm{div}\dot{u}\right] \\
& - \left[\mu(\nabla u + \nabla u^\top) : (\nabla u\nabla u + \nabla u^\top\nabla u^\top) + 2\lambda\mathrm{div}u\nabla u : \nabla u^\top\right].
\end{aligned} \tag{4.37}$$

Taking the L^2 inner product of this inequality with $\dot{\theta}$, we arrive at

$$\begin{aligned}
\frac{c_\nu}{2}\frac{d}{dt}\int\rho\dot{\theta}^2 - \kappa\int(\Delta\theta)^{\cdot}\dot{\theta} \le& C\int\rho\dot{\theta}^2|\nabla u| + C\int\rho|\dot{\theta}||\nabla u|^2 + C\int\rho|\dot{\theta}||\nabla\dot{u}| \\
& + C\int|\nabla u||\nabla\dot{u}||\dot{\theta}| + C\int|\nabla u|^3|\dot{\theta}|.
\end{aligned} \tag{4.38}$$

On the other hand, integrating by parts yields

$$\begin{aligned}
-\kappa\int(\Delta\theta)^{\cdot}\dot{\theta} =& -\kappa\int(\Delta\theta_t + u\cdot\nabla\Delta\theta)\dot{\theta} \\
=& \kappa\int\nabla\theta_t\nabla\dot{\theta} + \kappa\int\partial_i u^k\partial_{ki}\theta\dot{\theta} + \kappa\int u^k\partial_{ki}\theta\partial_i\dot{\theta} \\
=& \kappa\int|\nabla\dot{\theta}|^2 - \kappa\int\nabla\dot{\theta}\nabla u\nabla\theta + \kappa\int\nabla u : \nabla^2\theta\dot{\theta}.
\end{aligned} \tag{4.39}$$

Substituting (4.39) into (4.38), and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\frac{c_\nu}{2}\frac{d}{dt}\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + \kappa\|\nabla\dot{\theta}\|_{L^2}^2 \\
\le& C\int\rho\dot{\theta}^2|\nabla u| + C\int\rho|\dot{\theta}||\nabla u|^2 + C\int\rho|\dot{\theta}||\nabla\dot{u}| + C\int|\nabla u||\nabla\dot{u}||\dot{\theta}| \\
& + C\int|\nabla u|^3|\dot{\theta}| + \kappa\int|\nabla\dot{\theta}||\nabla u||\nabla\theta| + \kappa\int|\nabla u||\nabla^2\theta||\dot{\theta}| \\
\le& \frac{\kappa}{4}\|\nabla\dot{\theta}\|_{L^2}^2 + C\|\nabla u\nabla\theta\|_{L^2}^2 + \epsilon\|\dot{\theta}\nabla u\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 \\
& + C\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4.
\end{aligned} \tag{4.40}$$

Thanks to (4.33) and Lemma 2.2, we have

$$\epsilon\|\dot{\theta}\nabla u\|_{L^2}^2 \le \epsilon\|\nabla u\|_{L^4}^2\|\dot{\theta}\|_{L^4}^2 \le \epsilon C\left(\|\dot{\theta}\|_{L^2}^2 + \|\nabla\dot{\theta}\|_{L^2}^2\right) \le \epsilon C\|\nabla\dot{\theta}\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2, \tag{4.41}$$

and

$$\|\nabla u\nabla\theta\|_{L^2}^2 \le \|\nabla u\|_{L^3}^2\|\nabla\theta\|_{L^6}^2 \le C\|\nabla^2\theta\|_{L^2}^2. \tag{4.42}$$

Substituting (4.41) and (4.42) into (4.40), choosing a ϵ small enough, using (4.26), (4.31) and (4.19), we obtain (4.35). Then integrating (4.35) over $[0,t]$, using (3.2), (4.6) and (4.19), we get that for all $t>0$, there holds

$$\|(\sqrt{\rho}\dot{\theta})(t)\|_{L^2}^2+\int_0^t \|\nabla\dot{\theta}\|_{L^2}^2 dt' \le C. \tag{4.43}$$

Finally, we infer from (4.26), (4.6), (4.33) and (4.43) that

$$\sup_{t\ge 0}\|\nabla^2\theta(t)\|_{L^2}^2 \le C. \tag{4.44}$$

This completes the proof of Lemma 4.6. □

Before proceeding any further, to simplify the presentation, let us denote

$$\begin{aligned}
&A_1(t) := K_1\left(\rho\frac{|u|^2}{2}+R(1+\rho\ln\rho-\rho)+c_v\rho(\theta-\ln\theta-1)\right)-2\rho u\cdot\mathcal{B}[\rho-1],\\
&A_2(t) := K_2\int \rho|u|^4+\mu\|\nabla u\|_{L^2}^2+(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2+\frac{R^2}{\lambda+2\mu}\|\rho\theta-1\|_{L^2}^2-2R\int(\rho\theta-1)\mathrm{div}u,\\
&A_3(t) := \|\sqrt{\rho}\dot{u}\|_{L^2}^2+\kappa K_3\|\nabla\theta\|_{L^2}^2-K_3\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right]\theta,\\
&A_4(t) := c_v\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2,\\
&B_1(t) := \|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\rho-1\|_{L^2}^2,\\
&B_2(t) := \|\sqrt{\rho}u_t\|_{L^2}^2+\|u|\nabla u|\|_{L^2}^2,\\
&B_3(t) := \|\sqrt{\rho}\dot{\theta}\|_{L^2}^2+\|\nabla\dot{u}\|_{L^2}^2,\\
&B_4(t) := \|\nabla\dot{\theta}\|_{L^2}^2.
\end{aligned}$$

Now we give the proof of Proposition 4.1. All the constants $K_i, 4\le i\le 6$, $M_j, 6\le j\le 8$, C, $\tilde{a}$ and $\tilde{M}$ appearing in the proof below depend on $\mu,\lambda,R,\gamma,\kappa,\theta_*,\bar{\rho},\bar{\theta}$, $\|\rho_0\|_{H^1}$, $\|(u_0,\theta_0)\|_{H^2}$ and $\inf_{x\in\mathbb{T}^3}\rho_0$.

Proof of Proposition 4.1. Multiplying (4.20) by a sufficiently large constant K_4, and then adding the resulting inequality to (4.35), we find that, for some positive constant M_6, there holds

$$\frac{d}{dt}[K_4A_3(t)+A_4(t)]+M_6[B_3(t)+B_4(t)]\le C(B_1(t)+B_2(t)). \tag{4.45}$$

Next, choosing a positive constant K_5 so large that

$$\mu K_5\|\nabla u\|_{L^2}^2-K_4K_3\int\left[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right]\theta\ge\|\nabla u\|_{L^2}^2, \tag{4.46}$$

and

$$\frac{d}{dt}[K_5A_2(t)+K_4A_3(t)+A_4(t)]+M_7[B_2(t)+B_3(t)+B_4(t)]\le CB_1(t), \tag{4.47}$$

for some positive constant M_7. In view of (4.18), it is easy to see that

$$\begin{aligned}
&(\lambda+\mu)\|\mathrm{div}u\|_{L^2}^2+\frac{R^2}{\lambda+2\mu}\|\rho\theta-1\|_{L^2}^2-2R\int(\rho\theta-1)\mathrm{div}u\\
&\ge-\frac{R^2\mu}{(\lambda+\mu)(\lambda+2\mu)}\|\rho\theta-1\|_{L^2}^2\ge-C\left(\|\sqrt{\rho}(\theta-1)\|_{L^2}^2+\|\rho-1\|_{L^2}^2\right).
\end{aligned} \tag{4.48}$$

In view of (3.18), we can choose K_6 large enough, such that

$$\begin{aligned}&K_6A_1(t)+K_5A_2(t)+K_4A_3(t)+A_4(t)\\ \geq &C\Big(\|\sqrt{\rho}u\|_{L^2}^2+\|\rho-1\|_{L^2}^2+\|\sqrt{\rho}(\theta-1)\|_{L^2}^2+\|\rho^{\frac14}u\|_{L^4}^4\\ &+\|\nabla u\|_{L^2}^2+\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2\Big),\end{aligned} \tag{4.49}$$

and

$$\frac{d}{dt}\left[K_6A_1(t)+K_5A_2(t)+K_4A_3(t)+A_4(t)\right]+M_8\left[B_1(t)+B_2(t)+B_3(t)+B_4(t)\right]\leq 0, \tag{4.50}$$

for some positive constant M_8. On the other hand, by virtue of (4.33),

$$\|\rho^{\frac14}u\|_{L^4}^4\leq\|u\|_{L^\infty}^2\|\sqrt{\rho}u\|_{L^2}^2\leq C\|\nabla u\|_{L^2}^2.$$

Then

$$K_6A_1(t)+K_5A_2(t)+K_4A_3(t)+A_4(t)\leq C\left[B_1(t)+B_2(t)+B_3(t)+B_4(t)\right]. \tag{4.51}$$

It follows from (4.49)–(4.51) that

$$\|\sqrt{\rho}u\|_{L^2}+\|\rho-1\|_{L^2}+\|\sqrt{\rho}(\theta-1)\|_{L^2}+\|\rho^{\frac14}u\|_{L^4}^2+\|\nabla u\|_{L^2}+\|\sqrt{\rho}\dot{u}\|_{L^2}+\|\nabla\theta\|_{L^2}+\|\sqrt{\rho}\dot{\theta}\|_{L^2}\leq Ce^{-2\tilde{a}t}, \tag{4.52}$$

for some positive constant $\tilde{a}$. Next, by the Gagliardo-Nirenberg interpolation inequality, (3.11), (4.36) and (4.52), we find that

$$\begin{aligned}\|\theta-1\|_{L^\infty}\leq&C\|\theta-1\|_{L^6}^{\frac12}\|\theta-1\|_{W^{1,6}}^{\frac12}\\ \leq&C\left(\|\theta-1\|_{H^1}+\|\theta-1\|_{H^1}^{\frac12}\|\nabla^2\theta\|_{L^2}^{\frac12}\right)\\ \leq&C\left(\|\nabla u\|_{L^2}+\|\nabla\theta\|_{L^2}\right)+C\left(\|\nabla u\|_{L^2}+\|\nabla\theta\|_{L^2}\right)^{\frac12}\|\nabla^2\theta\|_{L^2}^{\frac12}\\ \leq&Ce^{-\tilde{a}t}.\end{aligned} \tag{4.53}$$

Combining these two inequalities above, and taking $\tilde{M}=2C$ with the constant C appearing in the last inequality of (4.53), we complete the proof of Proposition 4.1. □

5. Higher order uniform bounds

The purpose of this section is to obtain the uniform in time bounds of all derivatives of the solution.

5.1. Uniform bounds of ∇u

We begin this subsection by giving the exponential decay of $\rho-1$ in L^∞. It is in this lemma where the uniform lower bound of ρ starts playing its role.

Lemma 5.1. *Under the conditions of Theorem 1.2, there exist two positive constants C and c, depending on $\mu,\lambda,R,\gamma,\kappa,\rho_*,\theta_*,\bar{\rho},\bar{\theta}$, $\|\rho_0\|_{H^1}$, and $\|(u_0,\theta_0)\|_{H^2}$, such that*

$$\|\rho(t)-1\|_{L^\infty}\leq Ce^{-ct}. \tag{5.1}$$

Proof. The mass equation can be rewritten as

$$(\lambda+2\mu)(\rho_t+u\cdot\nabla\rho)+R\rho(\rho-1)=-\rho G-R\rho^2(\theta-1). \tag{5.2}$$

Let $X(t,y)$ be the particle path given by

$$\begin{cases}\frac{d}{dt}X(t,y)=u(t,X(t,y))\\ X(0,y)=y.\end{cases} \tag{5.3}$$

From (5.2), it is easy to see that $\rho(t, X(t,y))$ satisfies

$$(\lambda+2\mu)\frac{d}{dt}\rho(t,X(t,y)) + R\left(\rho(\rho-1)\right)(t,X(t,y)) = -\left(\rho G + R\rho^2(\theta-1)\right)(t,X(t,y)). \tag{5.4}$$

Multiplying this equality by $(\rho-1)(t,X(t,y))$, using the fact that $\rho \geq \rho_*$ and the Cauchy-Schwarz inequality, one deduces that there exist two positive constant $\beta = \beta(\rho_*, \lambda, \mu, R)$ and $C = C(\rho_*, \bar{\rho}, \lambda, \mu, R)$, such that

$$\frac{d}{dt}(\rho-1)^2(t,X(t,y)) + \beta(\rho-1)^2(t,X(t,y)) \leq C\left(\|G\|_{L^\infty}^2 + \|\theta-1\|_{L^\infty}^2\right), \tag{5.5}$$

i.e.,

$$\frac{d}{dt}\left(e^{\beta t}(\rho-1)^2(t,X(t,y))\right) \leq Ce^{\beta t}\left(\|G\|_{L^\infty}^2 + \|\theta-1\|_{L^\infty}^2\right). \tag{5.6}$$

Consequently,

$$\|\rho(t)-1\|_{L^\infty}^2 \leq Ce^{-\beta t}\|\rho_0-1\|_{L^\infty}^2 + C\int_0^t e^{\beta(t'-t)}\left(\|G\|_{L^\infty}^2 + \|\theta-1\|_{L^\infty}^2\right)dt'. \tag{5.7}$$

By the elliptic estimates, there hold

$$\|\nabla G\|_{L^6} \leq C\|\rho\dot{u}\|_{L^6} \leq C\|\dot{u}\|_{L^6} \leq C\left(\|\nabla\dot{u}\|_{L^2} + \|\sqrt{\rho}\dot{u}\|_{L^2}\right), \tag{5.8}$$

and

$$\begin{aligned}
\|G\|_{L^\infty}^2 + \|\theta-1\|_{L^\infty}^2 \leq & C\left(\|G\|_{L^6}\|G\|_{W^{1,6}} + \|\theta-1\|_{L^6}\|\theta-1\|_{W^{1,6}}\right) \\
\leq & C\left(\|G\|_{H^1}^2 + \|\theta-1\|_{H^1}^2 + \|G\|_{H^1}\|\nabla G\|_{L^6} + \|\theta-1\|_{H^1}\|\nabla^2\theta\|_{L^2}\right) \\
\leq & C\left(\|\rho\theta-1\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\theta-1\|_{H^1}^2\right) \\
& + C\left(\left(\|\rho\theta-1\|_{L^2} + \|\nabla u\|_{L^2} + \|\sqrt{\rho}\dot{u}\|_{L^2}\right)\|\nabla\dot{u}\|_{L^2} + \|\theta-1\|_{H^1}\|\nabla^2\theta\|_{L^2}\right).
\end{aligned} \tag{5.9}$$

Then in view of (3.2), (4.6) and (4.19), we obtain

$$\int_0^{\frac{t}{2}}\left(\|G\|_{L^\infty}^2 + \|\theta-1\|_{L^\infty}^2\right)dt' \leq C. \tag{5.10}$$

Moreover, using (4.52) and (4.19), we are led to

$$\begin{aligned}
\int_{\frac{t}{2}}^t\left(\|G\|_{L^\infty}^2 + \|\theta-1\|_{L^\infty}^2\right)dt' \leq & C\int_{\frac{t}{2}}^t e^{-4\tilde{a}t'}dt' + C\int_{\frac{t}{2}}^t e^{-2\tilde{a}t'}\left(\|\nabla\dot{u}\|_{L^2} + \|\nabla^2\theta\|_{L^2}\right)dt' \\
\leq & Ce^{-2\tilde{a}t} + C\left(\int_{\frac{t}{2}}^t e^{-4\tilde{a}t'}dt'\right)^{\frac{1}{2}}\left(\int_{\frac{t}{2}}^t\left(\|\nabla\dot{u}\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2\right)dt'\right)^{\frac{1}{2}} \\
\leq & Ce^{-\tilde{a}t}.
\end{aligned} \tag{5.11}$$

Accordingly,

$$
\begin{aligned}
&\int_0^t e^{\beta(t'-t)}\left(\|G\|_{L^\infty}^2+\|\theta-1\|_{L^\infty}^2\right)dt'\\
\leq& e^{-\frac{\beta}{2}t}\int_0^{\frac{t}{2}}\left(\|G\|_{L^\infty}^2+\|\theta-1\|_{L^\infty}^2\right)dt'+\int_{\frac{t}{2}}^t\left(\|G\|_{L^\infty}^2+\|\theta-1\|_{L^\infty}^2\right)dt'\\
\leq& C\exp\left\{-\min\left(\frac{\beta}{2},\tilde{a}\right)t\right\}.
\end{aligned}\tag{5.12}
$$

Substituting this inequality into (5.7), we conclude that (5.1) holds for $c=\frac{1}{2}\min\left\{\frac{\beta}{2},\tilde{a}\right\}$. This completes the proof of Lemma 5.1. □

Lemma 5.2. *Under the conditions of Theorem 1.2, there exists a positive constant C, depending on $\mu,\lambda,R,\gamma,\kappa,\rho_*$, θ_*, $\bar{\rho}$, $\bar{\theta}$, $\|\nabla\rho_0\|_{L^4}$, and $\|(u_0,\theta_0)\|_{H^2}$, such that*

$$
\sup_{t\geq 0}\|\nabla\rho(t)\|_{L^4}+\int_0^\infty\|\nabla\rho\|_{L^4}dt\leq C. \tag{5.13}
$$

Proof.

$$
\nabla\rho_t+\frac{R}{\lambda+2\mu}\rho\theta\nabla\rho+(u\cdot\nabla)\nabla\rho=-\nabla u\nabla\rho-\mathrm{div}u\nabla\rho-\frac{\rho\nabla G}{\lambda+2\mu}-\frac{R\rho^2\nabla\theta}{\lambda+2\mu}
$$

Taking the inner product of the above equation with $4|\nabla\rho|^2\nabla\rho$, and then integrating over $\mathbb{T}^3$, we see

$$
\frac{d}{dt}\|\nabla\rho\|_{L^4}^4+\frac{4R}{\lambda+2\mu}\int\rho\theta|\nabla\rho|^4\leq C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^4}^4+C\left(\|\nabla G\|_{L^4}+\|\nabla\theta\|_{L^4}\right)\|\nabla\rho\|_{L^4}^3, \tag{5.14}
$$

which implies that

$$
\frac{d}{dt}\|\nabla\rho\|_{L^4}+\frac{R\rho_*\theta_*}{\lambda+2\mu}\|\nabla\rho\|_{L^4}\leq C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^4}+C\left(\|\nabla G\|_{L^4}+\|\nabla\theta\|_{L^4}\right). \tag{5.15}
$$

To bound $\|\nabla u\|_{L^\infty}$, we first use (4.30) and (4.7) to deduce

$$
\Delta\nabla u=\frac{R}{\lambda+2\mu}\nabla^2\left(\rho\theta-\langle\theta\rangle_\rho\right)+\frac{1}{\lambda+2\mu}\nabla^2\left(G+R(\langle\theta\rangle_\rho-1)\right)-\nabla\mathrm{curlcurl}u.
$$

Let us decompose ∇u into three parts. More precisely, define w_i, $1\leq i\leq 3$ by the following three elliptic systems:

$$
\begin{aligned}
&\Delta w_1=-\nabla\mathrm{curlcurl}u,\quad \text{with}\quad \int_{\mathbb{T}^3}w_1=0,\\
&\Delta w_2=\frac{1}{\lambda+2\mu}\nabla^2\left(G+R(\langle\theta\rangle_\rho-1)\right),\quad \text{with}\quad \int_{\mathbb{T}^3}w_2=0,
\end{aligned}
$$

and

$$
\Delta w_3=\frac{R}{\lambda+2\mu}\nabla^2\left(\rho\theta-\langle\theta\rangle_\rho\right),\quad \text{with}\quad \int_{\mathbb{T}^3}w_3=0.
$$

Then the uniqueness implies that $\nabla u=w_1+w_2+w_3$. By the elliptic estimates, there holds

$$
\|\nabla w_1\|_{L^4}+\|\nabla w_2\|_{L^4}\leq C\left(\|\nabla\mathrm{curl}u\|_{L^4}+\|\nabla G\|_{L^4}\right)\leq C\|\rho\dot{u}\|_{L^4}. \tag{5.16}
$$

For w_3, noting that $[w_3]_{(0)}=0$, then by virtue of the Littlewood-Paley decomposition, and using Lemmas A.2 and A.3 in the Appendix, we derive

$$\begin{aligned}\|w_3\|_{L^\infty} &\le C\sum_{1\le j\le N-1}\|\Delta_j(\rho\theta-\langle\theta\rangle_\rho)\|_{L^\infty}+C\sum_{j\ge N}2^{-\frac{j}{4}}\|\Delta_j\nabla(\rho\theta)\|_{L^4}\\ &\le CN\|\rho\theta-\langle\theta\rangle_\rho\|_{L^\infty}+C2^{-\frac{N}{4}}\left(\|\nabla\rho\|_{L^4}+\|\nabla\theta\|_{L^4}\right)\\ &\le CN\left(\|\rho-1\|_{L^\infty}+\|\theta-1\|_{L^\infty}+\|\sqrt{\rho}u\|_{L^2}\right)+C2^{-\frac{N}{4}}\left(\|\nabla\rho\|_{L^4}+e\right),\end{aligned}$$

where we have used (1.6) and (4.36). Let $\epsilon_0\in(0,1)$ be a small constant that will be determined later. Take $N\in\mathbb{N}^+$ such that

$$2^{-\frac{N}{4}}(\|\nabla\rho\|_{L^4}+e)=\epsilon_0,$$

which implies

$$N\sim\ln\frac{\|\nabla\rho\|_{L^4}+e}{\epsilon_0}.$$

Accordingly

$$\|w_3\|_{L^\infty}\le C\ln\frac{\|\nabla\rho\|_{L^4}+e}{\epsilon_0}\left(\|\rho-1\|_{L^\infty}+\|\theta-1\|_{L^\infty}+\|\sqrt{\rho}u\|_{L^2}\right)+C\epsilon_0. \tag{5.17}$$

By the Sobolev embedding $W^{1,4}\hookrightarrow L^\infty$, (5.16) and (5.17) with ϵ_0 small enough, we find

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^4}+\bar{c}\|\nabla\rho\|_{L^4}\le& C\left(\|\dot{u}\|_{L^4}+\|\nabla\theta\|_{L^4}\right)+C\|\dot{u}\|_{L^4}\|\nabla\rho\|_{L^4}\\ &+C\left(\|\rho-1\|_{L^\infty}+\|\theta-1\|_{L^\infty}+\|\sqrt{\rho}u\|_{L^2}\right)\ln\frac{\|\nabla\rho\|_{L^4}+e}{\epsilon_0}\|\nabla\rho\|_{L^4}\end{aligned} \tag{5.18}$$

for some $\bar{c}>0$. Consequently,

$$\begin{aligned}\frac{d}{dt}\ln\frac{\|\nabla\rho\|_{L^4}+e}{\epsilon_0}\le& C\left(\|\dot{u}\|_{L^4}+\|\nabla\theta\|_{L^4}\right)\\ &+C\left(\|\rho-1\|_{L^\infty}+\|\theta-1\|_{L^\infty}+\|\sqrt{\rho}u\|_{L^2}\right)\ln\frac{\|\nabla\rho\|_{L^4}+e}{\epsilon_0}.\end{aligned} \tag{5.19}$$

It follows from (4.19), (4.36) and (4.52) that

$$\begin{aligned}&\int_0^\infty\|\dot{u}\|_{L^4}+\|\nabla\theta\|_{L^4}dt\\ \le& C\int_0^\infty\|\dot{u}\|_{L^2}^{\frac14}\|\dot{u}\|_{H^1}^{\frac34}+\|\nabla\theta\|_{L^2}^{\frac14}\|\nabla^2\theta\|_{L^2}^{\frac34}dt\\ \le& C\left(\int_0^\infty\|\sqrt{\rho}\dot{u}\|_{L^2}dt+\int_0^\infty\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac14}\|\nabla\dot{u}\|_{L^2}^{\frac34}dt+\int_0^\infty\|\nabla\theta\|_{L^2}^{\frac14}dt\right)\\ \le& C\left(1+\left(\int_0^\infty\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac25}dt\right)^{\frac58}\left(\int_0^\infty\|\nabla\dot{u}\|_{L^2}^2dt\right)^{\frac38}\right)\\ \le& C.\end{aligned} \tag{5.20}$$

Then (4.1), (5.1) and (5.20) enable us to use Gronwall's inequality to get

$$\ln\frac{\|\nabla\rho\|_{L^4}+e}{\epsilon_0}\le C, \tag{5.21}$$

and hence (5.13) holds. The proof of Lemma 5.2 is completed. □

Corollary 5.3. *Under the conditions of Theorem 1.2, there exists a positive constant C, depending on μ, λ, R, γ, κ, ρ_*, θ_*, $\bar\rho$, $\bar\theta$, $\|\nabla\rho_0\|_{L^4}$, and $\|(u_0,\theta_0)\|_{H^2}$, such that*

$$\sup_{t\ge 0}\|\nabla^2 u(t)\|_{L^2}^2+\int_0^\infty \|\nabla^2 u\|_{L^4}^2+\|\nabla u\|_{L^\infty}^2+\|\nabla^3\theta\|_{L^2}^2+\|\nabla\theta\|_{L^\infty}^2 dt\le C. \tag{5.22}$$

Proof. The momentum equation $(1.7)_2$ can be rewritten as

$$\mu\Delta u=-\frac{\lambda+\mu}{\lambda+2\mu}\nabla G+\frac{\mu}{\lambda+2\mu}\nabla P+\rho\dot u. \tag{5.23}$$

Then the elliptic estimate gives

$$\begin{aligned}\|\nabla^2 u\|_{L^2}^2\le& C\left(\|\nabla G\|_{L^2}^2+\|\nabla P\|_{L^2}^2+\|\rho\dot u\|_{L^2}^2\right)\\ \le& C\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2\right),\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}\|\nabla^2 u\|_{L^4}^2\le& C\left(\|\nabla G\|_{L^4}^2+\|\nabla P\|_{L^4}^2+\|\rho\dot u\|_{L^4}^2\right)\\ \le& C\left(\|\nabla\rho\|_{L^4}^2+\|\nabla\theta\|_{L^4}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla\dot u\|_{L^2}^2\right).\end{aligned} \tag{5.25}$$

Thus, it follows from (4.52), (4.19), (4.33), and (5.13) that

$$\sup_{t\ge 0}\|\nabla^2 u(t)\|_{L^2}^2+\int_0^\infty \|\nabla^2 u\|_{L^4}^2 dt\le C. \tag{5.26}$$

Next, similar to (5.16), using Poincaré inequality, (4.52), (4.19), (4.33), and (5.13) again, we have

$$\begin{aligned}\int_0^\infty \|\nabla u\|_{L^\infty}^2 dt\le& C\int_0^\infty \|\nabla u\|_{W^{1,4}}^2 dt\\ \le& C\int_0^\infty \|\nabla \mathrm{curl} u\|_{L^4}^2+\|\nabla G\|_{L^4}^2+\|\nabla(\rho\theta)\|_{L^4}^2 dt\\ \le& C\int_0^\infty \|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla\dot u\|_{L^2}^2+\|\nabla\rho\|_{L^4}^2+\|\nabla\theta\|_{L^4}^2 dt\\ \le& C.\end{aligned} \tag{5.27}$$

Finally, combining the elliptic estimate with (4.36), (4.52), (5.13), (5.26) and (5.27), we are led to

$$\begin{aligned}&\int_0^\infty \|\nabla^3\theta\|_{L^2}^2 dt\\ \le& C\int_0^\infty \|\nabla(\rho\dot\theta)\|_{L^2}^2+\|\nabla(P\mathrm{div} u)\|_{L^2}^2+\left\|\nabla\left(\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div} u)^2\right)\right\|_{L^2}^2 dt\\ \le& C\int_0^\infty \|\nabla\rho\dot\theta\|_{L^2}^2+\|\nabla\dot\theta\|_{L^2}^2+\|\nabla\theta\mathrm{div} u\|_{L^2}^2+\|\nabla\rho\mathrm{div} u\|_{L^2}^2+\|\nabla\mathrm{div} u\|_{L^2}^2+\left\|\nabla u\nabla^2 u\right\|_{L^2}^2 dt\end{aligned} \tag{5.28}$$

$$
\begin{aligned}
&\leq C\int_0^\infty \|\nabla\rho\|_{L^4}^2\|\dot\theta\|_{L^4}^2+\|\nabla\dot\theta\|_{L^2}^2+\left(\|\nabla\theta\|_{L^2}^2+\left\|\nabla^2u\right\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2\right)\|\nabla u\|_{L^\infty}^2+\|\nabla\mathrm{div}u\|_{L^2}^2dt\\
&\leq C\int_0^\infty \|\sqrt{\rho}\dot\theta\|_{L^2}^2+\|\nabla\dot\theta\|_{L^2}^2+\|\nabla u\|_{L^\infty}^2+\|\nabla\mathrm{div}u\|_{L^2}^2dt\\
&\leq C.
\end{aligned}
$$

Accordingly, by the Sobolev embedding, there holds

$$
\int_0^\infty \|\nabla\theta\|_{L^\infty}^2dt\leq C. \tag{5.29}
$$

This completes the proof of Corollary 5.3. □

Remark 5.4. Dealing with $\|\nabla u\|_{L^\infty}$ in the spirit of (5.20), it is easy to verify that the following estimate holds as well:

$$
\int_0^\infty \|\nabla u\|_{L^\infty}dt\leq C. \tag{5.30}
$$

5.2. *Uniform bounds of all derivatives of* (ρ,u,θ)

In this subsection, we will prove that the L^2 norm of all derivatives of ρ, u and θ are uniformly bounded in t. More precisely, we establish the following proposition.

Proposition 5.5. *Under the conditions of Theorem 1.2, there exists a positive constant C, depending on $\mu,\lambda,R,\gamma,\kappa$, $\rho_*,\theta_*,\bar\rho,\bar\theta$, and $\|(\rho_0,u_0,\theta_0)\|_{H^\infty}$, such that for all $k\geq 1$, there holds*

$$
\begin{aligned}
&\sup_{t\geq 0}\left(\|\nabla^k\rho(t)\|_{L^2}^2+\|\nabla^{k+1}u(t)\|_{L^2}^2+\|\nabla^{k+1}\theta(t)\|_{L^2}^2\right)\\
&+\int_0^\infty \|\nabla^k\rho\|_{L^2}^2+\|\nabla^k u_t\|_{L^2}^2+\|\nabla^k\theta_t\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}\theta\|_{L^2}^2dt\leq C.
\end{aligned}
\tag{5.31}
$$

Proof. Proposition 5.5 will be proved by induction. For $k=1$, (5.31) is a consequence of (4.19), (4.36), (5.13), and (5.22). Let $k\geq 2$ be an arbitrary integer. Assume that the following inequality holds:

$$
\begin{aligned}
&\sup_{t\geq 0}\left(\|\nabla^{k-1}\rho(t)\|_{L^2}^2+\|\nabla^{k}u(t)\|_{L^2}^2+\|\nabla^{k}\theta(t)\|_{L^2}^2\right)\\
&+\int_0^\infty \|\nabla^{k-1}\rho\|_{L^2}^2+\|\nabla^{k-1} u_t\|_{L^2}^2+\|\nabla^{k-1}\theta_t\|_{L^2}^2+\|\nabla^{k}u\|_{L^2}^2+\|\nabla^{k}\theta\|_{L^2}^2dt\leq C.
\end{aligned}
\tag{5.32}
$$

Next we shall show that (5.32) still holds with $k-1$ replaced by k. This will be achieved by the following three lemmas. □

We first improve the uniform bounds of ρ under the assumption (5.32).

Lemma 5.6. *Assume that the conditions in Theorem 1.2 hold, and in addition, (5.32) holds for an arbitrary integer $k\geq 2$. Then there exists a positive constant C, depending on $\mu,\lambda,R,\gamma,\kappa,\rho_*,\theta_*,\bar\rho,\bar\theta$, and $\|(\rho_0,u_0,\theta_0)\|_{H^\infty}$, such that*

$$\sup_{t\geq 0}\|\nabla^k\rho(t)\|_{L^2}^2+\int_0^\infty\|\nabla^k\rho\|_{L^2}^2dt\leq C. \tag{5.33}$$

Proof. Applying ∂^α to the mass equation (1.1)$_1$, using (4.7), we obtain

$$\partial^\alpha\rho_t+\frac{R}{\lambda+2\mu}\rho\theta\partial^\alpha\rho=-u\cdot\nabla\partial^\alpha\rho+[u,\partial^\alpha]\cdot\nabla\rho+[\rho,\partial^\alpha]\mathrm{div}u+\frac{R}{\lambda+2\mu}\rho[\theta,\partial^\alpha]\rho-\frac{1}{\lambda+2\mu}\rho\partial^\alpha G. \tag{5.34}$$

For $|\alpha|=k\geq 2$, taking the L^2 inner product with $\partial^\alpha\rho$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^k\rho\|_{L^2}^2+\tilde{c}\|\nabla^k\rho\|_{L^2}^2\\ \leq&C\|\nabla u\|_{L^\infty}\|\nabla^k\rho\|_{L^2}^2+C\|\nabla^kG\|_{L^2}\|\nabla^k\rho\|_{L^2}\\ &+C\sum_{|\alpha|=k}\left(\|[u,\partial^\alpha]\cdot\nabla\rho\|_{L^2}+\|[\rho,\partial^\alpha]\mathrm{div}u\|_{L^2}+\|[\theta,\partial^\alpha]\rho\|_{L^2}\right)\|\nabla^k\rho\|_{L^2},\end{aligned} \tag{5.35}$$

where $\tilde{c}:=\frac{R\rho_*\theta_*}{\lambda+2\mu}$. By virtue of (2.14) and the assumption (5.32), one deduces that

$$\|[\theta,\partial^\alpha]\rho\|_{L^2}\leq C\left(\|\nabla\theta\|_{L^\infty}\|\nabla^{k-1}\rho\|_{L^2}+\|\rho\|_{L^\infty}\|\nabla^k\theta\|_{L^2}\right)\leq C\left(\|\nabla\theta\|_{L^\infty}+\|\nabla^k\theta\|_{L^2}\right). \tag{5.36}$$

Next, we split the proof into two cases.

Case 1: $k=2$**.** Thanks to (5.13), direct calculations give

$$\begin{aligned}&\|[u,\partial^\alpha]\cdot\nabla\rho\|_{L^2}+\|[\rho,\partial^\alpha]\mathrm{div}u\|_{L^2}\\ \leq&C\sum_{|\alpha'|=1}\|\partial^{\alpha'}u\cdot\nabla\partial^{\alpha-\alpha'}\rho\|_{L^2}+\|\partial^\alpha u\cdot\nabla\rho\|_{L^2}+\|\partial^\alpha\rho\mathrm{div}u\|_{L^2}+C\sum_{|\alpha'|=1}\|\partial^{\alpha'}\rho\partial^{\alpha-\alpha'}\mathrm{div}u\|_{L^2}\\ \leq&C\|\nabla u\|_{L^\infty}\|\nabla^2\rho\|_{L^2}+C\|\nabla\rho\|_{L^4}\|\nabla^2u\|_{L^4}\leq C\|\nabla u\|_{L^\infty}\|\nabla^2\rho\|_{L^2}+C\|\nabla^2u\|_{L^4}.\end{aligned} \tag{5.37}$$

By the elliptic estimate and (5.13), it follows that

$$\begin{aligned}\|\nabla^2G\|_{L^2}\leq&C\|\nabla(\rho\dot{u})\|_{L^2}\\ \leq&C\left(\|\nabla\dot{u}\|_{L^2}+\|\dot{u}\|_{L^4}\|\nabla\rho\|_{L^4}\right)\\ \leq&C\left(\|\dot{u}\|_{L^2}+\|\nabla\dot{u}\|_{L^2}\right).\end{aligned} \tag{5.38}$$

Substituting (5.36)–(5.38) into (5.35), and using the Cauchy-Schwartz inequality, one deduces that

$$\begin{aligned}&\frac{d}{dt}\|\nabla^2\rho\|_{L^2}^2+\tilde{c}\|\nabla^2\rho\|_{L^2}^2\\ \leq&C\|\nabla u\|_{L^\infty}^2\|\nabla^2\rho\|_{L^2}^2+C\left(\|\nabla\theta\|_{L^\infty}^2+\|\nabla^2\theta\|_{L^2}^2+\|\nabla^2u\|_{L^4}^2+\|\dot{u}\|_{L^2}^2+\|\nabla\dot{u}\|_{L^2}^2\right).\end{aligned} \tag{5.39}$$

Then (4.19) and (5.22) enable us to use Gronwall's inequality to get (5.33).

Case 2: $k\geq 3$**.** Now it is easy to see that

$$\|\nabla\rho\|_{L^\infty}\leq C\|\nabla^k\rho\|_{L^2}, \tag{5.40}$$

which, together with the commutator estimate (2.14), yields

$$\begin{aligned}&\|[u,\partial^\alpha]\cdot\nabla\rho\|_{L^2}+\|[\rho,\partial^\alpha]\mathrm{div}u\|_{L^2}\\ \leq&C\left(\|\nabla u\|_{L^\infty}\|\nabla^k\rho\|_{L^2}+\|\nabla\rho\|_{L^\infty}\|\nabla^ku\|_{L^2}\right)\\ \leq&C\left(\|\nabla u\|_{L^\infty}+\|\nabla^ku\|_{L^2}\right)\|\nabla^k\rho\|_{L^2}.\end{aligned} \tag{5.41}$$

In view of the elliptic estimate and (5.32), we obtain

$$\begin{aligned}\|\nabla^k G\|_{L^2} \le& C\|\nabla^{k-1}(\rho\dot{u})\|_{L^2}\\ \le& C\left(\|\nabla^{k-1}\dot{u}\|_{L^2}+\|\dot{u}\|_{L^\infty}\|\nabla^{k-1}\rho\|_{L^2}\right)\\ \le& C\left(\|\dot{u}\|_{L^2}+\|\nabla^{k-1}u_t\|_{L^2}+\|\nabla^k u\|_{L^2}+\|\nabla u\|_{L^\infty}\right),\end{aligned} \tag{5.42}$$

where we have used the Sobolev embedding and Pincaré inequality to bound $\|\dot{u}\|_{L^\infty}$ as follows:

$$\begin{aligned}\|\dot{u}\|_{L^\infty} \le& C\left(\|\dot{u}\|_{L^2}+\|\nabla^2\dot{u}\|_{L^2}\right)\\ \le& C\left(\|\dot{u}\|_{L^2}+\|\nabla^2 u_t\|_{L^2}+\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla^3 u\|_{L^2}\right)\\ \le& C\left(\|\dot{u}\|_{L^2}+\|\nabla^{k-1}u_t\|_{L^2}+\|\nabla u\|_{L^\infty}+\|\nabla^k u\|_{L^2}\right).\end{aligned} \tag{5.43}$$

From (5.36), (5.41) and (5.42), we find that (5.35) implies

$$\begin{aligned}&\frac{d}{dt}\|\nabla^k\rho\|_{L^2}^2+\tilde{c}\|\nabla^k\rho\|_{L^2}^2\\ \le& C\left(\|\nabla u\|_{L^\infty}^2+\|\nabla^k u\|_{L^2}^2\right)\|\nabla^k\rho\|_{L^2}^2+C\left(\|\nabla\theta\|_{L^\infty}^2+\|\nabla^k\theta\|_{L^2}^2\right)\\ &+C\left(\|\dot{u}\|_{L^2}^2+\|\nabla^{k-1}u_t\|_{L^2}^2+\|\nabla^k u\|_{L^2}^2+\|\nabla u\|_{L^\infty}^2\right).\end{aligned} \tag{5.44}$$

Then (5.33) follows from (5.22) and (5.32) immediately. □

Corollary 5.7. *Under the conditions of Lemma 5.6, there exists a positive constant C, depending on $\mu, \lambda, R, \gamma, \kappa$, $\rho_*, \theta_*, \bar{\rho}, \bar{\theta}$, and $\|(\rho_0, u_0, \theta_0)\|_{H^\infty}$, such that*

$$\int_0^\infty \|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}\theta\|_{L^2}^2 dt < C. \tag{5.45}$$

Proof. Recalling the u satisfies the elliptic system (5.23), then it follows from (5.38) and (5.42) that

$$\begin{aligned}\|\nabla^{k+1}u\|_{L^2} \le& C\left(\|\nabla^k G\|_{L^2}+\|\nabla^k P\|_{L^2}+\|\nabla^{k-1}(\rho\dot{u})\|_{L^2}\right)\\ \le& C\left(\|\dot{u}\|_{L^2}+\|\nabla^{k-1}u_t\|_{L^2}+\|\nabla^k u\|_{L^2}+\|\nabla u\|_{L^\infty}+\|\nabla^k\rho\|_{L^2}+\|\nabla^k\theta\|_{L^2}\right).\end{aligned} \tag{5.46}$$

Thus, in view of (5.22), (5.32) and (5.33), there holds

$$\int_0^\infty \|\nabla^{k+1}u\|_{L^2}^2 dt < C. \tag{5.47}$$

Next, for the estimate of θ, since the case $k=2$ is included in (5.22), we just focus on the case $k \ge 3$. The estimate in (5.43) is applicable to $\dot{\theta}$ now. Accordingly, again by the elliptic estimate and (5.32), we have

$$\begin{aligned}\|\nabla^{k+1}\theta\|_{L^2} \le& C\left(\|\nabla^{k-1}(\rho\dot{\theta})\|_{L^2}+\|\nabla^{k-1}(P\mathrm{div}u)\|_{L^2}+\left\|\nabla^{k-1}\left(\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2\right)\right\|_{L^2}\right)\\ \le& C\left(\|\dot{\theta}\|_{L^2}+\|\nabla^{k-1}\theta_t\|_{L^2}+\|\nabla^k\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}+\|\nabla^k u\|_{L^2}+\|\nabla u\|_{L^\infty}\right).\end{aligned} \tag{5.48}$$

Combining this with the assumption (5.32) gives

$$\int_0^\infty \|\nabla^{k+1}\theta\|_{L^2}^2 dt < C. \tag{5.49}$$

This completes the proof of Corollary 5.7. □

Now the uniform estimates of u can be improved.

Lemma 5.8. *Under the conditions of Lemma 5.6, there exists a positive constant C, depending on $\mu, \lambda, R, \gamma, \kappa, \rho_*, \theta_*, \bar{\rho}, \bar{\theta}$, and $\|(\rho_0, u_0, \theta_0)\|_{H^\infty}$, such that*

$$\sup_{t\geq 0}\|\nabla^{k+1}u(t)\|_{L^2}^2 + \int_0^\infty \|\nabla^k u_t\|_{L^2}^2 dt \leq C. \tag{5.50}$$

Proof. Applying ∂^α to the momentum equation $(1.7)_2$ yields

$$\rho\partial^\alpha u_t - \mu\Delta\partial^\alpha u - (\lambda+\mu)\nabla\partial^\alpha \mathrm{div} u = [\rho, \partial^\alpha]u_t - \partial^\alpha(\rho u\cdot\nabla u) - \nabla\partial^\alpha P. \tag{5.51}$$

Taking the L^2 inner product of the above equation to $\partial^\alpha u_t$, integrating by parts, using (4.7), and summing with respect to $|\alpha| = k$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\mu\|\nabla^{k+1}u\|_{L^2}^2 + (\lambda+\mu)\|\nabla^k \mathrm{div} u\|_{L^2}^2\right) + \rho_*\|\nabla^k u_t\|_{L^2}^2 \\
\leq &\epsilon\|\nabla^k u_t\|_{L^2}^2 + C\sum_{|\alpha|=k}\left(\|[\rho,\partial^\alpha]u_t\|_{L^2}^2 + \|\partial^\alpha(\rho u\cdot\nabla u)\|_{L^2}^2\right) \\
&+ \sum_{|\alpha|=k}\left(\frac{d}{dt}\int \partial^\alpha P\partial^\alpha \mathrm{div} u - \frac{1}{2(\lambda+2\mu)}\frac{d}{dt}\|\partial^\alpha P\|_{L^2}^2 - \frac{1}{\lambda+2\mu}\int \partial^\alpha P_t\partial^\alpha G\right).
\end{aligned} \tag{5.52}$$

Noting that

$$-\frac{1}{\lambda+2\mu}\int \partial^\alpha P_t\partial^\alpha G = \frac{(-1)^{|\alpha|+1}}{\lambda+2\mu}\int P_t\partial^{2\alpha}G,$$

then similar to (4.10), integrating by parts, it is not difficult to verify that

$$\begin{aligned}
&-\frac{1}{\lambda+2\mu}\int \partial^\alpha P_t\partial^\alpha G \\
=&-\frac{\gamma-1}{\lambda+2\mu}\int \partial^{\alpha'}[\rho(u_t+(u\cdot\nabla)u)\cdot u]\partial_j\partial^\alpha G \\
&-\frac{\gamma}{\lambda+2\mu}\int \partial^\alpha(Pu)\cdot\nabla\partial^\alpha G + \frac{(\gamma-1)\kappa}{\lambda+2\mu}\int \nabla\partial^\alpha\theta\cdot\nabla\partial^\alpha G \\
&+\frac{\gamma-1}{\lambda+2\mu}\int\left(\lambda\partial^\alpha(\mathrm{div} u u)\cdot\nabla\partial^\alpha G + \mu\partial^\alpha\left[(\nabla u+\nabla u^\top)u\right]\cdot\nabla\partial^\alpha G\right),
\end{aligned} \tag{5.53}$$

where $|\alpha'| = |\alpha| - 1 = k-1$. Substituting this inequality into (5.52), we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\mu\|\nabla^{k+1}u\|_{L^2}^2 + (\lambda+\mu)\|\nabla^k \mathrm{div} u\|_{L^2}^2\right) + \rho_*\|\nabla^k u_t\|_{L^2}^2 \\
\leq &\epsilon\|\nabla^k u_t\|_{L^2}^2 + C\sum_{|\alpha|=k}\left(\|[\rho,\partial^\alpha]u_t\|_{L^2}^2 + \|\partial^\alpha(\rho u\cdot\nabla u)\|_{L^2}^2\right) \\
&+ C\left(\|\nabla^{k-1}[\rho(u_t+(u\cdot\nabla)u)\cdot u]\|_{L^2} + \|\nabla^k(Pu)\|_{L^2} + \|\nabla^{k+1}\theta\|_{L^2}\right)\|\nabla^{k+1}G\|_{L^2} \\
&+ C\left(\|\nabla^k(\mathrm{div} u u)\|_{L^2} + \|\nabla^k[(\nabla u+\nabla u^\top)u]\|_{L^2}\right)\|\nabla^{k+1}G\|_{L^2} \\
&+ \sum_{|\alpha|=k}\left(\frac{d}{dt}\int \partial^\alpha P\partial^\alpha \mathrm{div} u - \frac{1}{2(\lambda+2\mu)}\frac{d}{dt}\|\partial^\alpha P\|_{L^2}^2\right).
\end{aligned} \tag{5.54}$$

Using (5.32) and (5.33), the product rule (2.13) then gives

$$\|\nabla^k(\rho u\cdot\nabla u)\|_{L^2}^2 \leq C\left(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2\right), \tag{5.55}$$

$$\|\nabla^k (Pu)\|_{L^2} \le C\left(\|\nabla^k\rho\|_{L^2} + \|\nabla^k u\|_{L^2} + \|\nabla^k\theta\|_{L^2}\right), \tag{5.56}$$

$$\|\nabla^k(\mathrm{div}uu)\|_{L^2} + \|\nabla^k[(\nabla u + \nabla u^\top)u]\|_{L^2} \le C\left(\|\nabla^{k+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\right), \tag{5.57}$$

and

$$\|\nabla^{k-1}[\rho(u_t + (u\cdot\nabla)u)\cdot u]\|_{L^2} \le C\left(\|\nabla^{k-1}u_t\|_{L^2} + \|u_t\|_{L^\infty} + \|\nabla^k u\|_{L^2} + \|\nabla u\|_{L^\infty}\right). \tag{5.58}$$

By (5.33), similar to (5.42), one deduces that

$$\|\nabla^{k+1}G\|_{L^2} \le C\left(\|\dot u\|_{L^2} + \|\nabla^k u_t\|_{L^2} + \|\nabla^{k+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\right). \tag{5.59}$$

Case 1: $k=2$**.** Thanks to (5.13) and (5.33), direct calculations show that

$$\begin{aligned}\|[\rho,\partial^\alpha]u_t\|_{L^2}^2 \le& C\left(\|\nabla^2\rho\|_{L^2}^2\|u_t\|_{L^\infty}^2 + \|\nabla\rho\|_{L^4}^2\|\nabla u_t\|_{L^4}^2\right)\\ \le& C\left(\|u_t\|_{L^\infty}^2 + \|\nabla u_t\|_{L^4}^2\right).\end{aligned} \tag{5.60}$$

Collecting (5.54)–(5.60), using Cauchy-Schwartz inequality, we find that

$$\begin{aligned}&\frac{d}{dt}\left(\mu\|\nabla^3 u\|_{L^2}^2 + (\lambda+\mu)\|\nabla^2\mathrm{div}u\|_{L^2}^2\right) + \rho_*\|\nabla^2 u_t\|_{L^2}^2\\ \le& C\left(\|u_t\|_{L^\infty}^2 + \|\nabla u_t\|_{L^4}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla^3\theta\|_{L^2}^2\right)\\ &+ C\left(\|\dot u\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla^2\rho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2\right)\\ &+ \sum_{|\alpha|=2}\left(\frac{d}{dt}\int \partial^\alpha P\partial^\alpha \mathrm{div}u - \frac{1}{2(\lambda+2\mu)}\frac{d}{dt}\|\partial^\alpha P\|_{L^2}^2\right).\end{aligned} \tag{5.61}$$

By interpolations, there hold

$$\|u_t\|_{L^\infty}^2 \le C\|u_t\|_{L^2}^{\frac12}\|\nabla^2 u_t\|_{L^2}^{\frac32} + C\|u_t\|_{L^2}^2 \le \frac{\rho_*}{4}\|\nabla^2 u_t\|_{L^2}^2 + C\|u_t\|_{L^2}^2, \tag{5.62}$$

and

$$\|\nabla u_t\|_{L^4}^2 \le C\|\nabla u_t\|_{L^2}^{\frac12}\|\nabla^2 u_t\|_{L^2}^{\frac32} \le \frac{\rho_*}{4}\|\nabla^2 u_t\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2. \tag{5.63}$$

Substituting these two inequalities into (5.61), using (5.32), (5.33) and (5.45), integrating with respect to time t, then (5.50) follows.

Case 2: $k\ge 3$**.** From (5.33) and the commutator estimate (2.14), we have

$$\begin{aligned}\|[\rho,\partial^\alpha]u_t\|_{L^2}^2 \le& C\left(\|\nabla^k\rho\|_{L^2}^2\|u_t\|_{L^\infty}^2 + \|\nabla\rho\|_{L^\infty}^2\|\nabla^{k-1}u_t\|_{L^2}^2\right)\\ \le& C\left(\|u_t\|_{L^\infty}^2 + \|\nabla^{k-1}u_t\|_{L^2}^2\right).\end{aligned} \tag{5.64}$$

Similar to (5.61), it follows from (5.54)–(5.59) and (5.64) that

$$\begin{aligned}&\frac{d}{dt}\left(\mu\|\nabla^{k+1} u\|_{L^2}^2 + (\lambda+\mu)\|\nabla^k\mathrm{div}u\|_{L^2}^2\right) + \rho_*\|\nabla^k u_t\|_{L^2}^2\\ \le& C\left(\|u_t\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla^{k+1}\theta\|_{L^2}^2\right)\\ &+ C\left(\|\dot u\|_{L^2}^2 + \|\nabla^{k-1} u_t\|_{L^2}^2 + \|\nabla^k\rho\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k\theta\|_{L^2}^2\right)\\ &+ \sum_{|\alpha|=k}\left(\frac{d}{dt}\int \partial^\alpha P\partial^\alpha \mathrm{div}u - \frac{1}{2(\lambda+2\mu)}\frac{d}{dt}\|\partial^\alpha P\|_{L^2}^2\right),\end{aligned} \tag{5.65}$$

where we have used the following Sobolev embedding similar to (5.43):

$$\|u_t\|_{L^\infty} \le C\left(\|u_t\|_{L^2} + \|\nabla^{k-1}u_t\|_{L^2}\right). \tag{5.66}$$

Using (5.32), (5.33) and (5.45) again yields (5.50). We complete the proof of Lemma 5.8. □

Finally, we improve that uniform bounds of θ, and thus complete the proof of Proposition 5.5.

Lemma 5.9. *Under the conditions of Lemma 5.6, there exists a positive constant C, depending on $\mu, \lambda, R, \gamma, \kappa$, $\rho_*, \theta_*, \bar{\rho}, \bar{\theta}$, and $\|(\rho_0, u_0, \theta_0)\|_{H^\infty}$, such that*

$$\sup_{t\geq 0} \|\nabla^{k+1}\theta(t)\|_{L^2}^2 + \int_0^\infty \|\nabla^k \theta_t\|_{L^2}^2 dt \leq C. \tag{5.67}$$

Proof. Applying ∂^α to $(1.7)_3$ yields

$$c_\nu \rho \partial^\alpha \theta_t - \kappa \Delta \partial^\alpha \theta = c_\nu [\rho, \partial^\alpha]\theta_t - c_\nu \partial^\alpha (\rho u \cdot \nabla\theta) - \partial^\alpha (P \mathrm{div} u) - \partial^\alpha \left(\frac{\mu}{2} |\nabla u + \nabla u^\top|^2 + \lambda (\mathrm{div} u)^2 \right). \tag{5.68}$$

Arguing as (5.52), we are led to

$$\begin{aligned}
&\frac{\kappa}{2}\frac{d}{dt}\|\nabla^{k+1}\theta\|_{L^2}^2 + c_\nu \rho_* \|\nabla^k \theta_t\|_{L^2}^2 \\
\leq & \epsilon \|\nabla^k \theta_t\|_{L^2}^2 + C \sum_{|\alpha|=k} \|[\rho, \partial^\alpha]\theta_t\|_{L^2}^2 + C\|\nabla^k(\rho u \cdot \nabla\theta)\|_{L^2}^2 \\
&+ C\|\nabla^k (P \mathrm{div} u)\|_{L^2}^2 + C \left\| \nabla^k \left(\frac{\mu}{2} |\nabla u + \nabla u^\top|^2 + \lambda (\mathrm{div} u)^2 \right) \right\|_{L^2}^2 \\
\leq & \epsilon \|\nabla^k \theta_t\|_{L^2}^2 + C \sum_{|\alpha|=k} \|[\rho, \partial^\alpha]\theta_t\|_{L^2}^2 + C \left(\|\nabla\theta\|_{L^\infty}^2 + \|\nabla^{k+1}\theta\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 \right),
\end{aligned} \tag{5.69}$$

where we have used the following product estimates which are achieved with the aid of (5.32), (5.33) and (5.50):

$$\|\nabla^k(\rho u \cdot \nabla\theta)\|_{L^2}^2 \leq C \left(\|\nabla^{k+1}\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^\infty}^2 \right), \tag{5.70}$$

$$\|\nabla^k (P \mathrm{div} u)\|_{L^2}^2 \leq C \left(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 \right), \tag{5.71}$$

and

$$\left\| \nabla^k \left(\frac{\mu}{2} |\nabla u + \nabla u^\top|^2 + \lambda (\mathrm{div} u)^2 \right) \right\|_{L^2}^2 \leq C \|\nabla^{k+1}u\|_{L^2}^2 \|\nabla u\|_{L^\infty}^2 \leq C \|\nabla u\|_{L^\infty}^2. \tag{5.72}$$

Case 1: $k = 2$. Thanks to (5.13) and (5.33), direct calculations show that

$$\begin{aligned}
\|[\rho, \partial^\alpha]\theta_t\|_{L^2}^2 \leq & C \left(\|\theta_t\|_{L^\infty}^2 + \|\nabla\theta_t\|_{L^4}^2 \right) \\
\leq & C\epsilon \|\nabla^2 \theta_t\|_{L^2}^2 + C \left(\|\theta_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^2 \right).
\end{aligned} \tag{5.73}$$

Substituting (5.73) into (5.69), integrating the resulting inequality with respect to the time t, and using (5.45), we get (5.67) immediately.

Case 2: $k \geq 3$. From (5.33) and the commutator estimate (2.14), we have

$$\|[\rho, \partial^\alpha]\theta_t\|_{L^2}^2 \leq C \left(\|\theta_t\|_{L^2}^2 + \|\nabla^{k-1}\theta_t\|_{L^2}^2 \right). \tag{5.74}$$

Combining (5.69) with this inequality, by (5.32) and (5.45), we conclude that (5.67) holds.

The proof of Lemma 5.9 is completed. □

6. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. First of all, by (4.1), (4.33) and the Gagliardo-Nirenberg interpolation inequality, we have

$$\|u\|_{L^\infty} \leq C \|u\|_{L^6}^{\frac{1}{2}} \|\nabla u\|_{L^6}^{\frac{1}{2}} + \|u\|_{L^6} \leq C \|u\|_{H^1}^{\frac{1}{2}} \leq C e^{-\frac{\tilde{a}}{2}t}. \tag{6.1}$$

It follows from (6.1), (4.1) and (5.1) that

$$\|(\rho-1,u,\theta-1)\|_{L^\infty}\le Ce^{-bt}, \tag{6.2}$$

where $b:=\min\left(\frac{\tilde{a}}{2},c\right)$. Next, for $k\ge 1$, using (3.3), (5.31) and the Gagliardo-Nirenberg interpolation inequality, we are led to

$$\|(\nabla^k\rho,\nabla^k u,\nabla^k\theta)\|_{L^\infty}\le C\|(\rho-1,u,\theta-1)\|_{L^2}^{\frac{1}{2k+4}}\|(\nabla^{k+2}\rho,\nabla^{k+2}u,\nabla^{k+2}\theta)\|_{L^2}^{\frac{2k+3}{2k+4}}\le Ce^{-\frac{\bar{a}}{2k+4}t}. \tag{6.3}$$

Taking $a:=\min(\bar{a},4b)$, combining (6.2) with (6.3), we obtain (1.13). This completes the proof of Theorem 1.2.

Declaration of competing interest

The authors declare that there is no competing interest.

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Appendix A

In this section we give some elementary estimates in the Littlewood-Paley theory on $\mathbb{T}^3$ which are rarely seen in the literature. To begin with, we show an inequality that are frequently used in the multiplier estimates on torus. We would like to point out that the following lemma is just a generalization of the 1D case which can be found in [17].

Lemma A.1. *Let $\phi\in C_c(\mathbb{R}^3)$, $\ell>\frac{1}{2\pi}$ and*

$$f(x)=\sum_{n\in Z^3}\exp(in\cdot x)\phi\left(\frac{n}{\ell}\right).$$

Then there exists a constant $C=C(\phi)$, such that

$$\|f\|_{L^1(\mathbb{T}^3)}\le C. \tag{A.1}$$

Proof. We will follow the strategy of the proof of Lemma 1.1 in [17]. To simplify the presentation, for any smooth function g on $\mathbb{R}^3$, let us denote

$$g_\ell(x_1,x_2,\cdot)[x_3]:=g\left(\frac{x_1,x_2,x_3-2}{\ell}\right)-2g\left(\frac{x_1,x_2,x_3-1}{\ell}\right)+g\left(\frac{x_1,x_2,x_3}{\ell}\right),$$
$$g_\ell(x_1,\cdot,x_3)[x_2]:=g\left(\frac{x_1,x_2-2,x_3}{\lambda}\right)-2g\left(\frac{x_1,x_2-1,x_3}{\ell}\right)+g\left(\frac{x_1,x_2,x_3}{\ell}\right),$$
$$g_\ell(\cdot,x_2,x_3)[x_1]:=g\left(\frac{x_1-2,x_2,x_3}{\ell}\right)-2g\left(\frac{x_1-1,x_2,x_3}{\ell}\right)+g\left(\frac{x_1,x_2,x_3}{\ell}\right).$$

Using Abel's summation formula six times yields

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^3}\exp(in\cdot x)\phi\left(\frac{n}{\ell}\right)\\ &=\sum_{n'\in\mathbb{Z}^2}\exp(in'\cdot x')\sum_{n_3\in\mathbb{Z}}\exp(in_3x_3)\phi\left(\frac{n}{\ell}\right)\\ &=\sum_{n'\in\mathbb{Z}^2}\exp(in'\cdot x')\sum_{n_3\in\mathbb{Z}}\frac{\exp(in_3x_3)}{(\exp(ix_3)-1)^2}\phi_\ell(n_1,n_2,\cdot)[n_3]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n_1\in\mathbb{Z}} \exp(in_1x_1) \sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}} \frac{\exp(in_2x_2)}{(\exp(ix_2)-1)^2}\frac{\exp(in_3x_3)}{(\exp(ix_3)-1)^2} \\
&\quad \times (\phi_\ell(n_1,\cdot,n_3-2)[n_2] - 2\phi_\ell(n_1,\cdot,n_3-1)[n_2] + \phi_\ell(n_1,\cdot,n_3)[n_2]) \\
&= \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}} \frac{\exp(in_1x_1)}{(\exp(ix_1)-1)^2}\frac{\exp(in_2x_2)}{(\exp(ix_2)-1)^2}\frac{\exp(in_3x_3)}{(\exp(ix_3)-1)^2} \\
&\quad \times \Big\{ \phi_\ell(\cdot,n_2-2,n_3-2)[n_1] - 2\phi_\ell(\cdot,n_2-2,n_3-1)[n_1] + \phi_\ell(\cdot,n_2-2,n_3)[n_1] \\
&\quad -2\phi_\ell(\cdot,n_2-1,n_3-2)[n_1] + 4\phi_\ell(\cdot,n_2-1,n_3-1)[n_1] - 2\phi_\ell(\cdot,n_2-1,n_3)[n_1] \\
&\quad +\phi_\ell(\cdot,n_2,n_3-2)[n_1] - 2\phi_\ell(\cdot,n_2,n_3-1)[n_1] + \phi_\ell(\cdot,n_2,n_3)[n_1] \Big\} \\
&=: \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}} \frac{\exp(in_1x_1)}{(\exp(ix_1)-1)^2}\frac{\exp(in_2x_2)}{(\exp(ix_2)-1)^2}\frac{\exp(in_3x_3)}{(\exp(ix_3)-1)^2}\phi_\ell^{[3]}(n). \qquad \text{(A.2)}
\end{aligned}
$$

By Taylor's formula, there hold

$$
|\phi_\ell(n_1,n_2,\cdot)[n_3]| \le \frac{C}{\ell^2}, \qquad \text{(A.3)}
$$

$$
|\phi_\ell(n_1,\cdot,n_3-2)[n_2] - 2\phi_\ell(n_1,\cdot,n_3-1)[n_2] + \phi_\ell(n_1,\cdot,n_3)[n_2]| \le \frac{C}{\ell^4}, \qquad \text{(A.4)}
$$

and

$$
|\phi_\ell^{[3]}(n)| \le \frac{C}{\ell^6}. \qquad \text{(A.5)}
$$

To bound the L^1 norm of f, we write

$$
\int_{\mathbb{T}^3} |f(x)|dx = I + II + III + IV,
$$

where

$$
I := \int_0^{1/\ell}\int_0^{1/\ell}\int_0^{1/\ell} |f(x)|dx,
$$

$$
II := \left(\int_0^{1/\ell}\int_0^{1/\ell}\int_{1/\ell}^{2\pi} + \int_0^{1/\ell}\int_{1/\ell}^{2\pi}\int_0^{1/\ell} + \int_{1/\ell}^{2\pi}\int_0^{1/\ell}\int_0^{1/\ell} \right) |f(x)|dx,
$$

$$
III := \left(\int_0^{1/\ell}\int_{1/\ell}^{2\pi}\int_{1/\ell}^{2\pi} + \int_{1/\ell}^{2\pi}\int_0^{1/\ell}\int_{1/\ell}^{2\pi} + \int_{1/\ell}^{2\pi}\int_{1/\ell}^{2\pi}\int_0^{1/\ell} \right) |f(x)|dx,
$$

and

$$
IV := \int_{1/\ell}^{2\pi}\int_{1/\ell}^{2\pi}\int_{1/\ell}^{2\pi} |f(x)|dx.
$$

Clearly,

$$
I \le \frac{C}{\ell^3} \sum_{n\in\ell\,\mathrm{supp}\phi} \left|\phi\left(\frac{n}{\ell}\right)\right| \le C\|\phi\|_{L^\infty}. \qquad \text{(A.6)}
$$

Next, by virtue of the second equality in (A.2) and (A.3), we arrive at

$$\int_0^{1/\ell}\int_0^{1/\ell}\int_{1/\ell}^{2\pi} |f(x)|dx \le C\int_{1/\ell}^{2\pi}\frac{dx_3}{\ell x_3^2} \le C. \tag{A.7}$$

The third equality in (A.2) and (A.4) imply that

$$\int_0^{1/\ell}\int_{1/\ell}^{2\pi}\int_{1/\ell}^{2\pi} |f(x)|dx \le C\int_{1/\ell}^{2\pi}\frac{dx_2}{\ell x_2^2}\int_{1/\ell}^{2\pi}\frac{dx_3}{\ell x_3^2} \le C. \tag{A.8}$$

Since the three terms in II and III are of the same type respectively, the above two inequalities enable us to obtain

$$II + III \le C. \tag{A.9}$$

Finally, we infer from the last equality in (A.2) and (A.5) that

$$IV \le C\int_{1/\ell}^{2\pi}\frac{dx_1}{\ell x_1^2}\int_{1/\ell}^{2\pi}\frac{dx_2}{\ell x_2^2}\int_{1/\ell}^{2\pi}\frac{dx_3}{\ell x_3^2} \le C. \tag{A.10}$$

This completes the proof of Lemma A.1. □

Next we prove the Bernstein's inequality for functions whose Fourier transform are supported on an annulus. The easier side for functions with Fourier transform supported on a ball can be found in [17]. As a matter of fact, the proofs are similar to those on $\mathbb{R}^n$.

Lemma A.2 *(Bernsein's inequality). Let* $\mathcal{C} := \left\{\xi \in \mathbb{R}^3 : 0 < r_1 \le |\xi| \le r_2\right\}$ *with* $r_1 < r_2$, $\ell > \frac{1}{2\pi}$ *and* $r, p \in [0,\infty]$ *with* $r < p$. *If* v *is a periodic function on* $\mathbb{T}^3$ *such that*

$$\mathrm{supp}\hat{v} \subset \ell\mathcal{C},$$

then

$$\|v\|_{L^p} \le C\ell^{3(\frac{1}{r}-\frac{1}{p})-m}\|\nabla^m v\|_{L^r}. \tag{A.11}$$

Proof. Let $\tilde{\phi}$ be a smooth function supported on an annulus with value 1 on a neighborhood of $\mathcal{C}$. Then for any $k \in \mathbb{Z}^3$, there holds

$$\hat{v}(k) = \hat{v}(k)\tilde{\phi}\left(\frac{k}{\ell}\right)$$

Noting that

$$|k|^{2m} = \sum_{\alpha=m} A_\alpha(-ik)^\alpha(ik)^\alpha$$

for some positive constants $A_\alpha \in \mathbb{N}$, we write

$$\hat{v}(k) = \sum_{\alpha=m} A_\alpha\frac{(-ik)^\alpha}{|k|^{2m}}\tilde{\phi}\left(\frac{k}{\ell}\right)\widehat{\partial^\alpha v}(k) = \ell^{-m}\sum_{\alpha=m}\phi_\alpha\left(\frac{k}{\ell}\right)\widehat{\partial^\alpha v}(k), \tag{A.12}$$

where $\phi_\alpha(\xi) := A_\alpha\frac{(-i\xi)^\alpha}{|\xi|^{2m}}\tilde{\phi}(\xi) \in C_c(\mathbb{R}^3)$. Now let us define

$$f_\alpha(x) := \sum_{n\in\mathbb{Z}^3}\exp(in\cdot x)\phi_\alpha\left(\frac{n}{\ell}\right).$$

We infer from (A.12) that

$$v = \ell^{-m} \sum_{\alpha=m} f_\alpha * \partial^\alpha v.$$

It follows from Young's inequality and Lemma A.1 that

$$\|v\|_{L^p} \le C\ell^{-m}\|\nabla^m v\|_{L^p}.$$

Then (A.11) is a consequence of the above inequality and Lemma 1.2 in [17]. □

The following lemma will be used in the L^∞ estimate of ∇u.

Lemma A.3. *Let $p \in [1,\infty]$, and h be a periodic function in $L^p(\mathbb{T}^3)$. Assume that v is the unique solution of the following elliptic system*

$$\begin{cases} \Delta v = \nabla^2 h, \\ \int_{\mathbb{T}^3} v = 0. \end{cases} \tag{A.13}$$

Then for all dyadic blocks $\Delta_q, q \ge 1$ defined in (2.11), there holds

$$\|\Delta_q v\|_{L^p} \le C\|h\|_{L^p}, \tag{A.14}$$

where the constant C is independent of q.

Proof. Let $\tilde{\varphi}$ be a smooth function supported on an annulus with value 1 on a neighborhood of $\operatorname{supp}\varphi(|\cdot|)$, where φ is the cutoff function in (2.11). Then from (A.13), one deduces that for $\alpha, \beta \in \{1,2,3\}$,

$$\widehat{\Delta_q v^{\alpha\beta}}(k) = \frac{k_\alpha k_\beta}{|k|^2}\tilde{\varphi}\left(\frac{k}{2^q}\right)\widehat{\Delta_q h}(k) = \varphi_{\alpha\beta}\left(\frac{k}{2^q}\right)\widehat{\Delta_q h}(k),$$

where $\varphi_{\alpha\beta}(\xi) := \frac{\xi_\alpha\xi_\beta}{|\xi|^2}\tilde{\varphi}(\xi) \in C_c(\mathbb{R}^3)$. Define

$$f_{\alpha\beta}(x) := \sum_{n\in\mathbb{Z}^3} \exp(in\cdot x)\varphi_{\alpha\beta}\left(\frac{n}{2^q}\right).$$

Then

$$\Delta_q v^{\alpha\beta}(k) = f_{\alpha\beta} * \Delta_q h.$$

It follows from Young's inequality and Lemma A.1 that

$$\|\Delta_q v\|_{L^p} \le C\|\Delta_q h\|_{L^p}. \tag{A.15}$$

By definition,

$$\Delta_q h = f_q * h, \quad \text{with} \quad f_q(x) := \sum_{n\in Z^3} \exp(in\cdot x)\varphi\left(\frac{|n|}{2^q}\right).$$

Then (A.14) follows from (A.15), Young's inequality and Lemma A.1 immediately. □

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