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Regularity for diffuse reflection boundary problem to the stationary linearized Boltzmann equation in a convex domain

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Abstract

We investigate the regularity issue for the diffuse reflection boundary problem to the stationary linearized Boltzmann equation for hard sphere potential, cutoff hard potential, or cutoff Maxwellian molecular gases in a strictly convex bounded domain. We obtain pointwise estimates for first derivatives of the solution provided the boundary temperature is bounded differentiable and the solution is bounded. This result can be understood as a stationary version of the velocity averaging lemma and mixture lemma.

Keywords: Boltzmann equation; regularity; kinetic theory; stationary

1. Introduction

In this article, we consider the stationary linearized Boltzmann equation

$$\zeta \cdot \nabla f(x,\zeta) = L(f), \tag{1.1}$$

for $\zeta \in \mathbb{R}^3$ and $x \in \Omega$, where $\Omega \subset \mathbb{R}^3$ is a C^2 bounded strictly convex domain such that $\partial\Omega$ is of positive Gaussian curvature. Here, L represents the linearization of the collision operator. The collision operator in Boltzmann equation reads:

$$Q(F,G) = \int_{\mathbb{R}^3}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(F(\zeta')G(\zeta'_*) - F(\zeta)G(\zeta_*)\right) B(|\zeta_* - \zeta|, \theta)\, d\theta\, d\epsilon\, d\zeta_*, \tag{1.2}$$

where ζ, ζ_* and ζ', ζ'_* are pairs of velocities before and after the impact, and B is called the cross section, depending on interaction between particles. L is obtained by linearizing Q around the standard Maxwellian

$$M(\zeta) = \pi^{-\frac{3}{2}} e^{-|\zeta|^2} \tag{1.3}$$

in the fashion

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$$F = M + M^{\frac{1}{2}} f. \tag{1.4}$$

L reads

$$L(f) = M^{-\frac{1}{2}}(\zeta)[Q(M^{\frac{1}{2}} f, M) + Q(M, M^{\frac{1}{2}} f)]. \tag{1.5}$$

Notice that Q and L only act on functions of ζ variable, while x is considered as a parameter rather a variable. The widely used angular cutoff potential is a mathematical model introduced by Grad [14] by assuming

$$0 \le B(|\zeta - \zeta_*|, \theta) \le C|\zeta - \zeta_*|^\gamma \cos\theta \sin\theta. \tag{1.6}$$

In this article, we follow Grad's idea and assume

$$\begin{aligned} &B(|\zeta - \zeta_*|, \theta) = |\zeta - \zeta_*|^\gamma \beta(\theta), \\ &0 \le \beta(\theta) \le C \cos\theta \sin\theta, \\ &0 \le \gamma \le 1. \end{aligned} \tag{1.7}$$

The range of γ we consider corresponds to the hard sphere model, cutoff hard potential, and cutoff Maxwellian molecular gases. We shall discuss the properties of L under our assumption (1.7) in detail in Section 2.

The boundary condition under the consideration is the diffuse reflection boundary condition:

(1) First, there is no net flux on the boundary.
(2) Secondly, the velocity distribution function reflected from the boundary is in thermal equilibrium with the boundary temperature.

We use Γ_- to denote the incoming boundary:

$$\Gamma_- := \{(x, \zeta) | x \in \partial\Omega,\ \zeta \cdot n(x) < 0\}, \tag{1.8}$$

where $n(x)$ is the outward unit normal of $T_x(\partial\Omega)$. In the context of the linearized Boltzmann equation, the mathematical formula of the aforementioned diffuse reflection boundary condition could be described as: for $(x, \zeta) \in \Gamma_-$,

$$f(x, \zeta) = \psi(x) M^{\frac{1}{2}} + T(x)(|\zeta|^2 - 2) M^{\frac{1}{2}}, \tag{1.9}$$

$$\psi(x) = 2\sqrt{\pi} \int_{\zeta' \cdot n > 0} f(x, \zeta')|\zeta' \cdot n| M^{\frac{1}{2}} d\zeta'. \tag{1.10}$$

Here, $T(x)$ is the temperature on the boundary. To state our main goal of mathematical analysis, we define, for given $x \in \bar{\Omega}$,

$$\tau_-(x, \zeta) = \inf\left\{t \,\middle|\, t > 0,\ x - t\zeta \notin \Omega\right\}, \text{ and} \tag{1.11}$$

$$p(x, \zeta) = x - \tau_-(x, \zeta)\zeta. \tag{1.12}$$

Under the assumption (1.7), L can be decomposed into a multiplicative operator and an integral operator:

$$L(f) = -\nu(|\zeta|) f + K(f). \tag{1.13}$$

We take the integral operator K as the source term and rewrite (1.1) as

$$\zeta \cdot \nabla f(x, \zeta) + \nu(|\zeta|) f(x, \zeta) = K(f). \tag{1.14}$$

The corresponding integral form of the solution to (1.14) is

$$f(x, \zeta) = f(p(x, \zeta), \zeta) e^{-\nu(|\zeta|)\tau_-(x,\zeta)} + \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} K(f)(x - s\zeta, \zeta) ds. \tag{1.15}$$

We say f is a solution to (1.1) if f satisfies (1.15) almost everywhere. The existence of a solution to the presented problem has been established by Guiraud in 1970's [15,16]. Recently, Esposito, Guo, Kim, and Marra extended the result to non-convex domains in [12]. In particular, we notice that, under the assumption that $T(x)$ is bounded, Proposition 4.1 of [12] implies the existence of $L^\infty_{x,\zeta}$ solution to (1.14) supplemented with the boundary conditions (1.9)–(1.10). Concerning L^1 solutions, the linear case was done by Falk for convex domains [10], and the non-linear case was solved by Arkeryd and Nouri for slab geometry [1]. In the present article, we shall assume that $T(x)$ is bounded differentiable, i.e., $T(x)$ is differentiable and its first derivatives are bounded, and we shall aim at proving the interior differentiability of the solution to the problem (1.14) supplemented with (1.9)–(1.10), see Theorem 1.2.

It is worth mentioning that, in [12], they also proved that the solution is continuous away from the grazing set. For the higher regularity issue, by observing velocity averaging effect for the stationary linearized Boltzmann equation, the Hölder continuity up to $\frac{1}{2}$- away from the boundary was first established in [6] for inflow boundary value problems. In this article, we establish a pointwise estimate of the first derivatives of the solution. The main cruxes of this pointwise estimate are multifold. First, we need to overcome the difficulty brought by the diffuse reflection boundary condition. The diffuse reflection condition ((1.9) and (1.10)) involves the solution itself. Namely, inferring from (1.15) and (1.9)–(1.10), since we only know that the solution $f \in L^\infty_{x,\zeta}$, we cannot even take the formal derivative to f with respect to the space variable x. Secondly, we need to improve the regularity from Hölder continuity to differentiability. We shall discuss these issues in depth after introducing the main theorem. Regarding regularity issues for the time dependent Boltzmann equation, we refer the interested readers to [17–19].

We denote the distance of interior point x to $\partial\Omega$ by d_x, namely

$$d_x := \inf_{y\in\partial\Omega} |x-y|. \tag{1.16}$$

We would like to specify the domain we are dealing with.

Definition 1.1. We say an open bounded strictly convex set Ω in $\mathbb{R}^3$ satisfies the positive curvature condition if $\partial\Omega$ is C^2 and of positive Gaussian curvature.

The main result of this article is as follows.

Theorem 1.2. *Assume $\Omega \subset \mathbb{R}^3$ satisfies the positive curvature condition defined in Definition 1.1 above. Under the assumption (1.7), suppose $f \in L^\infty_{x,\zeta}$ is a solution to the stationary linearized Boltzmann equation (1.1) with the diffuse reflection boundary condition (1.9)–(1.10) such that the boundary temperature $T(x)$ (in (1.10)) is bounded differentiable, i.e., $T(x)$ is differentiable and its first derivatives are bounded. Then, for $\epsilon > 0$, we have*

$$\sum_{i=1}^{3}\left|\frac{\partial}{\partial x_i} f(x,\zeta)\right| + \sum_{i=1}^{3}\left|\frac{\partial}{\partial \zeta_i} f(x,\zeta)\right| \le C(1+d_x^{-1})^{\frac{4}{3}+\epsilon}, \tag{1.17}$$

where $x \in \Omega$ and $\zeta \in \mathbb{R}^3$.

Remark 1.3. Notice that the right-hand side of the above estimate diverges to infinity as x approaches the boundary. This hints the possibility that regularity of solutions to Boltzmann equation may become worse near boundary. Actually, it was observed in a simpler geometrical setting, a slab domain, by numerical evidence that a jump discontinuity and a logarithmic singularity occurs at boundary [5,21].

Now we briefly recall some ideas about the velocity averaging effect for the stationary linearized Boltzmann equation on bounded convex domains introduced in [6]. We iterate the integral equation (1.15) again and obtain

$$
\begin{aligned}
f(x,\zeta) &= f(p(x,\zeta),\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)} \\
&\quad + \int_0^{\tau_-(x,\zeta)} \int_{\mathbb{R}^3} e^{-\nu(|\zeta|)s} k(\zeta,\zeta')e^{-\nu(|\zeta'|)\tau_-(x-s\zeta,\zeta')} f\big(p(x-s\zeta,\zeta'),\zeta'\big)d\zeta' ds \\
&\quad + \int_0^{\tau_-(x,\zeta)} \int_{\mathbb{R}^3} \int_0^{\tau_-(x-s\zeta,\zeta')} e^{-\nu(|\zeta|)s} k(\zeta,\zeta')e^{-\nu(|\zeta'|)t} K(f)(x-s\zeta-t\zeta',\zeta')dtd\zeta' ds \\
&=: I(x,\zeta) + II(x,\zeta) + III(x,\zeta).
\end{aligned}
\tag{1.18}
$$

For I and II, the regularity of the boundary is preserved by the transport. The velocity averaging effect would play an important role in the improvement of regularity for III. Notice that, thanks to nice property of the integral kernel, (2.6), in fact $K(f)$ is bounded differentiable in ζ provide f is bounded, i.e.,

$$
\left\| \frac{\partial}{\partial \zeta_i} K(f) \right\|_{L^\infty_\zeta} \le C\|f\|_{L^\infty_\zeta}. \tag{1.19}
$$

For time dependent kinetic equations defined on the whole space, it is well-known that velocity averaging effect combining with transport effect can transfer regularity from velocity variables to space variables, e.g., famous Velocity Averaging Lemma [13] and Mixture Lemma [20]. As far as we know, there is no analogy result for the Mixture Lemma for stationary problems defined on bounded domains addressed elsewhere. To take care of the regularity of III, we change the variables ζ' to the spherical coordinates so that

$$
\zeta' = (\rho\cos\theta, \rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi). \tag{1.20}
$$

Also, we change the traveling time to the traveling distance:

$$
r = \rho t. \tag{1.21}
$$

Let $\hat{\zeta}' = \frac{\zeta'}{|\zeta'|}$. Then, we can rewrite III as

$$
\begin{aligned}
III = & \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} \int_0^\infty \int_0^\pi \int_0^{2\pi} \int_0^{|x-s\zeta-p(x-s\zeta,\zeta')|} \\
& k(\zeta,\zeta')e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x-s\zeta-r\hat{\zeta}',\zeta')\rho\sin\theta dr d\phi d\theta d\rho ds \\
=: & \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} G(x-s\zeta,\zeta)ds.
\end{aligned}
\tag{1.22}
$$

Notice that we can parametrize Ω by θ, ϕ, and r, thanks to the convexity of Ω. Therefore, by regrouping the integrals, we can change the formulation to contain an integral over space: Let $x_0 = x - s\zeta$ and $y = x - s\zeta - r\hat{\zeta}'$. We have

$$
G(x_0,\zeta) = \int_0^\infty \int_\Omega k\left(\zeta, \rho\frac{(x_0-y)}{|x_0-y|}\right) e^{-\nu(\rho)\frac{|x_0-y|}{\rho}} K(f)\left(y, \rho\frac{(x_0-y)}{|x_0-y|}\right) \frac{\rho}{|x_0-y|^2} dy d\rho. \tag{1.23}
$$

Notice that in the above formula, the velocity variables ζ' are replaced by the space variables x_0 and y, and therefore the regularity in velocity variables can be transferred to space variables. However, the singularity in the above integral formula does not allow us to differentiate $G(x_0,\zeta)$ with respect to x_0 directly. This is the reason why the result obtained in [6] is limited to the Hölder type continuity. In the present work, we overcome this obstacle by bootstrapping the regularity from Hölder continuity to differentiability with the help of divergence theorem (see Section 10). Notice that there is only a very narrow window that one can carry out this strategy. To this aim, we make big efforts to significantly refine Hölder type estimates in Sections 6, 7, and 8. In addition, we encounter not only the aforementioned cruxes but also the difficulties due to the diffuse reflection boundary condition.

Now, we shall give a brief account of the strategy that we employ to overcome all the aforementioned subtleties. First, plugging (1.9) into (1.15), we have

$$f(x,\zeta)=\big(\psi(p(x,\zeta))+T(p(x,\zeta))(|\zeta|^2-2)\big)M^{\frac{1}{2}}e^{-\nu(|\zeta|)\tau_-(x,\zeta)} + \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}K(f)(x-s\zeta,\zeta)ds. \tag{1.24}$$

Notice that, by (1.10), ψ is only a bounded function provided f is bounded. Hence, we cannot even take a formal derivative on the first term of the right hand side of (1.24). To deal with the differentiability of the first term of the right hand side of (1.24), we only need to take care of the differentiability of $\psi(p(x,\zeta))$. Plugging (1.24) into (1.10), we obtain

$$\begin{aligned}\psi(x) &= 2\sqrt{\pi}\int_{\zeta\cdot n(x)>0} T(p(x,\zeta))(|\zeta|^2-2)M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n(x)|d\zeta \\ &\quad + 2\sqrt{\pi}\int_{\zeta\cdot n(x)>0}\psi(p(x,\zeta))M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n(x)|d\zeta \\ &\quad + 2\sqrt{\pi}\int_{\zeta\cdot n(x)>0}\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}K(f)(x-s\zeta,\zeta)M^{\frac{1}{2}}(\zeta)|\zeta\cdot n(x)|dsd\zeta \\ &=: B_T+B_\psi+D_f.\end{aligned} \tag{1.25}$$

In the above expression, the domain of integration depends on the space variable x, which is not convenient for taking the formal derivatives. On the other hand, in the formula of B_ψ, the integrand ψ is a function of $p(x,\zeta)$. However, we only know that ψ is a bounded function. As we shall go into the details in Section 4, by using a sophisticated change of variables, we can rewrite B_ψ as follows

$$B_\psi(x)=\frac{2}{\pi}\int_0^\infty\int_{\partial\Omega}\psi(y)e^{-l^2|x-y|^2}e^{-\frac{\nu(l|x-y|)}{l}}[(x-y)\cdot n(x)]|(x-y)\cdot n(y)|l^3dA(y)dl. \tag{1.26}$$

We notice that with the above expression, the domain of integration is fixed and the formal derivative of $B_\psi(x)$ with respect to x variable does not involve ψ. This creates a startup for our regularity analysis. We use the same fashion to deal with the regularity of B_T.

For the term of D_f, applying a similar approach as in [6], we can convert D_f into the following formula

$$D_f=2\pi^{-\frac{1}{4}}\int_0^\infty\int_\Omega e^{-\frac{\nu(\rho)}{\rho}|x-y|}K(f)\left(y,\rho\frac{(x-y)}{|x-y|}\right)\frac{(x-y)\cdot n(x)}{|x-y|}e^{-\frac{\rho^2}{2}}\frac{\rho^2}{|x-y|^2}dyd\rho. \tag{1.27}$$

Similar to the treatment of III, the advantage we gain from the above transformation is that the regularity in velocity variables can be transferred to space variables. However, if we differentiate D_f with respect to x variables directly, the formula has a singularity which damages the integrability of the resulting formula. Therefore, one can only claim the Hölder type regularity by an argument similar to [6]. Nevertheless, we can in fact further bootstrap the regularity to differentiability. For the details of the treatment, see Section 9.

The organization of the rest part of this article is as follows. We recapitulate the important properties of the linearized collision operator L in Section 2. In Section 3, we prepare several useful auxiliary lemmas and propositions associated to the geometry of Ω which play crucial roles in the integrability arguments in the estimates of Section 4 – Section 9. In Section 10, we sum up the estimates from Section 4 to Section 9 and conclude the differentiability in x variables. Section 11 is devoted to the differentiability in ζ variables.

2. Properties of linearized collision operator

In this section, we summarize some known properties of the linearized collision operator L defined in (1.5) with a cross section satisfying our assumption (1.7) (see [2,7,14]). L can be decomposed into a multiplicative operator and an integral operator:

$$L(f) = -\nu(|\zeta|)f + K(f), \tag{2.1}$$

where

$$K(f)(x,\zeta) = \int_{\mathbb{R}^3} k(\zeta,\zeta_*)f(x,\zeta_*)d\zeta_* \tag{2.2}$$

is symmetric, i.e.,

$$k(\zeta,\zeta_*) = k(\zeta_*,\zeta).$$

The explicit expression of ν is

$$\nu(|\zeta|) = \beta_0 \int_{\mathbb{R}^3} e^{-|\eta|^2}|\eta-\zeta|^\gamma d\eta, \tag{2.3}$$

where $\beta_0 = \int_0^{\frac{\pi}{2}} \beta(\theta)d\theta$. Let $0<\delta<1$. The collision frequency $\nu(|\zeta|)$ and the collision kernel $k(\zeta,\zeta_*)$ satisfy

$$\nu_0(1+|\zeta|)^\gamma \le \nu(|\zeta|) \le \nu_1(1+|\zeta|)^\gamma, \tag{2.4}$$

$$|k(\zeta,\zeta_*)| \le C_1|\zeta-\zeta_*|^{-1}(1+|\zeta|+|\zeta_*|)^{-(1-\gamma)}e^{-\frac{1-\delta}{4}\left(|\zeta-\zeta_*|^2+(\frac{|\zeta|^2-|\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)}, \tag{2.5}$$

$$\left|\frac{\partial}{\partial\zeta_i}k(\zeta,\zeta_*)\right| \le C_2\frac{1+|\zeta|}{|\zeta-\zeta_*|^2}(1+|\zeta|+|\zeta_*|)^{-(1-\gamma)}e^{-\frac{1-\delta}{4}\left(|\zeta-\zeta_*|^2+(\frac{|\zeta|^2-|\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)}. \tag{2.6}$$

Here, the constants $0<\nu_0<\nu_1$ may depend on the potential and C_1 and C_2 may depend on δ and the potential. Notice that (2.5) was established in [2] and (2.6) can be concluded by the observation in [7] in case the cross section satisfies (1.7).

Related to the above estimates, the following proposition from [2] is crucial in our study.

Proposition 2.1. *For any* $\epsilon, a_1, a_2 > 0$*,*

$$\left|\int_{\mathbb{R}^3} \frac{1}{|\eta-\zeta_*|^{3-\epsilon}}e^{-a_1|\eta-\zeta_*|^2-a_2\frac{(|\eta|^2-|\zeta_*|^2)^2}{|\eta-\zeta_*|^2}}d\zeta_*\right| \le C_4(1+|\eta|)^{-1}, \tag{2.7}$$

where C_4 *may depend on* ϵ, a_1*, and* a_2*.*

Using the (2.6) and Proposition 2.1, we can conclude

$$\left\|\frac{\partial}{\partial\zeta_i}k(\zeta,\zeta_*)\right\|_{L^\infty_\zeta L^1_{\zeta_*}} < \infty, \quad \left\|\frac{\partial}{\partial\zeta_i}k(\zeta,\zeta_*)\right\|_{L^\infty_{\zeta_*} L^1_{\zeta}} < \infty. \tag{2.8}$$

Then, by Schur's test, we can conclude the following smoothing effect of K in velocity variable mentioned in [7].

Proposition 2.2. *For* $1 \le p \le \infty$*,*

$$\left\|\frac{\partial}{\partial\zeta_i}K(f)\right\|_{L^p_\zeta} \le C\|f\|_{L^p_\zeta}. \tag{2.9}$$

3. Geometric properties

In this section, we shall prove several important auxiliary lemmas and propositions which play important roles in our regularity theory. We first briefly recapitulate some important ingredients of differential geometry from [9] that we are going to employ.

Definition 3.1. Let M be a differentiable Riemannian manifold equipped with the Riemannian connection ∇. A parametrized curve $\phi(t):[0,r]\to M$ is called a geodesic curve if

$$\nabla_{\phi'(t)}\phi'(t)=0 \tag{3.1}$$

for $t\in(0,r)$.

If $|\phi'(t)|\equiv 1$, we call $\phi(t)$ a normalized geodesic. In this case, t is the arc length of the geodesic segment between $\phi(0)$ and $\phi(t)$. It is well-known that there exists a unique vector field on TM, the tangent bundle of M, whose trajectories are of the form $t\to(\phi(t),\phi'(t))$, where ϕ is a geodesic on M. This vector field is called the geodesic field on TM and its flow is called the geodesic flow on TM. The general existence theory of ODE systems implies the following property of geodesic flow.

Proposition 3.2. *Given $p\in M$, there exist a neighborhood V of p in M, a number $\epsilon>0$ and a C^∞ mapping $\phi:(-2,2)\times\mathbb{U}\to M$, $\mathbb{U}=\{(q,w)\in TM; q\in V, w\in T_qM, |w|<\epsilon\}$ such that $t\to\phi(t,q,w)$, $t\in(-2,2)$, is the unique geodesic of M which, at the instant $t=0$, passes through q with velocity w, for every $q\in V$ and for every $w\in T_qM$, with $|w|<\epsilon$.*

Let $p\in M$ and $\mathbb{U}\subset TM$ given by the above proposition. The exponential map on $\mathbb{U}$ is defined as

$$Exp_q(v)=\phi(1,q,v)=\phi(|v|,q,\frac{v}{|v|}),\ \ (q,v)\in\mathbb{U}. \tag{3.2}$$

In the context of our theory, we shall use the C^2 differentiable structure of the aforementioned geometric tools. The following lemma is our first main results of this section.

Lemma 3.3. *Suppose Ω satisfies the positive curvature condition defined in Definition 1.1. Then, there exists a constant C depending only on the geometry of domain Ω such that for any interior point $x\in\Omega$, we have*

$$\int\limits_{\partial\Omega}\frac{1}{|x-y|^2}dA(y)\le C(\big|\ln d_x\big|+1), \tag{3.3}$$

where $d_x=d(x,\partial\Omega)$ and $A(y)$ is the surface element of $\partial\Omega$ at point $y\in\partial\Omega$.

Lemma 3.3 is crucial in the proof of the refined Hölder type estimate (6.5) in Lemma 6.1 as well as Proposition 10.4. We may get a hint by directly calculate the integral over a bounded set on a plane. A special regular case of Lemma 3.3 is the case where $\partial\Omega$ is a sphere, for which one may prove Lemma 3.3 by direct calculation. To deal with the general case, we need the following proposition.

Proposition 3.4. *Suppose Ω satisfies the positive curvature condition defined in Definition 1.1. Then, there exists a constant r_1 (see (3.14)) depending only on Ω such that for any $x\in\Omega$ and $p_0\in\partial\Omega$ satisfying that (p_0-x) is parallel to $n(p_0)$, where $n(p_0)$ is the unit outward normal of $\partial\Omega$ at p_0, there holds the following inequality*

$$|x-p_0|^2+\frac{1}{2}|v|^2\le|Exp_{p_0}(v)-x|^2, \tag{3.4}$$
$$\text{for } 0\le|x-p_0|\le r_1 \text{ and } v\in T_{p_0}(\partial\Omega) \text{ with } 0\le|v|\le r_1.$$

Here, Exp_{p_0} is the exponential map from the tangent space $T_{p_0}(\partial\Omega)$ to $\partial\Omega$.

Since $\Omega \subset \mathbb{R}^3$ is a C^2 convex bounded domain such that $\partial\Omega$ is of positive Gaussian curvature, by continuity of curvatures and the compactness of $\partial\Omega$, we see that there are uniform positive upper and lower bounds for normal curvature and Gaussian curvature of $\partial\Omega$. Regarding the identity map as an immersion of $\partial\Omega$ into the Euclidean space $\mathbb{R}^3$, we equip $\partial\Omega$ with the induced metric and the corresponding the Riemannian connection (also known as Levi-Civita connection) structure. We use the notation

$$GB(p,r) := \left\{Exp_p(v)\big||v| < r\right\} \tag{3.5}$$

to denote the geodesic ball (which is also known as normal ball) on $\partial\Omega$ centered at $p \in \partial\Omega$ with geodesic radius r. Noting that

(1) $d(Exp_q)_0$ is the identity map of $T_q(\partial\Omega)$,
(2) $\partial\Omega$ is of positive Gaussian curvature,
(3) Gaussian curvature is the same as sectional curvature for two dimensional manifold,

and applying the Rauch theorem (see, for example, [3,9]), we see that there is a uniform radius, r_0, and a positive constant $a_0 < 1$ such that for every point $p \in \partial\Omega$ the exponential map $Exp_p : T_p(\partial\Omega) \to \partial\Omega$ is one-to-one and the Jacobian satisfies

$$a_0 \le \left|\det\left(\frac{\partial Exp_p}{\partial X}\right)\right| \le 1 \tag{3.6}$$

within the r_0-neighborhood of $T_p(\partial\Omega)$. With the above understanding, we now start to prove Proposition 3.4.

Proof. We are going to estimate the distance between $x \in \Omega$ and a point in the geodesic ball centered at p_0. We choose the coordinate such that $p_0 = (0,0,0)$, $x = (0,0,-d)$. Without loss of generality, we only need to consider the points on the normal geodesic $\phi(s) = (\phi_1(s), \phi_2(s), \phi_3(s))$ with $\phi(0) = (0,0,0)$ and $\phi'(0) = (1,0,0)$. Since normal curvature is bounded, there exist constants $0 < a < b$ independent of p_0 and ϕ such that

$$0 < a \le \left|\frac{d^2}{ds^2}\phi(s)\right| \le b, \tag{3.7}$$

for all $s \in (-r_0, r_0)$. By $\phi'(0) = (1,0,0)$, we derive from (3.7) that

$$1 - bs \le \phi_1'(s) \le 1 + bs, \tag{3.8}$$

$$-bs \le \phi_2'(s) \le bs, \tag{3.9}$$

$$-bs \le \phi_3'(s) \le bs. \tag{3.10}$$

Therefore,

$$s - \frac{1}{2}bs^2 \le \phi_1(s) \le s + \frac{1}{2}bs^2, \tag{3.11}$$

$$-\frac{1}{2}bs^2 \le \phi_2(s) \le \frac{1}{2}bs^2, \tag{3.12}$$

$$-\frac{1}{2}bs^2 \le \phi_3(s) \le \frac{1}{2}bs^2. \tag{3.13}$$

For further discussion, we define

$$r_1 := \min\left\{r_0, \frac{1}{4b}\right\}. \tag{3.14}$$

In the following analysis, we assume that

$$0 \le s \le r_1, \text{ and } 0 < d \le r_1.$$

Case 1: $d > \frac{1}{2}bs^2$. In this case, we have

$$\phi_3(s) + d > d - \frac{1}{2}bs^2,$$

and

$$
\begin{aligned}
|\phi(s)-x|^2 &\geq \phi_1^2(s)+\phi_2^2(s)+\left(d-\frac{1}{2}bs^2\right)^2 \\
&\geq \left(s-\frac{1}{2}bs^2\right)^2+0^2+\left(d-\frac{1}{2}bs^2\right)^2 \\
&= d^2+(1-bd-bs)s^2+\frac{1}{2}b^2s^4 \\
&\geq d^2+\frac{1}{2}s^2.
\end{aligned}
$$

Case 2: $d \leq \frac{1}{2}bs^2$. In this case, we see that

$$
\begin{aligned}
|\phi(s)-x|^2 &\geq \phi_1^2(s) \\
&\geq \left(s-\frac{1}{2}bs^2\right)^2 \\
&= \left(\frac{1}{2}bs^2\right)^2+(1-bs)s^2 \\
&\geq d^2+\frac{1}{2}s^2.
\end{aligned}
$$

Now, for $v \in T_{p_0}(\partial\Omega)$ with $0 \leq |v| \leq r_1$, by choosing the coordinate properly, we have $\phi(s) = Exp_{p_0}(v)$ with $s = |v|$. Summing up *Case 1* and *Case 2*, we conclude Proposition 3.4. □

Remark 3.5. Taking (3.14) into account, by (3.11), we see that

$$s \leq \frac{8}{7}|\phi(s)-\phi(0)|. \tag{3.15}$$

Proof of Lemma 3.3. Let $GB(p,r)$ be the geodesic ball on $\partial\Omega$ centered at p with geodesic radius r. We first take care of the case where $d_x \leq r_1$. We define

$$D_0 := \left\{p \in \partial\Omega \,\middle|\, (p-x) /\!\!/ n(p),\ d_x \leq |p-x| \leq r_1\right\}, \tag{3.16}$$

$$D_1 := \bigcup_{p\in D_0} GB\left(p, \frac{1}{10}r_1\right). \tag{3.17}$$

By the Vitali's covering lemma, we see that there exists a countable subcollection $\tilde{D}_0$ of D_0 such that $\bigcup_{p\in\tilde{D}_0} GB(p, \frac{1}{10}r_1)$ is a disjoint union of geodesic discs $GB(p, \frac{1}{10}r_1)$ and

$$D_1 \subset \bigcup_{p\in\tilde{D}_0} GB\left(p, \frac{5}{10}r_1\right). \tag{3.18}$$

On the other hand, due to (3.6), there is a uniform lower bound A_1 of the area of $GB(p, \frac{1}{10}r_1)$ for any $p \in \partial\Omega$. Since $\bigcup_{p\in\tilde{D}_0} GB(p, \frac{1}{10}r_1)$ is a disjoint union of $GB(p, \frac{1}{10}r_1)$ and the area of $\partial\Omega$ (denoted by A_2) is finite, the cardinality of $\tilde{D}_0$ satisfies

$$\#(\tilde{D}_0) \leq \frac{A_2}{A_1} < \infty. \tag{3.19}$$

We remark that, by the above argument, the upper bound of the cardinality $\#(\tilde{D}_0)$ is independent of the position of x. For the sake of convenience, we list all the elements of $\tilde{D}_0$ as follows

$$\tilde{D}_0 = \left\{p_1, p_2, \cdots, p_m\right\}, \tag{3.20}$$

where $m = \#(\tilde{D}_0)$, and define the sets

$$D_2 := \bigcup_{i=1}^{m} GB(p_i, r_1) \text{ and}$$
$$D_3 := \partial\Omega \setminus D_2.$$

Note that since D_2 is an open subset of $\partial\Omega$, D_3 is a compact set. Hence, there is a point $p' \in D_3$ that realizes the distance of x and D_3.

We claim that

$$|p' - x| = d(x, D_3) > \frac{r_1}{\sqrt{2}}. \tag{3.21}$$

Suppose, on the contrary, $|p' - x| = d(x, D_3) \leq \frac{r_1}{\sqrt{2}}$. We first observe that if p' is an interior point of D_3, due to $|p' - x| = d(x, D_3)$, we see that $(p' - x) /\!/ n(p')$ and hence $p' \in D_0$. This violates to $p' \in D_3$. On the other hand, if $p' \in \partial D_3$, then there exists $p_j \in \tilde{D}_0$ such that $p' \in \partial GB(p_j, r_1)$. Therefore, there exists $v \in T_{p_j}(\partial\Omega)$ with $|v| = r_1$ such that $Exp_{p_j}(v) = p'$. Applying Proposition 3.4, we derive that

$$|x - p_j|^2 + \frac{1}{2} r_1^2 \leq |p' - x|^2 \leq \frac{1}{2} r_1^2. \tag{3.22}$$

This implies $x = p_j$ which is a contradiction. We then conclude the claim.

It is easy to see that

$$\int_{D_3} \frac{1}{|x - y|^2} dA(y) \leq \frac{2}{r_1^2} \int_{\partial\Omega} dA(y) \leq \frac{2A_2}{r_1^2}. \tag{3.23}$$

On the other hand,

$$\begin{aligned} \int_{GB(p_i, r_1)} \frac{1}{|x - y|^2} dA(y) &\leq \int_0^{r_1} \int_0^{2\pi} \frac{1}{|x - p_i|^2 + \frac{1}{2} s^2} s ds d\theta \\ &\leq 2\pi \int_0^{\frac{1}{2} r_1^2} \frac{1}{d_x^2 + u} du \\ &\leq 4\pi \left| \ln d_x \right| + 2\pi \left| \ln \left(d_x^2 + \frac{1}{2} r_1^2 \right) \right| \\ &\leq C \left(1 + \left| \ln d_x \right| \right). \end{aligned} \tag{3.24}$$

Taking (3.23) and (3.24) into account, we prove Lemma 3.3 for the case where $d_x \leq r_1$. For the case where $d_x > r_1$, we may bound the left hand side of (3.3) by the right hand side of (3.23). This completes the proof of Lemma 3.3. □

Next, we investigate the estimates given in the following lemma.

Lemma 3.6. *Adopting the same geometric assumptions on Ω as stated in Definition 1.1, let r_1 be as defined by (3.14) in Lemma 3.3. Then, there exists a constant C such that for any $x, y \in \partial\Omega$ with $y \in GB(x, r_1)$, we have*

$$|n(x) \cdot (x - y)| \leq C|x - y|^2, \tag{3.25}$$

$$|n(y) \cdot (x - y)| \leq C|x - y|^2, \tag{3.26}$$

$$|n(y) \cdot v| = |n(y) \cdot (v - v')| \leq C|x - y|, \tag{3.27}$$

where $v \in T_x(\partial\Omega)$ is a unit vector and $v' \in T_y(\partial\Omega)$ is the parallel transport of v from $T_x(\partial\Omega)$ to $T_y(\partial\Omega)$.

The above lemma is crucial in the proofs of the Lemma 4.2 and Lemma 9.1. These geometric observations can resolve the difficulty from seemingly critical singularity (barely non-integrable) on a surface encountered in the proofs.

Proof. By choosing an appropriate coordinate system, we may assume $x = (0, 0, 0)$, $n(x) = (0, 0, 1)$ and $\phi(s)$ is the normal geodesic on $\partial\Omega$ connecting x and y within the geodesic disc $GB(x, r_1)$ such that

$$\begin{cases} \phi(0) = x, \\ \phi'(0) = (1, 0, 0), \\ Exp_x((\tau, 0, 0)) = \phi(\tau) = y. \end{cases}$$

Replacing s by τ in the estimates (3.11)–(3.13), we obtain

$$|x - y| \geq |\phi_1(\tau)| \geq \tau - \frac{1}{2}b\tau^2 \geq (1 - \frac{1}{2}br_1)\tau \geq \frac{1}{2}\tau, \quad \text{and}$$
$$|n(x) \cdot (x - y)| = |\phi_3(\tau)| \leq \frac{1}{2}b\tau^2 \leq 2b|x - y|^2.$$

This proves (3.25). By symmetry, (3.26) is derived from (3.25). To see (3.27), by replacing s by τ in (3.8)–(3.10), we obtain

$$\begin{aligned} |n(y) \cdot v| &= |n(y) \cdot (v - v')| \leq |v - v'| \\ &= |\phi'(0) - \phi'(\tau)| \leq \sqrt{3}b\tau \\ &\leq 2\sqrt{3}b|x - y|. \end{aligned}$$

Finally, we may choose $C = 4b$ so that (3.25)–(3.27) hold true. □

The next lemma is an important ingredient of the proof of the Hölder type estimate up to the boundary, Lemma 8.1.

Lemma 3.7. *Suppose Ω satisfies the positive curvature condition defined in Definition 1.1. Then, there exists $R_0 > 0$ depending only on Ω such that if $x \in \partial\Omega$, $y \in \Omega$, and*

$$d_y \leq R_0, \tag{3.28}$$

then a point $Y \in \partial\Omega$ such that $d(Y, y) = d_y$ is unique. Furthermore, there exist C_1', $C_2' > 0$ such that, for $y \in \Omega$ satisfying (3.28) and $x \in \partial\Omega$, if

$$n(Y) \cdot (x - y) \geq 0, \tag{3.29}$$

then

$$|x - y| \leq C_1' d_y^{\frac{1}{2}}, \tag{3.30}$$

or if

$$n(Y) \cdot (x - y) \leq 0, \tag{3.31}$$

then

$$|x - y| \geq C_2' d_y^{\frac{1}{2}}. \tag{3.32}$$

Proof. It is well known that there exists $R_1 > 0$ such that $d_y \leq R_1$ implies the existence of unique projection Y on $\partial\Omega$. The important task is to prove the second part of the lemma. Because of the assumption on Ω, there exist $R_3 > R_2 > 0$ such that for every point p on $\partial\Omega$ there exist a sphere $S_o(p)$ with radius R_3 and a sphere $S_i(p)$ with radius R_2 both tangent to $\partial\Omega$ at p and $S_o(p)$ contains the whole Ω and while $S_i(p)$ is contained completely within Ω. We let $R_0 = \min\{R_1, R_2\}$ and consider y with $d_y \leq R_0$. We name the centers of $S_o(Y)$ and $S_i(Y)$ as Y_o and Y_i respectively. Also, we name the plane perpendicular to $n(Y)$ passing through y as L. L divides $\partial\Omega$ into two components. If $n(Y) \cdot (x - y) \geq 0$ then x and Y fall in the same component. $\overrightarrow{yx}$ intersects $S_o(Y)$ at one point X'. Let L' be the plane passing x, y, and Y_o. There are two intersection points among L, L', and $S_o(Y)$. We name the one closer to X' as A. We can observe

$$|x - y| \leq |X' - y| \leq |A - y|. \tag{3.33}$$

Let $\theta = \angle AY_o y$. Then,

$$\cos\theta = \frac{|y - Y_o|}{|A - Y_o|} = \frac{R_3 - d_y}{R_3} = 1 - \frac{d_y}{R_3}. \tag{3.34}$$

Therefore,

$$\frac{d_y}{R_3} = 1 - \cos\theta = 2\sin^2\frac{\theta}{2}. \tag{3.35}$$

We obtain

$$\sin\frac{\theta}{2} = \sqrt{\frac{d_y}{2R_3}}. \tag{3.36}$$

On the other hand, we have

$$|A - y| = R_3 \sin\theta = 2R_3 \sin\frac{\theta}{2}\cos\frac{\theta}{2} \le \sqrt{2R_3}\sqrt{d_y}. \tag{3.37}$$

We finished the proof of (3.30). If $n(Y)\cdot(x - y) \le 0$, then x and Y are on different components. Let B be the intersection point between $\overline{xy}$ and $S_i(Y)$. We have

$$|x - y| > |y - B|. \tag{3.38}$$

We name the plane passing through y, Y_i, and x as L_1 and the plane perpendicular to $n(Y)$ passing through Y_i as L_2. There are two intersection points among L, L_1, and $S_i(Y)$. We denote the intersection point which is closer to B as U. If B lays between L and L_2, then

$$\angle BUy \ge \frac{\pi}{2}. \tag{3.39}$$

Therefore,

$$|B - y| \ge |U - y|. \tag{3.40}$$

Let $\theta' = \angle UY_i y$. Similarly, we have

$$\sin\frac{\theta'}{2} = \sqrt{\frac{d_y}{2R_2}} \tag{3.41}$$

Therefore,

$$|y - U| = R_2 \sin\theta' = 2R_2 \sin\frac{\theta'}{2}\cos\frac{\theta'}{2} \ge \sqrt{2}R_2 \sin\frac{\theta'}{2} \ge \sqrt{R_2 d_y}. \tag{3.42}$$

For the case Y and B are on different side of L_2,

$$|y - B| \ge R_2 \ge \sqrt{R_2 d_y}. \tag{3.43}$$

Hence, we finish the proof. □

4. Differentiability of B_ψ and B_T

By (1.10), the definition of ψ, we see that ψ is bounded whenever $f \in L^\infty_{x,\zeta}$. In this section, we shall further prove that the first derivatives of B_T and B_ψ are bounded provided T and ψ are bounded. By the differentiability on the boundary of Ω, we refer to the directional derivatives:

Definition 4.1. Let $x, \eta \in \mathbb{R}^3$ and D be a C^1 surface in $\mathbb{R}^3$ and $f : D \subset \mathbb{R}^3 \to \mathbb{R}$. Suppose $\phi : (-\epsilon, \epsilon) \to D$ is a smooth space curve such that

$$\phi(0) = x, \quad \frac{d}{dt}\phi(t)\bigg|_{t=0} = \eta. \tag{4.1}$$

We define

$$\nabla_\eta^x f(x) := \frac{d}{dt} f(\phi(t))\Big|_{t=0} \tag{4.2}$$

when the limit at right-hand-side exists.

Our first result in this section is the following lemma.

Lemma 4.2. *Suppose Ω satisfies the positive curvature condition defined in Definition 1.1 and $x \in \partial\Omega$. Suppose $T(x)$ and $\psi(x)$ are bounded. Then, the first derivatives of $B_T(x)$ and $B_\psi(x)$ are bounded.*

Recall that, for $x \in \partial\Omega$, we have

$$B_T(x) := 2\sqrt{\pi} \int\limits_{\zeta\cdot n>0} T(p(x,\zeta))(|\zeta|^2 - 2)M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n|d\zeta, \tag{4.3}$$

$$B_\psi(x) := 2\sqrt{\pi} \int\limits_{\zeta\cdot n>0} \psi(p(x,\zeta))M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n|d\zeta. \tag{4.4}$$

We shall only present the proof for B_ψ because the proof for B_T is similar. The following proposition gives a useful alternative formulation of B_ψ.

Proposition 4.3.

$$B_\psi(x) = \frac{2}{\pi}\int_0^\infty \int_{\partial\Omega} \psi(y)e^{-l^2|x-y|^2}e^{-\frac{\nu(l|x-y|)}{l}}[(x-y)\cdot n(x)]|(x-y)\cdot n(y)|l^3 dA(y)dl. \tag{4.5}$$

Proof. The idea of showing the equivalence between (4.4) and (4.5) is to do a change of coordinates. We first observe that, by the strictly convexity of Ω, for each ζ in the half space

$$H = \left\{\zeta \in \mathbb{R}^3 \middle| \zeta\cdot n(x) > 0\right\},$$

there exists exactly a unique pair $(y,l) \in \partial\Omega \times \mathbb{R}_+$ such that

$$\zeta = l(x-y). \tag{4.6}$$

Secondly, since the bounded set Ω is C^2 strictly convex, we can cover $\partial\Omega$ by finitely many local charts, i.e., for $1 \le i \le k$, there are

$$\begin{cases} \text{simply-connected open set } D_i \subset \mathbb{R}^2, \quad \text{and} \\ C^2 \text{ local diffeomorphism } \phi_i : D_i \to \partial\Omega \end{cases}$$

such that

$$\bigcup_{1\le i\le k} \phi_i(D_i) = \partial\Omega.$$

Summing up from the above observations, we can parametrize the half space H by a union of a finite number of cone domains. That is to plug $y = \phi_i(\alpha,\beta)$ into (4.6). This gives a coordinate change $\zeta_i : D_i \times (0,\infty) \to H$:

$$\zeta_i(\alpha,\beta,l) = l\big(x - \phi_i(\alpha,\beta)\big). \tag{4.7}$$

Denote $H_i := \zeta_i(D_i \times (0,\infty))$. We see that $H = \bigcup_{i=1}^k H_i$. On the other hand, direct calculation shows that the Jacobian of this coordinate change is given by

$$\left|\frac{\partial\zeta_i(\alpha,\beta,l)}{\partial(\alpha,\beta,l)}\right| = l^2\Big|\big(x - \phi_i(\alpha,\beta)\big)\cdot\big(\partial_\alpha\phi_i \times \partial_\beta\phi_i\big)\Big|. \tag{4.8}$$

We readily see that

$$\begin{aligned}
&2\sqrt{\pi}\int_{H_i}\psi(p(x,\zeta))M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n|d\zeta\\
&=\frac{2}{\pi}\int_0^\infty\int_{D_i}\psi(\phi_i(\alpha,\beta))e^{-l^2|x-\phi_i(\alpha,\beta)|^2}e^{-\frac{1}{l}\nu(l|x-\phi_i(\alpha,\beta)|)}\\
&\times[(x-\phi_i(\alpha,\beta))\cdot n(x)]|(x-\phi_i(\alpha,\beta))\cdot[\partial_\alpha\phi_i\times\partial_\beta\phi_i]|l^3d\alpha d\beta dl\\
&=\frac{2}{\pi}\int_0^\infty\int_{\phi_i(D_i)}\psi(y)e^{-l^2|x-y|^2}e^{-\frac{\nu(l|x-y|)}{l}}[(x-y)\cdot n(x)]|(x-y)\cdot n(y)|l^3dA(y)dl,
\end{aligned}\tag{4.9}$$

where $y=\phi(\alpha,\beta)$, $n(y)$ is the outward unit normal of $\partial\Omega$ at y and $A(y)$ is the surface element of $\partial\Omega$ at y.

Combining all the pieces and excluding the repetitions, we obtain the desired formula. □

Definition 4.4. For $x, y\in\Omega$ and $\zeta\in\mathbb{R}^3$, we define

$$\tau_-(x,\zeta):=\inf\{t>0|x-t\zeta\notin\Omega\},\tag{4.10}$$

$$p(x,\zeta):=x-\tau_-(x,\zeta)\zeta,\tag{4.11}$$

$$d_x:=\inf\big\{|x-y|\big|y\in\partial\Omega\big\},\tag{4.12}$$

$$d_{x,y}:=\min\{d_x,d_y\},\tag{4.13}$$

$$N(x,\zeta):=\frac{|n(p(x,\zeta))\cdot\zeta|}{|\zeta|}.\tag{4.14}$$

Now, we are ready to prove the Lemma 4.2

Proof of Lemma 4.2. Let $\alpha(t)$ be a normal geodesic and $v\in T_x(\partial\Omega)$, $|v|=1$ such that

$$\begin{aligned}
&\alpha(0)=x,\\
&\frac{d}{dt}\alpha(t)\Big|_{t=0}=v.
\end{aligned}\tag{4.15}$$

Then

$$\begin{aligned}
\nabla_v^x B_\psi(x)&:=\frac{d}{dt}B_\psi(\alpha(t))\Big|_{t=0}\\
&=\int_0^\infty\int_{\partial\Omega}\psi(y)e^{-l^2|x-y|^2}e^{-\frac{\nu(l|x-y|)}{l}}l^3\\
&\times\Big[\Big(-2l^2v\cdot(x-y)-\nu'(l|x-y|)\frac{v\cdot(x-y)}{|x-y|}\Big)(x-y)\cdot n(x)|(x-y)\cdot n(y)|\\
&+v\cdot n(x)|(x-y)\cdot n(y)|+(x-y)\cdot\frac{d}{dt}n(\alpha(t))\Big|_{t=0}|(x-y)\cdot n(y)|\\
&+(x-y)\cdot n(x)\mathrm{sgn}((x-y)\cdot n(y))(v\cdot n(y))\Big]dA(y)dl.
\end{aligned}\tag{4.16}$$

We note that due to the convexity of Ω, $\mathrm{sgn}((x-y)\cdot n(y))=-1$. Let $r_1>0$ be as defined by (3.14) in Proposition 3.4. Since Ω is a C^2, bounded and strictly convex domain, there exists a positive number r such that for each $x\in\partial\Omega$, we have

$$B_r(x)\cap\partial\Omega\subset GB(x,r_1).$$

We now break the domain of integration into two parts: $B_r(x) \cap \partial\Omega$ and $\partial\Omega \setminus B_r(x)$, and denote the corresponding integrals as $\nabla_v^x B_\psi^s$ and $\nabla_v^x B_\psi^l(x)$ respectively. First, we estimate $\nabla_v^x B_\psi^l(x)$. We notice that

(i) $v \cdot n(x) = 0$,
(ii) $|y - x| \geq r$, for $y \in \partial\Omega \setminus B_r(x)$,
(iii) $\frac{d}{dt} n(\alpha(t))$ is bounded due to smoothness and compactness of $\partial\Omega$,
(iv) $\nu'(l|x - y|) \leq C(1 + l|x - y|)^{\gamma - 1}$, which is uniformly bounded,
(v) as mentioned in the beginning of this section, ψ is bounded since $f \in L^\infty_{x,\xi}$ in our context.

Taking (i)–(v) into consideration, we obtain that

$$|\nabla_v^x B_\psi^l| \leq C \int_0^\infty \int_{\partial\Omega\setminus B_r(x)} e^{-r^2 l^2}(l^3 + l^5) dA(y) dl \leq C|\partial\Omega|. \tag{4.17}$$

Secondly, since $B_r(x) \cap \partial\Omega \subset GB(x, r_1)$, we may apply Lemma 3.6 to obtain

$$\begin{aligned} |\nabla_v^x B_\psi^s(x)| &\leq C \int_{GB(x,r_1)} \int_0^\infty e^{-l^2|x-y|^2}\big[l^5|x - y|^5 + l^3|x - y|^4 + l^3|x - y|^3\big] dl dA(y) \\ &\leq C \int_{GB(x,r_1)} \int_0^\infty e^{-z^2}\Big[\frac{z^5}{|x - y|} + z^3 + \frac{z^3}{|x - y|}\Big] dz dA(y) \\ &\leq C \int_{GB(x,r_1)} (1 + \frac{1}{|x - y|}) dA(y) \\ &\leq C \int_0^{r_1}\int_0^{2\pi} (1 + \frac{1}{r}) r d\theta dr \leq C. \end{aligned} \tag{4.18}$$

Notice that here $z = |x - y|l$ and (r, θ) are polar coordinates for the $T_x(\partial\Omega)$. Combining the estimates (4.17) and (4.18), the proof of Lemma 4.2 is complete. □

5. Hölder continuity of D_f

Recall that in (1.25) we define

$$D_f(x) := 2\sqrt{\pi} \int_{\zeta\cdot n>0} \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} K(f)(x - \zeta s, \zeta) M^{\frac{1}{2}}(\zeta)|\zeta \cdot n| ds d\zeta, \tag{5.1}$$

for $x \in \partial\Omega$. In this section, we shall prove the Hölder continuity of D_f. Let $\{n(x), e_2, e_3\}$ be an orthonormal basis of $T_x(\partial\Omega)$. We introduce spherical coordinates so that

$$\zeta = \rho\cos\theta n(x) + \rho\sin\theta\cos\phi e_2 + \rho\sin\theta\sin\phi e_3. \tag{5.2}$$

With the further coordinate change: $r = s\rho$, $\hat{\zeta} = \frac{\zeta}{|\zeta|}$, $y = x - r\hat{\zeta}$, we can rewrite D_f as

$$\begin{aligned} D_f &= 2\pi^{-\frac{1}{4}} \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{|\overline{p(x,\zeta)x}|} e^{-\frac{\nu(\rho)}{\rho} r} K(f)(x - r\hat{\zeta}, \zeta) e^{-\frac{|\zeta|^2}{2}} |\zeta \cdot n(x)| \rho \sin\theta dr d\phi d\theta d\rho \\ &= 2\pi^{-\frac{1}{4}} \int_0^\infty \int_\Omega e^{-\frac{\nu(\rho)}{\rho}|x-y|} K(f)(y, \rho\frac{(x - y)}{|x - y|}) \frac{(x - y)\cdot n(x)}{|x - y|} e^{-\frac{\rho^2}{2}} \frac{\rho^2}{|x - y|^2} dy d\rho. \end{aligned} \tag{5.3}$$

Lemma 5.1. *Suppose x_0 and x_1 are any two points on $\partial\Omega$. We have*

$$|D_f(x_0) - D_f(x_1)| \le C\|f\|_{L^\infty_{x,\zeta}} |x_0 - x_1| \left(1 + \left|\ln|x_0 - x_1|\right|\right). \tag{5.4}$$

Proof.

$$\begin{aligned} |D_f(x_0) - D_f(x_1)| \le& \left| 2\pi^{-\frac{1}{4}} \int_0^\infty \int_\Omega \left[K(f)(y, \rho\frac{(x_0-y)}{|x_0-y|}) - [K(f)(y, \rho\frac{(x_1-y)}{|x_1-y|}) \right] \right. \\ &\left. \times e^{-\frac{\nu(\rho)}{\rho}|x_0-y| - \frac{\rho^2}{2}} \rho^2 \frac{n(x_0)\cdot(x_0-y)}{|x_0-y|^3} dy d\rho \right| \\ &+ \left| 2\pi^{-\frac{1}{4}} \int_0^\infty \int_\Omega [K(f)(y, \rho\frac{(x_1-y)}{|x_1-y|}) \right. \\ &\left. \times \left[e^{-\frac{\nu(\rho)}{\rho}|x_0-y| - \frac{\rho^2}{2}} \rho^2 \frac{n(x_0)\cdot(x_0-y)}{|x_0-y|^3} - e^{-\frac{\nu(\rho)}{\rho}|x_1-y| - \frac{\rho^2}{2}} \rho^2 \frac{n(x_1)\cdot(x_1-y)}{|x_1-y|^3} \right] dy d\rho \right| \\ =:& \Delta D_{fK} + \Delta D_{fO}. \end{aligned} \tag{5.5}$$

We first estimate ΔD_{fK}. We break the domain of integration into two parts, $\Omega_1 = \Omega \cap B(x_0, 2|x_0 - x_1|)$ and $\Omega_2 := \Omega \setminus B(x_0, 2|x_0 - x_1|)$, and denote the corresponding integrals as ΔD^1_{fK} and ΔD^2_{fK} respectively. Because of smallness of the domain of integration, by (2.2) and (2.5), one may readily derive that

$$|\Delta D^1_{fK}| \le C\|f\|_{L^\infty_{x,\zeta}} |x_0 - x_1|. \tag{5.6}$$

To estimate D^2_{fK}, by employing the Lipschitz continuity of $K(f)$:

$$|K(f)(y, \zeta_1) - K(f)(y, \zeta_2)| \le C\|f\|_{L^\infty_{x,\zeta}} |\zeta_1 - \zeta_2|, \tag{5.7}$$

we get

$$\begin{aligned} &\left| K(f)(y, \rho\frac{(x_0-y)}{|x_0-y|}) - K(f)(y, \rho\frac{(x_1-y)}{|x_1-y|}) \right| \\ &\le C\|f\|_{L^\infty_{x,\zeta}} \left| \frac{\rho(x_0-y)}{|x_0-y|} - \frac{\rho(x_1-y)}{|x_1-y|} \right| \\ &\le C\rho\|f\|_{L^\infty_{x,\zeta}} \left| \frac{|x_1-y|(x_0-y) - |x_0-y|(x_1-y)}{|x_0-y||x_1-y|} \right| \\ &\le C\rho\|f\|_{L^\infty_{x,\zeta}} \left| \frac{|x_1-y|(x_0-x_1) + (|x_1-y| - |x_0-y|)(x_1-y)}{|x_0-y||x_1-y|} \right| \\ &\le C\frac{\rho|x_0-x_1|}{|x_0-y|} \|f\|_{L^\infty_{x,\zeta}}. \end{aligned} \tag{5.8}$$

Therefore,

$$\begin{aligned} |\Delta D^2_{fK}| &\le C|x_0-x_1|\|f\|_{L^\infty_{x,\zeta}} \int_0^\infty \int_{\Omega_2} e^{-\frac{\nu(\rho)}{\rho}|x_0-y| - \frac{\rho^2}{2}} \frac{\rho^3}{|x_0-y|^3} dy d\rho \\ &\le C|x_0-x_1|\|f\|_{L^\infty_{x,\zeta}} \int_0^\infty e^{-\frac{\rho^2}{2}} \rho^3 d\rho \int_{2|x_0-x_1|}^R \frac{1}{r} dr \\ &\le C|x_0-x_1|(1 + |\ln|x_0-x_1||)\|f\|_{L^\infty_{x,\zeta}}. \end{aligned} \tag{5.9}$$

Now, we proceed to estimate ΔD_{fO}. Suppose $0 < r < r_1$ as chosen in the proof of Lemma 4.2 so that for every $x \in \partial\Omega$, we have

$B_r(x) \cap \partial\Omega \subset GB(x, r_1)$.

CASE 1. If $2|x_0 - x_1| \geq r$, we see that

$$\frac{\Delta D_{fO}}{|x_0 - x_1|} \leq \frac{C\|f\|_{L^\infty_{x,\zeta}}}{r}. \tag{5.10}$$

CASE 2. In case of $2|x_0 - x_1| < r$, we split the domain of integration into two parts: $\Omega_1 = \Omega \cap B(x_0, 2|x_0 - x_1|)$ and $\Omega_2 := \Omega \setminus B(x_0, 2|x_0 - x_1|)$, and denote the corresponding integrals as ΔD^1_{fO} and ΔD^2_{fO} respectively. Due to smallness of domain of integration, we have

$$|\Delta D^1_{fO}| \leq C\|f\|_{L^\infty_{x,\zeta}} |x_0 - x_1|. \tag{5.11}$$

To estimate ΔD^2_{fO}, we let $x(t) \subset GB(x_0, r_1) \subset \partial\Omega$ be the normal geodesic connecting x_0 and x_1 with

$$x(0) = x_0, \ \ x(s) = x_1, \tag{5.12}$$

where s is the geodesic distance between x_0 and x_1 on $\partial\Omega$. We observe that

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{-\frac{\nu(\rho)}{\rho}|x(t)-y|}\frac{n(x(t))\cdot(x(t)-y)}{|x(t)-y|^3}\Big) \\
&\quad = -\frac{\nu(\rho)}{\rho}\frac{x'(t)\cdot(x(t)-y)}{|x(t)-y|}e^{-\frac{\nu(\rho)}{\rho}|x(t)-y|}\frac{n(x(t))\cdot(x(t)-y)}{|x(t)-y|^3} \\
&\qquad + \Big(\frac{(\frac{d}{dt}n(x(t)))\cdot(x(t)-y)}{|x(t)-y|^3} - 3\frac{\big(n(x(t))\cdot(x(t)-y)\big)\big(x'(t)\cdot(x(t)-y)\big)}{|x(t)-y|^5}\Big) \\
&\qquad\quad \times e^{-\frac{\nu(\rho)}{\rho}|x(t)-y|}.
\end{aligned}$$

Hence,

$$\Big|\frac{d}{dt}\Big(e^{-\frac{\nu(\rho)}{\rho}|x(t)-y|}\frac{n(x(t))\cdot(x(t)-y)}{|x(t)-y|^3}\Big)\Big| \leq Ce^{-\frac{\nu(\rho)}{\rho}|x(t)-y|}\Big(\frac{1}{|x(t)-y|^3} + (1+\frac{\nu(\rho)}{\rho})\frac{1}{|x(t)-y|^2}\Big).$$

By the fundamental theorem of calculus, we derive that

$$\begin{aligned}
\Delta D^2_{fO} &= \Big|2\pi^{-\frac{1}{4}}\int_0^\infty\int_{\Omega_2} K(f)(y, \frac{x_1-y}{|x_1-y|}\rho)\int_0^s \frac{d}{dt}\Big(e^{-\frac{\nu(\rho)}{\rho}|x(t)-y|-\frac{\rho^2}{2}}\rho^2\frac{n(x(t))\cdot(x(t)-y)}{|x(t)-y|^3}\Big)dtdyd\rho\Big| \\
&\leq C\|f\|_{L^\infty_{x,\zeta}}\Big|\int_0^s\int_0^\infty \rho(1+\rho)e^{-\frac{\rho^2}{2}}\int_{\Omega_2}(\frac{1}{|x(t)-y|^3}+\frac{1}{|x(t)-y|^2})dyd\rho dt\Big| \\
&\leq C\|f\|_{L^\infty_{x,\zeta}}\Big|\int_0^s\int_0^\infty \rho(1+\rho)e^{-\frac{\rho^2}{2}}\int_0^{2\pi}\int_0^{\pi}\int_{|x_0-x_1|}^{R}(1+\frac{1}{r})\sin\theta drd\theta d\phi d\rho dt\Big| \\
&\leq C\|f\|_{L^\infty_{x,\zeta}}|x_0-x_1|\big(1+\big|\ln|x_0-x_1|\big|\big),
\end{aligned} \tag{5.13}$$

where we have used the inequality

$$s \leq \frac{8}{7}|x_0 - x_1|$$

addressed in Remark 3.5. Combining the above estimates of ΔD^1_{fK}, ΔD^2_{fK}, ΔD^1_{fO} and ΔD^2_{fO}, the proof of Lemma 5.1 is complete. □

6. Hölder type estimates revisited

From the previous sections, we in fact can already claim interior Hölder continuity by an argument similar to [6]. In order to further bootstrap the regularity to differentiability, those estimates need to be significantly refined. In this section, we are going to prepare some estimates for I and II defined in (1.18). In the next section, Section 7, we will further improve the Mixture Lemma for stationary solution introduced in [6]. This series of discussions of Hölder continuity for Boltzmann equation was started from the observation for one spacial demensional problem in [4].

Our goal in this section is to prove the following lemma.

Lemma 6.1. *Assume that Ω satisfies the geometric assumptions defined in Definition 1.1. Suppose that for fixed $0<\epsilon<\frac{1}{6}$, there exist $a,\ M_0>0$ such that*

$$|f(X,\zeta)-f(Y,\zeta)|\leq M_0|X-Y|^{1-\epsilon}e^{-a|\zeta|^2}, \tag{6.1}$$

$$|f(X,\zeta)|\leq M_0e^{-a|\zeta|^2}, \tag{6.2}$$

for all $(X,\zeta),\ (Y,\zeta)\in\Gamma_-$. Then, there exists a constant C such that for any $x,\ y\in\Omega$,

$$|I(x,\zeta)-I(y,\zeta)|\leq C\frac{1}{d_{x,y}}|x-y|^{1-\epsilon}e^{-\frac{a}{2}|\zeta|^2}, \tag{6.3}$$

$$|I(x,\zeta)-I(y,\zeta)|\leq C\left(\frac{|x-y|^{1-\epsilon}}{N(x,\zeta)}+\frac{|x-y|}{N(x,\zeta)|\zeta|}+\frac{|x-y|^{1-\epsilon}}{N(y,\zeta)}+\frac{|x-y|}{N(y,\zeta)|\zeta|}\right)e^{-a|\zeta|^2} \tag{6.4}$$

$$\int_{\mathbb{R}^3}|k(\zeta,\zeta')||I(x)-I(y)|d\zeta'\leq C(1+d_{x,y}^{-1})^{\frac{1}{3}}(\left|\ln d_{x,y}\right|+1)|x-y|^{1-\epsilon}, \tag{6.5}$$

where I is defined in (1.18).

We need some observations in geometry to prove the above lemma.

Proposition 6.2. *Let x and y be interior points of Ω. We denote $p(x,\zeta)$ and $p(y,\zeta)$ by X and Y respectively. Then*

$$|x-X|\geq\frac{d_x}{N(x,\zeta)}. \tag{6.6}$$

Further more, if $|x-X|\leq|y-Y|$, then

$$|X-Y|\leq\frac{1}{N(x,\zeta)}|x-y|, \tag{6.7}$$

$$\big||x-X|-|y-Y|\big|\leq\frac{2}{N(x,\zeta)}|x-y|. \tag{6.8}$$

Proof. Let us first prove (6.6). Let F be the projection of x on the tangent plane $T_X(\partial\Omega)$. Because of convexity, $\overline{xF}$ intersects $\partial\Omega$ at one point F'. Then,

$$|x-X|N(x,\zeta)=|x-F|\geq|x-F'|\geq d_x, \tag{6.9}$$

which implies (6.6).

When $(x-y)/\!/\zeta$, (6.7) and (6.8) are trivial. If not, we let

$$e_1:=\frac{\zeta}{|\zeta|}, \tag{6.10}$$

$$e_3:=\frac{e_1\times(y-x)}{|e_1\times(y-x)|}, \tag{6.11}$$

$$e_2:=e_3\times e_1. \tag{6.12}$$

Also, we denote

$$n_1 = n(X) \cdot e_1, \tag{6.13}$$

$$n_2 = n(X) \cdot e_2, \tag{6.14}$$

$$n_3 = n(X) \cdot e_3 \tag{6.15}$$

$$n' = n_1 e_1 + n_2 e_2. \tag{6.16}$$

Notice that $n_1^2 + n_2^2 + n_3^2 = 1$ and $N(x, \zeta) = |n_1|$. Let E be the plane containing x, y, X and Y and $\Gamma^* = \partial\Omega \cap E$. We are going to discuss plane geometry on the plane E. Since $|X - x| \le |Y - y|$, the point $y^* := y + X - x$ lies on the line segment $\overline{yY}$. If $y^* = Y$, it is obvious that (6.7) and (6.8) hold true. In what follows, we assume $y^* \neq Y$. Due to the convexity of Ω, the tangent line of Γ^* passing X would intersect the half line $\overrightarrow{yY}$ at a single point Y^*. For the sake of convenience, we define

$$\theta_1 = \angle XYy^*, \quad \theta_2 = \angle XY^*Y, \quad \theta_3 = \angle Yy^*X.$$

By the law of sines, we see that

$$\frac{\overline{Xy^*}}{\sin\theta_1} = \frac{\overline{XY}}{\sin\theta_3} \qquad \text{and} \qquad \frac{\overline{Xy^*}}{\sin\theta_2} = \frac{\overline{XY^*}}{\sin\theta_3}.$$

In case $\theta_1 \ge \frac{\pi}{2}$ it is obvious that

$$\overline{XY} < \overline{Xy^*} = \overline{xy}. \tag{6.17}$$

In case $\theta_1 < \frac{\pi}{2}$, by monotonicity of sine function on the interval $[0, \frac{\pi}{2}]$ and the fact $\theta_1 > \theta_2$, we see

$$\overline{XY} = \frac{\sin\theta_2}{\sin\theta_1}\overline{XY^*} < \overline{XY^*} < \frac{1}{\sin\theta_2}\overline{Xy^*} = \frac{1}{\sin\theta_2}\overline{xy}. \tag{6.18}$$

On the other hand, one may readily see that

$$\sin\theta_2 = \left| e_1 \cdot \frac{n'}{|n'|} \right| = \left| \frac{n_1}{\sqrt{n_1^2 + n_2^2}} \right|.$$

Summing up of (6.17) and (6.18), in any case, we obtain that

$$|X - Y| \le \frac{1}{\sin\theta_2}|x - y| \le \frac{1}{|n_1|}|x - y| = \frac{1}{N(x, \zeta)}|x - y|. \tag{6.19}$$

Finally,

$$\big||x - X| - |y - Y|\big| \le |X - Y| + |x - y| \le \frac{2}{N(x, \zeta)}|x - y|. \tag{6.20}$$

This completes the proof of Proposition 6.2. □

Next, we shall prove the following proposition which has been mentioned in [6].

Proposition 6.3. *Let Ω be a C^1 bounded convex domain in $\mathbb{R}^3$. Suppose $x \in \Omega$, $X = p(x, \zeta) \in \partial\Omega$, and $z \in \overline{xX}$. Then,*

$$d_z \ge \frac{d_x}{R}|z - X|, \tag{6.21}$$

where R is the diameter of Ω.

Proof. We denote a point on $\partial\Omega$ that realizes d_z by Z. Let L_x be the plane passing x perpendicular to ζ. Let L_Z be the plane passing Z, z, and x. We denote the intersection point of L_x, L_Z, and $\partial\Omega$ on the same side with Z on L_Z to $\overline{Xx}$ by A. Let $\theta_1 = \angle ZXz$, $\theta_2 = \angle XZz$, and $\theta_1' = \angle AXz$. Due to the convexity of Ω, we have

$$1 \geq \sin\theta_1 \geq \sin\theta_1' = \frac{|A-x|}{|A-X|} \geq \frac{d_x}{R}, \tag{6.22}$$

where R is the diameter of Ω. By the law of sines,

$$d_z \geq d_z \sin\theta_2 = |X-z|\sin\theta_1 \geq |X-z|\frac{d_x}{R}. \tag{6.23}$$

This concludes the proposition. □

Now, we are ready to prove Lemma 6.1.

Proof. Without loss of generality, we may assume $|X-x| \leq |Y-y|$. Hence,

$$\begin{aligned} |I(x,\zeta) - I(y,\zeta)| &= |f(X,\zeta)e^{-\nu(|\zeta|)\frac{|X-x|}{|\zeta|}} - f(Y,\zeta)e^{-\nu(|\zeta|)\frac{|Y-y|}{|\zeta|}}| \\ &\leq |f(X,\zeta) - f(Y,\zeta)|e^{-\nu(|\zeta|)\frac{|X-x|}{|\zeta|}} + |f(Y,\zeta)|\left|e^{-\nu(|\zeta|)\frac{|X-x|}{|\zeta|}} - e^{-\nu(|\zeta|)\frac{|Y-y|}{|\zeta|}}\right| \\ &\leq Ce^{-a|\zeta|^2}\left((\frac{|x-y|}{N(x,\zeta)})^{1-\epsilon} + (\frac{\nu(|\zeta|)|x-y|}{N(x,\zeta)|\zeta|})\right)e^{-\nu(|\zeta|)\frac{d_x}{N(x,\zeta)|\zeta|}} \\ &\leq Ce^{-a|\zeta|^2}\left((\frac{|\zeta||x-y|}{d_x})^{1-\epsilon} + (\frac{|x-y|}{d_x})\right) \\ &\leq Cd_x^{-1}|x-y|^{1-\epsilon}e^{-\frac{a}{2}|\zeta|^2}. \end{aligned} \tag{6.24}$$

Notice that we have used the mean value theorem and Proposition 6.2 in the above estimate. We observe that the third line of (6.24) gives

$$|I(x,\zeta) - I(y,\zeta)| \leq C\left(\frac{|x-y|^{1-\epsilon}}{N(x,\zeta)} + \frac{|x-y|}{N(x,\zeta)|\zeta|}\right)e^{-a|\zeta|^2}. \tag{6.25}$$

Due to the symmetry of x and y, by (6.24) and (6.25), we obtain (6.3) and (6.4). To prove (6.5), we first divide the domain of integration into two:

$$B_0 := \{\zeta' \in \mathbb{R}^3 \,|\, |\zeta - \zeta'| < d_{x,y}^{\frac{1}{3}}\}, \tag{6.26}$$

$$B_0^c := \mathbb{R}^3 \setminus B_0. \tag{6.27}$$

We denote the corresponding integrals by $K\Delta I_1$ and $K\Delta I_2$ respectively. Using (6.3), we have

$$\begin{aligned} |K\Delta I_1| &\leq C\int_{B_0} \frac{|x-y|^{1-\epsilon}}{|\zeta-\zeta'|d_{x,y}}e^{-\frac{a}{2}|\zeta'|^2}d\zeta' \\ &\leq C\frac{|x-y|^{1-\epsilon}}{d_{x,y}}\int_0^{d_{x,y}^{\frac{1}{3}}}\int_0^{\pi}\int_0^{2\pi} r\sin\theta d\phi d\theta dr \\ &\leq Cd_{x,y}^{-\frac{1}{3}}|x-y|^{1-\epsilon}, \end{aligned} \tag{6.28}$$

where we used the spherical coordinates centered at ζ in the above inequality.

Using (6.4) and changing variable similar to Section 4, we have

$$
\begin{aligned}
|K\Delta I_2| &\le C\int_{B_0^c}\frac{e^{-a|\zeta'^2|}}{d_{x,y}^{\frac13}}\left(\frac{|x-y|^{1-\epsilon}}{N(x,\zeta')}+\frac{|x-y|}{N(x,\zeta')|\zeta'|}+\frac{|x-y|^{1-\epsilon}}{N(y,\zeta')}+\frac{|x-y|}{N(y,\zeta')|\zeta'|}\right)d\zeta' \\
&\le C\int_0^\infty\int_{\partial\Omega}\frac{e^{-al^2|x-z|^2}}{d_{x,y}^{\frac13}}\left(\frac{|x-z||x-y|^{1-\epsilon}}{|(x-z)\cdot n(z)|}+\frac{|x-y|}{|(x-z)\cdot n(z)|l}\right)l^2|(x-z)\cdot n(z)|dA(z)dl \\
&\quad +C\int_0^\infty\int_{\partial\Omega}\frac{e^{-al^2|y-z|^2}}{d_{x,y}^{\frac13}}\left(\frac{|y-z||x-y|^{1-\epsilon}}{|(y-z)\cdot n(z)|}+\frac{|x-y|}{|(y-z)\cdot n(z)|l}\right)l^2|(y-z)\cdot n(z)|dA(z)dl \\
&\le \frac{C}{d_{x,y}^{\frac13}}\int_{\partial\Omega}\int_0^\infty e^{-al^2|x-z|^2}\left(l^2|x-z||x-y|^{1-\epsilon}+l|x-y|)\right)dldA(z) \\
&\quad +\frac{C}{d_{x,y}^{\frac13}}\int_{\partial\Omega}\int_0^\infty e^{-al^2|y-z|^2}\left(l^2|y-z||x-y|^{1-\epsilon}+l|x-y|)\right)dldA(z) \\
&\le \frac{C}{d_{x,y}^{\frac13}}\int_{\partial\Omega}\int_0^\infty e^{-as^2}\left(s^2|x-y|^{1-\epsilon}+s|x-y|)\right)ds\left(\frac{1}{|x-z|^2}+\frac{1}{|y-z|^2}\right)dA(z) \\
&\le \frac{C}{d_{x,y}^{\frac13}}|x-y|^{1-\epsilon}\int_{\partial\Omega}\left(\frac{1}{|x-z|^2}+\frac{1}{|y-z|^2}\right)dA(z) \\
&\le \frac{C}{d_{x,y}^{\frac13}}|x-y|^{1-\epsilon}(\left|\ln d_x\right|+\left|\ln d_y\right|+1)\le\frac{C}{d_{x,y}^{\frac13}}|x-y|^{1-\epsilon}(\left|\ln d_{x,y}\right|+1).
\end{aligned}
\tag{6.29}
$$

Notice that, in the above estimates, we changed the variable $s=|x-z|l$, $s=|y-z|l$ and applied Lemma 3.3. The proof of Lemma 6.1 is complete. □

7. Regularity due to mixing

We shall elaborate the smoothing effect due to the combination of collision and transport in this section. In the following proposition, we improve the estimate in [6] from Hölder continuity with order $\frac12$- to almost Lipschitz continuous.

Proposition 7.1. *Suppose $f\in L^\infty_{x,\zeta}$ is a solution to the stationary linearized Boltzmann equation. Then, for all $x_0,x_1\in\Omega$ and $\zeta\in\mathbb{R}^3$,*

$$
|G(x_0,\zeta)-G(x_1,\zeta)|\le C\|f\|_{L^\infty_{x,\zeta}}|x_0-x_1|(1+\left|\ln|x_0-x_1|\right|). \tag{7.1}
$$

Proof. As in [6], we observe that

$$
\begin{aligned}
|G(x_0,\zeta)-G(x_1,\zeta)| &\le \left|\int_0^\infty\int_\Omega k\left(\zeta,\rho\frac{(x_0-y)}{|x_0-y|}\right)\frac{\rho e^{-\nu(\rho)\frac{|x_0-y|}{\rho}}}{|x_0-y|^2}\right. \\
&\quad\times\left.\left[K(f)\left(y,\rho\frac{(x_0-y)}{|x_0-y|}\right)-K(f)\left(y,\rho\frac{(x_1-y)}{|x_1-y|}\right)\right]dyd\rho\right| \\
&\quad+\left|\int_0^\infty\int_\Omega K(f)\left(y,\rho\frac{(x_1-y)}{|x_1-y|}\right)\right.
\end{aligned}
\tag{7.2}
$$

$$\times \left[k\left(\zeta, \rho\frac{(x_0-y)}{|x_0-y|}\right)\frac{\rho e^{-\nu(\rho)\frac{|x_0-y|}{\rho}}}{|x_0-y|^2} - k\left(\zeta, \rho\frac{(x_1-y)}{|x_1-y|}\right)\frac{\rho e^{-\nu(\rho)\frac{|x_1-y|}{\rho}}}{|x_1-y|^2} \right] dy d\rho \Bigg|$$

$$=: G_K + G_O.$$

The estimate for G_O has already been done in [6]. We only need to estimate G_K. We break the domain of integration into two, $\Omega_1 = \Omega \cap B(x_0, 2|x_0-x_1|)$ and $\Omega_2 := \Omega \setminus B(x_0, 2|x_0-x_1|)$, and name the corresponding integrals G_K^1 and G_K^2 respectively. Because of smallness of the domain of integration, we have

$$|G_K^1| \le C\|f\|_{L^\infty_{x,\zeta}}|x_0-x_1|. \tag{7.3}$$

To deal with G_K^2, we need to use the Lipschitz continuity of $K(f)$ (5.8),

$$\left| K(f)\left(y, \rho\frac{(x_0-y)}{|x_0-y|}\right) - K(f)\left(y, \rho\frac{(x_1-y)}{|x_1-y|}\right) \right| \le C\rho\frac{|x_0-x_1|}{|x_0-y|}\|f\|_{L^\infty_{x,\zeta}}.$$

Therefore, by taking the change of coordinates

$$y - x_0 = (r\cos\theta, r\sin\theta\cos\phi, r\sin\theta\sin\phi),$$

we see that

$$\begin{aligned} |G_K^2| &\le C|x_0-x_1|\|f\|_{L^\infty_{x,\zeta}} \int_0^\infty \int_{\Omega_2} \left| k\left(\zeta, \rho\frac{(x_0-y)}{|x_0-y|}\right)\right| \frac{\rho^2 e^{-\nu(\rho)\frac{|x_0-y|}{\rho}}}{|x_0-y|^3} dy d\rho \\ &\le C|x_0-x_1|\|f\|_{L^\infty_{x,\zeta}} \int_{\mathbb{R}^3} |k(\zeta,\zeta')| \int_{2|x_0-x_1|}^{R} \frac{1}{r} dr d\zeta' \\ &\le C|x_0-x_1|\left(1 + \big|\ln|x_0-x_1|\big|\right)\|f\|_{L^\infty_{x,\zeta}}. \quad \square \end{aligned} \tag{7.4}$$

With the above proposition, we can prove the following estimate.

Proposition 7.2. *Suppose* Ω *satisfies the geometric assumptions defined in Definition 1.1 and* $\epsilon > 0$. *Then, the following inequality holds*

$$|III(x,\zeta) - III(y,\zeta)| \le C\|f\|_{L^\infty_{x,\zeta}}(1 + d_{x,y}^{-1})|x-y|^{1-\epsilon}. \tag{7.5}$$

Proof. Let $X = p(x,\zeta)$ and $Y = p(y,\zeta)$. We will demonstrate the proof for the case when $|X-x| \le |Y-y|$. The other case can be proved in the same fashion. Noting that

$$\tau_-(x,\zeta) = \frac{|X-x|}{|\zeta|} \text{ and } \tau_-(y,\zeta) = \frac{|Y-y|}{|\zeta|},$$

we have

$$\begin{aligned} |III(x,\zeta) - III(y,\zeta)| &\le \int_0^{\frac{|X-x|}{|\zeta|}} e^{-\nu(|\zeta|)s}|G(x-s\zeta,\zeta) - G(y-s\zeta,\zeta)|ds \\ &\quad + \int_{\frac{|X-x|}{|\zeta|}}^{\frac{|Y-y|}{|\zeta|}} e^{-\nu(|\zeta|)s}|G(y-s\zeta,\zeta)|ds \\ &=: \Delta_1 + \Delta_2. \end{aligned} \tag{7.6}$$

By applying Proposition 7.1, we have

$$\Delta_1 \le C\|f\|_\infty |x-y|^{1-\epsilon}. \tag{7.7}$$

On the other hand

$$\begin{aligned}\Delta_2 &\le C\|f\|_\infty \int_{\frac{|X-x|}{|\zeta|}}^{\frac{|Y-y|}{|\zeta|}} e^{-\nu(|\zeta|)s} ds \\ &\le C\|f\|_\infty \left| e^{\frac{-\nu(|\zeta|)}{|\zeta|}|X-x|} - e^{-\frac{\nu(|\zeta|)}{|\zeta|}|Y-y|} \right| \\ &\le C\|f\|_\infty e^{\frac{-\nu(|\zeta|)}{|\zeta|}|X-x|} \left| \frac{\nu(|\zeta|)}{|\zeta|}(|Y-y|-|X-x|) \right|,\end{aligned} \tag{7.8}$$

where we have applied the mean value theorem and used the assumption $|X-x| \le |Y-y|$. We the apply Proposition 6.2 and get

$$\begin{aligned}\Delta_2 &\le C\|f\|_\infty e^{-\frac{\nu(|\zeta|)}{|\zeta|}\frac{d_x}{N(x,\zeta)}} \frac{\nu(|\zeta|)}{|\zeta|}\frac{|x-y|}{N(x,\zeta)} \\ &\le C\|f\|_\infty \frac{|x-y|}{d_x}.\end{aligned} \tag{7.9}$$

We can treat the case when $|Y-y| \le |X-x|$ similarly and conclude

$$\Delta_2 \le C\|f\|_\infty \frac{|x-y|}{d_{x,y}}. \tag{7.10}$$

By (7.7) and (7.10), we obtain (7.5) and the proof of Proposition 7.2 is complete. □

8. Behavior near the boundary

In this section, we investigate the behavior of f near the boundary. This is a preparation for proving the differentiability of D_f.

Lemma 8.1. *Assume Ω satisfied the positive curvature condition defined in Definition 1.1. Assume $f \in L^\infty_{x,\zeta}$ is a solution to the stationary linearized Boltzmann equation such that for fixed $0 < \epsilon < \frac{1}{6}$, there exist $0 < a,\ M_0 > 0$ such that*

$$|f(X,\zeta) - f(Y,\zeta)| \le M_0 |X-Y|^{1-\epsilon} e^{-a|\zeta|^2}, \tag{8.1}$$

$$|f(X,\zeta)| \le M_0 e^{-a|\zeta|^2} \tag{8.2}$$

for all $(X,\zeta),\ (Y,\zeta) \in \Gamma_-$. Then, for $x \in \partial\Omega$ and $y \in \Omega$,

$$|f(x,\zeta) - f(y,\zeta)| \le C\left(1 + \frac{1}{|\zeta|}\right)|x-y|^{\frac{1}{2}(1-\epsilon)}. \tag{8.3}$$

The above estimate not only gives a description on how singular f can be when ζ is small near the boundary but also plays an important role in proving the differentiability of boundary flux and therefore the solution itself, which will be elaborated in next section.

Proof. Without loss of generality, we may only consider the case $|x-y| < \min(\frac{1}{5}R_0, 1)$, where R_0 is introduced in Lemma 3.7. This case gives the unique projection of y on $\partial\Omega$ denoted by $Y_\perp$, i.e., $|y - Y_\perp| = d_y$.

STEP 1. We start with the case $\zeta \cdot n(Y_\perp) < 0$ and denote $Y_0 = p(y,\zeta)$. We observe that

$$|x - Y_0| \le |x-y| + |y - Y_0| \le C|x-y|^{\frac{1}{2}}, \tag{8.4}$$

where we have applied Lemma 3.7 by taking $x = Y_0 \in \partial\Omega$ in (3.30) and used the fact $|x-y| \le 1$.

Noting that $x, Y_0 \in \partial\Omega$, by (8.1) and the mean value theorem, we see that

$$\begin{aligned} |f(x,\zeta)-f(y,\zeta)| &= \left| f(x,\zeta) - f(Y_0,\zeta)e^{-\nu(|\zeta|)\frac{|y-Y_0|}{|\zeta|}} - \int_0^{\frac{|y-Y_0|}{|\zeta|}} e^{-\nu(|\zeta|)t} K(f)(y-\zeta t,\zeta)dt \right| \\ &\le |f(x,\zeta)-f(Y_0,\zeta)| + |f(Y_0,\zeta)| \left|1 - e^{-\nu(|\zeta|)\frac{|y-Y_0|}{|\zeta|}}\right| \\ &\quad + \left| \int_0^{\frac{|y-Y_0|}{|\zeta|}} e^{-\nu(|\zeta|)t} K(f)(y-\zeta t,\zeta)dt \right| \\ &\le M_0|x-Y_0|^{1-\epsilon} + C\|f\|_{L^\infty_{x,\zeta}} \frac{|y-Y_0|}{|\zeta|} \\ &\le C\left(1+\frac{1}{|\zeta|}\right)|x-y|^{\frac{1}{2}(1-\epsilon)}. \end{aligned} \tag{8.5}$$

Notice that we only used the fact $|e^{-a|\zeta|^2}| \le 1$ in the above estimate. The assumptions in Lemma 8.1 are the same with those in Lemma 6.1, which will be used in the proof.

STEP 2. In case $\zeta \cdot n(Y_\perp) \ge 0$, we define

$$D_0 = \{b \in \Omega | d(b,\partial\Omega) \ge 4|x-y|\}, \tag{8.6}$$

$$D_1 = \{b \in \Omega | d(b,\partial\Omega) \ge 5|x-y|\}. \tag{8.7}$$

Notice that D_0 is also a smooth convex domain. We denote $Y_0 = p(y,\zeta)$ and $X_0 = p(x,\zeta)$. If either $\overline{xX_0}$ or $\overline{yY_0}$ intersects ∂D_0 less than twice, we can conclude that

$$|x-X_0| \le C|x-y|^{\frac{1}{2}}, \ |y-Y_0| \le C|x-y|^{\frac{1}{2}}. \tag{8.8}$$

Therefore, we can show (8.3) holds as we proved in Step 1. Namely, if one of $\overline{xX_0}$ or $\overline{yY_0}$ intersects ∂D_0 less than twice, then both of $\overline{xX_0}$ and $\overline{yY_0}$ intersect ∂D_1 less than twice. Hence, by the proof of Lemma 3.7, we have

$$|x-X_0| \le C_1'(5|x-y|)^{\frac{1}{2}} \text{ and} \tag{8.9}$$

$$|y-Y_0| \le C_1'(5|x-y|)^{\frac{1}{2}}. \tag{8.10}$$

This implies (8.4), and hence (8.3) holds.

Next, we shall discuss the case that both line segments $\overline{xX_0}$ and $\overline{yY_0}$ intersect ∂D_0 twice. Let X_1 and X_2 be intersection points of $\overline{xX_0}$ and ∂D_0 in the order x, X_1, X_2, and X_0 on $\overline{xX_0}$. Also let Y_1 and Y_2 be intersection points of $\overline{yY_0}$ and ∂D_0 respectively. Let $X_{1\perp}$ be the unique projection of X_1 on $\partial\Omega$. Since $|X_1 - X_{1\perp}|$ realizes the distance between $\partial\Omega$ and ∂D_0, we have

$$n(X_{1\perp}) = n(X_1) = \frac{(X_{1\perp}-X_1)}{|X_{1\perp}-X_1|}. \tag{8.11}$$

Taking the tangent plane of ∂D_0 at X_1 into consideration and applying Lemma 3.7, we have

$$|X_1 - x| \le 4C_1'|x-y|^{\frac{1}{2}}. \tag{8.12}$$

By the same fashion, we have

$$|X_0 - X_2| \le 4C_1'|x-y|^{\frac{1}{2}}. \tag{8.13}$$

$$|Y_0 - Y_2| \le 4C_1'|x-y|^{\frac{1}{2}}, \tag{8.14}$$

$$|y - Y_1| \le 4C_1'|x-y|^{\frac{1}{2}}. \tag{8.15}$$

In what follows, we shall discuss estimates for the differences of I, II, and III defined on (1.18).

Regarding the estimate for I, we start with the following claim.

$$|X_0 - Y_0| \le C|x-y|^{\frac{1}{2}}, \tag{8.16}$$

$$\big||X_0 - x| - |Y_0 - y|\big| \le C|x-y|^{\frac{1}{2}}. \tag{8.17}$$

Without loss of generality, we may assume $|Y_0 - y| < |X_0 - x|$. Let $X' = Y_0 + (x - y)$. Notice that since $|x-y| < 4|x-y|$ and $|Y_0 - y| < |X_0 - x|$, we see that $X' \in \overline{X_2X_0}$. The unique projection of X' on $\partial\Omega$ is denoted by $X'_\perp$. Notice that by the convexity of the set $\{b|d(b,\partial\Omega) \ge d_{X'}\}$, we have $n(X'_\perp)\cdot(X_0 - X') \ge 0$. Therefore, we can apply Lemma 3.7 and prove the claim. Now, we have

$$\begin{aligned}
|I(x,\zeta) - I(y,\zeta)| &= \left| f(X_0,\zeta)e^{-\nu(|\zeta|)\frac{|X_0-x|}{|\zeta|}} - f(Y_0,\zeta)e^{-\nu(|\zeta|)\frac{|Y_0-y|}{|\zeta|}} \right| \\
&\le \left| f(X_0,\zeta)e^{-\nu(|\zeta|)\frac{|X_0-x|}{|\zeta|}} \right| \left| 1 - e^{-\nu(|\zeta|)\frac{|Y_0-y|-|X_0-x|}{|\zeta|}} \right| \\
&\quad + |f(X_0,\zeta) - f(Y_0,\zeta)| e^{-\nu(|\zeta|)\frac{|Y_0-y|}{|\zeta|}} \\
&\le C\left(1 + \frac{1}{|\zeta|}\right)|x-y|^{\frac{1}{2}(1-\epsilon)}.
\end{aligned} \tag{8.18}$$

Similarly, for III, without loss of generality we may assume $|Y_0 - y| \le |X_0 - x|$. By applying Proposition 7.1, we see that

$$\begin{aligned}
|III(x,\zeta) - III(y,\zeta)| &\le \left| \int_0^{\frac{|Y_0-y|}{|\zeta|}} e^{-\nu(|\zeta|)s}|G(x-s\zeta,\zeta)ds - G(y-s\zeta,\zeta)|ds \right| \\
&\quad + \left| \int_{\frac{|Y-y|}{|\zeta|}}^{\frac{|X_0-x|}{|\zeta|}} e^{-\nu(|\zeta|)s} G(x-s\zeta,\zeta)ds \right| \\
&\le C\|f\|_{L^\infty_{x,\zeta}} \left[\int_0^{\frac{|Y_0-y|}{|\zeta|}} e^{-\nu(|\zeta|)s}|x-y|^{1-\epsilon}ds + \frac{\big||X_0-x| - |Y_0-y|\big|}{|\zeta|} \right] \\
&\le C\left(1 + \frac{1}{|\zeta|}\right)|x-y|^{\frac{1}{2}}.
\end{aligned} \tag{8.19}$$

To estimate the difference of II, thanks to $X' = Y_0 + (x-y) \in \overline{X_2X_0}$, we may express

$$\begin{aligned}
|II(x,\zeta) - II(y,\zeta)| &= \left| \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)t} \int_{\mathbb{R}^3} k(\zeta,\zeta')I(x-t\zeta,\zeta')d\zeta'dt \right. \\
&\quad \left. - \int_0^{\tau_-(y,\zeta)} e^{-\nu(|\zeta|)t} \int_{\mathbb{R}^3} k(\zeta,\zeta')I(y-t\zeta,\zeta')d\zeta'dt \right| \\
&\le \int_0^{\frac{|x-X_1|}{|\zeta|}} e^{-\nu(|\zeta|)t} \int_{\mathbb{R}^3} |k(\zeta,\zeta')||I(x-t\zeta,\zeta')|d\zeta'dt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\frac{|x-X_1|}{|\zeta|}} e^{-\nu(|\zeta|)t} \int_{\mathbb{R}^3} |k(\zeta,\zeta')||I(y-t\zeta,\zeta')|d\zeta' dt \\
&+ \int_{\frac{|x-X_1|}{|\zeta|}}^{\frac{|x-X_2|}{|\zeta|}} e^{-\nu(|\zeta|)t} \int_{\mathbb{R}^3} |k(\zeta,\zeta')||I(x-t\zeta,\zeta')-I(y-t\zeta,\zeta')|d\zeta' dt \\
&+ \int_{\frac{|x-X_2|}{|\zeta|}}^{\frac{|x-X_0|}{|\zeta|}} e^{-\nu(|\zeta|)t} \int_{\mathbb{R}^3} |k(\zeta,\zeta')||I(x-t\zeta,\zeta')|d\zeta' dt \\
&+ \int_{\frac{|x-X_2|}{|\zeta|}}^{\frac{|y-Y_0|}{|\zeta|}} e^{-\nu(|\zeta|)t} \int_{\mathbb{R}^3} |k(\zeta,\zeta')||I(y-t\zeta,\zeta')|d\zeta' dt \\
&=: DII_1 + DII_2 + DII_3 + DII_4 + DII_5.
\end{aligned}
\tag{8.20}
$$

Notice that

$$
DII_1 + DII_2 + DII_4 + DII_5 \le C\frac{|x-y|^{\frac{1}{2}}}{|\zeta|} \tag{8.21}
$$

because of smallness of the domain of integration, (8.13)–(8.15). We are now focus on DII_3. Let $\hat{\zeta} = \frac{\zeta}{|\zeta|}$. We can rewrite

$$
DII_3 = \int_{|x-X_1|}^{|x-X_2|} \frac{1}{|\zeta|} e^{-\frac{\nu(|\zeta|)}{|\zeta|}r} \int_{\mathbb{R}^3} |k(\zeta,\zeta')||I(x-r\hat{\zeta},\zeta')-I(y-r\hat{\zeta},\zeta')|d\zeta' dr \tag{8.22}
$$

Let $z(r) = x - r\hat{\zeta}$ and $z'(r) = y - r\hat{\zeta}$. Notice that since $X' = Y_0 + (x-y)$ lies on $\overline{X_2X_0}$, we see that $z'(r) = y - r\hat{\zeta} \in \overline{yY_0}$ for $|x-X_1| \le r \le |x-X_2|$ and

$$
d_{z'} \ge \frac{1}{2}d_z + (\frac{1}{2}d_z - |z-z'|) \ge \frac{1}{2}d_z \ge 2|x-y|. \tag{8.23}
$$

Inferring from (6.5), we have

$$
\int_{\mathbb{R}^3} |k(\zeta,\zeta')||I(z(r),\zeta)-I(z'(r),\zeta)|d\zeta' \le C\left(1+d_{z(r)}^{-1}\right)^{(\frac{1}{3}+\epsilon')} |x-y|^{1-\epsilon}, \tag{8.24}
$$

for $0 < \epsilon' < 1$. Let Z be the midpoint of X_1 and X_2. Notice that $d_Z \ge 4|x-y|$.

Therefore, with the help of Proposition 6.3,

$$
\begin{aligned}
DII_3 &\le C\int_{|X_1-x|}^{|x-Z|} \frac{|x-y|^{1-\epsilon}}{|\zeta|}(1+\frac{1}{|x-y|r})^{\frac{1}{3}+\epsilon'} dr + C\int_{|X_2-X_0|}^{|X_0-Z|} \frac{|x-y|^{1-\epsilon}}{|\zeta|}(1+\frac{1}{|x-y|r})^{\frac{1}{3}+\epsilon'} dr \\
&\le C\frac{|x-y|^{\frac{2}{3}-\epsilon-\epsilon'}}{|\zeta|}
\end{aligned}
\tag{8.25}
$$

Choosing small enough ϵ', we can conclude the lemma. □

9. Differentiability of boundary flux

Let $g(t)$ be a normal geodesic on $\partial\Omega$ such that

$$g(0)=x, \tag{9.1}$$

$$g'(0)=v, \tag{9.2}$$

where $v\in T_x(\partial\Omega)$.

Recall that by definition

$$\nabla_v^x D_f(x)=\frac{d}{dt}D_f(g(t))\bigg|_{t=0}, \tag{9.3}$$

where

$$\begin{aligned} D_f(g(t))=2\pi^{-\frac{1}{4}}\int_0^\infty\int_\Omega\int_{\mathbb{R}^3} e^{-\frac{\nu(\rho)}{\rho}|g(t)-y|}k\left(\rho\frac{g(t)-y}{|g(t)-y|},\zeta'\right)\frac{(g(t)-y)\cdot n(g(t))}{|g(t)-y|^3} \\ \times e^{-\frac{\rho^2}{2}}\rho^2 f(y,\zeta')d\zeta' dy d\rho. \end{aligned} \tag{9.4}$$

We will devote this section to prove the following lemma.

Lemma 9.1. *There exists a constant C such that for any $v\in T_x\partial\Omega$ with $|v|=1$, we have*

$$|\nabla_v^x D_f(x)|\le C. \tag{9.5}$$

If we differentiate the formula (9.4) directly, we will obtain a singularity of $|x-y|^{-3}$, which is barely not integrable in Ω. However, by subtracting and adding $f(x,\zeta')$, we can use the local Hölder continuity in Lemma 8.1 to make it integrable. To deal with additional term we introduced, we observe that, in the integrand of formula (5.1), every x is paired up with y in the form of $x-y$, except for $n(x)$. Therefore, we can convert derivatives with respect to x to those with respect to y with a change of sign and rewrite all the terms except the term with derivative of $n(x)$ as a divergence form with respect to y and apply the divergence theorem. More precisely, we can write

$$\begin{aligned} 2^{-1}\pi^{\frac{1}{4}}\nabla_v^x D_f(x) &= \int_0^\infty\int_\Omega\int_{\mathbb{R}^3}\nabla_v^x\left(e^{-\frac{\nu(\rho)}{\rho}|x-y|}k\left(\rho\frac{x-y}{|x-y|},\zeta'\right)\frac{(x-y)\cdot n(x)}{|x-y|^3}\right) \\ &\quad\times\left[f(y,\zeta')-f(x,\zeta')\right]e^{-\frac{\rho^2}{2}}\rho^2 d\zeta' dy d\rho \\ &\quad-\int_0^\infty\int_{\mathbb{R}^3}\int_\Omega \mathrm{div}_y\left(\left[e^{-\frac{\nu(\rho)}{\rho}|x-y|}k\left(\rho\frac{x-y}{|x-y|},\zeta'\right)\frac{(x-y)\cdot n(x)}{|x-y|^3}e^{-\frac{\rho^2}{2}}\rho^2 f(x,\zeta')\right]v\right)dyd\zeta' d\rho \\ &\quad+\int_0^\infty\int_{\mathbb{R}^3}\int_\Omega e^{-\frac{\nu(\rho)}{\rho}|x-y|}k\left(\rho\frac{x-y}{|x-y|},\zeta'\right)\frac{(x-y)\cdot\nabla_v^x(n(x))}{|x-y|^3}e^{-\frac{\rho^2}{2}}\rho^2 f(x,\zeta')dyd\zeta' d\rho \\ &=:\nabla^x D_f^1+\nabla^x D_f^2+\nabla^x D_f^3. \end{aligned} \tag{9.6}$$

Notice that since $\partial\Omega$ is smooth, we can see that $\nabla^x D_f^3$ is bounded. Regarding $\nabla^x D_f^1$, by direct calculation, we have

$$
\begin{aligned}
\nabla_v^x D_f^1 = &\Big| \int_0^\infty \int_{\mathbb{R}^3} \int_\Omega e^{-\frac{\nu(\rho)}{\rho}|x-y|} k\left(\rho \frac{x-y}{|x-y|}, \zeta'\right) e^{-\frac{\rho^2}{2}} \rho^2 [f(y,\zeta') - f(x,\zeta')] \\
&\times \Big[-\frac{\nu(\rho)}{\rho} \frac{(x-y)\cdot v}{|x-y|} \frac{(x-y)\cdot n(x)}{|x-y|^3} + \frac{v \cdot n(x)}{|x-y|^3} + \frac{(x-y)\cdot \nabla_v^x n(x)}{|x-y|^3} \\
&- 3 \frac{(x-y)\cdot n(x)}{|x-y|^4} \frac{(x-y)\cdot v}{|x-y|} \Big] \\
&+ [f(y,\zeta') - f(x,\zeta')] e^{-\frac{\nu(\rho)}{\rho}|x-y|} \frac{(x-y)\cdot n(x)}{|x-y|^3} \\
&\times e^{-\frac{\rho^2}{2}} \rho^3 \left(\mathrm{grad}_\zeta k \left(\rho \frac{x-y}{|x-y|}, \zeta'\right)\right) \cdot \left(\frac{v}{|x-y|} - \frac{(x-y)\cdot v}{|x-y|^3}(x-y)\right) dy d\zeta' d\rho \Big| \\
\le\, & C \int_0^\infty \int_{\mathbb{R}^3} \int_\Omega \Big\{ \frac{1}{|\rho \frac{x-y}{|x-y|} - \zeta'|} e^{-\frac{1}{8}|\rho \frac{x-y}{|x-y|} - \zeta'|^2} \left[\frac{1}{|x-y|^{\frac{3}{2}+\frac{\epsilon}{2}}} + \frac{1}{|x-y|^{\frac{5}{2}+\frac{\epsilon}{2}}} \right] \rho^2 \\
&+ \frac{1 + |\rho \frac{x-y}{|x-y|}|}{|\rho \frac{x-y}{|x-y|} - \zeta'|^2} e^{-\frac{1}{8}|\rho \frac{x-y}{|x-y|} - \zeta'|^2} \frac{1}{|x-y|^{\frac{5}{2}+\frac{\epsilon}{2}}} \rho^3 \Big\} (1 + |\zeta'|^{-1}) e^{-\frac{\rho^2}{2}} dy d\zeta' d\rho \\
\le\, & C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^R \frac{1}{|\zeta - \zeta'|} e^{-\frac{1}{8}|\zeta-\zeta'|^2} \left(1 + \frac{1}{|\zeta'|}\right) e^{-\frac{|\zeta|^2}{2}} [r^{\frac{1}{2}-\frac{\epsilon}{2}} + r^{-\frac{1}{2}-\frac{\epsilon}{2}}] \\
&+ \left(\frac{(1+|\zeta|)^2}{|\zeta-\zeta'|^2} + \frac{1+|\zeta|}{|\zeta-\zeta'||\zeta'|} \right) e^{-\frac{1}{8}|\zeta-\zeta'|^2} e^{-\frac{|\zeta|^2}{2}} r^{-\frac{1}{2}-\frac{\epsilon}{2}} dr d\zeta' d\zeta \\
\le\, & C.
\end{aligned}
\tag{9.7}
$$

Notice that $grad_\zeta$ appears due to the chain rule. In the above derivation, in addition to (2.5) and (2.6), we have used the fact

$$
e^{-\frac{\nu(\rho)}{\rho}|x-y|} \frac{\nu(\rho)}{\rho} \le C|x-y|^{-1} \tag{9.8}
$$

and the triangle inequality $|\zeta| \le |\zeta'| + |\zeta' - \zeta|$. Next, we shall prove that $\nabla^x D_f^2$ is bounded in the senses of improper integral. We define $\Omega_\epsilon = \Omega \setminus B(x,\epsilon)$ and name the corresponding integral $\nabla^x D_f^{2,\epsilon}$. Applying the divergence theory, we have

$$
\begin{aligned}
\nabla^x D_f^{2,\epsilon} = &- \int_0^\infty \int_{\mathbb{R}^3} \int_{\partial\Omega \setminus B(x,\epsilon)} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k\left(\rho \frac{x-y}{|x-y|}, \zeta'\right) \frac{(x-y)\cdot n(x)}{|x-y|^3} \\
&\times e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta') [v \cdot n(y)] dA(y) d\zeta' d\rho \\
&- \int_0^\infty \int_{\mathbb{R}^3} \int_{\partial B(x,\epsilon)\cap\Omega} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k\left(\rho \frac{x-y}{|x-y|}, \zeta'\right) \frac{(x-y)\cdot n(x)}{|x-y|^3} \\
&\times e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta') \left[v \cdot \frac{x-y}{|x-y|} \right] dA(y) d\zeta' d\rho \\
=: &\, S^\epsilon + B^\epsilon.
\end{aligned}
\tag{9.9}
$$

For S^ϵ, we further break the domain of integration by $GB(x, r_1)$, where r_1 is as defined in Proposition 3.4. It is not hard to see that the integral outside $GB(x, r_1)$ is bounded. Inside the $GB(x, r_1)$, by applying Lemma 3.6, we have

$$\begin{aligned}
&\Big|\int_0^\infty\int_{\mathbb{R}^3}\int_{GB(x,r_1)\backslash B(x,\epsilon)} e^{-\frac{\nu(\rho)}{\rho}|x-y|}k\left(\rho\frac{x-y}{|x-y|},\zeta'\right)\frac{(x-y)\cdot n(x)}{|x-y|^3}\\
&\qquad\times e^{-\frac{\rho^2}{2}}\rho^2 f(x,\zeta')[v\cdot n(y)]dA(y)d\zeta' d\rho\Big| \qquad (9.10)\\
&\quad\le C\|f\|_{L^\infty_{x,\zeta}}\int_0^\infty\int_{GB(x,r_1)} e^{-\frac{\rho^2}{2}}\rho^2 dy d\rho \le C\|f\|_{L^\infty_{x,\zeta}}.
\end{aligned}$$

We are going to deal with B^ϵ and will see that it in fact forms a "residue". We introduce spherical coordinates on $B(x,\epsilon)$ so that $-n(x)$ is the north pole so that

$$\hat{\zeta} := \frac{x-y}{|x-y|} = \sin\theta\cos\phi v + \sin\theta\sin\phi(n(x)\times v) + \cos\theta n(x). \qquad (9.11)$$

We use D_ϵ to denote the domain in the chart that maps to $\partial B(x,\epsilon)\cap\Omega$. Let

$$D'_\epsilon := \{\rho[\sin\theta\cos\phi v + \sin\theta\sin\phi(n(x)\times v) + \cos\theta n(x))]|(\theta,\phi)\in D_\epsilon, \rho>0\}. \qquad (9.12)$$

We have

$$\begin{aligned}
B^\epsilon &= -\int_0^\infty\int_{\mathbb{R}^3}\int_0^\pi\int_0^{2\pi}\chi_{D_\epsilon}(\theta,\phi)e^{-\frac{\nu(\rho)}{\rho}\epsilon}k(\rho\hat{\zeta},\zeta')e^{-\frac{\rho^2}{2}}\rho^2 f(x,\zeta')\cos\theta\sin^2\theta\cos\phi d\phi d\theta d\zeta' d\rho\\
&= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\chi_{D'_\epsilon}(\zeta)e^{-\frac{\nu(|\zeta|)}{|\zeta|}\epsilon}k(\zeta,\zeta')e^{-\frac{|\zeta|^2}{2}}f(x,\zeta')\frac{\zeta\cdot n(x)}{|\zeta|}\frac{\zeta\cdot v}{|\zeta|}d\zeta d\zeta'
\end{aligned} \qquad (9.13)$$

We can conclude from the dominated convergence theorem that

$$\lim_{\epsilon\to 0} B^\epsilon = -\int_{\zeta\cdot n(x)\le 0}\int_{\mathbb{R}^3} k(\zeta,\zeta')e^{-\frac{|\zeta|^2}{2}}f(x,\zeta')\frac{\zeta\cdot n(x)}{|\zeta|}\frac{\zeta\cdot v}{|\zeta|}d\zeta d\zeta' \qquad (9.14)$$

We notice that

$$\begin{aligned}
|\nabla^x D_f^{2,\epsilon}| &\le C\|f\|_{L^\infty_{x,\zeta}}\int_0^\infty\int_\Omega\frac{1}{|x-y|^2}e^{-\frac{\rho^2}{2}}\rho^2 dy d d\rho\\
&\le C\|f\|_{L^\infty_{x,\zeta}}.
\end{aligned} \qquad (9.15)$$

We conclude the lemma.

10. Differentiability of f

The main result of this article is summarized in the following lemma.

Lemma 10.1. *Under the assumption of Theorem 1.2, regarding the solution of* (1.1)*, for* $\epsilon>0$*, we have the following estimates*

$$\left|\frac{\partial}{\partial x_i}f(x,\zeta)\right| \le C(1+d_x^{-1})^{\frac{4}{3}+\epsilon}, \qquad (10.1)$$

$$\left|\frac{\partial}{\partial \zeta_i}f(x,\zeta)\right| \le C(1+d_x^{-1})^{\frac{4}{3}+\epsilon}. \qquad (10.2)$$

This section is devoted to the proof of (10.1). We leave the proof of (10.2) to the next section. In view of (1.18), to prove (10.1), we shall proceed the estimates of $\frac{\partial}{\partial x_i}I(x,\zeta)$, $\frac{\partial}{\partial x_i}II(x,\zeta)$ and $\frac{\partial}{\partial x_i}III(x,\zeta)$ respectively. We first show that I and II defined in (1.18) preserve the regularity from boundary.

Lemma 10.2. *Let Ω be the domain introduced before. Suppose there exist constants $0 < a < \frac{1}{2}$, $M > 0$ such that*

$$|\nabla_\eta^x f(X,\zeta)| \le M|\eta| e^{-a|\zeta|^2}, \tag{10.3}$$

$$|f(X,\zeta)| \le M e^{-a|\zeta|^2}, \tag{10.4}$$

for all $(X,\zeta) \in \Gamma_-$ and $\eta \in T_X(\partial\Omega)$. Then, for $x \in \Omega$, the following estimates hold

$$\left|\frac{\partial}{\partial x_i} I(x,\zeta)\right| \le C d_x^{-1} e^{-\frac{a}{2}|\zeta|^2}, \tag{10.5}$$

$$\left|\frac{\partial}{\partial x_i} II(x,\zeta)\right| \le C(1+d_x^{-1})^{\frac{4}{3}+\epsilon'}, \tag{10.6}$$

$$\left|\frac{\partial}{\partial x_i} II(x,\zeta)\right| \le C[(1+d_x^{-1})^{\frac{1}{3}+\epsilon'}\left(1+\frac{1}{|\zeta|}\right) + \min(d_x^{-1}, \frac{1}{N(x,\zeta)|\zeta|}) e^{-\frac{a}{2}|\zeta|^2}]. \tag{10.7}$$

To prove Lemma 10.2, we have the following observation.

Proposition 10.3. *Let $\tau_-(x,\zeta)$ and $p(x,\zeta)$ be as defined in (1.11) and (1.12). Then*

$$|\frac{\partial}{\partial x_i} p(x,\zeta)| \le \frac{1}{N(x,\zeta)}, \tag{10.8}$$

$$|\frac{\partial}{\partial x_i} \tau_-(x,\zeta)| \le \frac{2}{N(x,\zeta)|\zeta|}, \tag{10.9}$$

$$\tau_-(x,\zeta) \ge \frac{d_x}{N(x,\zeta)|\zeta|}. \tag{10.10}$$

We notice that

(a) (10.8) is a direct result of (6.7) of Proposition 6.2.
(b) (10.9) is derived from (1.12), (10.8) and the fact that $N(x,\zeta) \le 1$.
(c) (10.10) is a direct result of (1.12) and (6.6) of Proposition 6.2.

We are now in a position to prove (10.5).

Proof of (10.5). Let e_i be the i-th unit vector in $\mathbb{R}^3$. Formal calculation gives

$$\begin{aligned}\frac{\partial}{\partial x_i} I(x,\zeta) &= \frac{d}{dt}\left(f(p(x+te_i,\zeta),\zeta)e^{-\nu(|\zeta|)\tau_-(x+te_i,\zeta)}\right)\Big|_{t=0} \\ &= \frac{d}{dt}(f(p(x+te_i,\zeta),\zeta))\Big|_{t=0} e^{-\nu(|\zeta|)\tau_-(x,\zeta)} \\ &\quad -\frac{\partial}{\partial x_i}\tau_-(x,\zeta)\ \nu(|\zeta|) f(p(x,\zeta),\zeta) e^{-\nu(|\zeta|)\tau_-(x,\zeta)}\end{aligned} \tag{10.11}$$

Note that by (10.3) and (10.8), we have

$$\left|\frac{d}{dt}(f(p(x+te_i,\zeta),\zeta))\Big|_{t=0}\right| \le \frac{M}{N(x,\zeta)} e^{-a|\zeta|^2}. \tag{10.12}$$

Therefore, summing up the above two equations and applying Proposition 10.3, we obtain

$$\left|\frac{\partial}{\partial x_i} I(x,\zeta)\right| \le C\Big(\frac{1}{N(x,\zeta)} e^{-\nu(|\zeta|)\frac{d_x}{N(x,\zeta)|\zeta|}} + \nu(|\zeta|)\frac{1}{N(x,\zeta)|\zeta|} e^{-\nu(|\zeta|)\frac{d_x}{N(x,\zeta)|\zeta|}}\Big) e^{-a|\zeta|^2} \tag{10.13}$$

$$\le C d_x^{-1}(|\zeta|+1) e^{-a|\zeta|^2} \le C d_x^{-1} e^{-\frac{a}{2}|\zeta|^2}. \quad \square \tag{10.14}$$

Taking the derivative on II with respect to x_i, we have

$$\begin{aligned}\frac{\partial}{\partial x_i} II(x,\zeta) = &\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} \int_{\mathbb{R}^3} k(\zeta,\zeta')\frac{\partial}{\partial x_i} I(x-s\zeta,\zeta')d\zeta' ds \\ &+ e^{-\nu(|\zeta|)\tau_-(x,\zeta)} \int_{\mathbb{R}^3} k(\zeta,\zeta') f(p(x,\zeta),\zeta')d\zeta' \frac{\partial}{\partial x_i}\tau_-(x,\zeta) \\ =:& II^A + II^B.\end{aligned} \tag{10.15}$$

By Proposition 10.3, we see that

$$|II^B| \leq C\|f\|_{L^\infty_{x,\zeta}} e^{-\nu_0 \frac{d_x}{N(x,\zeta)|\zeta|}} \frac{1}{N(x,\zeta)|\zeta|} e^{-\frac{a}{2}|\zeta|^2} \leq C\|f\|_{L^\infty_{x,\zeta}} d_x^{-1}. \tag{10.16}$$

By an analysis similar to Lemma 2 in [8], we have

$$|\int_{\mathbb{R}^3} k(\zeta,\zeta') f(p(x,\zeta),\zeta')d\zeta'| \leq Ce^{-\frac{a}{2}|\zeta|^2}. \tag{10.17}$$

Therefore, we also have

$$|II^B| \leq C\frac{1}{N(x,\zeta)|\zeta|} e^{-\frac{a}{2}|\zeta|^2}. \tag{10.18}$$

We let

$$H(x,\zeta) = \int_{\mathbb{R}^3} k(\zeta,\zeta') I(x,\zeta')d\zeta'. \tag{10.19}$$

Then, we have

$$II^A = \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} \frac{\partial}{\partial x_i} H(x-s\zeta,\zeta')ds \tag{10.20}$$

Concerning $\frac{\partial}{\partial x_i} H$, we have the following estimate.

Proposition 10.4. *Let H be as defined in (10.19). We have*

$$\left|\frac{\partial}{\partial x_i} H(x,\zeta)\right| \leq Cd_x^{-\frac{1}{3}} \left(\left|\ln d_x\right| + 1\right). \tag{10.21}$$

Proof. In order to get a good estimate near the boundary, we shall break the domain of integration into two, $B(\zeta, d_x^{\frac{1}{3}})$ and $\mathbb{R}^3 \setminus B(\zeta, d_x^{\frac{1}{3}})$, and name the corresponding integrals DH_s and DH_l respectively. For the estimate of DH_s, by applying the estimate (10.5), we obtain

$$\begin{aligned}|DH_s| \leq &\left| \int_{B(\zeta,d_x^{\frac{1}{3}})} k(\zeta,\zeta')\frac{\partial}{\partial x_i} I(x-s\zeta,\zeta')d\zeta' \right| \\ \leq &C\frac{1}{d_x} \int_{B(\zeta,d_x^{\frac{1}{3}})} |k(\zeta,\zeta')|d\zeta'\end{aligned}$$

$$\leq C\frac{1}{d_x}\int\limits_{B(\zeta,d_x^{\frac{1}{3}})}\frac{1}{|\zeta-\zeta'|}d\zeta' \qquad (10.22)$$

$$\leq C\frac{1}{d_x}\int_0^{\pi}\int_0^{d_x^{\frac{1}{3}}}\frac{1}{\rho}\rho^2\sin\theta d\rho d\theta$$

$$\leq Cd_x^{-\frac{1}{3}}.$$

Regarding the estimate of DH_l, we first notice that (10.13) and (2.4) imply

$$\left|\frac{\partial}{\partial x_i}I(x,\zeta')\right|\leq C\left(\frac{1}{N(x,\zeta')}+\frac{1}{N(x,\zeta')|\zeta|}\right)e^{-a|\zeta'|^2}, \qquad (10.23)$$

since $0\leq\gamma<1$. Next, we consider the coordinate change $\zeta'=l(y-x)$ as we employed in Section 1 together with the fact $|k(\zeta,\zeta')|<Cd_x^{-\frac{1}{3}}$ in the domain $\mathbb{R}^3\setminus B(\zeta,d_x^{\frac{1}{3}})$ to obtain

$$|DH_l|\leq Cd_x^{-\frac{1}{3}}\int_0^{\infty}\int_{\partial\Omega}\left|\frac{\partial}{\partial x_i}I(x,\zeta')\right|l^2|n(y)\cdot(x-y)|dA(y)dl$$

$$\leq Cd_x^{-\frac{1}{3}}\int_0^{\infty}\int_{\partial\Omega}\left(\frac{1}{N(x,\zeta')}+\frac{1}{N(x,\zeta')l|x-y|}\right)e^{-al^2|x-y|^2}l^2|n(y)\cdot(x-y)|dA(y)dl$$

$$\leq Cd_x^{-\frac{1}{3}}\int_0^{\infty}\int_{\partial\Omega}\left(\frac{|x-y|}{|n(y)\cdot(x-y)|}+\frac{1}{l|n(y)\cdot(x-y)|}\right)e^{-al^2|x-y|^2}l^2|n(y)\cdot(x-y)|dA(y)dl \qquad (10.24)$$

$$\leq Cd_x^{-\frac{1}{3}}\int_0^{\infty}\int_{\partial\Omega}e^{-al^2|x-y|^2}(l^2|x-y|+l)dA(y)dl$$

$$\leq Cd_x^{-\frac{1}{3}}\int_{\partial\Omega}\int_0^{\infty}e^{-az^2}(z^2+z)dz\frac{1}{|x-y|^2}dA(y)dz$$

$$\leq Cd_x^{-\frac{1}{3}}\int_{\partial\Omega}\frac{1}{|x-y|^2}dA(y)$$

$$\leq Cd_x^{-\frac{1}{3}}(|\ln d_x|+1).$$

Notice that we let $z=|x-y|l$ in the third last line in the above derivation and applied Lemma 3.3 to draw our conclusion in the last line. Combining (10.22) and (10.24), the proof of Proposition 10.4 is complete. □

Proof of (10.6) and (10.7). Now, we set $l=|x-p(x,\zeta)|$. Applying Proposition 6.3 and Proposition 10.4, we obtain

$$|II^A|\leq C\int_0^{\tau_-(x,\zeta)}e^{-\nu(|\zeta|)s}d_{(x-s\zeta)}^{-(\frac{1}{3}+\epsilon')}ds$$

$$\leq C\left(\left(\frac{d_x}{2}\right)^{-(\frac{1}{3}+\epsilon')}\int_0^{\frac{d_x}{2|\zeta|}}e^{-\nu(|\zeta|)s}ds+\int_{\frac{d_x}{2}}^{l}e^{-\frac{\nu(|\zeta|)r}{|\zeta|}}d_x^{-(\frac{1}{3}+\epsilon')}|l-r|^{-(\frac{1}{3}+\epsilon')}\frac{1}{|\zeta|}dr\right). \qquad (10.25)$$

We observe that

$$
\begin{aligned}
&\int_{\frac{d_x}{2}}^{l} e^{-\frac{\nu(|\zeta|)r}{|\zeta|}} d_x^{-(\frac{1}{3}+\epsilon')}|l-r|^{-(\frac{1}{3}+\epsilon')}\frac{1}{|\zeta|}dr \\
&\leq \begin{cases} C\int_{\frac{d_x}{2}}^{l} d_x^{-(\frac{1}{3}+\epsilon')}|l-r|^{-(\frac{1}{3}+\epsilon')}\frac{1}{r}dr \\ \int_{\frac{d_x}{2}}^{l} d_x^{-(\frac{1}{3}+\epsilon')}|l-r|^{-(\frac{1}{3}+\epsilon')}\frac{1}{|\zeta|}dr. \end{cases}
\end{aligned}
$$

Hence,

$$
|II^A| \leq \begin{cases} C(1+d_x^{-1})^{\frac{4}{3}+\epsilon'} \\ C(1+d_x^{-1})^{\frac{1}{3}+\epsilon'}\frac{1}{|\zeta|}. \end{cases} \tag{10.26}
$$

Finally, combining (10.16) and (10.26), we obtain (10.6) and (10.7). This completes the proof of Lemma 10.2. □

Combining Lemma 10.2 and Proposition 7.2, we have a refined estimate

$$
|f(x,\zeta)-f(y,\zeta)| \leq C\|f\|_{L^\infty_{x,\zeta}}(1+d_x^{-1})^{\frac{4}{3}+\epsilon}|x-y|^{1-\epsilon} \tag{10.27}
$$

in case $|x-y| < \frac{d_x}{2}$.

We are now in a position to perform bootstrapping the regularity.

Lemma 10.5. *Let $f \in L^\infty_{x,\zeta}$ be a stationary solution to the linearized Boltzmann equation and x be an interior point of Ω. Suppose that there exist $0<\sigma<1$, $0<\delta<d(x,\partial\Omega)$, and $M>0$ such that,*

$$
\sup_{\zeta\in\mathbb{R}^3} |f(x,\zeta)-f(y,\zeta)| \leq M|x-y|^\sigma, \tag{10.28}
$$

whenever $y\in B(x,\delta)$.

Then, G is differentiable at x. Furthermore,

$$
\left|\frac{\partial}{\partial x_i}G(x,\zeta)\right| \leq C(\|f\|_{L^\infty_{x,\zeta}}(1+|\ln\delta|)+M\delta^\sigma). \tag{10.29}
$$

Proof. Recall that

$$
G(x,\zeta) = \int_0^\infty\int_\Omega\int_{\mathbb{R}^3} k\left(\zeta,\rho\frac{(x-y)}{|x-y|}\right)e^{-\nu(\rho)\frac{|x-y|}{\rho}}k\left(\rho\frac{(x-y)}{|x-y|},\eta\right)f(y,\eta)\frac{\rho}{|x-y|^2}d\eta dy d\rho. \tag{10.30}
$$

We first formally differentiate the above formula with respect to x_i and divide the domain of integration into two parts: $B(x,\delta)$ and $\Omega\setminus B(x,\delta)$. We denote the corresponding integrals as g_s and g_l. Regarding the estimate of g_l, the typical term is

$$
\begin{aligned}
|g_{l1}| &:= \left|-2\int_0^\infty\int_{\Omega\setminus B(x,\delta)}\int_{\mathbb{R}^3} k\left(\zeta,\rho\frac{(x-y)}{|x-y|}\right)e^{-\nu(\rho)\frac{|x-y|}{\rho}}k\left(\rho\frac{(x-y)}{|x-y|},\eta\right)f(y,\eta)\frac{\rho}{|x-y|^3}\frac{x_i-y_i}{|x-y|}d\eta dy d\rho\right| \\
&\leq C\|f\|_{L^\infty_{x,\zeta}}\int_0^\infty\int_{\Omega\setminus B(x,\delta)} k\left(\zeta,\rho\frac{(x-y)}{|x-y|}\right)\frac{\rho}{|x-y|^3}dy d\rho
\end{aligned}
$$

$$\le C\|f\|_{L^\infty_{x,\zeta}} \int_0^\infty \int_{S^2} \int_\delta^R k(\zeta,\rho\omega)\frac{\rho}{r^3} r^2 dr d\omega d\rho \tag{10.31}$$
$$\le C\|f\|_{L^\infty_{x,\zeta}}(1+|\ln\delta|),$$

where R is the diameter of Ω. By using (2.8), in the same fashion, we readily see that

$$|g_l| \le C\|f\|_{L^\infty_{x,\zeta}}(1+|\ln\delta|). \tag{10.32}$$

Regarding the estimate of g_s, in order to utilize the Hölder continuity, we subtract and add $f(x,\eta)$ in the integrand as follows:

$$\begin{aligned} g_s(x,\zeta) &= \int_0^\infty \int_{B(x,\delta)} \int_{\mathbb{R}^3} \\ &(f(y,\eta)-f(x,\eta))\frac{\partial}{\partial x_i}\left[k\left(\zeta,\rho\frac{x-y}{|x-y|}\right)e^{-\nu(\rho)\frac{|x-y|}{\rho}}k\left(\rho\frac{x-y}{|x-y|},\eta\right)\frac{\rho}{|x-y|^2}\right]d\eta dy d\rho \\ &+\int_0^\infty \int_{B(x,\delta)} \int_{\mathbb{R}^3} f(x,\eta)\frac{\partial}{\partial x_i}\left[k\left(\zeta,\rho\frac{x-y}{|x-y|}\right)e^{-\nu(\rho)\frac{|x-y|}{\rho}}k\left(\rho\frac{x-y}{|x-y|},\eta\right)\frac{\rho}{|x-y|^2}\right]d\eta dy d\rho \\ &=: g_{s1}+g_{s2}. \end{aligned} \tag{10.33}$$

For g_{s1}, the Hölder continuity of f in space variables, see (10.28), makes the integrand integrable. We have

$$|g_{s1}| \le CM\delta^\sigma. \tag{10.34}$$

For g_{s2}, we first remove an ϵ-ball and integrate:

$$\begin{aligned} g_{s2}^\epsilon &:= \int_0^\infty \int_{B(x,\delta)\setminus B(x,\epsilon)} \int_{\mathbb{R}^3} f(x,\eta) \\ &\frac{\partial}{\partial x_i}\left[k\left(\zeta,\rho\frac{x-y}{|x-y|}\right)e^{-\nu(\rho)\frac{|x-y|}{\rho}}k\left(\rho\frac{x-y}{|x-y|},\eta\right)\frac{\rho}{|x-y|^2}\right]d\eta dy d\rho \\ &= \int_0^\infty \int_{B(x,\delta)\setminus B(x,\epsilon)} \int_{\mathbb{R}^3} f(x,\eta) \\ &\left(-\left(\frac{\partial}{\partial y_i}\left[k\left(\zeta,\rho\frac{x-y}{|x-y|}\right)e^{-\nu(\rho)\frac{|x-y|}{\rho}}k\left(\rho\frac{x-y}{|x-y|},\eta\right)\frac{\rho}{|x-y|^2}\right]\right)\right)d\eta dy d\rho \\ &= \int_0^\infty \int_{\mathbb{R}^3} \left(-\int_{S^2(x,\delta)} + \int_{S^2(x,\epsilon)}\right) f(x,\eta) \\ &\left[k\left(\zeta,\rho\frac{x-y}{|x-y|}\right)e^{-\nu(\rho)\frac{|x-y|}{\rho}}k\left(\rho\frac{x-y}{|x-y|},\eta\right)\frac{\rho}{|x-y|^2}\right]n_i(y)dA(y)d\eta d\rho \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x,\eta)k(\zeta,\zeta')\left(-e^{-\frac{\nu(|\zeta'|)}{|\zeta'|}\delta}+e^{-\frac{\nu(|\zeta'|)}{|\zeta'|}\epsilon}\right)k(\zeta',\eta)\frac{1}{|\zeta'|}\frac{\zeta_i'}{|\zeta'|}d\zeta' d\eta. \end{aligned} \tag{10.35}$$

Notice that the integrand in the last integral above is bounded by

$$2\|f\|_{L^\infty_{x,\zeta}} k(\zeta,\zeta')k(\zeta',\eta)|\zeta'|^{-1}, \tag{10.36}$$

which is integrable in (ζ',η). Therefore, we can pass the limit $\epsilon\to 0$ and, furthermore

$$|g_{s2}| \leq C\|f\|_{L^\infty_{x,\zeta}}. \tag{10.37}$$

Inferring from (10.32), (10.34) and (10.37), we obtain (10.29). □

Finally, to complete the proof of (10.1), we now estimate $\frac{\partial}{\partial x_i} III(x,\zeta)$. PROOF OF (10.1).

Proof. Differentiating III directly and applying (10.9), (10.10), (10.27) and (10.29), we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial x_i} III(x,\zeta)\right| &\leq \left|\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}\frac{\partial}{\partial x_i}G(x-s\zeta,\zeta)ds\right| \\ &\quad + \left|\frac{\partial}{\partial x_i}\tau_-(x,\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}G(p(x,\zeta),\zeta)\right| \\ &\leq C\|f\|_{L^\infty_{x,\zeta}}\left|\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}(1+d_{x-s\zeta}^{-\frac{1}{3}-2\epsilon})ds\right| + C\|f\|_{L^\infty_{x,\zeta}}d_x^{-1}.\end{aligned} \tag{10.38}$$

Letting $l=|x-p(x,\zeta)|$ and applying Proposition 6.3, we readily obtain

$$\begin{aligned}&\left|\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}(1+d_{x-s\zeta}^{-\frac{1}{3}-2\epsilon})ds\right| \\ &\leq C\left(1+d_x^{-\frac{1}{3}-2\epsilon}\right)\int_0^{\frac{d_x}{2|\zeta|}} e^{-\nu(|\zeta|)s}ds + \int_{\frac{d_x}{2}}^{l} e^{-\frac{\nu(|\zeta|)}{|\zeta|}r}\left(1+d_{x-r\hat{\zeta}}^{-\frac{1}{3}-2\epsilon}\right)\frac{1}{|\zeta|}dr \\ &\leq C\left(1+d_x^{-\frac{1}{3}-2\epsilon}\right) + Cd_x^{-1}\int_{\frac{d_x}{2}}^{l}\left(1+d_x^{-\frac{1}{3}-2\epsilon}|l-r|^{-\frac{1}{3}-2\epsilon}\right)dr \\ &\leq C\left(1+d_x^{-1}\right)^{\frac{4}{3}+2\epsilon}.\end{aligned} \tag{10.39}$$

Therefore, we see that

$$\left|\frac{\partial}{\partial x_i} III(x,\zeta)\right| \leq C\|f\|_{L^\infty_{x,\zeta}}\left(1+d_x^{-1}\right)^{\frac{4}{3}+2\epsilon}. \tag{10.40}$$

Taking (10.5), (10.6) and (10.40) into account, by (1.18), the proof of (10.1) is complete. □

Notice that

$$\int_{\frac{d_x}{2}}^{l} e^{-\frac{\nu(|\zeta|)}{|\zeta|}r}\left(1+d_{x-r\hat{\zeta}}^{-\frac{1}{3}-2\epsilon}\right)\frac{1}{|\zeta|}dr \leq \begin{cases} C\left(1+d_x^{-1}\right)^{\frac{1}{3}+2\epsilon}\dfrac{1}{|\zeta|} \\ Cd_x^{-1}\displaystyle\int_{\frac{d_x}{2}}^{l}\left(1+d_x^{-\frac{1}{3}-2\epsilon}|l-r|^{-\frac{1}{3}-2\epsilon}\right)dr. \end{cases}$$

Hence, if we allow the singularity at $\zeta=0$, we can estimate

$$\left|\frac{\partial}{\partial x_i} III(x,\zeta)\right| \leq C\left(1+d_x^{-1}\right)^{\frac{1}{3}+2\epsilon}\left(1+\frac{1}{|\zeta|}\right). \tag{10.41}$$

11. Derivative with respect to velocity

In this section, we shall discuss the derivative of $f(x,\zeta)$ with respect to microscopic velocity ζ, i.e., (10.2). Differentiating the integral equation (1.15) directly, we have

$$\begin{aligned}\frac{\partial}{\partial\zeta_i}f(x,\zeta) = e^{-\nu(|\zeta|)\tau_-(x,\zeta)}\left[\nabla^x_{\frac{\partial}{\partial\zeta_i}p(x,\zeta)}f(p(x,\zeta),\zeta)+\frac{\partial}{\partial\zeta_i}f(p(x,\zeta),\zeta)\right]\\
&+ f(p(x,\zeta),\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}\left[-\nu'(|\zeta|)\frac{\zeta_i}{|\zeta|}\tau_-(x,\zeta)-\nu(|\zeta|)\frac{\partial\tau_-(x,\zeta)}{\partial\zeta_i}\right]\\
&+\frac{\partial\tau_-(x,\zeta)}{\partial\zeta_i}e^{-\nu(|\zeta|)\tau_-(x,\zeta)}K(f)(p(x,\zeta),\zeta)\\
&+\int_0^{\tau_-(x,\zeta)}-\nu'(|\zeta|)\frac{\zeta_i}{|\zeta|}e^{-\nu(|\zeta|)s}K(f)(x-s\zeta,\zeta)s\,ds\\
&-\int_0^{\tau_-(x,\zeta)}e^{-\nu(|\zeta|)s}\frac{\partial}{\partial x_i}K(f)(x-s\zeta,\zeta)s\,ds\\
&+\int_0^{\tau_-(x,\zeta)}e^{-\nu(|\zeta|)s}\frac{\partial}{\partial\zeta_i}K(f)(x-s\zeta,\zeta)ds\\
&=: D_v^1+D_v^2+D_v^3+D_v^4+D_v^5+D_v^6.\end{aligned} \tag{11.1}$$

From the fact $\nu'(|\zeta|)$ is bounded, we can conclude that D_v^4 is bounded. Using the estimate

$$\left\|\frac{\partial}{\partial\zeta_i}K(f)\right\|_{L^\infty_\zeta}\le C\|f\|_{L^\infty_\zeta}, \tag{11.2}$$

we can prove that D_v^6 is bounded. Using (10.21), (10.7), and (10.41), we have

$$\left|\frac{\partial}{\partial x_i}K(f)(x-s\zeta,\zeta)\right|\le C\left(1+d_{x-s\zeta}^{-1}\right)^{\frac13+\epsilon}. \tag{11.3}$$

Then, applying Proposition 6.3, we have

$$\begin{aligned}|D_v^5| &\le C\int_0^{\tau_-(x,\zeta)}e^{-\nu(|\zeta|)s}\left(1+[d_x|\zeta|(\tau_-(x,\zeta)-s)]^{-\frac13-\epsilon}\right)s\,ds\\
&\le C\int_0^l e^{-\frac{\nu(|\zeta|)}{|\zeta|}r}\left(1+[d_x|(l-r))]^{-\frac13-\epsilon}\right)\frac{r}{|\zeta|^2}dr\\
&\le C\left(1+d_x^{-1}\right)^{\frac13+\epsilon}\left(1+\frac{1}{|\zeta|}\right).\end{aligned} \tag{11.4}$$

If we do not want the singularity in the expression, we can have

$$|D_v^5|\le C(1+d_x^{-1})^{\frac43+\epsilon}. \tag{11.5}$$

To deal with the rest of terms, we need to discuss the derivative of τ_- and p with respect to ζ.

Proposition 11.1. *Suppose that Ω is a C^1 bounded convex domain in $\mathbb{R}^3$, $x \in \Omega$, and ζ, $\zeta' \in \mathbb{R}^3$. Then,*

$$\left|\frac{\partial}{\partial \zeta_i}\tau_-(x,\zeta)\right| \le \frac{\tau_-(x,\zeta)}{N(x,\zeta)|\zeta|} \tag{11.6}$$

$$\left|\frac{\partial}{\partial \zeta_i}p(x,\zeta)\right| \le \tau_-(x,\zeta)\left(1+\frac{1}{N(x,\zeta)}\right). \tag{11.7}$$

The above proposition is a direct consequence from the explicit formula in Lemma 2 in [11].

Let

$$\eta = \frac{\frac{\partial}{\partial \zeta_i}p(x,\zeta)}{\left|\frac{\partial}{\partial \zeta_i}p(x,\zeta)\right|}. \tag{11.8}$$

Then,

$$\nabla^x_{\frac{\partial}{\partial \zeta_i}p(x,\zeta)} f(p(x,\zeta),\zeta) = \nabla^x_\eta f(p(x,\zeta),\zeta)|\frac{\partial}{\partial \zeta_i}p(x,\zeta)| \le C|\frac{\partial}{\partial \zeta_i}p(x,\zeta)|. \tag{11.9}$$

Notice that

$$e^{-\nu(|\zeta|)\tau_-(x,\zeta)}\frac{\tau_-(x,\zeta)}{N(x,\zeta)} \le C d_x^{-1}. \tag{11.10}$$

We conclude the lemma.

Conflict of interest statement

There is no conflict of interest.

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