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A new path to the non blow-up of incompressible flows

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Abstract

One of the most challenging questions in fluid dynamics is whether the three-dimensional (3D) incompressible Navier-Stokes, 3D Euler and two-dimensional Quasi-Geostrophic (2D QG) equations can develop a finite-time singularity from smooth initial data. Recently, from a numerical point of view, Luo & Hou presented a class of potentially singular solutions to the Euler equations in a fluid with solid boundary [1,2]. Furthermore, in two recent papers [3,4], Tao indicates a significant barrier to establishing global regularity for the 3D Euler and Navier-Stokes equations, in that any method for achieving this, must use the finer geometric structure of these equations. In this paper, we show that the singularity discovered by Luo & Hou which lies right on the boundary is not relevant in the case of the whole domain $\mathbb{R}^3$. We reveal also that the translation and rotation invariance present in the Euler, Navier-Stokes and 2D QG equations are the key for the non blow-up in finite time of the solutions. The translation and rotation invariance of these equations combined with the anisotropic structure of regions of high vorticity allowed to establish a new geometric non blow-up criterion which yield us to the non blow-up of the solutions in all the Kerr's numerical experiments and to show that the potential mechanism of blow-up introduced in [5] cannot lead to the blow-up in finite time of solutions of Euler equations.

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1. Introduction

The Navier-Stokes and Euler equations describe the motion of a fluid in the three-dimensional space. These fundamental equations were derived over 250 years ago by Euler and since then have played a major role in fluid dynamics. They have enriched many branches of mathematics, were involved in many areas outside mathematical activity from weather prediction to exploding supernova (see for instance the surveys [6], [7]) and present important open physical and mathematical problems (see [6]). Regarding the 2D Quasi-Geostrophic (2D QG) equation, it appears in atmospheric studies. It describes the evolution of potential temperature u on the two dimensional boundary of a rapidly rotating half space with small Rossby and Ekman numbers, for the case of special solutions with constant potential

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vorticity in the interior and constant buoyancy frequency (normalized to one), where equations in the bulk are compressible Euler or Navier-Stokes equations coupled with temperature equation, continuity equation, and equation of state.

In the case of Navier-Stokes equations, for a long time ago, a global weak solution $u \in L^\infty(0,\infty; L^2(\mathbb{R}^3))^3$ and $\nabla u \in L^2(\mathbb{R}^3 \times (0,\infty))^3$ was built by Leray [8]. In particular, Leray introduced a notion of weak solutions for the Navier-Stokes equations, and proved that, for every given $u_0 \in L^2(\mathbb{R}^3)^3$, there exists a global weak solution $u \in L^\infty([0,+\infty[; L^2(\mathbb{R}^3))^3 \cap L^2([0,\infty[; \dot{H}^1(\mathbb{R}^3))^3$. Hopf has proved the existence of a global weak solution in the general case $\mathbb{R}^d$, $d \geq 2$, [9]. Meanwhile the regularity and the uniqueness of this weak solution has been known for a long time ago for the two-dimensional case (see [10], [11], [12], [13]), in the three-dimensional case the problem remains widely open in spite of great efforts made. On the uniqueness many works have been done (see [14], [15], [16], [17], [18]). Concerning the regularity of weak solutions, in [19], it is proved that if u is a Leray-Hopf weak solution belonging to $L^q(]0,T]; L^q(\mathbb{R}^3))^3$ with $\frac{2}{q}+\frac{3}{q} \leq 1$, $2 < p < \infty$, $3 < q < \infty$, then the solution $u \in C^\infty(\mathbb{R}^3 \times]0,T])^3$. In [20] and [21], it is showed that if u is a weak solution in $C([0,T]; L^3(\mathbb{R}^3))^3$, then $u \in C^\infty(\mathbb{R}^3 \times]0,T])^3$. The limit case of $L^\infty([0,T]; L^3(\mathbb{R}^3))^3$ has been solved in [22]. Other criterion regularity can also be found in [23,24,15,25–30].

In the case of Euler Equations, in the two dimension case, uniqueness and existence of classical solutions have been known for a long time ago (see [31–34,10]). However for the full three space dimensions, little is known about smooth solutions apart from classical short-time existence and uniqueness. Moreover, weak solutions are known to be badly behaved from the point of view of Hadamard's well-posedness theory (see for instance the surveys [35,36]). Considerable efforts have been devoted to the study of the regularity properties of the 3D Euler equations. The main difficulty in the analysis lies in the presence of the nonlinear vortex stretching term and the lack of a regularization mechanism. Despite these difficulties, a few important partial results concerning the regularity of 3D Euler equations have been obtained over the years (see [37–43]).

In the case of 2D QG equation, besides its direct physical significance [44,45], the 2D QG equation has very interesting features of resemblance to the 3D Euler equation, being also an outstanding open problem of the finite time blow-up issue. In particular, one can derive a necessary and sufficient blow-up condition for the 2D QG equation similar to the well-known Beale-Kato-Majda (BKM) criterion (Beale-Kato-Majda [37]). More precisely, the solution to the 2D QG equation (11) becomes singular at time T^* if and only if $\int_0^{T^*} \|\nabla^\perp u(t)\|_{L^\infty}\, dt = +\infty$ (see [46]). Thus, $\nabla^\perp u$ plays a role similar to the vorticity ω in the 3D Euler equations. In the recent years, the 2D QG equation has been the focus of intense mathematical research [46–52].

Unfortunately despite of considerable efforts devoted to the regularity issue of the 3D Euler, 3D Navier-Stokes and 2D QG equations, standard scaling heuristics have long indicated to the experts that the identity energy, together with the harmonic analysis estimates available for the heat equation and for the Euler bilinear operator, are not sufficient by themselves if one wishes to improve the theory on the Cauchy problem for these equations. It seems crucial to use the specific structure of the nonlinear term in these equations, as well as the divergence free assumption. Indeed, some finite time blowup results have been established for various Navier-Stokes type equations (see [53–57]). Nevertheless, for all of these Navier-Stokes type equations, the cancellation property of the Euler bilinear operator did not hold and for some, the energy identity did not hold (see [53–55]).

However, recently it was shown also in [3], a finite time blow up solution to an averaged three-dimensional Navier-Stokes equations of type $\partial_t u = \Delta u + \tilde{B}(u,u)$, where $\tilde{B}$ is an averaged version of the Euler bilinear operator B, acting also on divergence free vector fields u and obeying as B to the cancellation property $\langle \tilde{B}(u,u), u\rangle = 0$. This result suggests that any successful method to affirmatively answer to the Existence and Smoothness problem must either use finer structure of B or else must rely crucially on some estimate or other property of the Euler bilinear operator B that is not shared by the averaged operator $\tilde{B}$. Such additional structure exists for instance, the Euler equation has a vorticity formulation involving only differential operators rather than pseudo-differential ones.

However, even this vorticity formulation is not a barrier to get a finite time blow up solution. Indeed, it was shown in [4], finite time blow-up solutions in the *class* of generalised Euler equations sharing with the Euler equation its main features such as: *vorticity formulation, energy conservation, Kelvin circulation theorem, vorticity-vector potential formulation viewed as the Generalised Biot-Savart, function space estimates for the vector potential operator*. Then, it seems that there is no room left to establish global regularity of solutions of 3D Euler equations. However, as it is mentioned in [4], there are two properties of the Euler equations which are not obeyed by the generalised Euler equations, namely translation invariance and rotation invariance.

Further, these symmetries basically determined the usual Biot-Savart law (see [58,59]) which are thus not shared by the Generalised Euler equations introduced in [4]. Furthermore, the use of Biot-Savart law allows to rewrite the vortex stretching term for Euler ($\nu = 0$) and Navier-Stokes ($\nu > 0$) equations as follows (see [30,60]):

$$(\omega \cdot \nabla)\boldsymbol{v} \cdot \omega = \alpha |\omega|^2, \tag{1}$$

where (see Equation (7) in [30])

$$\alpha(x,t) = \frac{3}{4\pi} P.V. \int_{\mathbb{R}^3} (\hat{y} \cdot \xi(x,t)) \det(\hat{y}, \xi(x+y,t), \xi(x,t)) |\omega(x+y,t)| \frac{dy}{|y|^3}, \tag{2}$$

with $\hat{y} = \dfrac{y}{|y|}$, $\xi = \dfrac{\omega}{|\omega|}$ and $\det(a,b,c)$ is the determinant of the matrix with columns a, b, c in that order. We thus notice from the expression of α that if the direction of the vorticity, ξ varies mildly within a small region around x, then the singularity of the integrand in (2) will be mild.

In this paper, we bring new insights which shed light on the mechanisms involved in the non blow-up of the solutions. We highlight through new geometric non blow-up criteria how the geometric regularity of the direction of vorticity combined with the anisotropic structure of the localized regions containing the positions where the maximum of the magnitude of the vorticity are reached, should prevent the formation of singularities. The novelty in the results of this paper lies on the use of the these two features in obtaining geometric non blow-up criteria using the finer structure of the Euler bilinear operator B. Up to now, many progress had been made to better take into account the geometrical properties and flow structures in the non blow-up criteria (see e.g. [30,61,62,60,41,63,42,64,65]). However none of these non blow-up criteria integrated both the geometric regularity of the direction of vorticity and the anisotropic structure of localized regions containing the positions where the maximum of the magnitude of the vorticity are reached. The most advanced non blow-up criteria were given in [42,64,65] and were established by using the Lagrangian formulation of the vorticity equation of the 3D Euler and 2D QG equations.

However the results obtained in [4] suggest that even the most advanced non blow-up criterion [42,64] do not capture the finest structures of the Euler bilinear operator B since it was shown in [4] that there exist generalised Euler equations sharing the same property than Euler equations as the Lagrangian formulation for their vorticity equations and for which their solutions blow up in finite time. Indeed, from [42,64], one can observe that the Deng-Hou-Yu non-blowup criterion can be applied to all the *class* of generalised Euler equations introduced in [4].

Then, in order to bring new insights in the investigation of whether the 3D incompressible Navier-Stokes, Euler and 2D QG equations can develop a finite-time singularity from smooth initial data, it was crucial to establish new non blow-up criteria which take into account the special structure of these equations not shared by the Generalised Euler equations.

Then in our Theorem 7.1, under mild assumptions based on the anisotropic structure of regions of high vorticity, we show that the solutions of 3D Euler, 3D Navier-Stokes and 2D QG equations cannot blow up at a finite time T^* if

$$\int_0^{T^*} \mathbf{A}_d(t) \left(1 + \log^+\left(\frac{\|\omega(t)\|_\infty}{\Omega(t)}\right)\right) dt < \infty,$$

where the functions $\mathbf{A}_d$ and Ω satisfy:

$$\begin{aligned} &\mathbf{A}_d(t) \leq \|\nabla\xi(t)\|_\infty \\ &\Omega(t) = \frac{(T^* - t)^{-1}}{1 + \log^+((T^* - t)\|u(t)\|_\infty \mathbf{A}_0(t))} \\ &\mathbf{A}_0(t) \leq \|\nabla\xi(t)\|_\infty. \end{aligned}$$

Note that $\xi(t)$ is well defined only on $\mathcal{O}(t)$ the set of points x of $\mathbb{R}^d$ where $\omega(x,t) \neq 0$ and then $\|\nabla\xi(t)\|_\infty$ must be understood as $\|\nabla\xi(t)\|_{L^\infty(\mathcal{O}(t))}$.

In the case of 3D Euler equations and 2D QG equations by using their Lagrangian formulation, in Theorem 7.2 we go further in the non blow-up criteria by showing under mild assumptions based on the anisotropic structure of regions of high vorticity, that their solutions do not blow up at a finite time T^* if

$$\int_0^{T^*} \mathbf{A}_d(t)\,dt < \infty.$$

These results are obtained after a fine analysis of the term α defined by (2) combined with some results based on the anisotropic structure of regions of high vorticity. Our analysis starts by considering at each time $t \in]0, T^*[$ the regions containing the positions where the maximum of the magnitude of the vorticity are reached and shrinking to zero as time tends to T^* the alleged time of singularity. More precisely, these regions are balls of radius $\rho_0(t) = O((T^* - t)\|u(t)\|_\infty)$ and of centre the position of a point where the maximum of the magnitude of the vorticity is reached. Inside these regions, we then consider the regions of high vorticity for which the magnitude of the vorticity is greater than some function $\Omega(t)$ such that $\Omega(t) \gtrsim \dfrac{(T^*-t)^{-1}}{1+\log^+(\rho_0(t)\|\nabla\xi(t)\|_\infty)}$.

In our analysis, to track in time the positions where the maximum of the magnitude of the vorticity is reached, we had to overcome the obstruction that we do not know if there exists an isolated absolute maximum for the vorticity achieved along a smooth curve in time as it was assumed in Proposition 2.1 of [46] and also in [42,64,65] (which assume that the position where the maximum of vorticity is reached, is advected with the flow). Moreover, recent numerical experiments show that it is not always the case (see [66], see also section 5.4.5 in [67]). We thus overcome this difficulty by using a result of Pshenichnyi concerning directional derivatives of the function of maximum and the structure of a set of supporting functionals [68].

Our analysis led first to the non blow-up criterion given by our Theorem 5.1, namely, the solutions of 3D Euler, 3D Navier-Stokes and 2D QG equations cannot blow up at a finite time T^* if

$$\int_0^{T^*} \mathbf{A}_d(t)\pi(t)\,dt < \infty, \tag{3}$$

where the function π is given by:

$$\pi(t) \stackrel{\text{def}}{=} \sup_{x\in\Theta(t)} \sup_{0<R\le\rho_0(t)} \frac{1}{R^{d-1}} \int_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz, \tag{4}$$

with $\Theta(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d;\ |\omega(x,t)| = \|\omega(t)\|_\infty\}$ and $\mathcal{V}(t) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d;\ |\omega(z,t)| \ge \Omega(t)\}$.

In our Lemma 6.1, we thus derive a straightforward estimate of the function $\pi(t)$, that is

$$\pi(t) \le 3\|\omega(t)\|_\infty \sup_{x\in\Theta(t)} |\mathcal{V}(t) \cap B(x,\rho_0(t))|^{\frac{1}{d}}. \tag{5}$$

Thanks to the non blow-up criterion (3) and (5), we show the non blow-up in finite time of the solutions of Euler equations for Kerr's numerical experiments [69–71] without additional numerical tests as it was the case in [42, 64], just by using the anisotropic structure of regions of high vorticity whose the features are described in [69–71]. Moreover, we show that the potential mechanism of blow-up introduced in [5] cannot lead to blow-up in finite time for Euler equations. To go further in our estimate of the function π, we use some assumptions characterizing the anisotropic structure of regions of high vorticity whose the justifications are given at the beginning of subsection 7.2 and we show in Proposition 7.1 that

$$\pi(t) \lesssim 1 + \log^+\left(\frac{\|\omega(t)\|_\infty}{\Omega(t)}\right), \tag{6}$$

which yield to Theorem 7.1.

In the case of Euler equations and 2D QG equations by using their Lagrangian formulation, after a fine and sharp analysis of the expression of the function π (4) led thanks to our Lemmata 7.2,7.3 and 7.4, in Proposition 7.2 we go further in the non blow-up criteria by showing under mild assumptions based on the anisotropic structure of regions of high vorticity, that

$$\pi(t) = O(1). \tag{7}$$

We emphasize that according the 'thickness' of the structure of regions of high vorticity that these two estimates (6) and (7) can be much better. Indeed from the analysis led in [72,73] for the study of collapse of vortex lines and agrees with numerical experiments [74,75], we could expect that (see Remark 7.1)

$$\pi(t) \lesssim \Omega(t)^{-\frac{1}{2}},$$

and then obtain in this case, the non blow-up in finite time of the solutions of Euler equations if

$$\int_0^{T^*} \mathbf{A}_d(t)\Omega(t)^{-\frac{1}{2}}\,dt < \infty. \tag{8}$$

We point out also that our geometric non blow-up criterion reveals the role of the geometric structures of the Incompressible flows in the non blow-up in finite time of the solutions and presents the advantage to be established in an Eulerian setting in comparison with all the recent geometric non blow-up criteria [41,42,64,46] using the Lagrangian formulation of Incompressible Inviscid Flows, which requires much more computational effort as it is mentioned in [76] and in section 5.4.5 of [67]. Furthermore, due to the existence of hyperbolic-saddle singularities suggested by the generation of strong fronts in geophysical/meteorology observations (see [46,47]), and antiparallel vortex line pairing observed in numerical simulations and physical experiments, it was important to take them into account in our geometric non blow-up criterion. This is performed thanks to the term $\mathbf{D}_d(\hat{y}, \xi(x+y,t), \xi(x,t))$ (see (27), (28)) involved in the definition of the function $\mathbf{A}_d$ given at (39).

Then, the paper is organized as follows:

- In section 2, we give some notations and definitions.
- In section 3, we recall some results about the local regularity of solutions of Navier-Stokes, Euler and 2D QG equations.
- In section 4, we give the reason for which we can assume for any time t that $\|\omega(t)\|_\infty > 0$ without loss of generality.
- In section 5, in Theorem 5.1, we establish a new geometric criterion for the non blow-up in finite time of the solutions of 3D Navier-Stokes, 3D Euler and 2D QG equations. We show that their solutions cannot blow up at a finite time T^* if $\int_0^{T^*} \mathbf{A}_d(t)\pi(t)\,dt < \infty$, where $\mathbf{A}_d(t)$ is based on the regularity of the direction of the vorticity ξ in regions shrinking to zero as time tends to T^* and containing the positions where the maximum of the magnitude of the vorticity is reached (see definition of $\mathbf{A}_d$ at (39)).
- In section 6, we show the non-blowup in finite time of the solutions of the Euler equations in the numerical experiments considered these last years, by using inequality (5) about the function π (4) and the anisotropic structure of regions of high vorticity described in [69–71].
- In section 7, we show the estimates (5), (6) and (7) concerning the function π defined by (4), and obtain new non blow-up criteria in Theorems 7.1 and 7.2.

Let us now introduce the 3D Navier-Stokes and Euler equations given by,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p - \nu\Delta u = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{9}$$

in which $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$, $p = p(x,t) \in \mathbb{R}$ and $\nu \geq 0$ ($\nu = 0$ corresponds to the Euler equations) denote respectively the unknown velocity field, the scalar pressure function of the fluid at the point $(x,t) \in \mathbb{R}^3 \times [0,\infty[$ and the viscosity of the fluid,

with initial conditions,

$$u(x,0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3, \tag{10}$$

where the initial data u_0 is a divergence free vector field on $\mathbb{R}^3$.

Regarding the 2D QG equation in $\mathbb{R}^2$, it is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + v\cdot\nabla u = 0, \\ v = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}u, \end{cases} \tag{11}$$

with initial data,

$$u(x,0) = u_0. \tag{12}$$

Here $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. For v we have also the following representation

$$v = R^{\perp}u, \tag{13}$$

where we have used the notation, $R^{\perp}u = (-R_2u, R_1u)$ with R_j, $j = 1, 2$, for the 2D Riesz transform defined by (see e.g. [77])

$$R_j(u)(x,t) = \frac{1}{2\pi} P.V \int_{\mathbb{R}^2} \frac{(x_j - y_j)}{|x-y|^3} u(y,t)\,dy.$$

2. Some notations and definitions

In this section, we assume that $d \in \mathbb{N}$, $d \geq 2$.

For any vector $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$, we denote by $|x|$ the euclidean norm of x given by $|x| = \sqrt{\sum_{i=1}^{d} |x_i|^2}$. For any $y \in \mathbb{R}^d$, $y \neq 0$, we denote by $\hat{y}$ the unit vector $\hat{y} = \frac{y}{|y|}$. For any m-dimensional subset A of $\mathbb{R}^d$, $1 \leq m \leq d$, we denote by $|A|$ its measure. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of real square matrices of size d. We denote by Id the identity matrix of $\mathcal{M}(\mathbb{R}^d)$. For any vector field v defined from $\mathbb{R}^d$ to $\mathbb{R}^d$, we denote by ∇v the gradient matrix of v, the matrix of $\mathcal{M}(\mathbb{R}^d)$ with ij-component, $\dfrac{\partial v_i}{\partial x_j}$ for all $1 \leq i, j \leq d$. For any real a, we denote by a^+ the real defined by $a^+ \stackrel{\text{def}}{=} \max(a, 0)$. For any function φ defined on $\mathbb{R}^d \times [0, +\infty[$, for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on $\mathbb{R}^d$ by $x \longmapsto \varphi(x,t)$. We denote by $C_c^{\infty}(\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support in $\mathbb{R}^d$. We denote by BC the class of bounded and continuous functions and by BC^m the class of bounded and m times continuously derivable functions.

For any $R > 0$ and $x_0 \in \mathbb{R}^d$, we denote by $B(x_0, R)$, the ball of $\mathbb{R}^d$ of centre x_0 and radius R. For any $R > 0$, we denote by B_R, the ball of $\mathbb{R}^d$ of centre 0 and radius R.

We denote by div the differential operator given by, $\text{div} = \sum_{i=1}^{d} \dfrac{\partial}{\partial x_i}$.

We denote $A \lesssim B$, $B \gtrsim A$ or $A = O(B)$ the estimate $A \leq c\,B$ where $c > 0$ is an absolute constant. If we need c to depend on a parameter, we shall indicate this by subscripts, thus for instance $A \lesssim_s$ denotes the estimate $A \leq c_s B$ for some c_s depending on s. We use $A \sim B$ as shorthand for $A \lesssim B \lesssim A$.

For any $f \in L^p(\mathbb{R}^d)$ (resp. $L^p(\mathbb{R}^d)^d$ or $L^p(\mathbb{R}^d)^{d\times d}$) with $1 \leq p \leq +\infty$, we denote by $\|f\|_p$ and $\|f\|_{L^p}$, the L^p-norm of f.

We denote by $H^s(\mathbb{R}^d)$ the Sobolev space $J^{-s}L^2(\mathbb{R}^d)$ where $J = (1-\Delta)^{\frac{1}{2}}$. We denote by $H^s_\sigma(\mathbb{R}^3)$ the Sobolev space $H^s_\sigma(\mathbb{R}^3) \stackrel{\text{def}}{=} \{\psi \in H^s(\mathbb{R}^3)^3 : \text{div}\psi = 0\}$. In order to unify our notations with the two dimensional case 2D QG, we denote by $H^s_\sigma(\mathbb{R}^2)$ the Sobolev space $H^s(\mathbb{R}^2)$.

We denote by $\mathbb{P}$ the well-known 3D matrix Leray's projection operator with components,

$$\mathbb{P}_{i,j} = \delta_{i,j} - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Delta^{-1} = \delta_{i,j} - R_j R_k, \tag{14}$$

where R_j are the Riesz transform given by $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}} = \frac{1}{4\pi}\frac{x_j}{|x|^4}\star$ (see [77] for more details), Δ^{-1} is the inverse of Laplace operator given by $\Delta^{-1} = -\frac{1}{4\pi|x|}\star$, with $\star$ the convolution operator.

3. Local regularity of the solutions

In this section, we deal with the main result on local regularity of 3D Navier-Stokes and Euler equations in its general form. By introducing $\mathbb{P}$ the matrix Leray operator, Euler equations (9)–(10) can be re-written as follows,

$$\frac{\partial u}{\partial t} + \mathbb{P}(u\cdot\nabla)u = 0, \tag{15}$$

with initial conditions,

$$u(0) = u_0. \tag{16}$$

For u solution of (15)–(16), $\omega = \nabla\times u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u - \nu\Delta\omega = 0, \tag{17}$$

with initial conditions,

$$\omega(0) = \omega_0,$$

where $\omega_0 = \nabla\times u_0$ is the vorticity of u_0.

In the case of 2D QG equation, we get for u solution of (11), $\omega = \nabla^{\perp}u$ the vorticity of u formally satisfies the vorticity equation,

$$\frac{\partial \omega}{\partial t} + (v\cdot\nabla)\omega - (\omega\cdot\nabla)v = 0, \tag{18}$$

with initial conditions,

$$\omega(0) = \omega_0,$$

where $\omega_0 = \nabla^{\perp}u_0$ is the vorticity of u_0.

In the region where $|\omega| > 0$, we define ξ the direction of the vorticity by $\xi = \frac{\omega}{|\omega|}$.

3.1. Local regularity for 3D Navier-Stokes or 3D Euler equations

Assuming $u_0 \in H^r_\sigma(\mathbb{R}^3)$ with $r > \frac{5}{2}$, thanks to Theorem 3.5 in [78], Theorem 1 in [79] (see also Theorem I in [80] and the results obtained in [37]), we deduce that there exists a time $T > 0$ such that there exists an unique strong solution $u \in C([0,T[, H^r_\sigma(\mathbb{R}^3)) \cap C^1([0,T[, H^{r-2}_\sigma(\mathbb{R}^3))$ to the Navier-Stokes or Euler equations (15)–(16) and the energy equality holds for u, that means for all $t \in [0,T[$,

$$\|u(t)\|_2 + 2\nu\int_0^t \|\nabla u(s)\|_2^2\,ds = \|u_0\|_2. \tag{19}$$

Moreover, if $u \notin C([0,T], H^r_\sigma(\mathbb{R}^3))$, then we get (see [37,78,81]),

$$\int_0^T \|\omega(t)\|_\infty dt = +\infty. \tag{20}$$

Notice thanks to Remark 3.7 in [78], in the case of Euler equations, we get in addition that $u \in C^1([0,T[, H^{r-1}_\sigma(\mathbb{R}^3))$. We retrieve the pressure p from the velocity u with the formula,

$$p = -\Delta^{-1}\mathrm{div}((u \cdot \nabla)u).$$

Furthermore, we get the local estimate (21). Indeed, thanks to remark 4.4 in [78], we get

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ with } t_0 < t < T, \tag{21}$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

3.2. Local regularity for 2D QG equation

This subsection is devoted to the local well-posedness of the 2D QG equation with a characterization of the maximal time existence of strong solutions. By using the same arguments as the proof of Proposition 4.2 in [82], we get that the H^s-norm of u is controlled by the integral in time of the maximum magnitude of the vorticity of u. A such Proposition has been proved in [46] for any integer $s \geq 3$, but here we extend this result to all real $s > 2$. This improvement is obtained by using the logarithmic Sobolev inequality proved in [81,78] which requires only that $s > 2$ instead of using the one proved in [37] as it is the case in [46] and which requires integer $s \geq 3$. Then by using the same arguments as the proof of Proposition 4.3 in [82], we get the following result which gives an improvement in comparison with Theorem 2.1 in [46]:

Assuming $u_0 \in H^r(\mathbb{R}^2)$ with $r > 2$, we get that there exists a time $T > 0$ such that there exists an unique strong solution $u \in C([0, T[, H^r(\mathbb{R}^2))$ to the 2D QG equation (11)–(12) and the energy equality holds for u, that means for all $p \in [2, \infty]$ and $t \in [0, T[$,

$$\|u(t)\|_p = \|u_0\|_p. \tag{22}$$

Moreover, if $u \notin C([0, T], H^r(\mathbb{R}^2))$, then

$$\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty. \tag{23}$$

Owing to $u \in C([0, T[, H^r(\mathbb{R}^2))$ and thanks to Lemma X4 in [78], from 2D QG (11), we get $u \in C^1([0, T[, H^{r-1}(\mathbb{R}^2))$.

Similarly as in (21), we have

$$\|u(t)\|_{H^r} \leq \frac{\|u(t_0)\|_{H^r}}{1 - c\|u(t_0)\|_{H^r}(t - t_0)} \text{ for } t_0 < t < T, \tag{24}$$

provided that $1 - c\|u(t_0)\|_{H^r}(t - t_0) > 0$, where $c > 0$ is a constant.

4. Assumption on the maximum vorticity

Let $d \in \{2, 3\}$, $r > \frac{d}{2} + 1$ and $u_0 \in H^r_\sigma(\mathbb{R}^d)$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler or 2D QG equations (9)–(10) in the class

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-2}(\mathbb{R}^d)).$$

Thanks to the results of the section 3, a such time T^* exists.

In this paper, we are concerned with the non blowup in finite time of the solutions u at times such T^*. Then, without loss of generality, in the whole of this paper, we consider only times of existence T^* such that for all $t \in [0, T^*[$,

$$\|\omega(t)\|_\infty > 0. \tag{25}$$

Indeed, let us assume that there exists $t_0 \in [0, T^*[$ such that $\|\omega(t_0)\|_\infty = 0$.

In the case of 2D QG equations (11), we get that $\omega(t_0) \equiv 0$ and then $\nabla u(t_0) \equiv 0$. Since $x \mapsto u(t_0, x)$ vanishes at infinity, then we get $u(t_0) \equiv 0$. Then by using inequality (24) concerning the local regularity, we deduce that $u(t) \equiv 0$ for all $t \in [t_0, T^*[$ and no blowup can occur at the time T^*.

By following step by step the proof of Lemma 4 given in [42] but keeping the term $\|u(t)\|_{L^2(\mathbb{R}^d)}$ after using the Cauchy-Schwarz inequality, we obtain for all $t \in [0, T^*[$,

$$\begin{aligned} \|u(t)\|_\infty &\lesssim \|u(t)\|_2^{\frac{2}{d+2}} \|\omega(t)\|_\infty^{\frac{d}{d+2}} \\ &\leq \|u_0\|_2^{\frac{2}{d+2}} \|\omega(t)\|_\infty^{\frac{d}{d+2}}, \end{aligned} \tag{26}$$

where we have used (19) for the last inequality. Then thanks to (26) used with $d = 3$, we obtain that $\|u(t_0)\|_\infty \equiv 0$ which implies that $u(t_0) \equiv 0$. Then by using the inequality (21) of local regularity, we deduce $u(t) \equiv 0$ for all $t \in [t_0, T^*[$ and thus no blowup can occur at the time T^*.

5. Geometric properties for non blow-up of the solutions

Historically, non blow-up criteria for the incompressible Euler equations and 2D QG equations commonly focus on global features of the flow, such as norms of the velocity or the vorticity fields. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy for overcoming such shortcomings was established by focusing more on geometrical properties and flow structures (see e.g. [41,83]), such as vortex tubes or vortex lines.

In particular, in [41,46] the authors showed that local geometric regularity of the unit vorticity vector can lead to depletion of the vortex stretching. They prove that if there is up to time T an $O(1)$ region in which the vorticity vector is smoothly directed, i.e., the maximum norm of $\nabla\xi$ (here $\xi = \dfrac{\omega}{|\omega|}$, ω the vorticity) in this region is L^2 integrable in time from 0 to T, and the maximum norm of velocity in some $O(1)$ neighbourhood of this region is uniformly bounded in time, then no blow-up can occur in this region up to time T.

However, this theorem dealt with $O(1)$ regions in which the vorticity vector is assumed to have some regularity, while in numerical computations, the regions that have such regularity and contain maximum vorticity are all shrinking with time (see [84,85,71,70,86,87]).

Inspired by the work of [41,46], in [42,64,65] the authors showed that geometric regularity of Lagrangian vortex filaments, even in an extremely localized region containing the maximum of vorticity which may shrink with time, can lead to depletion of the nonlinear vortex stretching, thus avoiding finite time singularity formation of the 3D Euler equations and 2D QG equations.

However, all the recent geometric constraints for non blow-up criteria of Euler and 2D QG equations based on local geometric regularity of Lagrangian vortex filaments [42,64,65] make the assumption that the position where the maximum of vorticity is reached, is advected with the flow, however it is not always the case, as described in [66] (see also section 5.4.5 of [67]).

Then in our Theorem 5.1, we establish in an Eulerian setting a new geometric non blow-up criterion for the Navier-Stokes, Euler and 2D QG equations based on the regularity of the direction of the vorticity in extremely localized regions containing the positions where the maximum of the magnitude of the vorticity are reached and shrinking to zero as time increase to some T^* the alleged time of singularity. Our Eulerian geometric non blow-up criterion should give also new impetus to the numerical experiments due to their ease of implementation in comparison with Lagrangian geometric non blow-up criteria (see [76], see also section 5.4.5 of [67]). Moreover our geometric non blow-up criterion is also valid for the Navier-Stokes equations that is not the case for the existing geometric non blow-up criteria obtained in [41,42,64,65] based on a Lagrangian formulation of Incompressible Inviscid Flows.

To obtain our Theorem 5.1, we begin with Lemma 5.1.

Lemma 5.1. *Let $d \in \mathbb{N}^*$, $T > 0$ and $f \in C([0, T]; BC(\mathbb{R}^d))$ such that $\inf_{t\in[0,T]} \|f(t)\|_\infty > 0$ and for any $t \in [0, T]$, $|f(x, t)| \to 0$ as $|x| \to +\infty$. Then there exists $R > 0$ such that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x\in B_R} |f(x, t)|$.*

Proof. We set $a = \inf_{t\in[0,T]} \|f(t)\|_\infty > 0$. Since $t \mapsto f(t)$ is a continuous function from the compact $[0, T]$ into the metric space $L^\infty(\mathbb{R}^d)$ then it is uniformly continuous. Hence, there exists $N \in \mathbb{N}^*$ such that for all $t, t' \in [0, T]$, $|t - t'| \leq \frac{T}{N}$ we have $\|f(t) - f(t')\|_\infty \leq \dfrac{a}{4}$. We introduce the subdivision $\{t_i\}_{\{i\in[\![0,N]\!]\}}$ of $[0, T]$ defined by $t_i = i\frac{T}{N}$

for $i \in [\![0, N]\!]$. Since for any $t \in [0, T]$, $|f(x,t)| \to 0$ as $|x| \to +\infty$, then for each $i \in [\![0, N]\!]$, there exists $R_i > 0$ such that for all $|x| \geq R_i$, $|f(x,t_i)| \leq \frac{a}{4}$. We set $R = \max_{i \in [\![0,N]\!]} R_i$. Let $t \in [0, T]$ then there exists $j \in [\![0, N]\!]$ such that $|t - t_j| \leq \frac{T}{N}$ and hence for all $|x| \geq R \geq R_j$, we have $|f(x,t)| \leq |f(x,t) - f(x,t_j)| + |f(x,t_j)| \leq \frac{a}{2} \leq \frac{\|f(t)\|_\infty}{2}$. Then, we infer that for all $t \in [0, T]$, $\|f(t)\|_\infty = \sup_{x \in B_R} |f(x,t)|$, which concludes the proof. □

Before to prove Theorem 5.1, we need to introduce the following function $\mathbf{D}_d$ defined from $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}$ with $d \in \{2, 3\}$ as follows: for $d = 3$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3)\,\mathrm{Det}(a_1, a_2, a_3).$$

The Det in $\mathbf{D}_d$ is the determinant of the matrix whose columns are the three unit column vectors a_1, a_2, a_3. We observe that $\mathrm{Det}(a_1, a_2, a_3) = a_1 \cdot (a_2 \times a_3)$, then, we get

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3)\; a_1 \cdot (a_2 \times a_3), \tag{27}$$

and for $d = 2$,

$$\mathbf{D}_d(a_1, a_2, a_3) = (a_1 \cdot a_3^\perp)\,(a_2 \cdot a_3^\perp), \tag{28}$$

where for any $z = (z_1, z_2) \in \mathbb{R}^2$, $z^\perp = (-z_2, z_1)$. We can notice that for $d \in \{2, 3\}$ the function $\mathbf{D}_d$ is linear from its second variable.

From (27) and (28) we get $\mathbf{D}_d(a_1, a_3, a_3) = 0$ then we deduce that for any $a_1, a_2, a_3 \in B(0, 1)$,

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq |a_2 - a_3|, \tag{29}$$

and we get also

$$|\mathbf{D}_d(a_1, a_2, a_3)| \leq 1. \tag{30}$$

Now, we turn to the proof of our Theorem.

Theorem 5.1. *Let $d \in \{2, 3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-2}(\mathbb{R}^d)). \tag{31}$$

Let ρ_0 be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by

$$\rho_0(t) \stackrel{\text{def}}{=} 36(T^* - t)\|u(t)\|_\infty. \tag{32}$$

Let $\mathbf{A}_0$ be the function defined from $[0, T^*[$ to $]0, +\infty[$ for all $t \in [0, T^*[$ by:

$$\mathbf{A}_0(t) \stackrel{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0, \rho_0(t)) \setminus \{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y,t), \xi(x,t))^+}{|y|}, \tag{33}$$

where for any $t \in [0, T^*[$

$$\Theta(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d;\, |\omega(x,t)| = \|\omega(t)\|_\infty\}. \tag{34}$$

Let Ω be the function defined from $[0, T^*[$ to $]0, +\infty[$ by:

$$\Omega(t) \stackrel{\text{def}}{=} \frac{(T^* - t)^{-1}}{8(1 + \log^+(4\rho_0(t)\mathbf{A}_0(t)))}. \tag{35}$$

We introduce also the set of high vorticity regions defined for all $t \in [0, T^*[$ by

$$\mathcal{V}(t) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^d;\, |\omega(z,t)| \geq \Omega(t)\}. \tag{36}$$

Let π be the function defined from $[0, T^*[$ to $[0, +\infty[$, for all $t \in [0, T^*[$ by

$$\pi(t) \stackrel{\text{def}}{=} \sup_{x\in\Theta(t)} \sup_{0<R\leq\rho_0(t)} \frac{1}{R^{d-1}} \int\limits_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz. \tag{37}$$

Let ρ be the function defined from $[0, T^*[$ to $[0, +\infty[$ for all $t \in [0, T^*[$ by

$$\rho(t) \stackrel{\text{def}}{=} 4(d+1)c_d(T^* - t)\pi(t), \tag{38}$$

where $c_d = \frac{3}{4\pi}$ if $d = 3$, $c_d = \frac{1}{2\pi}$ else.

Let $\mathbf{A}_d$ be the function defined from $[0, T^*[$ to $[0, +\infty[$ for all $t \in [0, T^*[$ by

$$\mathbf{A}_d(t) \stackrel{\text{def}}{=} \sup_{x\in\Theta(t)} \sup_{y\in B(0,\rho(t))\setminus\{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y,t), \xi(x,t))^+}{|y|}. \tag{39}$$

Then if there exists $t_1 \in [0, T^*[$ such that

$$\int\limits_{t_1}^{T^*} \mathbf{A}_d(t)\pi(t) < +\infty, \tag{40}$$

then the solution u cannot blowup at the finite time T^*.

Moreover, we have for all $t \in [0, T^*[$ and $x \in \Theta(t)$,

$$\nabla|\omega|(x,t) = 0 \text{ and } \nabla\cdot\xi(x,t) = 0.$$

Proof. Let $0 < T < T^*$. We want first to apply Lemma 5.1 to the function ω, then we check that the hypotheses of the Lemma are satisfied.

Since $u \in C([0,T]; H^r(\mathbb{R}^d)) \cap C^1([0,T]; H^{r-2}(\mathbb{R}^d))$, then we infer that $\omega \in C([0,T]; H^{r-1}(\mathbb{R}^d)) \cap C^1(]0,T]; H^{r-3}(\mathbb{R}^d))$. Thanks to the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow BC^m(\mathbb{R}^d)$ for $s > \frac{d}{2} + m$, $m \in \mathbb{N}$ and since $r > \frac{d}{2} + 3$ we deduce that $\omega \in C([0,T]; BC^2(\mathbb{R}^d)) \cap C^1([0,T]; BC(\mathbb{R}^d))$. Thanks to (25), we get that $\inf_{t\in[0,T]} \|\omega(t)\|_\infty > 0$. Moreover, since $\omega \in C([0,T]; H^{r-1}(\mathbb{R}^d))$ with $r > \frac{d}{2} + 3$, we have for any $t \in [0,T]$, $|\omega(x,t)| \to 0$ as $|x| \to +\infty$, the proof follows immediately by using the density of $C_0^\infty(\mathbb{R}^d)$ in $H^{r-1}(\mathbb{R}^d)$ and the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $r > \frac{d}{2} + 3$.

Then thanks to Lemma 5.1, there exists $R > 0$ such that for all $t \in [0,T]$, $\|\omega(t)\|_\infty = \sup_{x\in B_R} |\omega(x,t)|$. Then for all $t \in [0,T]$, the set $\Theta(t)$ defined by (34) can be rewritten as follows:

$$\Theta(t) = \{x \in B_R;\, |\omega(x,t)| = \|\omega(t)\|_\infty\}. \tag{41}$$

We introduce the direction of the vorticity $\xi = \frac{\omega}{|\omega|}$ defined on the non empty open set $\mathcal{O} \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{R}^d \times [0,T];\, |\omega(x,t)| > 0\}$.

We set $\boldsymbol{v} = u$ in the case of 3D Navier-Stokes or 3D Euler equations and $\boldsymbol{v} = R^\perp u$ with $\nu = 0$ in the case of 2D QG equation.

Then by multiplying (17) or (18) by ξ, we get that for all $(x,t) \in \mathcal{O}$,

$$\begin{aligned} &\frac{\partial|\omega|}{\partial t}(x,t) + \boldsymbol{v}(x,t)\cdot\nabla|\omega|(x,t) - (\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\cdot\xi(x,t) \\ &\qquad - \nu\Delta|\omega|(x,t) + \nu|\omega(x,t)||\nabla\xi(x,t)|^2 = 0. \end{aligned} \tag{42}$$

We introduce the function φ defined for all $t \in [0,T]$ by

$$\varphi(t) \stackrel{\text{def}}{=} \sup_{x\in B_R} |\omega(x,t)|$$

and we search the expression of its derivative. For this, we use the main Theorem obtained in [68] or Theorem 1 in [88] after verifying that the hypotheses of the Theorem are satisfied.

Since $\omega \in C([0,T]; BC^2(\mathbb{R}^d)) \cap C^1([0,T]; BC(\mathbb{R}^d))$, then we deduce that $|\omega| \in BC(\mathcal{O})$, $\frac{\partial|\omega|}{\partial t} \in BC(\mathcal{O})$ and $\nabla^2|\omega| \in BC(\mathcal{O})$. Since for any $t \in [0,T]$, $\Theta(t) \subset \mathcal{O} \times \{t\}$, then, thanks to the results obtained in [68] (see also Theorem 1 in [88]), by using also (41) we obtain the expression of the derivative of φ given for any $t \in [0,T]$ by,

$$\varphi'(t) = \sup_{x\in\Theta(t)} \frac{\partial|\omega|}{\partial t}(x,t). \tag{43}$$

Further for all $x \in \Theta(t) \subset B_R$, we have $|\omega(x,t)| = \varphi(t) = \|\omega(t)\|_\infty$, we thus infer that

$$\nabla|\omega|(x,t) = 0 \text{ and } \Delta|\omega|(x,t) \le 0. \tag{44}$$

Therefore, we have for all $x \in \Theta(t)$,

$$\begin{aligned}\frac{\partial|\omega|}{\partial t}(x,t) &= \frac{\partial|\omega|}{\partial t}(x,t) + \boldsymbol{v}(x,t)\cdot\nabla|\omega|(x,t)\\ &= (\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\cdot\xi(x,t) + \nu\Delta|\omega|(x,t) - \nu|\omega(x,t)||\nabla\xi(x,t)|^2\\ &\le (\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\cdot\xi(x,t),\end{aligned} \tag{45}$$

where we have used (42) for the second equality and (44) for the last inequality. We can notice that we get equality for (45) in the case of 3D Euler or 2D QG equations, since for these equations we have not the terms $\nu\Delta|\omega|(x,t)$ and $\nu|\omega(x,t)||\nabla\xi(x,t)|^2$.

Then using (45), from (43), we obtain,

$$\varphi'(t) \le \sup_{x\in\Theta(t)} (\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\cdot\xi(x,t),$$

which means that

$$\frac{d}{dt}\|\omega(t)\|_\infty \le \sup_{x\in\Theta(t)} (\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\cdot\xi(x,t), \tag{46}$$

where equality holds in the case of 3D Euler or 2D QG equations. We use now the function α introduced in [60,30] for the 3D Navier-Stokes or 3D Euler equations and in [46] for the 2D QG equation, defined for all $(x,t) \in \mathcal{O}$ by,

$$\alpha(x,t) = c_d P.V.\int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \xi(x+y,t), \xi(x,t))\,|\omega(x+y,t)|\,\frac{dy}{|y|^d}, \tag{47}$$

where $\hat{y} = \frac{y}{|y|}$ and in the case of 3D Navier-Stokes or 3D Euler equations for which $d=3$, $c_d = \frac{3}{4\pi}$ and in the case of 2D QG equation for which $d=2$, $c_d = \frac{1}{2\pi}$. We use the fact that $|\omega(x+y,t)|\xi(x+y,t) = \omega(x+y,t)$ and the fact that $\mathbf{D}_d$ is linear in comparison with its second variable, to rewrite (47) as follows:

$$\alpha(x,t) = c_d P.V.\int_{\mathbb{R}^d} \mathbf{D}_d(\hat{y}, \omega(x+y,t), \xi(x,t))\,\frac{dy}{|y|^d}. \tag{48}$$

By using the Biot-Savart law (see [89]) for which in the case of Euler and Navier-Stokes equations, we have

$$\boldsymbol{v}(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y)dy,$$

and in the case of 2D QG equations, we get an equivalent formula

$$\boldsymbol{v}(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{|y|}\omega(x+y,t)dy,$$

we deduce as in [60,30] and [46] that for all $(x,t) \in \mathcal{O}$

$$(\omega(x,t)\cdot\nabla)\boldsymbol{v}(x,t)\ \cdot \xi(x,t) = \alpha(x,t)|\omega(x,t)|.$$

Therefore, from (46), we deduce that for all $t \in [0,T]$,

$$\begin{aligned}\frac{d}{dt}\|\omega(t)\|_\infty &\le \sup_{x\in\Theta(t)} \alpha(x,t)|\omega(x,t)| \\ &= \left(\sup_{x\in\Theta(t)} (\alpha(x,t)\right)\|\omega(t)\|_\infty,\end{aligned} \tag{49}$$

where we have used the fact that for all $x \in \Theta(t)$, $|\omega(x,t)| = \|\omega(t)\|_\infty$. Let us estimate now $\alpha(x,t)$ for any $t \in [0,T]$ and $x \in \Theta(t)$. For this purpose, let us take $t \in [0,T]$ and $x \in \Theta(t)$, then we decompose the term $\alpha(x,t)$ as the sum of three terms,

$$\alpha(x,t) = I_1 + I_2 + I_3 \tag{50}$$

where,

$$I_1 = c_d \int_{B(0,\min(\rho(t),\rho_0(t)))} \mathbf{D}_d(\hat{y}, \omega(x+y,t), \xi(x,t))\,\frac{dy}{|y|^d}, \tag{51}$$

$$I_2 = c_d \int_{B(0,\rho_0(t))\cap B(0,\min(\rho(t),\rho_0(t)))^c} \mathbf{D}_d(\hat{y}, \omega(x+y,t), \xi(x,t))\,\frac{dy}{|y|^d} \tag{52}$$

and

$$I_3 = c_d \int_{B(0,\rho_0(t))^c} \mathbf{D}_d(\hat{y}, \omega(x+y,t), \xi(x,t))\frac{dy}{|y|^d}. \tag{53}$$

Then, we estimate the three terms I_1, I_2 and I_3. For the term I_1, from (51) we get

$$\begin{aligned} I_1 &= c_d \int_{B(0,\min(\rho(t),\rho_0(t)))} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y,t), \xi(x,t))}{|y|}|\omega(x+y,t)|\,\frac{dy}{|y|^{d-1}} \\ &\le c_d\mathbf{A}_d(t) \int_{B(0,\min(\rho(t),\rho_0(t)))} \frac{|\omega(x+y,t)|}{|y|^{d-1}}\,dy \\ &= c_d\mathbf{A}_d(t) \int_{B(x,\min(\rho(t),\rho_0(t)))\cap\mathcal{V}(t)^c} \frac{|\omega(z,t)|}{|x-z|^{d-1}}\,dz \\ &\quad + c_d\mathbf{A}_d(t) \int_{B(x,\min(\rho(t),\rho_0(t))))\cap\mathcal{V}(t)} \frac{|\omega(z,t)|}{|x-z|^{d-1}}\,dz \\ &\le c_d\mathbf{A}_d(t)\Omega(t) \int_{B(0,\rho(t))} \frac{dy}{|y|^{d-1}} + c_d\mathbf{A}_d(t) \int_{B(x,\min(\rho(t),\rho_0(t))))\cap\mathcal{V}(t)} \frac{|\omega(z,t)|}{|z-x|^{d-1}}\,dz. \end{aligned}$$

Furthermore, we have

$$\int_{B(0,\rho(t))} \frac{dy}{|y|^{d-1}} = |B(0,1)|\rho(t).$$

Therefore, by using the fact that $c_d|B(0,1)| \le 1$ (since $|B(0,1)| = \dfrac{4\pi}{3}$ for $d=3$ and $|B(0,1)| = \pi$ for $d=2$), we deduce

$$I_1 \le \mathbf{A}_d(t)\Omega(t)\rho(t) + \mathbf{A}_d(t) I_{1,1}, \tag{54}$$

where

$$I_{1,1} \stackrel{\text{def}}{=} c_d \int\limits_{B(x,\min(\rho(t),\rho_0(t)))\cap\mathcal{V}(t)} \frac{|\omega(z,t)|}{|z-x|^{d-1}}\, dz. \tag{55}$$

Let $\varepsilon(t) \stackrel{\text{def}}{=} \dfrac{\pi(t)}{\|\omega(t)\|_\infty}$, then we have

$$I_{1,1} \le c_d \int\limits_{B(x,\varepsilon(t))} \frac{|\omega(z,t)|}{|z-x|^{d-1}}\, dz + c_d I_{1,2}, \tag{56}$$

with

$$I_{1,2} \stackrel{\text{def}}{=} \int\limits_{B(x,\varepsilon(t))^c \cap B(x,\min(\rho(t),\rho_0(t)))\cap\mathcal{V}(t)} \frac{|\omega(z,t)|}{|z-x|^{d-1}}\, dz. \tag{57}$$

On one hand, we have

$$\begin{aligned} c_d \int\limits_{B(x,\varepsilon(t))} \frac{|\omega(z,t)|}{|z-x|^{d-1}}\, dz &\le c_d \|\omega(t)\|_\infty \int\limits_{B(x,\varepsilon(t))} \frac{dz}{|z-x|^{d-1}} \\ &= c_d |B(0,1)| \|\omega(t)\|_\infty\, \varepsilon(t) \\ &\le \pi(t). \end{aligned} \tag{58}$$

On the other hand, we estimate $I_{1,2}$.

If $\mathcal{V}(t) = \emptyset$ or $\min(\rho(t), \rho_0(t)) \le \varepsilon(t)$ then from (57) we infer that $I_{1,2} = 0$. Let us assume now that $\mathcal{V}(t) \ne \emptyset$ and $\min(\rho(t), \rho_0(t)) > \varepsilon(t)$. Then, from (57) we get

$$\begin{aligned} I_{1,2} &= \int\limits_{\varepsilon(t)}^{\min(\rho(t),\rho_0(t))} \left(\int\limits_{\partial B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)| \frac{d\gamma(z)}{|z-x|^{d-1}} \right) dR \\ &= \int\limits_{\varepsilon(t)}^{\min(\rho(t),\rho_0(t))} \frac{1}{R^{d-1}} \left(\int\limits_{\partial B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\, d\gamma(z) \right) dR. \end{aligned} \tag{59}$$

We introduce the function F_t defined from $[0, \rho_0(t)]$ to $[0, +\infty[$ for all $0 \le R \le \rho_0(t)$ by

$$F_t(R) \stackrel{\text{def}}{=} \int\limits_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\, dz. \tag{60}$$

We observe that $\dfrac{dF_t}{dR}(R) = \displaystyle\int\limits_{\partial B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\, d\gamma(z)$, then from (59) we get $I_{1,2} = \displaystyle\int\limits_{\varepsilon(t)}^{\min(\rho(t),\rho_0(t))} \frac{1}{R^{d-1}} \frac{dF_t}{dR}(R)\, dR$ and by using an integration by parts we obtain

$$I_{1,2} = \frac{F_t(\min(\rho(t),\rho_0(t)))}{\min(\rho(t),\rho_0(t))^{d-1}} - \frac{F_t(\varepsilon(t))}{\varepsilon(t)^{d-1}} + (d-1) \int\limits_{\varepsilon(t)}^{\min(\rho(t),\rho_0(t))} \frac{F_t(R)}{R^d}\, dR. \tag{61}$$

From (60), we notice that $\pi(t) = \sup\limits_{0<R\le\rho_0(t)} \dfrac{F_t(R)}{R^{d-1}}$, then from (61) we deduce

$$I_{1,2} \le \pi(t) + (d-1)\pi(t) \int_{\varepsilon(t)}^{\min(\rho(t),\rho_0(t))} \frac{dR}{R}$$
$$= \pi(t)\left(1 + (d-1)\log\left(\frac{\min(\rho(t),\rho_0(t))}{\varepsilon(t)}\right)\right).$$

Therefore whatever the case we get

$$I_{1,2} \le \pi(t)\left(1 + (d-1)\log^+\left(\frac{\min(\rho(t),\rho_0(t))}{\varepsilon(t)}\right)\right). \tag{62}$$

Since $\min(\rho(t),\rho_0(t)) \le \rho(t)$ and thanks to (38) we get

$$\frac{\min(\rho(t),\rho_0(t))}{\varepsilon(t)} \le 4(d+1)c_d(T^*-t)\|\omega(t)\|_\infty$$
$$\le 4(T^*-t)\|\omega(t)\|_\infty.$$

Therefore from (62) we infer

$$I_{1,2} \le \pi(t)\left(1 + (d-1)\log^+\left(4(T^*-t)\|\omega(t)\|_\infty\right)\right). \tag{63}$$

Thanks to (55)–(58) and (63), we get

$$I_{1,1} \le \pi(t) + c_d\pi(t)(1 + (d-1)\log^+(4(T^*-t)\|\omega(t)\|_\infty)). \tag{64}$$

Then thanks to (64), from (54) we obtain

$$I_1 \le \mathbf{A}_d(t)(\Omega(t)\rho(t) + \pi(t) + c_d\pi(t)(1 + (d-1)\log^+(4(T^*-t)\|\omega(t)\|_\infty))). \tag{65}$$

By using the definition of the functions ρ and Ω, for which we have $\Omega(t)\rho(t) \le 4c_d(d+1)\pi(t)$, from (65) we deduce

$$\begin{aligned} I_1 &\le \mathbf{A}_d(t)\pi(t)(4c_d(d+1) + 1 + c_d(1 + (d-1)\log^+(4(T^*-t)\|\omega(t)\|_\infty))) \\ &\le 6\mathbf{A}_d(t)\pi(t)(1 + \log^+(4(T^*-t)\|\omega(t)\|_\infty)). \end{aligned} \tag{66}$$

For the term I_2 given by (52), after using the change of variables $z = x + y$, we decompose I_2 as the sum of two terms $I_{2,1}$ and $I_{2,2}$ defined by

$$I_{2,1} = c_d \int_{B(x,\rho_0(t))\cap B(x,\min(\rho(t),\rho_0(t)))^c\cap\mathcal{V}(t)^c} \mathbf{D}_d(\widehat{z-x}, \omega(z,t), \xi(x,t)) \frac{dz}{|z-x|^d} \tag{67}$$

and

$$I_{2,2} = c_d \int_{B(x,\rho_0(t))\cap B(x,\min(\rho(t),\rho_0(t)))^c\cap\mathcal{V}(t)} \mathbf{D}_d(\widehat{z-x}, \omega(z,t), \xi(x,t)) \frac{dz}{|z-x|^d}. \tag{68}$$

Let us estimate the terms $I_{2,1}$ and $I_{2,2}$. For this purpose, we introduce the function γ defined for all $t \in [0, T^*[$ by $\gamma(t) = \min\left(\frac{1}{4A_0(t)\Omega(t)(T^*-t)}, \rho_0(t)\right)$. From (67), we observe

$$\begin{aligned} I_{2,1} &\le c_d \int_{B(x,\rho_0(t))\cap\mathcal{V}(t)^c} \mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))^+ \frac{|\omega(z,t)|}{|z-x|^d}\,dz \\ &\le c_d\Omega(t) \int_{B(x,\rho_0(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))^+}{|z-x|^d}\,dz \\ &= c_d\Omega(t) \int_{B(x,\gamma(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))^+}{|z-x|^d}\,dz \\ &\quad + c_d\Omega(t) \int_{B(x,\rho_0(t))\cap B(x,\gamma(t))^c} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))^+}{|z-x|^d}\,dz. \end{aligned} \tag{69}$$

Furthermore, on one hand, by (33) we have for all $z \in B(x, \gamma(t))$,

$$\mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))^+ \le A_0(t)|z-x|,$$

and hence, we obtain

$$\begin{aligned}\Omega(t) \int\limits_{B(x,\gamma(t))} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))^+}{|z-x|^d} dz &\le A_0(t)\Omega(t) \int\limits_{B(x,\gamma(t))} \frac{dz}{|z-x|^{d-1}} \\ &= |B(0,1)|\, A_0(t)\Omega(t)\gamma(t) \\ &\le \frac{|B(0,1)|}{4(T^*-t)},\end{aligned} \tag{70}$$

where we have used the definition of the function γ. On the other hand, since $|\mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))| \le 1$, we get

$$\begin{aligned}&\int\limits_{B(x,\rho_0(t))\cap B(x,\gamma(t))^c} \frac{\mathbf{D}_d(\widehat{z-x}, \xi(z,t), \xi(x,t))^+}{|z-x|^d} dz \\ &\le \int\limits_{B(x,\rho_0(t))\cap B(x,\gamma(t))^c} \frac{dz}{|z-x|^d} \\ &= |B(0,1)| \int\limits_{\gamma(t)}^{\rho_0(t)} \frac{ds}{s} \\ &= |B(0,1)| \log\left(\frac{\rho_0(t)}{\gamma(t)}\right) \\ &\le |B(0,1)| \log^+\left(4\rho_0(t)A_0(t)\Omega(t)(T^*-t)\right),\end{aligned} \tag{71}$$

where we have used again the definition of the function γ. Owing to (70) and (71), from (69), we deduce

$$I_{2,1} \le c_d|B(0,1)|\left(\frac{1}{4(T^*-t)} + \Omega(t)\log^+\left(4\rho_0(t)A_0(t)\Omega(t)(T^*-t)\right)\right). \tag{72}$$

For the term $I_{2,2}$, if $\mathcal{V}(t) = \emptyset$ or if $\rho(t) \ge \rho_0(t)$ then from (68) we infer that $I_{2,2} = 0$. Let us assume now that $\mathcal{V}(t) \ne \emptyset$ and $\rho(t) < \rho_0(t)$. Then, from (68) we get

$$\begin{aligned}I_{2,2} &= c_d \int\limits_{\rho(t)}^{\rho_0(t)} \left(\int\limits_{\partial B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)| \frac{d\gamma(z)}{|z-x|^d}\right) dR \\ &= c_d \int\limits_{\rho(t)}^{\rho_0(t)} \frac{1}{R^d} \left(\int\limits_{\partial B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\, d\gamma(z)\right) dR.\end{aligned} \tag{73}$$

By using (60) and an integration by parts we obtain

$$I_{2,2} = c_d\left(\frac{F_t(\rho_0(t))}{\rho_0(t)^d} - \frac{F_t(\rho(t))}{\rho(t)^d} + d\int\limits_{\rho(t)}^{\rho_0(t)} \frac{F_t(R)}{R^{d+1}} dR\right). \tag{74}$$

Since $\pi(t) = \sup\limits_{0<R\le\rho_0(t)} \dfrac{F_t(R)}{R^{d-1}}$, then from (74) we deduce

$$I_{2,2} \le c_d\left(\frac{\pi(t)}{\rho_0(t)} + d\pi(t)\int\limits_{\rho(t)}^{\rho_0(t)} \frac{dR}{R^2}\right)$$

$$\le c_d(d+1)\frac{\pi(t)}{\rho(t)},$$

where we have used the fact that $\rho(t) < \rho_0(t)$. Therefore whatever the case, we obtain that

$$I_{2,2} \le c_d(d+1)\frac{\pi(t)}{\rho(t)}. \tag{75}$$

By using (38) the definition of the function ρ, from (75) we deduce

$$I_{2,2} \le \frac{1}{4(T^*-t)}. \tag{76}$$

Owing to (72) and (76), since also that $I_2 = I_{2,1} + I_{2,2}$ and $c_d|B(0,1)| \le 1$ we thus obtain

$$I_2 \le \frac{1}{2(T^*-t)} + \Omega(t)\log^+\left(4\rho_0(t)A_0(t)\Omega(t)(T^*-t)\right). \tag{77}$$

For the term I_3 given by (53), to obtain a precise non blowup criterion for 3D the Navier-Stokes, 3D Euler and 2D QG equations that could be used easily in numerical experiments, it was important to explicit the constant involved in the estimate of the term I_3. For this purpose, we deal first with the case of the 3D the Navier-Stokes and 3D Euler equations, then after we consider the case of the 2D QG equations.

In the case of the 3D Euler or 3D Navier-Stokes equations for which $d=3$, we get $\mathbf{D}_3(\hat{y}, \omega(x+y,t), \xi(x,t)) = (\hat{y}\cdot\xi(x,t))\det(\hat{y}, \omega(x+y,t), \xi(x,t))$. Since $\det(\hat{y}, \omega(x+y,t), \xi(x,t)) = (\xi(x,t)\times\hat{y})\cdot\omega(x+y,t)$ and $\omega(x+y,t) = \nabla_y \times u(x+y,t)$, we deduce

$$\mathbf{D}_3(\hat{y}, \omega(x+y,t), \xi(x,t)) = (\hat{y}\cdot\xi(x,t))(\xi(x,t)\times\hat{y})\cdot\nabla_y\times u(x+y,t).$$

Then, after using an integration by parts, from (53), we deduce,

$$\begin{aligned} I_3 = c_3 &\int\limits_{B(0,\rho_0(t))^c} \nabla_y \times \left(\frac{(\hat{y}\cdot\xi(x,t))(\xi(x,t)\times\hat{y})}{|y|^3}\right)\cdot u(x+y,t)\,dy \\ &- c_3 \int\limits_{\partial B(0,\rho_0(t))} \left(\frac{(\hat{y}\cdot\xi(x,t))(\xi(x,t)\times\hat{y})}{|y|^3}\right)\cdot\hat{y}\times u(x+y,t)\,d\gamma(y). \end{aligned} \tag{78}$$

After setting $\psi(y) \equiv \dfrac{(\hat{y}\cdot\xi(x,t))}{|y|^3}$ and $\mathbf{V}(y) \equiv (\xi(x,t)\times\hat{y})$, by using the following vectorial identity $\nabla\times(\psi\mathbf{V}) = \nabla\psi\ \times\mathbf{V} + (\nabla\times\mathbf{V})\psi$, we obtain after elementary computations, that for all $y \neq 0$,

$$\left|\nabla_y \times \left(\frac{(\hat{y}\cdot\xi(x,t))(\xi(x,t)\times\hat{y})}{|y|^3}\right)\right| \le \left|\nabla\left(\frac{\hat{y}}{|y|^3}\right)\right| + \frac{|\nabla_y\times(\xi(x,t)\times\hat{y})|}{|y|^3}.$$

We have $\left|\nabla\left(\dfrac{\hat{y}}{|y|^3}\right)\right| \le \dfrac{3}{|y|^4}$. Furthermore, we have $\nabla_y\times(\xi(x,t)\times\hat{y}) = (\nabla_y\cdot\hat{y})\xi(x,t) - (\xi(x,t)\cdot\nabla_y)\hat{y}$ and then we deduce $|\nabla_y\times(\xi(x,t)\times\hat{y})| \le |\nabla\cdot\hat{y}| + |\nabla\hat{y}| \le \dfrac{3}{|y|}$. After gathering these results, we obtain that for all $y\neq 0$,

$$\left|\nabla_y \times \left(\frac{(\hat{y}\cdot\xi(x,t))(\xi(x,t)\times\hat{y})}{|y|^3}\right)\right| \le \frac{6}{|y|^4}.$$

Therefore, from (78) we obtain

$$I_3 \le 6c_3 \int\limits_{B(0,\rho_0(t))^c} \frac{|u(x+y,t)|}{|y|^4}\,dy + c_3 \int\limits_{\partial B(0,\rho_0(t))} |u(x+y,t)|\frac{dy}{|y|^3}d\gamma(y). \tag{79}$$

In the case of 2D QG equations for which $d=2$, we get $\mathbf{D}_2(\hat{y}, \omega(x+y,t), \xi(x,t)) = (\hat{y}\cdot\xi(x,t)^\perp)(\omega(x+y,t)\cdot\xi(x,t)^\perp)$. Since $\omega(x+y,t) = \nabla_y^\perp u(x+y,t)$, we deduce

$$\mathbf{D}_2(\hat{y}, \omega(x+y,t), \xi(x,t)) = (\hat{y}\cdot\xi(x,t)^\perp)\xi(x,t)^\perp\cdot\nabla_y^\perp u(x+y,t).$$

Then, after using an integration by parts, from (53), we deduce,

$$\begin{aligned} I_3 = & -c_2 \int\limits_{B(0,\rho_0(t))^c} \nabla_y^{\perp} \cdot \left(\frac{(\hat{y}\cdot\xi(x,t)^{\perp})\xi(x,t)^{\perp}}{|y|^2} \right) u(x+y,t)\, dy \\ & + c_2 \int\limits_{\partial B(0,\rho_0(t))} \left(\frac{(\hat{y}\cdot\xi(x,t)^{\perp})\xi(x,t)^{\perp}}{|y|^2} \right) \cdot y^{\perp} u(x+y,t). \end{aligned} \tag{80}$$

After setting $\psi(y) \equiv \dfrac{(\hat{y}\cdot\xi(x,t)^{\perp})}{|y|^2}$ and $\mathbf{V}(y) \equiv \xi(x,t)^{\perp}$, by using the following vectorial identity $\mathrm{curl}(\psi\mathbf{V}) = \nabla^{\perp}\psi \cdot \mathbf{V} + \psi\, \mathrm{curl}\mathbf{V}$, we obtain after elementary computations, that for all $y \neq 0$,

$$\begin{aligned} \left| \nabla_y^{\perp} \left(\frac{(\hat{y}\cdot\xi(x,t)^{\perp})\xi(x,t)^{\perp}}{|y|^2} \right) \right| & \leq \left| \nabla \left(\frac{\hat{y}}{|y|^2} \right) \right| \\ & \leq \frac{2}{|y|^3}\,. \end{aligned}$$

Therefore, from (80) we obtain

$$I_3 \leq 2c_2 \int\limits_{B(0,\rho_0(t))^c} \frac{|u(x+y,t)|}{|y|^3}\, dy + c_2 \int\limits_{\partial B(0,\rho_0(t))} \frac{|u(x+y,t)|}{|y|^2}\, d\gamma(y). \tag{81}$$

Therefore, whatever the case considered, 3D Navier-Stokes, 3D Euler or 2D QG equations, from (79) and (81) we get

$$I_3 \leq d(d-1)c_d \int\limits_{B(0,\rho_0(t))^c} \frac{|u(x+y,t)|}{|y|^{d+1}}\, dy + c_d \int\limits_{\partial B(0,\rho_0(t))} \frac{|u(x+y,t)|}{|y|^d}\, d\gamma(y). \tag{82}$$

Then from (82), we obtain

$$\begin{aligned} I_3 & \leq c_d \|u(t)\|_\infty \left(d(d-1) \int\limits_{B(0,\rho_0(t))^c} \frac{dy}{|y|^{d+1}} + \int\limits_{\partial B(0,\rho_0(t))} \frac{d\gamma(y)}{|y|^d} \right) \\ & = c_d \|u(t)\|_\infty \left(d(d-1)|B(0,1)| \int_{\rho_0(t)}^{+\infty} \frac{ds}{s^2} + \frac{|\partial B(0,1)|}{\rho_0(t)} \right) \\ & \leq 9 \frac{\|u(t)\|_\infty}{\rho_0(t)}. \end{aligned} \tag{83}$$

Then, owing to (66), (77) and (83), from (50) we deduce that for any $t \in [0,T]$ and $x \in \Theta(t)$,

$$\begin{aligned} \alpha(x,t) \leq & 6\mathbf{A}_d(t)\pi(t)(1+\log^+(4(T^*-t)\|\omega(t)\|_\infty))) \\ & + \frac{1}{2(T^*-t)} + \Omega(t)\log^+\left(4\rho_0(t)A_0(t)\Omega(t)(T^*-t)\right) + 9\frac{\|u(t)\|_\infty}{\rho_0(t)}. \end{aligned} \tag{84}$$

By using (32), we deduce that for any $t \in [0,T]$ and $x \in \Theta(t)$,

$$\begin{aligned} \alpha(x,t) \leq & 6\mathbf{A}_d(t)\pi(t)(1+\log^+(4(T^*-t)\|\omega(t)\|_\infty))) \\ & + \frac{3}{4(T^*-t)} + \Omega(t)\log^+\left(4\rho_0(t)A_0(t)\Omega(t)(T^*-t)\right). \end{aligned} \tag{85}$$

Furthermore, thanks to (35) we get $(T^*-t)\Omega(t)\log^+(4\rho_0(t)\mathbf{A}_0(t)) \leq \dfrac{1}{8}$ and $(T^*-t)\Omega(t) < 1$ which implies that $\log^+((T^*-t)\Omega(t)) = 0$ and hence we obtain that for all $t \in [0,T^*[$,

$$
\begin{aligned}
&(T^*-t)\Omega(t)\log^+\big(4\rho_0(t)\mathbf{A}_0(t)\Omega(t)(T^*-t)\big)\\
&\quad\le (T^*-t)\Omega(t)(\log^+(4\rho_0(t)\mathbf{A}_0(t))+\log^+\big(\Omega(t)(T^*-t)\big))\\
&\quad=(T^*-t)\Omega(t)\log^+(4\rho_0(t)\mathbf{A}_0(t))\\
&\quad\le\frac{1}{8}.
\end{aligned}
$$

Therefore from (85), we deduce that for any $t\in[0,T]$ and $x\in\Theta(t)$,

$$
\alpha(x,t)\le 6\mathbf{A}_d(t)\pi(t)(1+\log^+(4(T^*-t)\|\omega(t)\|_\infty))+\frac{7}{8(T^*-t)}. \tag{86}
$$

Then from (49) we deduce

$$
\frac{d}{dt}\|\omega(t)\|_\infty\le\left(6\mathbf{A}_d(t)\pi(t)(1+\log^+(4(T^*-t)\|\omega(t)\|_\infty))+\frac{7}{8(T^*-t)}\right)\|\omega(t)\|_\infty, \tag{87}
$$

which is valid for all $t\in[0,T]$ and $T<T^*$ and then inequality (87) is valid for all $t\in[0,T^*[$. Let $t_0\in[0,T^*[$ such that

$$
4(T^*-t_0)\le 1. \tag{88}
$$

Then we get that for all $t\in[t_0,T^*[$, $4(T^*-t)\le 1$ and hence

$$
\begin{aligned}
\log^+(4(T^*-t)\|\omega(t)\|_\infty))&\le\log^+(4(T^*-t))+\log^+\|\omega(t)\|_\infty\\
&=\log^+\|\omega(t)\|_\infty.
\end{aligned} \tag{89}
$$

Owing to (89), from (87) we deduce that for all $t\in[t_0,T^*[$

$$
\frac{d}{dt}\|\omega(t)\|_\infty\le\left(6\mathbf{A}_d(t)\pi(t)(1+\log^+\|\omega(t)\|_\infty)+\frac{7}{8(T^*-t)}\right)\|\omega(t)\|_\infty. \tag{90}
$$

Thanks to Grönwall inequality, from (90) we deduce that for all $t\in[t_0,T^*[$,

$$
\begin{aligned}
\|\omega(t)\|_\infty&\le\|\omega(t_0)\|_\infty e^{\int_{t_0}^t\left(6\mathbf{A}_d(\tau)\pi(\tau)(1+\log^+\|\omega(\tau)\|_\infty)+\frac{7}{8(T^*-\tau)}\right)d\tau}\\
&=\left(\frac{T^*-t_0}{T^*-t}\right)^{\frac{7}{8}}\|\omega(t_0)\|_\infty e^{\int_{t_0}^t 6\mathbf{A}_d(\tau)\pi(\tau)(1+\log^+\|\omega(\tau)\|_\infty)\,d\tau}.
\end{aligned} \tag{91}
$$

Since the function $z\mapsto\log^+(z)$ is non-decreasing on $]0,+\infty[$, then after applying the function $1+\log^+$ to the inequality (91) and using the fact that $\log^+(ab)\le\log^+a+\log^+b$, we thus obtain that for all $t\in[t_0,T^*[$

$$
\begin{aligned}
1+\log^+\|\omega(t)\|_\infty&\le 1+\log\left(\left(\frac{T^*-t_0}{T^*-t}\right)^{\frac{7}{8}}\right)+\log^+\|\omega(t_0)\|_\infty\\
&\qquad+6\int_{t_0}^t\mathbf{A}_d(\tau)\pi(\tau)(1+\log^+\|\omega(\tau)\|_\infty)\,d\tau.
\end{aligned} \tag{92}
$$

Since the function $t\mapsto\log\left(\left(\frac{T^*-t_0}{T^*-t}\right)^{\frac{7}{8}}\right)$ is increasing over $[t_0,T^*[$ then thanks to Grönwall Lemma, by (92) we deduce that for all $t\in[t_0,T^*[$,

$$
1+\log^+\|\omega(t)\|_\infty\le\left(\log\left(\left(\frac{T^*-t_0}{T^*-t}\right)^{\frac{7}{8}}\right)+1+\log^+\|\omega(t_0)\|_\infty\right)e^{6\int_{t_0}^t\mathbf{A}_d(\tau)\pi(\tau)\,d\tau}. \tag{93}
$$

Since $\log\|\omega(t)\|_\infty\le 1+\log^+\|\omega(t)\|_\infty$, then from (93) we infer that for all $t\in[t_0,T^*[$,

$$\|\omega(t)\|_\infty \le \exp\left(\left(\log\left(\left(\frac{T^*-t_0}{T^*-t}\right)^{\frac{7}{8}}\right)+1+\log^+\|\omega(t_0)\|_\infty\right)e^{6\int_{t_0}^t \mathbf{A}_d(\tau)\pi(\tau)\,d\tau}\right)$$
$$=\exp\left(\left(1+\log^+\|\omega(t_0)\|_\infty\right)e^{6\int_{t_0}^t \mathbf{A}_d(\tau)\pi(\tau)\,d\tau}\right)\left(\frac{T^*-t_0}{T^*-t}\right)^{\frac{7}{8}e^{6\int_{t_0}^t \mathbf{A}_d(\tau)\pi(\tau)\,d\tau}}. \tag{94}$$

Let us assume that there exists $t_1 \in [0, T^*[$ such that $\int_{t_1}^{T^*} \mathbf{A}_d(\tau)\pi(\tau)\,d\tau < +\infty$. Then, in addition of (88), we choose $t_0 \in [t_1, T^*[$ such that $M_{t_0} \overset{\text{def}}{=} \int_{t_0}^{T^*} \mathbf{A}_d(\tau)\pi(\tau)\,d\tau < \frac{1}{6}\log\left(\frac{8}{7}\right)$. We thus get for all $t \in [t_0, T^*[$,

$$\frac{7}{8}e^{6\int_{t_0}^t \mathbf{A}_d(\tau)\pi(\tau)\,d\tau} \le \frac{7}{8}e^{6M_{t_0}} < 1.$$

Therefore with $\eta_{t_0} \overset{\text{def}}{=} \frac{7}{8}e^{6M_{t_0}}$, from (94) we deduce that for all $t \in [t_0, T^*[$

$$\|\omega(t)\|_\infty \le \exp\left(\left(1+\log^+\|\omega(t_0)\|_\infty\right)e^{6M_{t_0}}\right)\left(\frac{T^*-t_0}{T^*-t}\right)^{\eta_{t_0}}. \tag{95}$$

Since $\eta_{t_0} < 1$, from (95) we thus deduce that $\int_{t_0}^{T^*} \|\omega(t)\|_\infty\,dt < +\infty$. Since $u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d))$ and thanks to the Sobolev embedding $H^r(\mathbb{R}^d) \hookrightarrow BC^3(\mathbb{R}^d)$ due to $r > \dfrac{d}{2}+3$, we infer that $\omega \in C([0, T^*[; BC^2(\mathbb{R}^d))$ which implies that $\int_0^{t_0} \|\omega(t)\|_\infty\,dt < +\infty$. Therefore we deduce that $\int_0^{T^*} \|\omega(t)\|_\infty\,dt < +\infty$. If u blows up at the finite time T^* then thanks to (20) and (23) we have $\int_0^{T^*} \|\omega(t)\|_\infty\,dt = +\infty$ which leads to a contradiction. Then, we deduce that u cannot blow up at the time T^* which concludes the first part of proof.

Thanks to (44), we have already $\nabla|\omega|(x,t) = 0$. Since $\nabla\cdot\omega = 0$ and $\omega = |\omega|\xi$, then we get

$$0 = \nabla\cdot\omega = |\omega|\nabla\cdot\xi + \xi\cdot\nabla|\omega|. \tag{96}$$

However, for all $x \in \Theta(t)$, $|\omega(x,t)| = \|\omega(t)\|_\infty > 0$ and from (44), we have $\nabla|\omega|(x,t) = 0$. Therefore, from (96), we deduce that for all $t \in [0, T]$ and $x \in \Theta(t)$,

$$\nabla\cdot\xi(x,t) = 0, \tag{97}$$

which completes the proof. □

6. No blow up in finite time for numerical experiments

In this section, we show the non-blowup in finite time of the solutions of the 3D Euler equations in the numerical experiments considered these last years.

First, we emphasize that the singularity discovered in [2] which lies right on the boundary is not relevant in the case of the whole domain $\mathbb{R}^3$. Indeed recently, the authors found a convincing numerical evidence for a singular solution to the Euler equations in a fluid with periodic boundary condition along the axial direction and no-flow boundary condition on the solid wall [2] (see also [1]), for which the point of the potential singularity, which is also the point of the maximum vorticity, is always located at the solid boundary. However thanks to Theorem 5.1, we deduce that such singularity can not exist in the whole domain $\mathbb{R}^3$. Indeed, in the whole domain of $\mathbb{R}^3$ at any point of the maximum vorticity, $q_0 \in \mathbb{R}^3$, thanks to Theorem 5.1 we get $\nabla|\omega|(q_0,t) = 0$ for any time t before the alleged time of singularity T^*, then this result combined with the fact that the vorticity ω is a divergence-free vector field, yields to get $\nabla\cdot\xi(q_0,t) = 0$ in Theorem 5.1. However in [2], the presence of a solid boundary and the fact that q_0 the point of the maximum vorticity is always located on the solid boundary, prevent to get $\nabla|\omega|(q_0,t) = 0$ and this allows to get $\nabla\cdot\xi(q_0,t) \sim (T^*-t)^{-2.9165} \neq 0$ as it is observed in their numerical test. This latter is the main element used to invalidate the Deng-Hou-Yu non-blowup criterion [42,64].

There have been many computational attempts to find finite-time singularities of the 3D Euler and Navier-Stokes equations: see, e.g. [90–92,84,93,94,69]. One example that has been studied extensively in these numerical investigations is the interaction of two perturbed antiparallel vortex tubes. All the subsequent calculations assumed an anti-parallel geometry, for which there are two symmetry planes. One in $y-z$ is between the vortices and was called the 'dividing plane'. The other in $x-z$ is at the position of maximum perturbation and was called the 'symmetry plane'. The difficulty faced in each computational attempts cited was to find a better initial condition within this geometry (see [95]). From these computational attempts, a numerical controversy takes place around the question to know whether or not there is finite-time blow-up of the solutions of Euler equations (see [95]).

In this section, we propose an answer to this controversy by using our Theorem 5.1. By using the anisotropic structure of regions of high vorticity described in [69,70], we show straightforward thanks to our Theorem 5.1 that the solutions of Euler equations cannot blow up in finite time in these numerical experiments [69–71].

For this purpose, we give a first bound of the function π defined by (37) in the following Lemma. The bound given in Lemma 6.1 of the function π is not a sharp bound but obtained without assumptions.

Lemma 6.1. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2}+3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0,T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0,T^*[; H^{r-2}(\mathbb{R}^d)).$$

Under the definitions (32)–(37) in the Theorem 5.1, we have the following estimate: for all $t \in [0,T^*[$

$$\pi(t) \le 3\|\omega(t)\|_\infty \sup_{x\in\Theta(t)} |\mathcal{V}(t)\cap B(x,\rho_0(t))|^{\frac{1}{d}}.$$

Proof. For any $t \in [0,T^*[$, $x \in \Theta(t)$ and $0 < R \le \rho_0(t)$, we get

$$\frac{1}{R^{d-1}} \int\limits_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz \le \frac{1}{R^{d-1}}\|\omega(t)\|_\infty |\mathcal{V}(t)\cap B(x,R)|.$$

Furthermore, we have

$$\begin{aligned}
|\mathcal{V}(t)\cap B(x,R)| &= |\mathcal{V}(t)\cap B(x,R)|^{\frac{1}{d}}|\mathcal{V}(t)\cap B(x,R)|^{\frac{d-1}{d}}\\
&\le |\mathcal{V}(t)\cap B(x,R)|^{\frac{1}{d}}|B(x,R)|^{\frac{d-1}{d}}\\
&= |B(0,1)|^{\frac{d-1}{d}}|\mathcal{V}(t)\cap B(x,R)|^{\frac{1}{d}}R^{d-1}\\
&\le 3|\mathcal{V}(t)\cap B(x,R)|^{\frac{1}{d}}R^{d-1}
\end{aligned}$$

where we have used the fact that $|B(0,1)|^{\frac{2}{3}} = \left(\frac{4\pi}{3}\right)^{\frac{2}{3}}$ if $d=3$ or $|B(0,1)|^{\frac{1}{2}} = \pi^{\frac{1}{2}}$ if $d=2$.

Then, we deduce for that any $t \in [0,T^*[$, $x \in \Theta(t)$ and $0 < R \le \rho_0(t)$,

$$\frac{1}{R^{d-1}} \int\limits_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz \le 3\|\omega(t)\|_\infty |\mathcal{V}(t)\cap B(x,R)|^{\frac{1}{d}}. \tag{98}$$

Owing to (98), we thus conclude the proof. □

Now, we can show straightforward thanks to Theorem 5.1 and Lemma 6.1 that the solutions of Euler equations cannot blow up in finite time in the numerical experiments [69–71].

For this purpose, we recall that in the numerical experiments [69,70], the author show that the blow-up rates at some time T^* the alleged time of singularity, to be considered for $\|\omega(t)\|_\infty$, $\|u(t)\|_\infty$ and $\|\nabla\xi(t)\|_\infty$ in [69,70] are

$$\|\omega(t)\|_\infty \sim (T^*-t)^{-1},\ \|u(t)\|_\infty \sim (T^*-t)^{-\frac{1}{2}} \text{ and } \|\nabla\xi(t)\|_\infty \sim (T^*-t)^{-\frac{1}{2}}, \tag{99}$$

for time $t \in [t_0,T^*[$ with $t_0 \in [0,T^*[$ sufficiently close to T^*. Moreover for time $t \in [t_0,T^*[$ with t_0 sufficiently close to T^*, the author showed that the support of the maximum vorticity

$$\mathcal{E}(t) = \{x \in \mathbb{R}^3, |\omega(x,t)| \sim \|\omega(t)\|_\infty\}$$

is characterized by two length scales $(T^* - t)$ and $(T^* - t)^{\frac{1}{2}}$ and its volume is bounded by

$$|\mathcal{E}(t)| \lesssim (T^* - t)^2. \tag{100}$$

Thanks to the blow-up rates (99), from (32) we get that for all $t \in [t_0, T^*[$

$$\rho_0(t) \sim (T^* - t)^{\frac{1}{2}}.$$

From (33), thanks again to (99) we have that for all $t \in [t_0, T^*[$

$$\begin{aligned}\mathbf{A}_0(t) &\le \sup_{x\in\Theta(t)} \|\nabla\xi(t)\|_{L^\infty(B(x,\rho_0(t)))} \\ &\lesssim (T^* - t)^{-\frac{1}{2}}.\end{aligned}$$

Then we deduce that for all $t \in [t_0, T^*[$

$$\mathbf{A}_0(t)\rho_0(t) \lesssim 1. \tag{101}$$

Therefore thanks to (101), from the definition (35) of the function Ω, we deduce for all $t \in [t_0, T^*[$

$$\Omega(t) \gtrsim (T^* - t)^{-1}. \tag{102}$$

Owing to (102) and since for all $t \in [t_0, T^*[$, $\|\omega(t)\|_\infty \sim (T^* - t)^{-1}$ thanks to (99), for the set $\mathcal{V}(t)$ defined by (36) we deduce that for all $t \in [t_0, T^*[$

$$\mathcal{V}(t) = \{x \in \mathbb{R}^3, |\omega(x,t)| \sim \|\omega(t)\|_\infty\}.$$

Then thanks to (100) we get that for all $t \in [t_0, T^*[$

$$|\mathcal{V}(t)| \lesssim (T^* - t)^2. \tag{103}$$

Thanks to Lemma 6.1, inequality (103) and the fact that $\|\omega(t)\|_\infty \sim (T^* - t)^{-1}$ thanks to (99), for the function π defined by (37), we get that for all $t \in [t_0, T^*[$

$$\pi(t) \lesssim (T^* - t)^{-\frac{1}{3}}. \tag{104}$$

Furthermore, thanks to (99), for the function $\mathbf{A}_3$ defined by (39) for $d = 3$, we get that for all $t \in [t_0, T^*[$

$$\mathbf{A}_3(t) \le \|\nabla\xi(t)\|_\infty \lesssim (T^* - t)^{-\frac{1}{2}}. \tag{105}$$

Owing to (104) and (105) we deduce that for all $t \in [t_0, T^*[$

$$\mathbf{A}_3(t)\pi(t) \lesssim (T^* - t)^{-\frac{5}{6}}. \tag{106}$$

Then, we deduce

$$\int_{t_0}^{T^*} \mathbf{A}_3(t)\pi(t) \lesssim (T^* - t_0)^{\frac{1}{6}} < +\infty. \tag{107}$$

Therefore, thanks to (107) and Theorem 5.1, we deduce that the solutions of the Euler equations considered for the numerical experiments [69,70] cannot blow-up in finite time at the alleged time of singularity T^*. If one considers the plausible scenario of blow up proposed in [5], one observed that we get also the blow-up rates (99) and the estimate (100) (see [96, section 4]), hence the potential mechanism proposed for the blow-up in finite time of solutions of Euler equations in [5] cannot in fact lead to the blow-up in finite time of the solutions of Euler equations.

7. Toward the non blowup in finite time of the solutions

In this section, under mild assumptions deriving from the structure of the regions of high vorticity, we obtain the non-blowup in finite time at some time T^* of the solutions of 2D QG, 3D Euler and 3D Navier-Stokes equations in the case where

$$\|\nabla\xi(t)\|_\infty \sim (T^* - t)^{-\gamma_\xi},\ 0 \le \gamma_\xi < 1.$$

In the previous section, we have outlined that the estimate obtained in Lemma 6.1 for the function π defined by (37) is not sharp, then in the subsection 7.2 we propose a better estimate for the function π and go further in the non blow-up criteria. However, before to deal with new non blow-up criteria in the subsection 7.2 we need to introduce in the subsection 7.1 the Lagrangian flow map X and the definitions of vortex lines and vortex tubes in order to justify the assumption (121) used in Proposition 7.1 and for their use in Lemmata 7.2, 7.3, 7.4 and in the Proposition 7.2.

7.1. Lagrangian flow map, vortex lines and vortex tubes

Let $d \in \{2, 3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \dfrac{d}{2} + 2$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-1}(\mathbb{R}^d)). \tag{108}$$

Solutions in this class exist thanks to section 3. We set $\boldsymbol{v} = u$ in the case of 3D Euler equations and $\boldsymbol{v} = R^\perp u$ in the case of 2D QG equation.

Owing to $u \in C([0, T^*[, H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[, H^{r-1}(\mathbb{R}^d))$ with $r > \dfrac{d}{2} + 2$ and thanks to the L^2-boundedness of the Riesz transforms, we infer that $\boldsymbol{v} \in C([0, T^*[, H^r(\mathbb{R}^d)) \cap C^1([0, T^*[, H^{r-1}(\mathbb{R}^d))$ with $r > \dfrac{d}{2} + 2$. Then by using the Sobolev embedding, $H^m(\mathbb{R}^d) \hookrightarrow BC^n(\mathbb{R}^d)$ with $n = [m - \frac{d}{2}]$ and $m > \frac{d}{2}$, we deduce that for any $0 < T < T^*$,

$$u \in BC^1(\mathbb{R}^d \times [0, T]),\ \nabla u \in C([0, T^*[; BC^1(\mathbb{R}^d)) \tag{109}$$

and

$$\boldsymbol{v} \in BC^1(\mathbb{R}^d \times [0, T]),\ \nabla \boldsymbol{v} \in C([0, T^*[; BC^1(\mathbb{R}^d)). \tag{110}$$

In Proposition 7.1, in the case of 3D Euler equations and 2D QG equations, we give an estimate of the function π defined by (37). For this purpose, we need to give the definition of a vortex line and recall some results about the Lagrangian flow map.

We thus introduce the flow map $X(\alpha, \tau, t)$ the particle path that passes by $\alpha \in \mathbb{R}^d$ at time $\tau \in [0, T^*[$. That is for $\tau \in [0, T^*[$ fixed, $X(\alpha, \tau, t)$ solves on $[0, T^*[$ (see chapter 4 in [97] for more details on the flow map X)

$$\begin{aligned} &\frac{\partial X(\alpha, \tau, t)}{\partial t} = \boldsymbol{v}(X(\alpha, \tau, t), t), \\ &X(\alpha, \tau, \tau) = \alpha \in \mathbb{R}^d, \end{aligned} \tag{111}$$

Thanks to Cauchy-Lipschitz Theorem (see Theorems 2.2 and 2.13 in [98]), for any $\alpha \in \mathbb{R}^d$ and $\tau \in [0, T^*[$ thanks to (110) we get that there exists an unique solution $X(\alpha, \tau, \cdot) \in C^1([0, T^*[)$ to equation (111). For all $t \in [0, T^*[$ and $\tau \in [0, T^*[$, the map $X(\cdot, \tau, t)$ defined by equation (111) is a volume preserving C^1-diffeomorphism from $\mathbb{R}^d$ on itself. Indeed thanks to (110) and the Theorems 2.2, 2.10 and 2.13 in [98], we deduce that for any $t \in [0, T^*[$ and $\tau \in [0, T^*[$, $X(\cdot, \tau, t)$ is a C^1-diffeomorphism from $\mathbb{R}^d$ on itself with inverse $X(\cdot, t, \tau)$, we notice also $X \in C^1(\mathbb{R}^d \times [0, T^*[\times [0, T^*[)$. Moreover for any $t \in [0, T^*[$ and $\tau \in [0, T^*[$, $X(\cdot, \tau, t)$ is a volume preserving mapping thanks to Proposition 1.4 in [97], for which we get

$$\det(\nabla_\alpha X(\alpha, \tau, t)) = 1. \tag{112}$$

Furthermore, we have for all $\tau \in [0, T^*[$, $t \in [0, T^*[$ and $\alpha \in \mathbb{R}^d$ (see Proposition 1.8 in [97] or Proposition page 24 in [89] for Euler equations and see [65] for 2D QG equation),

$$\omega(X(\alpha,\tau,t),t) = \nabla_\alpha X(\alpha,\tau,t)\omega(\alpha,\tau). \tag{113}$$

Recall that a vortex line in a fluid is an arc on an integral curve of the vorticity $\omega(x,t)$ for fixed t, and a vortex tube is a tubular neighbourhood in $\mathbb{R}^d$, $d \in \{2,$ arising as a union of vortex lines. In what follows, we give a parametrization of vortex lines and vortex tubes.

We set $\mathcal{O} \stackrel{\text{def}}{=} \{(x,\tau') \in \mathbb{R}^d \times [0,T^*[;\, |\omega(x,\tau')| > 0\}$ and for any $t \in [0,T^*[$, $\mathcal{O}(t) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d;\, |\omega(x,t)| > 0\}$. From (25), we get that for any $t \in [0,T^*[$, $\mathcal{O}(t)$ is nonempty. Thanks to (109), we get that ω is continuous on $\mathbb{R}^d \times [0,T^*[$ and then we deduce that $\mathcal{O}$ is an open subset of $\mathbb{R}^d \times [0,T^*[$ and also that for all $t \in [0,T^*[$, $\mathcal{O}(t)$ is an open subset of $\mathbb{R}^d$. Notice thanks again to (109) that ξ and $\nabla\xi$ are continuous on $\mathcal{O}$. Then, we get that for all $t \in [0,T^*[$, $\xi(\cdot,t) \in C^1(\mathcal{O}(t))$.

Then, for all $t \in [0,T^*[$ and $\alpha \in \mathcal{O}(t)$, we denote by $\mathbf{x}_t(\alpha,\cdot) : \mathfrak{J}_{\alpha,t} \longmapsto \mathbb{R}^d$ the vortex line that passes through α at the time t and defined by the ordinary differential equation:

$$\begin{aligned} &\frac{\partial \mathbf{x}_t(\alpha,s)}{\partial s} = \xi(\mathbf{x}_t(\alpha,s),t), \\ &\mathbf{x}_t(\alpha,0) = \alpha. \end{aligned} \tag{114}$$

The set $\mathfrak{J}_{\alpha,t} \subset \mathbb{R}$ not reduced to $\{0\}$ denotes the maximal interval of existence of the unique solution $\mathbf{x}_t(\alpha,\cdot)$ of (114), this is ensured thanks to Cauchy-Lipschitz Theorem (see e.g. Theorems 2.2 and 2.13 in [98]). For any $t \in [0,T^*[$, we introduce $U_t = \{(\alpha,s) \in \mathcal{O}(t) \times \mathbb{R};\ \mathbf{x}_t(\alpha,s) \in \mathcal{O}(t)\}$ the set of definition of the function $\mathbf{x}_t$. For any $t \in [0,T^*[$ since $\xi(\cdot,t) \in C^1(\mathcal{O}(t))$, then from Theorem 2.9 in [98], we get that $\mathbf{x}_t$ is continuous on U_t. We notice that $U_t = \mathbf{x}_t^{-1}(\mathcal{O}(t))$ and hence we obtain that U_t is an open subset of $\mathcal{O}(t) \times \mathbb{R}$.

From Theorem 2.10 in [98], we obtain that

$$\mathbf{x}_t \in C^1(U_t). \tag{115}$$

Any vortex tube $\mathfrak{T}$ at a time $t \in [0,T^*[$ can be defined as $\mathfrak{T} \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha,s);\, \alpha \in \mathcal{A}_0,\ s \in I_{\alpha,t} \subset \mathfrak{J}_{\alpha,t}\}$ where $\mathcal{A}_0$ is a bounded smooth surface (resp. curve) of $\mathbb{R}^3$ (resp. of $\mathbb{R}^2$) and for each $\alpha \in \mathcal{A}_0$, $I_{\alpha,t}$ is an interval of $\mathbb{R}$ containing 0.

7.2. Anisotropic structure for the improvement of non blow-up criteria

In this subsection, in Proposition 7.1 we propose to show that the function $\pi(t)$ defined by (37) involved Theorem 5.1 is bounded by $C\left(1+\log^+\left(\frac{\|\omega(t)\|_\infty}{\Omega(t)}\right)\right)$ by using assumptions related to the anisotropic scaling in the collapse of regions of high vorticity containing the positions of the maximum vorticity. In Proposition 7.2, in the case of the Euler and 2D QG equations, we improve logarithmically the result obtained in Proposition 7.1 by showing that the function π is bounded.

These results come from the special feature of the structure of regions of high vorticity surrounding the peak of vorticity $\{y \in \mathbb{R}^d;\, |\omega(y,t)| \gtrsim \|\omega(t)\|_\infty\}$ observed in the numerical experiments [70,69,74,75] and from analytical models [75, section 3], [97, sections 1.4 and 1.5], namely they are pancake-like structure characterized by two length scales whose one of it is bounded by $O\left(\frac{1}{\|\omega(t)\|_\infty}\right)$ and plays the role of the thickness of the pancake-like structure. This suggests that for any $t \in [t_0,T^*[$ with $t_0 \in [0,T^*[$, $x \in \Theta(t)$, $\lambda \geq \Omega(t)$ the set $\{y \in \mathbb{R}^d;\, |\omega(y,t)| \geq \lambda\}$ may be characterized by three length scales whose one of them is of order $\frac{1}{\lambda}$, where Ω and Θ are respectively defined by (35) and (34). Since for any $0 < R \leq \rho_0(t)$ the set $\mathcal{V}_{\lambda,R}(t) \stackrel{\text{def}}{=} \{y \in B(x,R);\, |\omega(y,t)| \geq \lambda\} \subset B(x,R)$, we thus expect that the set $\mathcal{V}_{\lambda,R}(t)$ may have two of its length scales of order R and the third one of order $\frac{1}{\lambda}$. Then we expect that for any $t \in [t_0,T^*[$, $x \in \Theta(t)$, $\lambda \geq \Omega(t)$ and $0 < R \leq \rho_0(t)$,

$$|\mathcal{V}_{\lambda,R}(t)| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}. \tag{116}$$

In Lemma 7.1 we give an argument in favour of the assumption (116) in the case of 3D Euler equations and 2D QG equation by using their Lagrangian structure. In Lemma 7.1, the property (P1) expresses the fact that we expect the length of any segment of a vortex line included in the structure $\mathcal{V}_{\lambda,R}(t)$ is bounded by $O(R)$, since $\mathcal{V}_{\lambda,R}(t) \subset B(x,R)$. Furthermore property (P2) expresses the pancake structure of regions of high vorticity observed in numerical experiments. Indeed for the case of 3D Euler equations, if one assumes that the set $\mathcal{V}^0_{\lambda,R}(t) \stackrel{\text{def}}{=} X^{-1}(\mathcal{V}_{\lambda,R}(t), t_0, t)$ is characterized by three length scales $\ell^0_1, \ell^0_2, \ell^0_3$ associated to three main directions orthogonal between them pairwise, then we should have for one of these length scales $\ell^0_1 \lesssim \ell^0_{\lambda,R}(t)$ or $\ell^0_2 \lesssim \ell^0_{\lambda,R}(t)$ or $\ell^0_3 \lesssim \ell^0_{\lambda,R}(t)$ (117). Let us say that $\ell^0_1 \lesssim \ell^0_{\lambda,R}(t)$. Assuming that during the time between t_0 and $t \in]t_0, T^*[$ the set $\mathcal{V}^0_{\lambda,R}(t)$ becomes a pancake-shaped structure, we thus expect that $\ell^0_2 \lesssim R$ or $\ell^0_3 \lesssim R$ since $\mathcal{V}_{\lambda,R}(t) \subset B(x,R)$. Let us say that $\ell^0_2 \lesssim R$. For the last length scale ℓ^0_3, we just expect that $\ell^0_3 = O(1)$.

In the case of 3D Euler equations, we thus expect that $|\mathcal{V}^0_{\lambda,R}(t_0)| \lesssim \ell^0_{\lambda,R}(t)R$.

In the case of 2D QG equations, we will have only the two length scales ℓ^0_1 and ℓ^0_3 and then we expect that $|\mathcal{V}^0_{\lambda,R}(t_0)| \lesssim \ell^0_{\lambda,R}(t)$. Then Lemma 7.1 gives an explanation of assumption (116).

Lemma 7.1. *Under the definitions (31)–(35) in the Theorem 5.1, we assume that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $0 < R \le \rho_0(t)$ and $\lambda \ge \Omega(t)$ the sets $\mathcal{V}_{\lambda,R}(t) \stackrel{\text{def}}{=} \{y \in B(x,R); |\omega(y,t)| \ge \lambda\}$ and $\mathcal{V}^0_{\lambda,R}(t) \stackrel{\text{def}}{=} X^{-1}(\mathcal{V}_{\lambda,R}(t), t_0, t)$ satisfy:*

(P1) $|L_t \cap \mathcal{V}_{\lambda,R}(t)| \lesssim R$ for any vortex line L_t at time t,
(P2) $|\mathcal{V}^0_{\lambda,R}(t)| \lesssim \ell^0_{\lambda,R}(t) R^{d-2}$ where

$$\ell^0_{\lambda,R}(t) \stackrel{\text{def}}{=} \sup_{L_{t_0} \subset \mathcal{T}(t_0)} |L_{t_0} \cap \mathcal{V}^0_{\lambda,R}(t)| \tag{117}$$

and $\mathcal{T}(t_0)$ denotes the set of all vortex lines L_{t_0} at time t_0.

Then, we get that for any $t \in [t_0, T^*[$

$$|\mathcal{V}_{\lambda,R}(t)| \lesssim \|\omega(t_0)\|_\infty \frac{R^{d-1}}{\lambda}.$$

Proof. Let us take $t \in [t_0, T^*[$. If $|\mathcal{V}_{\lambda,R}(t)| = 0$ the proof follows immediately. Then we assume that $|\mathcal{V}_{\lambda,R}(t)| > 0$. We have $|\mathcal{V}_{\lambda,R}(t)| = |\mathcal{V}^0_{\lambda,R}(t)|$ since $X(\cdot, t_0, t)$ is a volume preserving C^1-diffeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$, then we get $|\mathcal{V}^0_{\lambda,R}(t)| > 0$ and Property (P2) yields to

$$|\mathcal{V}_{\lambda,R}(t)| \lesssim \ell^0_{\lambda,R}(t) R^{d-2}. \tag{118}$$

Furthermore for any vortex line L_{t_0} at time t_0, we notice $X(L_{t_0} \cap \mathcal{V}^0_{\lambda,R}(t), t_0, t) = L_t \cap \mathcal{V}_{\lambda,R}(t)$ where L_t is the vortex line at time t defined by $L_t = X(L_{t_0}, t_0, t)$. Then, for any vortex line L_{t_0} at time t_0 such that $|L_{t_0} \cap \mathcal{V}^0_{\lambda,R}(t)| > 0$, by using the equation just below of (3.12) in [42] and the definition of the set $\mathcal{V}_{\lambda,R}(t)$, we obtain

$$\frac{|L_t \cap \mathcal{V}_{\lambda,R}(t)|}{|L_{t_0} \cap \mathcal{V}^0_{\lambda,R}(t)|} \ge \frac{\lambda}{\|\omega(t_0)\|_\infty}. \tag{119}$$

By using Property (P1), for any vortex line L_{t_0} at time t_0 such that $|L_{t_0} \cap \mathcal{V}^0_{\lambda,R}(t)| > 0$, we deduce from (119) that $\dfrac{R}{|L_{t_0} \cap \mathcal{V}^0_{\lambda,R}(t)|} \gtrsim \dfrac{\lambda}{\|\omega(t_0)\|_\infty}$ which implies

$$\frac{R}{\ell^0_{\lambda,R}(t)} \gtrsim \frac{\lambda}{\|\omega(t_0)\|_\infty}. \tag{120}$$

Therefore by using (120), from (118) we thus infer $|\mathcal{V}_{\lambda,R}(t)| \lesssim \|\omega(t_0)\|_\infty \dfrac{R^{d-1}}{\lambda}$, which concludes the proof. □

Thanks to assumption (116), we obtain the following Proposition 7.1.

Proposition 7.1. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-2}(\mathbb{R}^d)).$$

Under the definitions (32)–(37) in the Theorem 5.1, we assume that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$, we get that for all $\lambda \geq \Omega(t)$ and $0 < R \leq \rho_0(t)$

$$|\{y \in B(x, R); |\omega(y, t)| \geq \lambda\}| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}. \tag{121}$$

Then we get that for all $t \in [t_0, T^*[$

$$\pi(t) \lesssim_{t_0} 1 + \log^+\left(\frac{\|\omega(t)\|_\infty}{\Omega(t)}\right).$$

Proof. We have for all $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$

$$\begin{aligned}
\int\limits_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz &= \int\limits_{B(x,R)\cap\mathcal{V}(t)} \int\limits_0^{|\omega(z,t)|} d\lambda\,dz \\
&= \Omega(t)|B(x,R)\cap\mathcal{V}(t)| + \int\limits_{B(x,R)\cap\mathcal{V}(t)} \int\limits_{\Omega(t)}^{|\omega(z,t)|} d\lambda\,dz \\
&= \Omega(t)|B(x,R)\cap\mathcal{V}(t)| + \int\limits_{\{z\in B(x,R),\ \Omega(t)<\lambda<|\omega(z,t)|\}} d\lambda dz \\
&= \Omega(t)|B(x,R)\cap\mathcal{V}(t)| \\
&\quad + \int\limits_{[\Omega(t),\|\omega(t)\|_\infty]} |\{z \in B(x,R);\ |\omega(z,t)| > \lambda\}|\,d\lambda,
\end{aligned}$$

where we have used Fubini-Tonelli Theorem. Thanks to (121) we deduce that for all $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \leq \rho_0(t)$

$$\begin{aligned}
\int\limits_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz &\lesssim_{t_0} R^{d-1}\left(1 + \int\limits_{[\Omega(t),\|\omega(t)\|_\infty]} \frac{d\lambda}{\lambda}\right) \\
&= R^{d-1}\left(1 + \log^+\left(\frac{\|\omega(t)\|_\infty}{\Omega(t)}\right)\right).
\end{aligned} \tag{122}$$

Owing to (122), we thus conclude the proof. □

Remark 7.1. The analysis led in [72,73] for the study of collapse of vortex lines and agrees with numerical experiments [74,75] suggests that the thickness of the regions of high vorticity $\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}$ is $\frac{1}{\lambda^{\frac{3}{2}}}$ and since these regions are included in the ball $B(x,R)$, we expect that $|\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}| \lesssim_{t_0} \frac{R^{d-1}}{\lambda^{\frac{3}{2}}}$. Then under this assumption and by using the same arguments as previously, we obtain

$$\pi(t) \lesssim \Omega(t)^{-\frac{1}{2}}.$$

Thanks to Theorem 5.1 and Proposition 7.1 we obtain Theorem 7.1.

Theorem 7.1. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-2}(\mathbb{R}^d)).$$

Under the definitions (32)–(35) in the Theorem 5.1, we assume that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$, we get that for all $\lambda \geq \Omega(t)$ and $0 < R \leq \rho_0(t)$

$$|\{y \in B(x,R); |\omega(y,t)| \geq \lambda\}| \lesssim_{t_0} \frac{R^{d-1}}{\lambda}.$$

Then if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t)\left(1 + \log^+\left(\frac{\|\omega(t)\|_\infty}{\Omega(t)}\right)\right) dt < +\infty,$$

then the solution u cannot blowup at the finite time T^* where

$$\mathbf{A}_d(t) \stackrel{\text{def}}{=} \sup_{x\in\Theta(t)} \sup_{y\in B(0,\rho(t))\setminus\{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y,t), \xi(x,t))^+}{|y|}$$

$$\rho(t) = O\left((T^* - t)\left(1 + \log^+\left(\frac{\|\omega(t)\|_\infty}{\Omega(t)}\right)\right)\right).$$

Remark 7.2. Under the considerations of Remark 7.1, the non blow-up of the solutions of Euler equations is obtained if there exists $t_1 \in [t_0, T^*[$ such that

$$\int_{t_1}^{T^*} \mathbf{A}_d(t)\Omega(t)^{-\frac{1}{2}}\, dt < +\infty.$$

Now, in the case of Euler equations and 2D QG equations by using their Lagrangian formulation, after a fine and sharp analysis of the expression of π (37) we go further in the non blow-up criteria by showing in Proposition 7.2 under mild assumptions based on the anisotropic structure of regions of high vorticity, that $\pi(t) = O(1)$. For this purpose, we need the Lemmata 7.2, 7.3 and 7.4.

Lemma 7.2. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-2}(\mathbb{R}^d)).$$

Let $\mathcal{A}_0$ be a smooth surface of $\mathbb{R}^3$ with boundary if $d = 3$ or a curve of $\mathbb{R}^2$ if $d = 2$. For any $t_0 \in [0, T^*[$, let $\mathcal{A}(t_0, t) \stackrel{\text{def}}{=} X(\mathcal{A}_0, t_0, t)$ be the evolution of $\mathcal{A}_0$ through the flow map X from the time t_0 to t, for any $t \in [0, T^*[$. For any $t_0 \in [0, T^*[$ and $t \in [0, T^*[$, let

$$\Gamma(t_0, t) \stackrel{\text{def}}{=} \int_{\mathcal{A}(t_0,t)} |\omega(y,t) \cdot \boldsymbol{n}_t(y)|\, d\sigma(y),$$

where $\boldsymbol{n}_t(\cdot)$ denotes a unit normal vector of $\mathcal{A}(t_0, t)$.

Then for any $t_0 \in [0, T^*[$ we get that $\Gamma(t_0, \cdot)$ is constant over $[0, T^*[$, that is to say for all $t \in [t_0, T^*[$,

$$\Gamma(t_0, t) = \Gamma(t_0, t_0).$$

Proof. Thanks to Lemma 5 and Remark 3 of [99] (see also (4.9) chapter 9 in [100]) and Lemma 7 of [99], we infer that for any $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\int_{\mathcal{A}(t_0,t)} |\omega(y,t)\cdot \boldsymbol{n}_t(y)|\, d\sigma(y) = \int_{\mathcal{A}_0} |\omega(X(\alpha,t_0,t),t)\cdot (\nabla X(\alpha,t_0,t))^{-T}\boldsymbol{n}(\alpha)|\, d\sigma(\alpha),$$

where $\boldsymbol{n}$ is a unit normal vector of $\mathcal{A}_0$. Thanks to (113), we infer that for any $\alpha \in \mathcal{A}_0$, $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\omega(X(\alpha,t_0,t),t)\cdot (\nabla X(\alpha,t_0,t))^{-T}\boldsymbol{n}(\alpha) = \omega(\alpha,t_0)\cdot \boldsymbol{n}(\alpha).$$

Therefore, we thus obtain that for any $t_0 \in [0, T^*[$ and $t \in [0, T^*[$

$$\int_{\mathcal{A}(t_0,t)} |\omega(y,t)\cdot \boldsymbol{n}_t(y)|\, d\sigma(y) = \int_{\mathcal{A}_0} |\omega(\alpha,t_0)\cdot \boldsymbol{n}(\alpha)|\, d\sigma(\alpha),$$

which concludes the proof. □

Lemma 7.3. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2}+3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0,T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0,T^*[; H^{r-2}(\mathbb{R}^d)).$$

Let $t \in [0, T^*[$ and $\mathfrak{T}_t$ a vortex tube at this time. Let $\mathcal{A}_t$ and $\mathcal{B}_t$ be two connected smooth orientable surfaces of $\mathbb{R}^3$ (resp. curves of $\mathbb{R}^2$ if $d = 2$) with boundary such their boundary encircle the vortex tube $\mathfrak{T}_t$ and such that any vortex line of the vortex tube $\mathfrak{T}_t$ intersects both $\mathcal{A}_t$ and $\mathcal{B}_t$ once each of them.

Then we get

$$\int_{\mathcal{A}_t} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y) = \int_{\mathcal{B}_t} |\omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y)| d\tilde{\sigma}(y),$$

where $\boldsymbol{n}_t$ and $\tilde{\boldsymbol{n}}_t$ are respectively the unit normal vector varying smoothly on the surfaces (resp. curves if $d = 2$) $\mathcal{A}_t$ and $\mathcal{B}_t$, oriented to be outward to the portion of the tube $\mathfrak{T}_t$ delimited by $\mathcal{A}_t$ and $\mathcal{B}_t$.

Proof. For any $x \in \mathcal{A}_t$, we denote by $\mathfrak{T}_t(x)$ the vortex line passing through x at time t, we get $\mathfrak{T}_t(x) \subset \mathfrak{T}_t$ and there exists an unique $y_{x,t} \in \mathcal{B}_t$ such that $\mathfrak{T}_t(x) \cap \mathcal{B}_t = \{y_{x,t}\}$. We thus introduce the function $\boldsymbol{\Phi}_t$ defined from $\mathcal{A}_t$ to $\mathcal{B}_t$ for all $x \in \mathcal{A}_t$ by $\boldsymbol{\Phi}_t(x) = y_{x,t}$. Since any vortex line of the vortex tube $\mathfrak{T}_t$ intersects both $\mathcal{A}_t$ and $\mathcal{B}_t$ once each of them and since also $\mathcal{A}_t$ and $\mathcal{B}_t$ are smooth surfaces (smooth curves if $d = 2$), we infer thanks also to (115) that the function $\boldsymbol{\Phi}_t$ is a homeomorphism from $\mathcal{A}_t$ to $\mathcal{B}_t$.

We introduce the pairwise disjoints subsets of $\mathcal{A}_t$, namely $\mathcal{A}_t^+ \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y,t)\cdot \boldsymbol{n}_t(y) > 0\}$, $\mathcal{A}_t^- \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y,t)\cdot \boldsymbol{n}_t(y) < 0\}$ and $\mathcal{A}_t^0 \stackrel{\text{def}}{=} \{y \in \mathcal{A}_t; \omega(y,t)\cdot \boldsymbol{n}_t(y) = 0\}$. By the Sobolev embedding $H^{r-1}(\mathbb{R}^d) \hookrightarrow BC^{m_r}(\mathbb{R}^d)$, $m_r = [r - 1 - \frac{d}{2}] \geq 2$, we get $\omega(t) \in BC^{m_r}(\mathbb{R}^d)$. Then $\mathcal{A}_t^+$ and $\mathcal{A}_t^-$ are open subsets of the surface (curve if $d = 2$) $\mathcal{A}_t$ and thus they are also smooth surfaces (smooth curves if $d = 2$). On one hand, we have

$$\begin{aligned}\int_{\mathcal{A}_t} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y) &= \int_{\mathcal{A}_t^+} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y) + \int_{\mathcal{A}_t^-} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y) \\ &\quad + \int_{\mathcal{A}_t^0} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y) \\ &= \int_{\mathcal{A}_t^+} \omega(y,t)\cdot \boldsymbol{n}_t(y) d\sigma(y) - \int_{\mathcal{A}_t^-} \omega(y,t)\cdot \boldsymbol{n}_t(y) d\sigma(y),\end{aligned} \tag{123}$$

where we have used the definition of the sets $\mathcal{A}_t^+$, $\mathcal{A}_t^-$ and $\mathcal{A}_t^0$. Since $\boldsymbol{\Phi}_t$ is a homeomorphism from $\mathcal{A}_t$ to $\mathcal{B}_t$, we get that $\boldsymbol{\Phi}_t(\mathcal{A}_t^+) \subset \mathcal{B}_t$, $\boldsymbol{\Phi}_t(\mathcal{A}_t^-) \subset \mathcal{B}_t$ and $\boldsymbol{\Phi}_t(\mathcal{A}_t^+) \cap \boldsymbol{\Phi}_t(\mathcal{A}_t^-) = \emptyset$. On the other hand, thanks to Helmholtz's first vortex Theorem (see e.g. [101, chapter 2]), we have

$$\int_{\mathcal{A}_t^+} \omega(y,t)\cdot \boldsymbol{n}_t(y) d\sigma(y) = -\int_{\boldsymbol{\Phi}_t(\mathcal{A}_t^+)} \omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y) d\tilde{\sigma}(y), \tag{124}$$

and

$$\int_{\mathcal{A}_t^-} \omega(y,t)\cdot \boldsymbol{n}_t(y) d\sigma(y) = -\int_{\boldsymbol{\Phi}_t(\mathcal{A}_t^-)} \omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y) d\tilde{\sigma}(y). \tag{125}$$

Then owing to (124) and (125), from (123) we deduce

$$\begin{aligned}\int_{\mathcal{A}_t} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y) = &-\int_{\boldsymbol{\Phi}_t(\mathcal{A}_t^+)} \omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y) d\tilde{\sigma}(y)\\ &+\int_{\boldsymbol{\Phi}_t(\mathcal{A}_t^-)} \omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y) d\tilde{\sigma}(y),\end{aligned}$$

which implies

$$\int_{\mathcal{A}_t} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y) \leq \int_{\mathcal{B}_t} |\omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y)| d\tilde{\sigma}(y). \tag{126}$$

It remains to show that

$$\int_{\mathcal{B}_t} |\omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y)| d\tilde{\sigma}(y) \leq \int_{\mathcal{A}_t} |\omega(y,t)\cdot \boldsymbol{n}_t(y)| d\sigma(y). \tag{127}$$

By introducing the pairwise disjoints subsets of $\mathcal{B}_t$, namely $\mathcal{B}_t^+ \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y) > 0\}$, $\mathcal{B}_t^- \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y) < 0\}$ and $\mathcal{B}_t^0 \stackrel{\text{def}}{=} \{y \in \mathcal{B}_t; \omega(y,t)\cdot \tilde{\boldsymbol{n}}_t(y) = 0\}$ and using the fact that $\boldsymbol{\Phi}_t^{-1}$ is a homeomorphism from $\mathcal{B}_t$ to $\mathcal{A}_t$, we deduce with the same arguments used to get (126), inequality (127). Then, owing to (126) and (127) we conclude the proof. □

Before to turn to the proof of Lemma 7.4, Proposition 7.2 and Theorem 7.2, we need to introduce some definitions. Let $r > \frac{d}{2} + 3$ and $T^* > 0$ be such that there exists an unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class

$$u \in C([0,T^*[; H_\sigma^r(\mathbb{R}^d)) \cap C^1([0,T^*[; H^{r-2}(\mathbb{R}^d)).$$

For any $t \in [0,T^*[$ and any vortex tube $\mathfrak{T}_t$ at time t, we define by

$\mathcal{S}(\mathfrak{T}_t)$ the set of the connected smooth orientable surfaces of $\mathbb{R}^3$ (curves of $\mathbb{R}^2$ if $d=2$) with boundary that is intersected only once by any vortex line of $\mathfrak{T}_t$ and such that their boundary encircle the vortex tube $\mathfrak{T}_t$.

We define also the function $\Gamma_{\mathfrak{T}_t}$ defined from $\mathcal{S}(\mathfrak{T}_t)$ to $[0,+\infty[$ for all $\mathcal{A} \in \mathcal{S}(\mathfrak{T}_t)$ by

$$\Gamma_{\mathfrak{T}_t}(\mathcal{A}) \stackrel{\text{def}}{=} \int_{\mathcal{A}} |\omega(y,t)\cdot \boldsymbol{n}(y)| d\sigma(y). \tag{128}$$

Thanks to Lemma 7.3, we deduce that for any $t \in [0,T^*[$ and any vortex tube $\mathfrak{T}_t$ at time t

$$\Gamma_{\mathfrak{T}_t} \text{ is constant over } \mathcal{S}(\mathfrak{T}_t). \tag{129}$$

Owing to (129), for any $t \in [0,T^*[$ and any vortex tube $\mathfrak{T}_t$ at time t, we define $\boldsymbol{\Gamma}_{\text{abs}}(\mathfrak{T}_t)$ that we call the absolute strength of the vortex tube $\mathfrak{T}_t$ by

$$\mathbf{\Gamma}_{\rm abs}(\mathfrak{T}_t) \overset{\rm def}{=} \Gamma_{\mathfrak{T}_t}(\mathcal{A}_0), \tag{130}$$

with $\mathcal{A}_0$ an arbitrary element of $\mathcal{S}(\mathfrak{T}_t)$. As vortex tube moves with the fluid characterized by the flow map X (thanks to Helmholtz's first vortex Theorem), then for any vortex tube $\mathfrak{T}_t$ at a time $t \in [0, T^*[$, we deduce that $X(\mathfrak{T}_t, t, \tau)$ is a vortex tube at time τ for any $\tau \in [0, T^*[$.

Thanks to Lemma 7.2, we infer that for any $t \in [0, T^*[$ and any vortex tube $\mathfrak{T}_t$ at time t,

$$\mathbf{\Gamma}_{\rm abs}(\mathfrak{T}_t) = \mathbf{\Gamma}_{\rm abs}(X(\mathfrak{T}_t, t, \tau)) \text{ for any } \tau \in [0, T^*[, \tag{131}$$

which means that the absolute strength of any vortex tube $\mathfrak{T}_t$ at a time $t \in [0, T^*[$ moving with the fluid does not change with the time.

Lemma 7.4. *Let $d \in \{2, 3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-2}(\mathbb{R}^d)).$$

Let $t \in [0, T^*[$ and $\mathfrak{T}_t$ a vortex tube at time t defined by $\mathfrak{T}_t \overset{\rm def}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_t, s \in J_t\}$ with $\mathcal{A}_t$ a connected smooth orientable surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if $d = 2$) with boundary and J_t an interval of $\mathbb{R}$ containing 0 such that

- $J_t \subset \bigcap_{\alpha \in \mathcal{A}_t} \mathfrak{J}_{\alpha,t}$,
- *any vortex line of the tube $\mathfrak{T}_t$ intersects $\mathcal{A}_t$ only once, i.e.*

$$\forall \beta \in \mathcal{A}_t,\ \{\mathbf{x}_t(\beta, s); s \in \mathfrak{J}_{\beta,t}\} \cap \mathcal{A}_t = \{\beta\}. \tag{132}$$

Then we have

$$\int_{\mathfrak{T}_t} |\omega(z, t)|\, dz = |J_t|\, \mathbf{\Gamma}_{\rm abs}(\mathfrak{T}_t).$$

Proof. If $J_t = 0$ then the result follows immediately. Therefore we assume that $J_t \neq \{0\}$. For any $s \in J_t$, we define the smooth surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if $d = 2$) with boundary,

$$\mathcal{A}_t(s) \overset{\rm def}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}_t\}.$$

Due to the definition of the vortex tube $\mathfrak{T}_t$, we get that for any $s \in J_t$, the boundary of $\mathcal{A}_t(s)$ encircles the vortex tube $\mathfrak{T}_t$. Thanks to (114), we get

$$\int_{\mathfrak{T}_t} |\omega(z, t)|\, dz = \int_{s \in J_t} \int_{\mathcal{A}_t(s)} |\omega(y, t)|\ |\boldsymbol{n}_t(s) \cdot \xi(y, t)|\ d\sigma(\alpha) ds,$$

where $\boldsymbol{n}_t(s)$ is a unit normal vector of $\mathcal{A}_t(s)$. Since $\omega(y, t) = |\omega(y, t)|\xi(y, t)$ then we obtain

$$\begin{aligned} \int_{\mathfrak{T}_t} |\omega(z, t)|\, dz &= \int_{s \in J_t} \int_{\mathcal{A}_t(s)} |\omega(y, t) \cdot \boldsymbol{n}_t(s)|\ d\sigma(\alpha) ds \\ &= \int_{s \in J_t} \Gamma_{\mathfrak{T}_t}(\mathcal{A}_t(s))\, ds. \end{aligned} \tag{133}$$

We show now that for any $s_0 \in J_t$, any vortex line of the vortex tube $\mathfrak{T}_t$ intersects $\mathcal{A}_t(s_0)$ only once. For this purpose, let $\alpha_1 \in \mathcal{A}_t(s_0)$. Thanks to Cauchy-Lipschitz Theorem (see e.g. Theorem 2.2 in [98]) used for (114), we deduce that there exists an unique $\beta_1 \in \mathcal{A}_t$ such that $\alpha_1 = \mathbf{x}_t(\beta_1, s_0)$. Suppose for a contradiction that

$$\{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1,t}\} \cap \mathcal{A}_t(s_0) \neq \{\alpha_1\}.$$

Then there exists $\alpha_2 \neq \alpha_1$ such that $\alpha_2 \in \{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1,t}\} \cap \mathcal{A}_t(s_0)$. Therefore we get that $\alpha_2 = \mathbf{x}_t(\beta_1, s_2)$ with $s_2 \in \mathfrak{J}_{\beta_1,t}$, $s_2 \neq s_0$ since $\alpha_2 \neq \alpha_1$. We get also that there exists an unique $\beta_2 \in \mathcal{A}_t$ such that $\alpha_2 = \mathbf{x}_t(\beta_2, s_0)$ where $\beta_2 \neq \beta_1$ since $\alpha_2 \neq \alpha_1$. We thus infer that

$$\mathbf{x}_t(\beta_1, s_2) = \mathbf{x}_t(\beta_2, s_0). \tag{134}$$

By the maximality of $\mathbf{x}_t$, from (134) we infer that $s_2 - s_0 \in \mathfrak{J}_{\beta_1,t}$ and $\beta_2 = \mathbf{x}_t(\beta_1, s_2 - s_0)$ which implies $\{\beta_1, \beta_2\} \subset \{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1,t}\} \cap \mathcal{A}_t$. This latter contradicts (132). Therefore, we deduce that

$$\{\mathbf{x}_t(\beta_1, s); s \in \mathfrak{J}_{\beta_1,t}\} \cap \mathcal{A}_t(s_0) = \{\alpha_1\}.$$

This means that the vortex line of the vortex tube $\mathfrak{T}_t$ passing through $\alpha_1 \in \mathcal{A}_t(s_0)$ intersects $\mathcal{A}_t(s_0)$ only once, which matches to our desired result. Then we get that for any $s \in J_t$, $\mathcal{A}_t(s) \in \mathcal{S}(\mathfrak{T}_t)$ and hence thanks to (129) and (130), from (133) we obtain $\displaystyle\int_{\mathfrak{T}_t} |\omega(z,t)|\,dz = |J_t|\,\mathbf{\Gamma}_{\text{abs}}(\mathfrak{T}_t)$. Then we conclude the proof. □

Proposition 7.2. *Let $d \in \{2, 3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \dfrac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0, T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0, T^*[; H^{r-2}(\mathbb{R}^d)).$$

Under the definitions (32)–(37) in the Theorem 5.1, we assume that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \le \rho_0(t)$ there exists a vortex tube $\mathfrak{T}^R_{x,t}$ defined by $\mathfrak{T}^R_{x,t} \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha, s); \alpha \in \mathcal{A}^R_{x,t}, s \in I^R_{x,t}\}$ with $\mathcal{A}^R_{x,t}$ a connected smooth orientable surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if $d = 2$) and $I^R_{x,t}$ an interval of $\mathbb{R}$ containing 0 such that:

- *(P1) $\mathcal{V}(t) \cap B(x, R) \subset \mathfrak{T}^R_{x,t}$.*
- *(P2) any vortex line of the tube $\mathfrak{T}^R_{x,t}$ intersects $\mathcal{A}^R_{x,t}$ only once.*
- *(P3) $|I^R_{x,t}| \lesssim R$*
- *(P4) $\mathbf{\Gamma}_{\text{abs}}(\mathfrak{T}^R_{x,t}) \lesssim \overline{\boldsymbol{v}}(t_0)\, R^{d-2}$ where $\overline{\boldsymbol{v}}(t_0) > 0$ is a real which depend only on t_0 (and have the characteristic of a velocity).*

Then we get that for all $t \in [t_0, T^*[$

$$\pi(t) \lesssim \overline{\boldsymbol{v}}(t_0).$$

Proof. Let $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \le \rho_0(t)$. Thanks to property (P1) we have

$$\int_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz \le \int_{\mathfrak{T}^R_{x,t}} |\omega(z,t)|\,dz. \tag{135}$$

Furthermore, thanks to property (P2) and Lemma 7.4 we get

$$\int_{\mathfrak{T}^R_{x,t}} |\omega(z,t)|\,dz = |I^R_{x,t}|\,\mathbf{\Gamma}_{\text{abs}}(\mathfrak{T}^R_{x,t}). \tag{136}$$

Thanks to the properties (P3) and (P4), from (136) we deduce

$$\int_{\mathfrak{T}^R_{x,t}} |\omega(z,t)|\,dz \lesssim R^{d-1}\overline{\boldsymbol{v}}(t_0). \tag{137}$$

Owing to (137), from (135) we infer

$$\int_{B(x,R)\cap\mathcal{V}(t)} |\omega(z,t)|\,dz \lesssim R^{d-1}\overline{\boldsymbol{v}}(t_0). \tag{138}$$

From the definition (37) of the function π, thanks to (138) we thus deduce that for all $t \in [t_0, T^*[$

$$\pi(t) \lesssim \overline{\boldsymbol{v}}(t_0),$$

which concludes the proof. □

In the two following Remarks, we give explicit values for $\overline{\boldsymbol{v}}(t_0)$.

Remark 7.3. In the case of 2D QG equation for which $d = 2$, we have that for all $t \in [t_0, T^*[$,

$$\pi(t) \lesssim \|u_0\|_\infty,$$

if we replace the hypothesis (P4) by the assumption that the real-valued function $\omega(\cdot,t)\cdot\boldsymbol{n}$ keeps a constant sign over $\mathcal{A}^R_{x,t}$, where $\boldsymbol{n}$ is a unit normal vector varying smoothly on $\mathcal{A}^R_{x,t}$.

Indeed in this case, we get $\int_{\mathcal{A}^R_{x,t}} |\omega(\alpha,t)\cdot\boldsymbol{n}(\alpha)|\,d\alpha = \left|\int_{\mathcal{A}^R_{x,t}} \omega(\alpha,t)\cdot\boldsymbol{n}(\alpha)\,d\alpha\right|$ and furthermore thanks to Stokes Theorem we have $\int_{\mathcal{A}^R_{x,t}} \omega(\alpha,t)\cdot\boldsymbol{n}(\alpha)\,d\alpha = u(\alpha_2,t) - u(\alpha_1,t)$ where α_2 and α_1 are the two endpoints of the line segment $\mathcal{A}^R_{x,t}$. We thus infer $\int_{\mathcal{A}^R_{x,t}} |\omega(\alpha,t)\cdot\boldsymbol{n}(\alpha)|\,d\alpha \le 2\|u(t)\|_\infty = 2\|u_0\|_\infty$ thanks to (22) and then we take $\overline{\boldsymbol{v}}(t_0) = \|u_0\|_\infty$. Then with the properties (P1)–(P3), we thus obtain that for all $t \in [t_0, T^*[$, $\pi(t) \lesssim \|u_0\|_\infty$.

Remark 7.4. For any $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \le \rho_0(t)$ let us assume that there exists $t_1 \in [0, t_0]$ depending on t, x and R such that for the vortex tube $\mathfrak{T}^R_{x,t_1} \stackrel{\text{def}}{=} X(\mathfrak{T}^R_{x,t}, t, t_1)$ at time t_1 we have

$$\inf_{\mathcal{A}\in\mathcal{S}(\mathfrak{T}^R_{x,t_1})} |\mathcal{A}| \lesssim R^{d-2},$$

then Property (P4) holds with $\overline{\boldsymbol{v}}(t_0) = \|\omega\|_{L^\infty(\mathbb{R}^d\times[0,t_0])}$. Indeed thanks to (131) we have

$$\boldsymbol{\Gamma}_{\text{abs}}(\mathfrak{T}^R_{x,t}) = \boldsymbol{\Gamma}_{\text{abs}}(\mathfrak{T}^R_{x,t_1}).$$

Furthermore, thanks to (129) and (130), we deduce that

$$\begin{aligned}\boldsymbol{\Gamma}_{\text{abs}}(\mathfrak{T}^R_{x,t_1}) &= \inf_{\mathcal{A}\in\mathcal{S}(\mathfrak{T}^R_{x,t_1})} \int_{\mathcal{A}} |\omega(y,t_1)\cdot\boldsymbol{n}(y)|\,d\sigma(y)\\ &\le \|\omega(t_1)\|_\infty \inf_{\mathcal{A}\in\mathcal{S}(\mathfrak{T}^R_{x,t_1})} |\mathcal{A}|\\ &\lesssim \|\omega\|_{L^\infty(\mathbb{R}^d\times[0,t_0])}\, R^{d-2}.\end{aligned}$$

Therefore, we deduce that $\boldsymbol{\Gamma}_{\text{abs}}(\mathfrak{T}^R_{x,t}) \lesssim \|\omega\|_{L^\infty(\mathbb{R}^d\times[0,t_0])}\, R^{d-2}$ which matches with Property (P4) for $\overline{\boldsymbol{v}}(t_0) = \|\omega\|_{L^\infty(\mathbb{R}^d\times[0,t_0])}$.

Then thanks to Theorem 5.1 and Proposition 7.2 we deduce Theorem 7.2.

Theorem 7.2. *Let $d \in \{2,3\}$, $u_0 \in H^r_\sigma(\mathbb{R}^d)$ with $r > \frac{d}{2} + 3$. Let $T^* > 0$ be such that there exists a unique strong solution u to the 3D Navier-Stokes, 3D Euler equations (9)–(10) or 2D QG equations (11)–(12) in the class*

$$u \in C([0,T^*[; H^r_\sigma(\mathbb{R}^d)) \cap C^1([0,T^*[; H^{r-2}(\mathbb{R}^d)).$$

Under the definitions (32)–(36) in the Theorem 5.1, we assume that there exists $t_0 \in [0, T^*[$ such that for any $t \in [t_0, T^*[$, $x \in \Theta(t)$ and $0 < R \le \rho_0(t)$ there exists a vortex tube $\mathfrak{T}^R_{x,t}$ defined by $\mathfrak{T}^R_{x,t} \stackrel{\text{def}}{=} \{\mathbf{x}_t(\alpha,s); \alpha \in \mathcal{A}^R_{x,t}, s \in I^R_{x,t}\}$ with $\mathcal{A}^R_{x,t}$ a connected smooth orientable surface of $\mathbb{R}^3$ (curve of $\mathbb{R}^2$ if $d = 2$) and $I^R_{x,t}$ an interval of $\mathbb{R}$ containing 0 such that:

(P1) $\mathcal{V}(t) \cap B(x, R) \subset \mathfrak{T}^R_{x,t}$.
(P2) $|I^R_{x,t}| \lesssim R$
(P3) *any vortex line of the tube* $\mathfrak{T}^R_{x,t}$ *intersects* $\mathcal{A}^R_{x,t}$ *only once.*
(P4) $\mathbf{\Gamma}_{\text{abs}}(\mathfrak{T}^R_{x,t}) \lesssim \overline{\boldsymbol{v}}(t_0) R^{d-2}$ *where* $\overline{\boldsymbol{v}}(t_0) > 0$ *is a real depending only on* t_0 *(and have the characteristic of a velocity).*

Then if there exists $t_1 \in [t_0, T^*[$ *such that*

$$\int_{t_1}^{T^*} \mathbf{A}_d(t)\, dt < +\infty,$$

then the solution u cannot blowup at the finite time T^* *with*

$$\mathbf{A}_d(t) \overset{\text{def}}{=} \sup_{x \in \Theta(t)} \sup_{y \in B(0,\rho(t))\setminus\{0\}} \frac{\mathbf{D}_d(\hat{y}, \xi(x+y,t), \xi(x,t))^+}{|y|}$$
$$\rho(t) = O((T^* - t)\overline{\boldsymbol{v}}(t_0)).$$

Declaration of Competing Interest

The author declares that there is no competing interest.

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