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Asymptotic behavior of spreading fronts in the anisotropic Allen–Cahn equation on $\mathbb{R}^n$ ☆

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Abstract

We consider the Cauchy problem for the anisotropic (unbalanced) Allen–Cahn equation on $\mathbb{R}^n$ with $n \geq 2$ and study the large time behavior of the solutions with spreading fronts. We show, under very mild assumptions on the initial data, that the solution develops a well-formed front whose position is closely approximated by the expanding Wulff shape for all large times. Such behavior can naturally be expected on a formal level and there are also some rigorous studies in the literature on related problems, but we will establish approximation results that are more refined than what has been known before. More precisely, the Hausdorff distance between the level set of the solution and the expanding Wulff shape remains uniformly bounded for all large times. Furthermore, each level set becomes a smooth hypersurface in finite time no matter how irregular the initial configuration may be, and the motion of this hypersurface is approximately subject to the anisotropic mean curvature flow $V_\gamma = \kappa_\gamma + c$ with a small error margin. We also prove the eventual rigidity of the solution profile at the front, meaning that it converges locally to the traveling wave profile everywhere near the front as time goes to infinity. In proving this last result as well as the smoothness of the level surfaces, an anisotropic extension of the Liouville type theorem of Berestycki and Hamel (2007) for entire solutions of the Allen–Cahn equation plays a key role.

Résumé

Nous considérons le problème de Cauchy pour l'équation d'Allen–Cahn (de moyenne non nulle) anisotropique dans $\mathbb{R}^n$ avec $n \geq 2$, et étudions le comportement en temps grand des solutions propageantes. Nous montrons, sous des hypothèses assez faibles sur la donnée initiale, que la solution développe un véritable front de propagation dont la position peut être approchée d'assez près, en temps grand, par une forme de Wulff en expansion. Un tel comportement peut être attendu formellement, et il existe aussi dans la littérature certaines études rigoureuses sur des problèmes analogues. Le principal objectif de cet article est d'établir des résultats d'approximation plus fins que ce qui était connu auparavant. Plus précisément, la distance de Hausdorff entre un ensemble de niveau de la solution et la forme de Wulff en expansion reste bornée uniformément en temps grand. De plus, chaque ensemble de niveau devient en temps fini une hypersurface régulière, quelque soit l'irrégularité de sa configuration initiale, et le mouvement

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de cette hypersurface est régi (approximativement) par le flot de courbure moyenne anisotropique $V_\gamma = \kappa_\gamma + c$, avec une marge d'erreur petite. Nous prouvons aussi la rigidité asymptotique du profil de la solution, c'est-à-dire qu'il converge, à proximité du front et quand le temps tend vers l'infini, vers le profil de l'onde progressive. Une extension au cas anisotropique d'un théorème de type Liouville de Berestycki et Hamel (2007), portant sur les solutions entières de l'équation d'Allen–Cahn, joue un rôle clé dans la preuve de ce dernier résultat, ainsi que de la régularité des ensembles de niveau.

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1. Introduction

In this paper, we consider the asymptotic behavior of spreading fronts in an anisotropic Allen–Cahn type equation. More precisely, we consider the following Cauchy problem:

$$\begin{cases} u_t = \operatorname{div} a_p(\nabla u) + f(u), & x \in \mathbb{R}^n,\ t > 0, \quad \text{(a)} \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \quad \text{(b)} \end{cases} \tag{1}$$

where $n \geq 2$, $a_p(p)$ denotes $\nabla_p a(p)$, and $u_0(x)$ is bounded and sufficiently smooth. The function f is a bistable type nonlinearity that is smooth, say C^2, and has exactly three zeros $0 < \alpha < 1$ with

$$f'(0) < 0, \quad f'(\alpha) > 0, \quad f'(1) < 0, \tag{2}$$

and satisfies

$$\int_0^1 f(s)ds > 0. \tag{3}$$

The function $a : \mathbb{R}^n \to \mathbb{R}^+$ in (1a) is strictly convex and homogeneous of degree two, namely $a(\lambda p) = \lambda^2 a(p)$, $({}^\forall \lambda \in \mathbb{R})$ and belongs to $C^2(\mathbb{R}^n \setminus \{0\}) \cap C^1(\mathbb{R}^n)$. Actually one can relax this condition slightly by requiring a to be only positively homogeneous, namely, $a(\lambda p) = \lambda^2 a(p)$, $({}^\forall \lambda > 0)$; see Appendix B. We further assume that there exists a positive constant Λ such that, for any $p \in \mathbb{R}^n \setminus \{0\}$,

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{p_i p_j}(p)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^n. \tag{4}$$

The goal of the present paper is to study the spreading fronts of the solution of (1) and to give a rather detailed picture of their behavior, not only determining their rough asymptotic shape (i.e., the Wulff shape, as one would naturally expect) but also showing the eventual smoothness of the level surfaces and proving the asymptotic rigidity of the transition layers at the front position, neither of which has been known before except in very limited situations. Note that we derive these refined estimates not for a specially chosen class of initial data but for a rather large class of initial data (that are possibly sign-changing) as specified in (17)–(19). Thus our results will confirm the validity and universality of fine asymptotics of spreading fronts that are expected from formal analysis.

Now, for the convenience of the reader who are not familiar with anisotropic diffusion equations of the form (1a), let us make a few remarks on the term $a(p)$. Condition (4) implies that $a(p)$ is strictly convex, but the converse is not necessarily true. Equation (1a) can be rewritten formally as

$$u_t = \sum_{i,j=1}^n a_{p_i p_j}(\nabla u)u_{x_i x_j} + f(u).$$

Therefore condition (4) is needed to ensure that equation (1a) be uniformly parabolic. Note that the functions $a_{p_i p_j}$ are homogeneous of degree zero and are therefore bounded on $\mathbb{R}^n \setminus \{0\}$, but are not necessarily continuous at $p = 0$

(see Subsection 2.1). Thus, one cannot expect the solution $u(x,t)$ to be classical. Hereafter we will deal with weak solutions of (1); see Definition 2.4.

The function $a(p)$ describes the anisotropic nature of the diffusion. The function $\gamma(p) := \sqrt{2a(p)}$ is positively homogeneous of degree one and satisfies $\gamma(p) = \gamma(-p)$. Its restriction $\gamma_0 = \gamma|_{S^{n-1}} : S^{n-1} \to \mathbb{R}^+$ is called the *energy density function*. Similar anisotropic energy functions appear in the study of crystal growth and anisotropic mean curvature flows, and are associated with the notion of the Wulff shape, which we define below; see, for instance, [15,29,28]. Conversely, if an energy density function $\gamma_0 : S^{n-1} \to \mathbb{R}^+$ satisfying $\gamma(\nu) = \gamma(-\nu)$ is given, one can reconstruct the function $a(p)$ by first extending it as a positively homogeneous function of degree one as:

$$\gamma(p) = \gamma_0\left(\frac{p}{|p|}\right)|p|, \quad p \in \mathbb{R}^n,$$

and then setting $a(p) = (\gamma(p))^2/2$. The resulting function $a(p)$ is homogeneous of degree two and satisfies

$$\gamma(p) = \sqrt{2a(p)}. \tag{5}$$

The energy density function γ_0 and its extension γ lead to the following two notions: the *Frank diagram* $F_1 \subset \mathbb{R}^n$ and the *Wulff shape* $W_1 \subset \mathbb{R}^n$. They are defined by

$$F_1 = \{p \in \mathbb{R}^n \mid \gamma(p) \le 1\},$$
$$W_1 = \{x \in \mathbb{R}^n \mid \gamma^*(x) \le 1\},$$

where $\gamma(p)$ is as in (5) and $\gamma^*(x)$ is the dual of $\gamma(p)$ defined by

$$\gamma^*(x) = \sup\{x \cdot p \mid p \in F_1\} = \sup_{p\in\mathbb{R}^n} \frac{x \cdot p}{\gamma(p)} = \max_{\nu \in S^{n-1}} \frac{x \cdot \nu}{\gamma(\nu)}. \tag{6}$$

This function γ^* is non-negative, convex and positively homogeneous of degree one, and satisfies $\gamma^*(x) = \gamma^*(-x)$ by the definition. Thus the Wulff shape is always convex, while the Frank diagram is convex if and only if $a(p)$ is a convex function. In the present paper, since $a(p)$ is assumed to be strictly convex in order for equation (1a) to be parabolic, the Frank diagram F_1 and the Wulff shape W_1 are both strictly convex. We note that the definition (6) is equivalent to $\gamma^*(x) = \sqrt{2a^*(x)}$, where $a^*(x)$ is the convex conjugate of $a(p)$, see Lemma A.1 in Appendix A.

Since $\gamma^*(x)$ is a norm, it defines an anisotropic metric on $\mathbb{R}^n$ through the anisotropic distance function

$$\gamma^*(x-y), \quad x, y \in \mathbb{R}^n.$$

Then the Wulff shape W_1 is the unit ball with respect to this anisotropic distance. We note that the equation (1a) reduces to the usual *isotropic* Allen–Cahn equation $u_t = \Delta u + f(u)$ in the case $a(p) = |p|^2/2$. In this case, we have $\gamma(p) = |p|$ and $\gamma^*(x) = |x|$, therefore the corresponding Frank diagram F_1 and Wulff shape W_1 are both the usual unit ball in the Euclidean distance. As we will mention in Remark 1.6, some of the results in the present paper are new even in the isotropic case.

For later discussions, we introduce the notion of anisotropic signed distance function. Let Ω be a bounded domain with smooth boundary $\partial\Omega$. The *anisotropic signed distance function* for Ω is defined by

$$d_\gamma(x;\Omega) = \begin{cases} \min\limits_{y\in\partial\Omega} \gamma^*(x-y) & \text{if } x \notin \Omega, \quad \text{(a)} \\ -\min\limits_{y\in\partial\Omega} \gamma^*(x-y) & \text{if } x \in \Omega. \quad \text{(b)} \end{cases} \tag{7}$$

Then $d_\gamma(x;\Omega) = 0$ if and only if $x \in \partial\Omega$, and $d_\gamma(x;\Omega) > 0$ (resp. < 0) if and only if x lies outside (resp. inside) of Ω. Next we define the *expanding Wulff shape* $W_R(t) \subset \mathbb{R}^n$ by

$$W_R(t) = \rho(t;R)\, W_1,$$

where $\rho(t;R)$ is the solution of

$$\rho'(t;R) = -\frac{n-1}{\rho(t;R)} + c, \quad \rho(0;R) = R > \frac{n-1}{c}, \tag{8}$$

and c is the positive constant defined in (14) below, which represents the speed of the traveling wave of the one-dimensional Allen–Cahn equation. The condition $R > (n-1)/c$ in (8) guarantees that $\rho(t; R) \to \infty$ as $t \to \infty$. It is easily seen that $\rho(t; R)$ satisfies

$$\rho(t; R) = R + ct - ((n-1)/c)\log t + o(\log t), \tag{9}$$

see also Lemma 3.5. The boundary of the set $W_R(t)$, denoted by $\partial W_R(t)$, coincides with the sphere of radius $\rho(t; R)$ in the anisotropic distance $\gamma^*(x)$ defined in (6), namely,

$$\partial W_R(t) = \{x \in \mathbb{R}^n \mid \gamma^*(x) = \rho(t; R)\}.$$

Note that the anisotropic signed distance for $W_R(t)$, which is defined by (7) with $\Omega = W_R(t)$, can simply be written as

$$d_\gamma(x; W_R(t)) = \gamma^*(x) - \rho(t; R). \tag{10}$$

The above identity follows from the trigonometric inequality $|\gamma^*(x) - \gamma^*(y)| \le \gamma^*(x-y)$ and the fact that γ^* is homogeneous of degree one. It is well known, and as we will see in Subsection 2.1, $d_\gamma(x; W_R(t))$ satisfies

$$\partial_t d_\gamma = \operatorname{div} a_p(\nabla d_\gamma) - c \quad \text{on } \partial W_R(t). \tag{11}$$

Here, by the definition (10), the term $-\partial_t d_\gamma$ coincides with the anisotropic outward normal velocity of the surface $\partial W_R(t)$ denoted by V_γ, while, as we will explain in Subsection 2.1, the term $-\operatorname{div} a_p(\nabla d_\gamma)$ represents the anisotropic mean curvature of $\partial W_R(t)$ denoted by κ_γ. Thus equation (11) implies that the motion of the surface $\partial W_R(t)$ is subject to the anisotropic mean curvature flow

$$V_\gamma = \kappa_\gamma + c. \tag{12}$$

Furthermore, $\partial W_R(t)$ is an expanding self-similar solution of equation (12).

As we mentioned earlier, the goal of the present paper is to show that, under very mild assumptions on the initial value u_0, the level surface $\Gamma(t)$ of the solution of (1) is well approximated by $\partial W_R(t)$, the boundary of the expanding Wulff shape. Such a result can naturally be anticipated formally if one pays attention to the close relation between equation (1a) and the anisotropic mean curvature flow (12), which can be established by a singular perturbation argument. Once the problem is reduced (formally) to (12), the approximation of solutions of (12) by $\partial W_R(t)$ can easily be shown by the comparison argument. However, the usual argument that establishes the link between (1a) and (12) works only on a finite time interval, therefore one has to be careful in using (12) as a replacement for (1a) if the main focus is the long time behavior as $t \to \infty$. Our method in the present paper allows us to establish results on fine approximation of $\Gamma(t)$ by $\partial W_R(t)$ — including its smoothness and the direction of its normals — for a large class of solutions of (1).

Before presenting our main results, let us introduce some more notation. Let $u(x,t) = \Phi(x-ct)$ denote the traveling wave of the one-dimensional Allen–Cahn equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t > 0. \tag{13}$$

Here Φ and c satisfy

$$\begin{cases} \Phi''(z) + c\,\Phi'(z) + f(\Phi(z)) = 0, \quad z \in \mathbb{R}, & \text{(a)} \\ \Phi(-\infty) = 1,\ \Phi(+\infty) = 0, & \text{(b)} \\ \Phi(0) = \alpha. & \text{(c)} \end{cases} \tag{14}$$

Since f is a bistable type nonlinearity, the pair (c, Φ) is determined uniquely under the normalization condition (14c), where α is the constant defined in (2). It is also known that $\Phi'(z) < 0$ for $z \in \mathbb{R}$ and that $c > 0$ if f satisfies (3). Moreover, there exist positive constants C_Φ and λ such that

$$|\Phi(z)|,\ |\Phi'(z)|,\ |\Phi''(z)| \le C_\Phi e^{-\lambda z}, \quad z \ge 0, \tag{15}$$

$$|\Phi(z) - 1|,\ |\Phi'(z)|,\ |\Phi''(z)| \le C_\Phi e^{\lambda z}, \quad z \le 0. \tag{16}$$

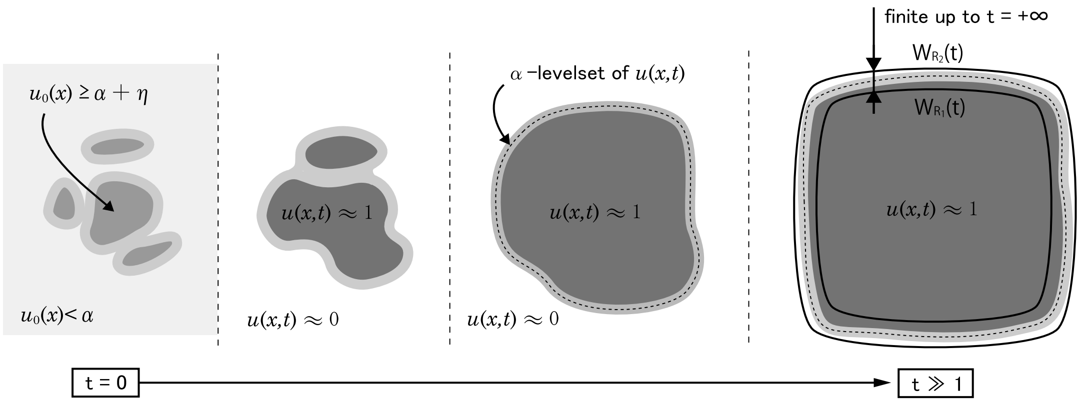

Fig. 1. Generation and asymptotic behavior of the front in the problem (1).

In the Cauchy problem for equation (13), the traveling wave $\Phi(x - ct)$ is exponentially stable in $L^\infty(\mathbb{R})$ under an appropriate class of initial perturbations. For details, see [9,14] for example. Throughout this paper, $\Phi(z)$ and c will denote the function and the constant defined in (14), and λ, C_Φ will denote the positive constants defined in (15)–(16).

Our main results are the following.

Theorem 1.1 *(Main theorem). For each $m > 0$ and $\eta > 0$, there exists a positive constant L such that, if the initial value u_0 belongs to $C^{2+\theta}(\mathbb{R}^n)$ for some $\theta \in (0, 1)$ and satisfies*

$$\inf_{x \in \mathbb{R}^n} u_0(x) \geq -m, \tag{17}$$

$$\min_{|x| \leq L} u_0(x) \geq \alpha + \eta, \tag{18}$$

$$\limsup_{|x| \to \infty} u_0(x) < \alpha, \tag{19}$$

for the constant α defined in (2), then there exist positive constants R, T, and a bounded smooth function $l : S^{n-1} \times [T, \infty) \to \mathbb{R}$ such that the solution $u(x,t)$ of (1) and its level set

$$\Gamma(t) = \{x \in \mathbb{R}^n \mid u(x,t) = \alpha\}, \tag{20}$$

satisfy

$$x \in \Gamma(t) \ \text{ if and only if } \ d_\gamma(x; W_R(t)) = l\left(\frac{x}{|x|}, t\right), \quad t \geq T, \tag{21}$$

where $d_\gamma(x; W_R(t))$ is as defined in (10). Moreover, one has

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}^n \setminus \{0\}} \left| u(x,t) - \Phi\left(d_\gamma(x; W_R(t)) - l\left(\frac{x}{|x|}, t\right)\right)\right| = 0, \tag{22}$$

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}^n \setminus \{0\}} \left| \nabla u(x,t) - \Phi'\left(d_\gamma(x; W_R(t)) - l\left(\frac{x}{|x|}, t\right)\right) \nabla \gamma^*(x)\right| = 0, \tag{23}$$

$\lim_{t \to \infty} u(0,t) \to 1$, *and* $\lim_{t \to \infty} \nabla u(0,t) = 0$.

We note that, since $d_\gamma(x; W_R(t)) = \gamma^*(x) - \rho(t; R)$ by definition, we have

$$\nabla \Phi\left(d_\gamma(x; W_R(t))\right) = \Phi'\left(d_\gamma(x; W_R(t))\right) \nabla \gamma^*(x).$$

Thus (23) implies that $\nabla u(x,t)$ approaches $\nabla \Phi(d_\gamma(x; W_R(t)))$ with a positional perturbation of $l(x/|x|, t)$. Hence (22) and (23) are consistent.

The conditions (18)–(19) are basically the same as those that guarantee occurrence of spreading fronts in the classical (isotropic) Allen–Cahn equation, see [3,21] for instance. Theorem 1.1 implies that, under such standard assumptions on the initial value $u_0(x)$, and with no other extra hypotheses, the solution $u(x,t)$ develops a spreading front whose position and shape are well approximated by the boundary of the expanding Wulff shape $\partial W_R(t)$ for all large times (see Fig. 1). More precisely, we have the following corollary:

Corollary 1.2 *(Smoothness and location of the front). Let the assumptions of Theorem 1.1 hold and let $T>0$ be as in Theorem 1.1. Then, for each $t\ge T$, the α-level set $\Gamma(t)$ defined by (20) is a smooth hypersurface and the region enclosed by $\Gamma(t)$ is star-shaped with respect to the origin. Furthermore, the Hausdorff distance*

$$d_{\mathcal{H}}(\Gamma(t),\partial W_R(t))$$

remains uniformly bounded for all $t\ge T$.

Proof. The smoothness of $\Gamma(t)$ follows from (21) and the smoothness of $l(\nu,t)$. (Or it is a direct consequence of the fact that $\nabla u\neq 0$ around $\Gamma(t)$, which follows from (23) and $\Phi'(0)\neq 0$.) The boundedness of $d_{\mathcal{H}}(\Gamma(t),\partial W_R(t))$ follows from (21) and the uniform boundedness of $l(\nu,t)$. The star-shapedness is a consequence of the fact that $\nu\mapsto l(\nu,t)$ is a single-valued well-defined function on S^{n-1}. □

Corollary 1.3 *(Convergence of the normals of $\Gamma(t)$). Let the assumptions of Theorem 1.1 hold. For each $\nu\in S^{n-1}$, let $x_\nu(t)$ denote the intersection point between $\Gamma(t)$ and the half-line $\{\xi\nu \mid \xi>0\}$ for large $t>0$. Then the Euclidean outward unit normal of $\Gamma(t)$ at $x_\nu(t)\in\Gamma(t)$, denoted by $n(x_\nu(t))$, satisfies*

$$\lim_{t\to\infty} n(x_\nu(t))=\lim_{t\to\infty}\frac{\nabla\gamma^*(x_\nu(t))}{|\nabla\gamma^*(x_\nu(t))|}=\frac{\nabla\gamma^*(\nu)}{|\nabla\gamma^*(\nu)|},\quad \textit{uniformly in } \nu\in S^{n-1}, \tag{24}$$

the right-hand side being equal to the (Euclidean) outward unit normal to the Wulff shape $W_1=\{x\in\mathbb{R}^n \mid \gamma^*(x)\le 1\}$.

Proof. When $x=x_\nu(t)\in\Gamma(t)$, we have $d_\gamma(x;W_R(t))-l(x/|x|,t)=0$ from (21). Thus, (23) gives

$$\lim_{t\to\infty}\left|\nabla u(x_\nu(t),t)-\Phi'(0)\nabla\gamma^*(x_\nu(t))\right|=0,\quad \text{uniformly in } \nu\in S^{n-1},$$

and hence (24) follows. □

As we will see in the proof of Theorem 1.1 in Section 4, the smoothness of $\Gamma(t)$ (and that of the function l) follows from the fact that $\nabla u(x,t)\neq 0$ around $\Gamma(t)$ for all large t. This fact also implies that the coefficient $a_{pp}(\nabla u)$ is smooth around $\Gamma(t)$, and therefore the solution $u(x,t)$ belongs to (and is bounded) in $C^{2+\theta,1+\theta/2}$ there. Consequently, the convergence in (22) takes place in the sense of $C^{2,1}$ around $\Gamma(t)$. This implies, in particular, that u_t is uniformly bounded and uniformly positive around $\Gamma(t)$ for all sufficiently large t (see Lemma 4.8). These observations lead to the following result:

Theorem 1.4 *(Asymptotic behavior of the front). Let the assumptions of Theorem 1.1 hold and let $\Gamma(t)$ be the α-level set of the solution $u(x,t)$ defined by (20). Then the following hold:*

(i) *$\Gamma(t)$ is monotonically expanding for all large t;*
(ii) *there exists a constant $C>0$ such that*

$$d_{\mathcal{H}}(\Gamma(t+\tau),\Gamma(t))\le C\tau \ \ \textit{for all sufficiently large } t \textit{ and all } \tau\ge 0;$$

(iii) *the function $l: S^{n-1}\times[T,\infty)\to\mathbb{R}$ defined in Theorem 1.1 satisfies $\partial_t l(\nu,t)\to 0$ as $t\to\infty$ uniformly in $\nu\in S^{n-1}$;*
(iv) *$V_\gamma\to c$ and $\kappa_\gamma\to 0$ as $t\to\infty$ uniformly on $\Gamma(t)$, where V_γ and κ_γ denote, respectively, the anisotropic normal velocity and the anisotropic mean curvature of $\Gamma(t)$ that appear in (12).*

Remark 1.5. The statement (iv) of the above theorem asserts that the law of motion of Γ_t is asymptotically given by $V_\gamma = c$ as $t \to \infty$, and that the effect of the curvature κ_γ on the motion of Γ_t becomes nearly negligible for large t. However, the long-time effect of the curvature is by no means negligible. Indeed, the boundary of the expanding Wulff shape $\partial W_R(t)$ evolves by the equation $V_\gamma = \kappa_\gamma + c$, and the presence of κ_γ in the above interface equation gives rise to the positional drift of order $log(t)$ as shown in (9). The fact that Γ_t remains uniformly close to $\partial W_R(t)$ implies that the long-time curvature effect is non-negligible on the motion of Γ_t.

As we will show in Appendix A.3, for each $\nu \in S^{n-1}$, equation (1a) has planar wave solution given in the form

$$u(x,t) = \Phi(\nabla\gamma^*(\nu)\cdot x - ct). \tag{25}$$

The front of this solution propagates in the direction parallel to $\nabla\gamma^*(\nu)$ with the speed $c/|\nabla\gamma(\nu)|$ in the Euclidean distance. Note that the statement (22) in Theorem 1.1 can be rewritten as

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^n\setminus\{0\}}\left|u(x,t) - \Phi\left(\nabla\gamma^*(x)\cdot x - \rho(t;R) - l\left(\frac{x}{|x|},t\right)\right)\right| = 0,$$

since $\gamma^*(x) = \nabla\gamma^*(x)\cdot x$ by the homogeneity of γ^*. We also note that

$$\rho(t;R) + l(\nu,t) = ct + o(t) \ \text{ as } \ t\to\infty,$$

since $\rho(t;R) = R + ct - ((n-1)/c)\log t + o(\log t)$ as mentioned before and since $l(\nu,t)$ is bounded. In view of these, we see that (22) implies that the profile of the solution at the front converges to a planar wave solution of the form (25).

Remark 1.6. If the equation is isotropic and if the initial value satisfies $u_0 \geq 0$ and has compact support, then one can use the reflection argument of [22] to show that $\nabla u \neq 0$ outside the convex hull of the support of u_0; hence $\Gamma(t)$ is smooth there. Furthermore, the same reflection argument shows that the inward normal lines to $\Gamma(t)$ always hit the convex hull of the support of u_0, therefore the shape of $\Gamma(t)$ becomes more and more spherically symmetric as it expands toward infinity. However, such a reflection argument does not work in anisotropic equations. Furthermore we are not assuming that u_0 has compact support, nor do we assume that $u_0 \geq 0$. Therefore the method of [22] cannot be applied to the present problem. Thus the results in Corollaries 1.2 and 1.3, as well as Theorem 1.4, are new even in the isotropic case.

We next give a simple example of the Frank diagram and the Wulff shape. Actually the equation in the following example reduces to the isotropic Allen–Cahn equation $u_t = \Delta u + f(u)$ by linear rescaling of coordinates, so this is a trivial case.

Example 1.7 *(Linear anisotropy).* Consider the equation

$$u_t = Au_{xx} + Bu_{yy} + f(u), \quad (x,y)\in\mathbb{R}^2, t>0,$$

namely, the case where $a(p,q) = Ap^2/2 + Bq^2/2$. Then the convex conjugate of $a(p,q)$ is $a^*(x,y) = x^2/(2A) + y^2/(2B)$, see Appendix A. The Frank diagram F_1 and the Wulff shape W_1 are both ellipses given by

$$F_1 = \left\{(p,q)\in\mathbb{R}^2 \,\middle|\, \gamma(p,q) = \sqrt{Ap^2 + Bq^2} \leq 1\right\},$$

$$W_1 = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \gamma^*(x,y) = \sqrt{\frac{x^2}{A} + \frac{y^2}{B}} \leq 1\right\}.$$

We note that, if $a(p)$ is homogeneous of degree two and belongs to $C^2(\mathbb{R}^n)$, it has to be a quadratic form as in Example 1.7. Therefore the corresponding equation is linear and hence has trivial anisotropy. This means that whenever we deal with nontrivial anisotropy, the derivatives $(a_{p_ip_j}(p))_{i,j}$ are necessarily discontinuous at $p=0$ (at least for some i,j). This is the reason why we assume somewhat weaker regularity: $a\in C^2(\mathbb{R}^n\setminus\{0\})\cap C^1(\mathbb{R}^n)$. As a consequence, we have to consider weak solutions of (1), see Definition 2.4 and Proposition 2.5 in Subsection 2.2 for details.

In Theorem 1.1, we did not specify how the function $l(\nu, t)$ behaves for large t. At present, we do not know if $l(\nu, t)$ has a limit $l_*(\nu) = \lim_{t\to\infty} l(\nu, t)$ in general. This question is still open, while a partial answer was obtained in [31], where the author proves that such a limit exists in the isotropic equation if the initial data is a small perturbation (in $H^1(\mathbb{R}^n)$) to the radially symmetric well-formed front.

Let us now make a brief (and very partial) review of related results. Spreading fronts for the isotropic equation

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^n, t > 0, \tag{26}$$

where $f(u)$ is either bistable or monostable type, was studied systematically in the pioneering paper [3], where they introduced the notion of "spreading speed" and proved that it coincides with the (minimal) speed of traveling waves. More precisely, they proved that, for any $\varepsilon > 0$, the solution $u(x, t)$ converges to 1 as $t \to \infty$ uniformly in the region $|x| \le (c^* - \varepsilon)t$, while it converges to 0 as $t \to \infty$ uniformly in the region $|x| \ge (c^* + \varepsilon)t$, where c^* denotes the spreading speed. The paper [22] studies the isotropic Allen–Cahn equation (26) and, under the assumption that u_0 is nonnegative and compactly supported and that $u \to 1$ as $t \to \infty$, proves that the spreading front becomes spherically symmetric in the C^1 sense as $t \to \infty$, and that the profile of u around the spreading front converges to that of the traveling wave. This result, which is proved by the plane reflection method, is similar to (22)–(23) of our Theorem 1.1 and Corollary 1.3, though we do not need to assume that u_0 be nonnegative nor compactly supported. The papers [31,34] also deal with the isotropic Allen–Cahn equation (26) and show that the initial spherical asymmetry does not necessarily disappear as $t \to \infty$. (In our terminology, their results give an example in which the function $l(\nu, t)$ in (22) does not converge to a constant as $t \to \infty$.) The paper [27] considers the equation on the hyperbolic space: $u_t = \Delta_H u + f(u)$ in $\mathbb{H}^n$, and studies the effect of the curvature of the underlying space $\mathbb{H}^n$ on the speed of spreading fronts. The paper [30] studies spreading fronts for spatially periodic reaction–diffusion equation on $\mathbb{R}^n$ (both for bistable and monostable nonlinearity) and derives the so-called Freidlin–Gärtner formula for the spreading speed, which implies that the asymptotic shape of the spreading front roughly converges to the Wulff shape associated with the homogenized equation. Unlike the spatially homogeneous problems, whether the solution develops a well-formed transition layer around the front or not is not known yet. The paper [8] introduces the notion of generalized transition fronts in a general unbounded domain in $\mathbb{R}^n$, and [19] gives classification of generalized fronts for bistable equations on $\mathbb{R}^n$.

The asymptotic behavior of solutions of (1) is closely related to suitable singular limits of the anisotropic Allen–Cahn equation. Consider the following equation with a small parameter $\varepsilon > 0$:

$$u^\varepsilon_t = \varepsilon \operatorname{div} a_p(\nabla u^\varepsilon) + \frac{1}{\varepsilon} f(u^\varepsilon), \quad x \in \mathbb{R}^n, t \in [0, T], \tag{27}$$

where f is a bistable nonlinearity. When f is of unbalanced type such that $\int_0^1 f(s)ds \ne 0$, the case we treat in this paper, the front motion of the solution of (27) is governed by the equation $V_\gamma = c$ in the sharp interface limit as $\varepsilon \to 0$. Note also that the equation (27) is converted to (1a) through the rescaling $\tilde{x} = \varepsilon^{-1}x$, $\tilde{t} = \varepsilon^{-1}t$. On the other hand, when f is of balanced or slightly unbalanced type (namely, it depends on ε in such a way that $\int_0^1 f_\varepsilon(s)ds = O(\varepsilon)$), one needs to use a different scaling, namely

$$u^\varepsilon_t = \operatorname{div} a_p(\nabla u^\varepsilon) + \frac{1}{\varepsilon^2} f(u^\varepsilon), \tag{28}$$

and the corresponding interface equation in the sharp interface limit is given by $V_\gamma = \kappa_\gamma$ or $V_\gamma = c + \kappa_\gamma$, respectively, where c is a suitable constant. Note that, in this case, the equation is converted to (1a) by the rescaling $\tilde{x} = \varepsilon^{-1}x$, $\tilde{t} = \varepsilon^{-2}t$. The singular limit problem for (28) has been studied in [1,12,13,18]. However, these results are valid only for finite time intervals, and hence they give no precise information about the asymptotic behavior of spreading fronts of equation (1a) as $t \to \infty$, which is the main theme of the present paper.

As far as the authors know, there is no earlier rigorous study of the long-time behavior of solutions of the anisotropic Allen–Cahn equation on $\mathbb{R}^n$. On the other hand, there are many results on the long-time behavior of the corresponding surface evolution equations and related problems. The papers [29,33] prove that the asymptotic shape of closed hypersurfaces that evolve by the crystal growth $V_\gamma = c$ in $\mathbb{R}^n$ is characterized by the Wulff shape. This result can be obtained by the comparison argument. Note that, for this type of equation, the Frank diagram need not be convex for the equation to be well-posed, so the corresponding Wulff shape may have corners, which adds extra subtlety to the comparison argument. In [29,33], this technical subtlety is handled by converting the equation to the Hamilton–Jacobi equation and applying the comparison principle for viscosity solutions of the Hamilton–Jacobi equation. The papers [20,32] study the equation $V_\gamma = \kappa_\gamma + c$ in $\mathbb{R}^n$ or a more generalized version of this equation and prove that the region $\Omega(t)$ enclosed by the evolving hypersurface satisfies $\Omega(t)/t \to CW_1$ as $t \to \infty$, where C is some positive constant and W_1 is the Wulff shape determined by γ. More precisely, the paper [32] derives this result by assuming the function γ to be C^2 and strictly convex (as in the present paper) and applying the comparison argument for viscosity solutions, from which one obtains

$$W_{R_1}(t) \subset \Omega(t) \subset W_{R_2}(t), \quad t > 0. \tag{29}$$

In [20], convergence results similar to [32] are proved for a more general equation of the form $V_\gamma = -\mathrm{tr}\,[E(n)\nabla n] + c$, where n denotes the Euclidean outward unit normal to $\partial\Omega(t)$ and $E(n)$ is an arbitrary symmetric matrix that is positive semi-definite and continuous in $n \in S^{n-1}$. Here the convexity of γ is not necessarily assumed, but the equation is still (degenerate) parabolic because of the positive semi-definiteness of $E(n)$. Note that $-\mathrm{tr}\,[E(n)\nabla n] = \kappa_\gamma$ when $E(n) = \nabla^2\gamma(n)$; in this case γ has to be convex for $E(n)$ to be positive semi-definite. The proofs in [20] also rely on the comparison argument for viscosity solutions.

We note that comparison arguments alone cannot tell whether or not the boundary $\partial\Omega(t)$ eventually become adequately regular. Nor do they show if the normals to $\partial\Omega(t)$ converge to those of $W_R(t)$ as $t \to \infty$. The results of the present paper answer these questions for the level surfaces of solutions of (1); see Theorem 1.1 and Corollary 1.3.

Finally, we remark that the surface evolution equations discussed above can also be obtained as sharp interface limits of models other than the Allen–Cahn equation. In [16], the authors study a stochastic interface model, — the so-called the Ginzburg–Landau $\nabla\phi$-interface model —, and show that the interface dynamics become deterministic in a suitable large scale limit and are described by anisotropic mean curvature flows. There is also an extensive literature on the phase field models with anisotropic free energy and their sharp interface limit; see, for example, [10,11,17] and the references therein.

The rest of the paper is organized as follows. In Section 2, we make some preparations. In Subsection 2.1, we recall the notion of the anisotropic mean curvature flow and its basic properties. In Subsection 2.2, we define weak solutions of (1) and prove their existence and uniqueness for each initial value u_0 (Proposition 2.5), and establish the comparison principle for weak solutions (Proposition 2.6).

In Section 3, we give relatively refined upper and lower bounds for the solution $u(x,t)$ by constructing a pair of comparison functions whose level sets are the expanding Wulff shapes (Proposition 3.1). This immediately implies that the level surface $\Gamma(t)$ of the solution u remains within bounded distance from $\partial W_R(t)$ for all large t. To prove this proposition, we first give a rough estimate of the solution near the origin and near the infinity (Lemma 3.2) in Subsection 3.1. In Subsection 3.2, we construct a fine set of super-solutions and sub-solutions (Lemma 3.7). In Subsection 3.3, we give the proof of Proposition 3.1.

In Section 4, we complete the proof of Theorems 1.1 and 1.4 by showing the fine structure of transition layers around the spreading front. First, in Subsection 4.1, we prove the local convergence of the solution to the planar waves in the topology of $BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R})$ (Lemma 4.1). Next, in Subsection 4.2, we show the strict monotonicity of the solution around the front (Lemma 4.3). Finally, in Subsections 4.3 and 4.4, we complete the proof of Theorems 1.1 and 1.4, respectively.

The appendix is devoted to giving supplementary results concerning anisotropy, and to proving Lemma 4.2, which states that any entire solution of (1a) that lies between two planar waves for all $t \in \mathbb{R}$ is a planar wave. This lemma generalizes Theorem 3.1 of [7] to anisotropic equations.

We end this section by summarizing notation for function spaces in the present paper. Let D be a domain in $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. $C^k(D)$ denotes the set of functions defined on D whose derivatives up to the k-th order are continuous, and $W^{k,\infty}(D)$ is the set of functions whose weak derivatives up to the k-th order belong to $L^\infty(D)$. For any non-negative integer k, we define $BC^k(D) := C^k(D) \cap W^{k,\infty}(D)$ and $\|\cdot\|_{BC^k(D)} := \|\cdot\|_{W^{k,\infty}(D)}$. For $\theta \in (0,1)$, $C^\theta(D)$ denotes the Hölder space, namely, the set of functions defined on D that are bounded and uniformly Hölder continuous with exponent θ. The set of functions in $BC^k(D)$ whose k-th order derivatives belong to $C^\theta(D)$ is denoted by $C^{k+\theta}(D)$.

Let $u(x,t)$ be a function defined on a domain Q in $\mathbb{R}^n \times \mathbb{R}$. By $u \in C^{1,0}(Q)$ and $u \in C^{2,1}(Q)$, we mean $u, u_{x_i} \in C^0(Q)$ and $u, u_{x_i}, u_{x_i x_j}, u_t \in C^0(Q)$ for all $1 \le i, j \le n$, respectively. We define $BC^{1,0}(Q)$ and $BC^{2,1}(Q)$ in a similar way. For $\theta \in (0,1)$, $C^{\theta,\theta/2}(Q)$ denotes the set of functions defined on Q that are bounded and uniformly Hölder continuous with exponent θ and $\theta/2$ with respect to x and t, respectively. By $u \in C^{1+\theta,\theta/2}(Q)$ and $u \in C^{2+\theta,1+\theta/2}(Q)$, we mean $u, u_{x_i} \in C^{\theta,\theta/2}(Q)$ and $u, u_{x_i}, u_{x_i x_j}, u_t \in C^{\theta,\theta/2}(Q)$ for all $1 \le i, j \le n$, respectively.

2. Preliminaries

In this section, we make preparations for discussions in later sections. In Subsection 2.1, we recall basic properties of homogeneous functions and define notions concerning the anisotropic mean curvature flow. In Subsection 2.2, we define weak solutions of the problem (1), and sketch the proof of their unique existence and regularity. We also recall the comparison principle for weak sub- and super-solutions of (1), which is a main tool for our later analysis.

2.1. *Anisotropic mean curvature flow and related notions*

We first summarize basic properties of $a(p)$. Throughout this paper, we assume that $a(p)$ in equation (1a) belongs to $C^2(\mathbb{R}^n \setminus \{0\}) \cap C^1(\mathbb{R}^n)$ and is strictly convex and homogeneous of degree two. Namely

$$a(\lambda p) = \lambda^2 a(p), \quad \lambda \in \mathbb{R}, p \in \mathbb{R}^n.$$

Then $a_p : \mathbb{R}^n \to \mathbb{R}^n$ and $a_{pp} : \mathbb{R}^n \to \mathbb{R}^{n^2}$ are homogeneous of degree one and zero, respectively:

$$a_p(\lambda p) = \lambda a_p(p), \quad a_p(p) \cdot p = 2a(p), \quad \lambda \in \mathbb{R}, p \in \mathbb{R}^n,$$
$$a_{pp}(\lambda p) = a_{pp}(p), \quad a_{pp}(p)p = a_p(p), \quad \lambda \in \mathbb{R} \setminus \{0\}, p \in \mathbb{R}^n \setminus \{0\}.$$

The function $a_{pp} = (a_{p_i p_j})$ is bounded on $\mathbb{R}^n \setminus \{0\}$, but we do not assume continuity of a_{pp} at $p = 0$. In fact, as we mentioned earlier, a_{pp} being continuous at $p = 0$ implies that a_{pp} is constant on $\mathbb{R}^n$, which implies that $a(p)$ is a quadratic polynomial and the equation (1a) has linear (hence trivial) anisotropy. We next summarize basic properties of $\gamma(p)$ and its dual $\gamma^*(x)$ defined by (6). They are frequently used in our computations.

Remark 2.1 *(Basic properties of $\gamma(p)$ and $\gamma^*(x)$).* By the definition, both $\gamma(p)$ and its dual $\gamma^*(x)$ are positively homogeneous of degree one and symmetric in the sense that $\gamma(p) = \gamma(-p)$ and $\gamma^*(x) = \gamma^*(-x)$, or equivalently

$$\gamma(\lambda p) = |\lambda|\gamma(p), \quad \gamma^*(\lambda x) = |\lambda|\gamma^*(x), \quad \lambda \in \mathbb{R}, \ p, x \in \mathbb{R}^n. \tag{30}$$

Consequently, $\nabla\gamma(p)$ and $\nabla\gamma^*(x)$ are positively homogeneous of degree zero and satisfy $\nabla\gamma(-p) = -\nabla\gamma(p)$ and $\nabla\gamma^*(-x) = -\nabla\gamma(x)$. Thus,

$$\nabla\gamma(\lambda p) = \frac{\lambda}{|\lambda|}\nabla\gamma(p), \quad \nabla\gamma^*(\lambda x) = \frac{\lambda}{|\lambda|}\nabla\gamma^*(x), \tag{31}$$
$$\nabla\gamma(p) \cdot p = \gamma(p), \quad \nabla\gamma^*(x) \cdot x = \gamma^*(x), \tag{32}$$

for any $\lambda \in \mathbb{R} \setminus \{0\}$ and any $p, x \in \mathbb{R}^n \setminus \{0\}$. Moreover, we have

$$\gamma^*(\nabla\gamma(p)) = 1, \quad \gamma(\nabla\gamma^*(x)) = 1, \tag{33}$$
$$\gamma(p)\nabla\gamma^*(\nabla\gamma(p)) = p, \quad \gamma^*(x)\nabla\gamma(\nabla\gamma^*(x)) = x, \tag{34}$$

for any $p, x \in \mathbb{R}^n \setminus \{0\}$. For the proof of (33)–(34), see Remark A.6 in Subsection A.1. See also [5,6].

Next we define *anisotropic normal vector, anisotropic mean curvature,* and *anisotropic normal velocity*. First we recall the definition of anisotropic signed distance function $d_\gamma(x; \Omega(t))$, where $\Omega(t)$ is any moving domain with smooth boundary $\partial\Omega(t)$. Then it is smooth in a tubular neighborhood of $\partial\Omega(t)$ and (33) implies

$$\gamma(\nabla d_\gamma(x; \Omega(t))) = 1, \quad x \in \partial\Omega(t). \tag{35}$$

The proof of (35) is also found in [5,6]. Now we are ready to define the anisotropic mean curvature and related notions. These are basically the same as found in [1,4,5]; see also Appendix A for more details.

Definition 2.2. At each point of $\partial\Omega(t)$, the Euclidean outward normal vector n and the anisotropic outward normal vector ν_γ are given by

$$n = \frac{\nabla d_\gamma}{|\nabla d_\gamma|}, \quad \nu_\gamma = \nabla\gamma(n) \ \big(=: \nabla\gamma(p)|_{p=n}\big). \tag{36}$$

The Euclidean normal velocity V, the anisotropic normal velocity V_γ, and the anisotropic mean curvature κ_γ on $\partial\Omega(t)$ are defined by

$$V = -\frac{\partial_t d_\gamma}{|\nabla d_\gamma|}, \quad V_\gamma = \frac{V}{\gamma(n)}, \quad \kappa_\gamma = -\operatorname{div}\nu_\gamma, \tag{37}$$

where n is the Euclidean outward normal vector given in (36) and "div" in (37) is taken in the usual Euclidean sense in $\mathbb{R}^n$. (See Fig. 3 in Section A.2 for the geometric interpretation of the anisotropic normal vector ν_γ.)

We now recall the notion of the anisotropic mean curvature flow. First, we note that the anisotropic mean curvature κ_γ and the anisotropic normal velocity V_γ defined in (37) are rewritten as

$$\kappa_\gamma = -\operatorname{div} a_p(\nabla d_\gamma), \quad V_\gamma = -\partial_t d_\gamma, \quad \text{on } \partial\Omega(t). \tag{38}$$

Indeed, since $a_p(p) = \gamma(p)\nabla\gamma(p)$ by the definition, we have

$$a_p(\nabla d_\gamma) = \gamma(\nabla d_\gamma)\nabla\gamma(\nabla d_\gamma) = \nabla\gamma(\nabla d_\gamma) = \nabla\gamma\left(\frac{\nabla d_\gamma}{|\nabla d_\gamma|}\right) = \nu_\gamma,$$

where the second equality follows from (35) and the last equality follows from (36). Thus the former identity in (38) follows. The latter identity in (38) is shown as follows:

$$V_\gamma = \frac{V}{\gamma(n)} = -\frac{\partial_t d_\gamma}{|\nabla d_\gamma|\gamma\left(\frac{\nabla d_\gamma}{|\nabla d_\gamma|}\right)} = -\frac{\partial_t d_\gamma}{\gamma(\nabla d_\gamma)} = -\partial_t d_\gamma,$$

where the last equality comes from (35). Consequently, the anisotropic mean curvature flow (with a constant driving term) $V_\gamma = \kappa_\gamma + c$ is represented in terms of the anisotropic signed distance function d_γ by

$$\partial_t d_\gamma = \operatorname{div} a_p(\nabla d_\gamma) - c, \quad \text{on } \partial\Omega(t). \tag{39}$$

Remark 2.3 *(Wulff shape and the mean curvature flow).* For the expanding Wulff shape $W_R(t)$, the anisotropic signed distance function for $\partial W_R(t)$ is given by $d_\gamma(x; W_R(t)) = \gamma^*(x) - \rho(t; R)$ as in (10). Thus the anisotropic normal vector for $\partial W_R(t)$ is given by $\nu_\gamma = \nabla\gamma(\nabla\gamma^*(x))$. This and (34) yield

$$\kappa_\gamma = -\operatorname{div}\big(\nabla\gamma(\nabla\gamma^*(x))\big) = -\operatorname{div}\left(\frac{x}{\gamma^*(x)}\right) \qquad = -\frac{n-1}{\gamma^*(x)} \quad \text{on } \partial W_R(t).$$

Since $\gamma^*(x) = \rho(t; R)$ on $\partial W_R(t)$, this implies

$$\kappa_\gamma = -\frac{n-1}{\rho(t; R)} \quad \text{on } \partial W_R(t).$$

Hence the anisotropic mean curvature is constant on $W_R(t)$. On the other hand, since $\rho(t; R)$ satisfies (8), the second identity in (38) gives

$$V_\gamma = -\partial_t\big(\gamma^*(x) - \rho(t; R)\big) = \rho'(t; R) = -\frac{n-1}{\rho(t; R)} + c.$$

Thus $d_\gamma(x; W_R(t))$ satisfies $V_\gamma = \kappa_\gamma + c$ (and (39)). This means that $\partial W_R(t)$ is an expanding self-similar solution of the anisotropic mean curvature flow.

2.2. Weak solutions and the comparison principle

In this subsection, we first state the definitions of weak solutions, weak sub- and super-solutions to the problem (1). We then show the unique existence and regularity of weak solutions, and prove the comparison principle for weak sub- and super-solutions. There is an extensive literature on weak solutions of parabolic equations; see for instance, [23,24]. However, those existence results are mainly for problems on bounded domains therefore they do not apply directly to our problem. Thus, for the convenience of the reader, we state and prove basic properties of weak solutions of (1) in this subsection. The notation for the function spaces used here are given at the end of Introduction.

Definition 2.4 *(Weak solutions).* By a weak solution to the problem (1), we mean a function $u(x,t) \in BC^{1,0}(\mathbb{R}^n \times [0,\infty))$ that satisfies $\mathcal{L}_w[u] = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, where $\mathcal{L}_w$ is defined by

$$\mathcal{L}_w[u] = -\int_{\mathbb{R}^n} u\varphi \, dx\Big|_{t=0} + \int_0^\infty \int_{\mathbb{R}^n} -u\varphi_t + a_p(\nabla u) \cdot \nabla\varphi - f(u)\varphi \, dx \, dt. \tag{40}$$

A function $u(x,t) \in BC^{1,0}(\mathbb{R}^n \times [0,\infty))$ is called a weak sub-solution (resp. weak super-solution) if it satisfies $\mathcal{L}_w[u] \le 0$ (resp. $\mathcal{L}_w[u] \ge 0$) for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$ with $\varphi \ge 0$.

As we see in Definition 2.4 above, throughout this paper, we only consider weak solutions that are bounded on $\mathbb{R}^n \times [0,\infty)$. We note that if $u \in C^{2,1}(\mathbb{R}^n \times [0,\infty)) \cap BC^{1,0}(\mathbb{R}^n \times [0,\infty))$ is a solution (resp. sub-solution, super-solution) of the equation (1a) in the usual classical sense, it is also a weak solution (resp. weak sub-solution, weak super-solution) in the sense of Definition 2.4.

To show the existence of the weak solution $u(x,t)$ of (1), we first consider an approximate equation in which $a(p)$ is replaced by its mollified one. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a smooth non-negative function that has compact support and satisfies $\int_{\mathbb{R}^n} \eta(x)dx = 1$. We define the smooth function $a^\varepsilon(p)$ with $\varepsilon > 0$ by

$$a^\varepsilon(p) = (\eta_\varepsilon * a)(p), \quad \eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right). \tag{41}$$

By replacing $a(p)$ in (1a) by $a^\varepsilon(p)$, we consider the Cauchy problem of the form

$$\begin{cases} u_t^\varepsilon = \operatorname{div} a_p^\varepsilon(\nabla u^\varepsilon) + f(u^\varepsilon), & x \in \mathbb{R}^n,\ t > 0, \quad \text{(a)} \\ u^\varepsilon(x,0) = u_0(x), & x \in \mathbb{R}^n. \quad \text{(b)} \end{cases} \tag{42}$$

We note that the function $a^\varepsilon(p)$ satisfies

$$\lim_{\varepsilon \to 0} \|a^\varepsilon - a\|_{W^{1,\infty}(\mathbb{R}^n)} = 0,$$

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^n a^\varepsilon_{p_i p_j}(p)\xi_i\xi_j \le \Lambda|\xi|^2, \quad p, \xi \in \mathbb{R}^n, \varepsilon > 0,$$

where Λ is the constant given in (4). It is well known, see for instance [23,24], that the problem (42) has a unique classical solution $u^\varepsilon \in C^{2+\theta,1+\theta/2}(\mathbb{R}^n \times [0,\infty))$ when $u_0 \in C^{2+\theta}(\mathbb{R}^n)$. The following proposition gives a weak solution u of (1) as the limit of a sequence of the classical solutions $\{u^\varepsilon\}$ of (42).

Proposition 2.5 *(Unique existence and regularity of weak solutions). Assume $u_0 \in C^{2+\theta}(\mathbb{R}^n)$ for some $\theta \in (0,1)$. Then the following hold:*

(i) *There exists a unique weak solution $u(x,t)$ of (1) in the sense of Definition 2.4 that satisfies*

$$u \in C^{1+\theta,\theta/2}(\mathbb{R}^n \times [0,\infty)), \quad u \in C^{2+\theta,1+\theta/2}_{loc}(\mathbb{R}^n \times [0,\infty) \setminus \partial Q),$$

where $Q = \{(x,t) \in \mathbb{R}^n \times [0,\infty) \mid \nabla u(x,t) = 0\}$ and ∂Q denotes its boundary.

(ii) *There exists a sequence* $\varepsilon_1 > \varepsilon_2 > \cdots \to 0$ *such that the sequence of the classical solutions* $\{u^{\varepsilon_i}\}$ *of* (42) *with the same initial value* u_0 *as that of* (1) *satisfies*

$$u^{\varepsilon_i} \to u \text{ in } BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty)) \text{ as } i \to \infty.$$

Proof. The problem (42) has a unique classical solution $u^\varepsilon \in C^{2+\theta,1+\theta/2}(\mathbb{R}^n \times [0,\infty))$, since a^ε and u_0 in (42) are both sufficiently smooth by the assumptions, where $\|u^\varepsilon\|_{C^{2+\theta,1+\theta/2}(\mathbb{R}^n\times[0,\infty))}$ may blow up as $\varepsilon \to 0$, however, it satisfies

$$\|u^\varepsilon\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))} \le C, \tag{43}$$

with a constant C that is independent of $\varepsilon > 0$. For details, see Theorems 1.1 and 3.1 in Chapter V of [23] for instance. Therefore the sequence $\{u^\varepsilon\}$ is relatively compact in $BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty))$. Hence, there exist a sequence $\{\varepsilon_i\}$ with $\varepsilon_1 > \varepsilon_2 > \cdots \to 0$ and a function $u(x,t)$ defined on $\mathbb{R}^n \times [0,\infty)$ such that

$$u^{\varepsilon_i} \to u \text{ in } BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty)) \text{ as } i \to \infty.$$

Moreover, it is clear from the definition of the norm in $C^{1+\theta,\theta/2}$ that

$$\|u\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))} \le C,$$

where C is the constant in (43). Since u^{ε_i} satisfies the problem (42) also in the weak sense of Definition 2.4, we have $\mathcal{L}_w[u^{\varepsilon_i}] = 0$ for any $\varphi \in C^\infty_0(\mathbb{R}^n \times [0,\infty))$. By passing to the limit as $i \to \infty$, we obtain $\mathcal{L}_w[u] = 0$ for any $\varphi \in C^\infty_0(\mathbb{R}^n \times [0,\infty))$, since the support of φ is compact and the convergences of $u^{\varepsilon_i} \to u$ and $a^{\varepsilon_i} \to a$ are in the senses of $BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty))$ and $W^{1,\infty}(\mathbb{R}^n)$, respectively. Thus, $u(x,t)$ is the weak solution of (1) in the sense of Definition 2.4.

Next, we show $u \in C^{2+\theta,1+\theta/2}_{loc}(\mathbb{R}^n \times [0,\infty) \setminus \partial Q)$. At any interior point of Q, since $\nabla u = 0$ in its neighborhood, u is spatially constant and therefore satisfies $u_t = f(u)$, with f being smooth. Thus u is locally a smooth function of t alone; hence, in particular, $u \in C^{2+\theta,1+\theta/2}$ in that neighborhood. At any exterior point of Q, the function $a_p(\nabla u)$ is sufficiently regular in its neighborhood, since $a(p)$ is C^2 in $\mathbb{R}^n \setminus \{0\}$. Consequently, we have $u \in C^{2+\theta,1+\theta/2}$ there, by the usual interior Schauder estimate.

Finally, the uniqueness of the weak solution follows from the comparison principle for weak solutions given by Proposition 2.6 below. This completes the proof. □

Proposition 2.6 *(Comparison principle). Let* $u^-(x,t)$ *and* $u^+(x,t)$ *be a weak sub-solution and a weak super-solution of* (1) *in the sense of Definition* 2.4*, respectively. Assume that they belong to* $C^{1+\theta,\theta/2}(\mathbb{R}^n \times [0,\infty))$ *for some* $\theta \in (0,1)$*, and that* $u^-(x,0) \le u^+(x,0)$ *for* $x \in \mathbb{R}^n$*. Then*

$$u^-(x,t) \le u^+(x,t), \quad x \in \mathbb{R}^n,\ t \ge 0.$$

The comparison principle for weak solutions on a bounded domain is rather standard, but the one for an unbounded domain does not follow immediately from the one for bounded domains. Therefore, for the convenience of the reader, we prove the above proposition. Our proof is based on the strong maximum principle for weak solutions and a sliding argument. We begin with the following auxiliary lemma.

Lemma 2.7. *Let* $u(x,t)$ *be a weak sub- (resp. super-) solution of* (1) *in the sense of Definition* 2.4*. Suppose that it belongs to* $C^{1+\theta,\theta/2}(\mathbb{R}^n \times [0,\infty))$ *for some* $\theta \in (0,1)$*. Then, for any sequence* $\{x_i\}_{i=1,2,\cdots} \subset \mathbb{R}^n$*, there exist a subsequence* $\{x_{k_i}\}$ *and a weak sub- (resp. super-) solution* $w(x,t)$ *in the sense of Definition* 2.4 *such that*

$$u(x + x_{k_i}, t) \to w(x,t) \text{ in } BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty)) \text{ as } i \to \infty.$$

Proof. Define $u^i(x,t) = u(x + x_i, t)$. Then, we have

$$\|u^i\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))} = \|u\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))} < \infty, \quad \text{for } i = 1,2,\cdots.$$

Thus the sequence $\{u^i\}$ is relatively compact in $BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty))$. Hence, there exist a subsequence $\{u^{k_i}\}$ of $\{u^i\}$ and a function $w(x,t)$ defined on $\mathbb{R}^n \times [0,\infty)$ such that

$$u^{k_i} \to w \text{ in } BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty)) \text{ as } i \to \infty.$$

On the other hand, since the convergence to w is in the sense of $BC^{1,0}_{loc}$ and since a_p is continuous, by passing to the limit as $i \to \infty$, $\mathcal{L}_w[u^{k_i}] \le 0$ (resp. $\mathcal{L}_w[u^{k_i}] \ge 0$) gives $\mathcal{L}_w[w] \le 0$ (resp. $\mathcal{L}_w[w] \ge 0$) for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$. This completes the proof. □

We are now ready to prove Proposition 2.6.

Proof of Proposition 2.6. Since $u^\pm \in C^{1+\theta,\theta/2}(\mathbb{R}^n \times [0,\infty))$ by the assumption, they are both bounded on $\mathbb{R}^n \times [0,\infty)$. Thus, the constant m below is well-defined

$$m = \sup_{x\in\mathbb{R}^n, t\in[0,\infty)} e^{-Mt}(u^- - u^+).$$

Assume $m > 0$. Then there exists a sequence $\{(x_i,t_i)\} \subset \mathbb{R}^n \times [0,\infty)$ such that

$$m = \lim_{i\to\infty} e^{-Mt_i}(u^-(x_i,t_i) - u^+(x_i,t_i)).$$

Since we have $\limsup_{t\to\infty} \sup_{x\in\mathbb{R}^n} e^{-Mt}(u^- - u^+) \le 0$ by the boundedness of $u^\pm$, we find that $\{t_i\}$ is bounded. Thus, we assume without the loss of generality that $\lim_{i\to\infty} t_i = t_*$ holds for some $t_* > 0$. From Lemma 2.7, by choosing a subsequence, which we again denote by $\{(x_i,t_i)\}$, we have a weak sub-solution $v^-(x,t)$ and a weak super-solution $v^+(x,t)$ such that

$$u^-(x + x_i, t) \to v^-(x,t) \text{ and } u^+(x + x_i, t) \to v^+(x,t),$$

in $BC^{1,0}_{loc}(\mathbb{R}^n \times [0,\infty))$ as $i \to \infty$. Define the function $w = e^{-Mt}(v^- - v^+)$. Then it satisfies

$$w_t - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(A_{ij}(x,t) w_{x_j}\right) - (B(x,t) - M)\, w \le 0,$$

in the weak sense, where A_{ij} and B are defined by

$$A_{ij}(x,t) = \int_0^1 a_{p_i p_j}(\nabla v^+ + \theta \nabla(v^- - v^+)) d\theta, \quad B(x,t) = \int_0^1 f'(v^+ + \theta(v^- - v^+)) d\theta.$$

We note that $A_{ij} \in L^\infty(\mathbb{R}^n)$ and is positively definite. Since $v^\pm$ are both bounded by the assumption, we can choose M large enough to satisfy $B(x,t) - M \le 0$. Thus, we can apply the strong maximum principle for weak solutions given by Theorem 6.25 in [24], and obtain a contradiction. Indeed, w satisfies

$$w(0,t_*) = e^{-Mt_*}(v^-(0,t_*) - v^+(0,t_*)) = \lim_{i\to\infty} e^{-Mt_i}(u^-(x_i,t_i) - u^+(x_i,t_i)) = m > 0,$$

and satisfies $w(x,0) \le 0$ for $x \in \mathbb{R}^n$. This contradiction implies $m \le 0$. □

The following lemma is a slight modification of Lemma 2.7. This will be used in Section 4.

Lemma 2.8 *(Construction of weak entire solution). Let $u(x,t)$ be a weak solution in the sense of Definition 2.4 with an initial value $u_0 \in C^{2+\theta}(\mathbb{R}^n)$ for some $\theta \in (0,1)$. Then, for any sequence $\{(x_i,t_i)\} \subset \mathbb{R}^n \times [0,\infty)$ with $0 < t_1 < t_2 < \cdots \to \infty$, there exist a subsequence $\{(x_{k_i}, t_{k_i})\}$ and a function $w(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}$ such that*

(i) *$u(x + x_{k_i}, t + t_{k_i}) \to w(x,t)$ in $BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R})$ as $i \to \infty$.*

(ii) *$w(x,t)$ is a weak entire solution of $w_t = \operatorname{div} a_p(\nabla w) + f(w)$, namely, it satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} -w\varphi_t + a_p(\nabla w)\cdot\nabla\varphi - f(w)\varphi\,dx\,dt = 0, \tag{44}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. Moreover, one has $w \in C^{1+\theta,\theta/2}(\mathbb{R}^n \times \mathbb{R})$.

Proof. Define $u^i(x,t) = u(x+x_i, t+t_i)$. Since $u \in C^{1+\theta,\theta/2}(\mathbb{R}^n \times [0,\infty))$ from Proposition 2.5, we have

$$\|u^i\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times[-t_i,\infty))} = \|u\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))} < \infty, \quad \text{for } i = 1,2,\cdots.$$

Thus the sequence $\{u^i\}$ is relatively compact in $BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R})$. Hence, there exist a subsequence $\{u^{k_i}\}$ of $\{u^i\}$ and a function $w(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}$ such that

$$u^{k_i} \to w \text{ in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}) \text{ as } i \to \infty.$$

Moreover, $w \in C^{1+\theta,\theta/2}(\mathbb{R}^n \times \mathbb{R})$ holds, since it is clear from the definition of the norm in $C^{1+\theta,\theta/2}$ that $\|w\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times\mathbb{R})} = \|u\|_{C^{1+\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))}$. On the other hand, u^{k_i} satisfies

$$-\int_{\mathbb{R}^n} u^{k_i}\varphi\,dx\Big|_{t=-t_{k_i}} + \int_{-t_{k_i}}^{\infty}\int_{\mathbb{R}^n} -u^{k_i}\varphi_t + a_p(\nabla u^{k_i})\cdot\nabla\varphi - f(u^{k_i})\varphi\,dx\,dt = 0,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [-t_{k_i},\infty))$. Since the convergence to w is in the sense of $BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R})$ and since a_p is continuous, by passing to the limit as $i \to \infty$, we have (44) for any $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. This completes the proof. □

3. Approximation by the Wulff shape

The aim of this section is to prove Proposition 3.1 below, which gives an upper and a lower bound for the solution $u(x,t)$ of (1) at large time. It roughly states that the solution $u(x,t)$ is sandwiched between two functions whose level sets both coincide with the expanding Wulff shapes at slightly different time phases. This fact will be important for the analyzes in Section 4. We also note that part of Corollary 1.2 follows immediately from Proposition 3.1.

Proposition 3.1 *(Upper and lower bounds for $u(x,t)$). For each $m > 0$ and $\eta > 0$, there exists a positive constant L such that, if the initial value u_0 belongs to $C^{2+\theta}(\mathbb{R}^n)$ for some $\theta \in (0,1)$ and satisfies*

$$\inf_{x\in\mathbb{R}^n} u_0(x) \geq -m, \tag{45}$$

$$\min_{|x|\leq L} u_0(x) \geq \alpha + \eta, \tag{46}$$

$$\limsup_{|x|\to\infty} u_0(x) < \alpha, \tag{47}$$

for the constant α defined in (2), then there exist positive constants T, R, and K such that the solution $u(x,t)$ of (1) satisfies

$$W_-(x,t) \leq u(x, T+t) \leq W_+(x,t), \quad x \in \mathbb{R}^n,\ t \geq 0,$$

for the functions

$$W_\pm(x,t) = \Phi(d_\gamma(x; W_R(t)) \mp K) \pm 2(1+t)^{-\frac{3}{2}},$$

where $d_\gamma(x, W_R(t))$ and Φ are defined in (10) and (14), respectively.

3.1. Rough generation of the front

In this subsection, we prove Lemma 3.2 below, which gives a preliminary estimate for front propagation. It roughly states that, under the assumptions on u_0 as in Proposition 3.1, the solution $u(x,t)$ becomes very close to 1 on a large area around the origin, and very close to 0 faraway from the origin, after a certain lapse of time.

Lemma 3.2 *(Rough generation of the front). For any positive constants m, η, δ and R, there exist positive constants L, T, and K such that, if the initial value u_0 belongs to $C^{2+\theta}(\mathbb{R}^n)$ for some $\theta \in (0,1)$ and satisfies*

$$\inf_{x\in\mathbb{R}^n} u_0(x) \ge -m, \tag{48}$$

$$\min_{|x|\le L} u_0(x) \ge \alpha + \eta, \tag{49}$$

$$\limsup_{|x|\to\infty} u_0(x) < \alpha, \tag{50}$$

for the constant α defined in (2), then the solution $u(x,t)$ of (1) satisfies

$$-\delta \le u(x,T) \le 1+\delta, \quad x\in\mathbb{R}^n, \tag{51}$$

$$u(x,T) \le \delta, \quad |x| \ge K, \tag{52}$$

$$u(x,T) \ge 1-\delta, \quad |x| \le R. \tag{53}$$

Before proving Lemma 3.2, we introduce two auxiliary lemmas. Lemma 3.3 below is used to prove the estimate (52) in Lemma 3.2.

Lemma 3.3 *(Super-solutions with one-dimensional profiles). Let α_* and M be any given constants satisfying $\alpha_* \in (0,\alpha)$ and $M > 1$. Then there exist positive constants σ and β such that, for any $\nu \in S^{n-1}$ and any $K > 0$, the function $u^+(x,t;\nu)$ defined by*

$$u^+(x,t;\nu) = M\Phi(\nu\cdot x - \sigma t - K) + \alpha_* e^{-\beta t},$$

satisfies $\mathcal{L}[u^+] := u_t^+ - \operatorname{div} a_p(\nabla u^+) - f(u^+) \ge 0$ in the classical sense.

Proof. It is known (see Lemma 2.2 of [26] for instance) that there exists a positive constant k such that

$$|\Phi''(s)| \le -k\Phi'(s), \quad s\in\mathbb{R}. \tag{54}$$

Set $\delta_0 = \min\{\alpha-\alpha_*, M-1\}/2$, where we note that $\alpha_*+\delta_0 < \alpha < 1 < M-\delta_0$. Since f is of bistable type as in (2), we can choose positive constants μ_1 and μ_2 such that

$$-f(s) \ge \mu_1 s, \quad 0 \le s \le \alpha_* + \delta_0, \tag{55}$$

$$-f(s) \ge \mu_2, \quad M-\delta_0 \le s \le M+\alpha_*. \tag{56}$$

Set $\beta = \min\{\mu_1, \mu_2/\alpha_*\}$ and let σ be a positive constant satisfying

$$\min_{\nu\in S^{n-1}} \sigma - k(\gamma(\nu))^2 > 0. \tag{57}$$

We compute $\mathcal{L}[u^+]$. By noting that $\nabla u^+ = M\Phi'\nu$ with $M\Phi' < 0$, it follows from (30) and (31) (with $\lambda < 0$) in Remark 2.1 that

$$\gamma(\nabla u^+) = \gamma(M\Phi'\nu) = -M\Phi'\gamma(\nu),$$
$$\nabla\gamma(\nabla u^+) = \nabla\gamma(M\Phi'\nu) = -\nabla\gamma(\nu).$$

Thus, since $a_p(p) = \gamma(p)\nabla\gamma(p)$ by the definition, we have

$$\operatorname{div} a_p(\nabla u^+) = \operatorname{div}\big(M\Phi'\gamma(\nu)\nabla\gamma(\nu)\big) = M\Phi''\gamma(\nu)\nabla\gamma(\nu)\cdot\nu = M\Phi''(\gamma(\nu))^2.$$

This, together with (54), implies

$$\mathcal{L}[u^+] = -\sigma M\Phi' - \beta\alpha_* e^{-\beta t} - (\gamma(\nu))^2 M\Phi'' - f(M\Phi + \alpha_* e^{-\beta t})$$
$$\geq -(\sigma - k(\gamma(\nu))^2)M\Phi' - \beta\alpha_* e^{-\beta t} - f(M\Phi + \alpha_* e^{-\beta t}).$$

When $M\Phi \in (0, \delta_0]$, by using (55) and (57), we have

$$\mathcal{L}[u^+] \geq -\beta\alpha_* e^{-\beta t} + \mu_1(M\Phi + \alpha_* e^{-\beta t}) \geq (\mu_1 - \beta)\alpha_* e^{-\beta t}.$$

Hence $\mathcal{L}[u^+] \geq 0$, since $\beta \leq \mu_1$. Next, when $M\Phi \in [M - \delta_0, M)$, by using (56) and (57), we have

$$\mathcal{L}[u^+] \geq -\beta\alpha_* e^{-\beta t} + \mu_2.$$

Hence again $\mathcal{L}[u^+] \geq 0$ since $\beta \leq \mu_2/\alpha_*$. Finally, when $M\Phi \in (\delta_0, M - \delta_0)$, we have

$$\mathcal{L}[u^+] \geq (\sigma - k(\gamma(\nu))^2) \inf_{\delta_0 < M\Phi < M-\delta_0} (-M\Phi') - \beta + \min_{\delta_0 \leq s \leq M-\delta_0+\alpha_*} f(s).$$

Since $\inf_{\delta_0 < M\Phi < M-\delta_0}(-M\Phi') > 0$ holds from $\Phi' < 0$, by choosing σ large enough if necessary, we have $\mathcal{L}[u^+] \geq 0$. This completes the proof. □

The following lemma is used to prove the estimate (53) in Lemma 3.2. The key argument in the proof of Lemma 3.4 is the upper estimate for fundamental solutions given by [2], which allows us to derive an estimate that is independent of ε in a^ε in (42a).

Lemma 3.4. *Let $u^\varepsilon(x,t)$ be the classical solution of (42). For any positive constants m, η, δ, R, and T_*, there exist constants $L > 0$ and $T \geq T_*$ that are both independent of ε such that, if the initial value $u_0 \in C^{2+\theta}(\mathbb{R}^n)$ satisfies*

$$\inf_{x\in\mathbb{R}^n} u_0(x) \geq -m, \tag{58}$$

$$\min_{|x|\leq L} u_0(x) \geq \alpha + \eta, \tag{59}$$

for the constant α defined as in (2), then one has

$$u^\varepsilon(x,T) \geq 1 - \delta, \quad |x| \leq R. \tag{60}$$

Proof. We assume $u_0 \leq \alpha + \eta + 1$ without the loss of generality. Indeed, for a general u_0 satisfying (58)–(59), we choose $\tilde{u}_0$ that satisfies (58)–(59) and $\tilde{u}_0 \leq u_0$, $\tilde{u}_0 \leq \alpha + \eta + 1$. Once we prove the lower estimate (60) for such $\tilde{u}_0$, the result for the general u_0 follows immediately from the comparison theorem.

Let L be a positive constant to be determined later. By the assumptions on u_0, we can choose a function $v_0 \in C^{2+\theta}(\mathbb{R}^n)$ that satisfies

$$v_0(x) = u_0(x), \quad |x| \leq L,$$
$$u_0(x) \leq v_0(x) \leq \alpha + \eta + 1, \quad x \in \mathbb{R}^n,$$
$$\inf_{x\in\mathbb{R}^n} v_0(x) \geq \alpha + \frac{\eta}{2}.$$

Fix $\varepsilon > 0$ arbitrarily and let $a^\varepsilon(p)$ be the function defined as in (41). We consider the problem of the form

$$\begin{cases} v_t^\varepsilon = \operatorname{div} a_p^\varepsilon(\nabla v^\varepsilon) + f(v^\varepsilon), & x \in \mathbb{R}^n,\ t > 0, \\ v^\varepsilon(x,0) = v_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Consider the function $U(t)$ that satisfies $U' = f(U)$ with the initial condition $U(0) = \alpha + \eta/2$. Since $U(0) > \alpha$, we have $U(t) \to 1$ as $t \to \infty$. Thus, by virtue of the usual comparison principle for the classical solutions, we can choose a constant $T \geq T_*$ that is independent of $\varepsilon > 0$ such that

$$v^\varepsilon(x,T) \geq U(T) \geq 1 - \frac{\delta}{2}, \quad x \in \mathbb{R}^n. \tag{61}$$

We next consider the function $w^\varepsilon(x,t) = v^\varepsilon(x,t) - u^\varepsilon(x,t)$. Then it satisfies

$$\begin{cases} w_t^\varepsilon = \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(A_{ij}^\varepsilon(x,t) w_{x_j}^\varepsilon\right) + B^\varepsilon(x,t) w^\varepsilon, & x \in \mathbb{R}^n,\ t>0, \\ w^\varepsilon(x,0) = v_0(x) - u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $A_{ij}^\varepsilon(x,t)$ and $B^\varepsilon(x,t)$ are given by

$$A_{ij}^\varepsilon(x,t) = \int_0^1 a_{p_i p_j}^\varepsilon(\nabla u^\varepsilon + \theta \nabla w^\varepsilon) d\theta, \quad B^\varepsilon(x,t) = \int_0^1 f'(u^\varepsilon + \theta w^\varepsilon) d\theta.$$

As we have $w^\varepsilon(x,0) \geq 0$ by the definition of v_0, the usual maximum principle for the classical solutions gives $w^\varepsilon(x,t) \geq 0$ on $\mathbb{R}^n \times [0,\infty)$. Thus, since u^ε and v^ε are both bounded as $-m \leq u^\varepsilon, v^\varepsilon \leq \alpha + \eta + 1$, there exists a constant $M > 0$ independent of ε such that $B^\varepsilon(x,t) w^\varepsilon \leq M w^\varepsilon$. Thus, by considering the solution $h^\varepsilon(x,t)$ of the problem

$$\begin{cases} h_t^\varepsilon = \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(A_{ij}^\varepsilon(x,t) h_{x_j}^\varepsilon\right) + M h^\varepsilon, & x \in \mathbb{R}^n,\ t>0, & \text{(a)} \\ h^\varepsilon(x,0) = v_0(x) - u_0(x), & x \in \mathbb{R}^n, & \text{(b)} \end{cases} \tag{62}$$

the comparison principle gives $w^\varepsilon(x,t) \leq h^\varepsilon(x,t)$ on $\mathbb{R}^n \times [0,\infty)$. On the other hand, since (62a) is uniformly parabolic because of the strict convexity of $a^\varepsilon(x)$, we have the expression

$$h^\varepsilon(x,t) = e^{Mt} \int_{\mathbb{R}^n} Z^\varepsilon(x,\xi,t,0)(v_0(\xi) - u_0(\xi))\, d\xi,$$

where $Z^\varepsilon(x,\xi,t,\tau)$ is the fundamental solution of $h_t^\varepsilon = \sum_{i,j=1}^n (A_{ij}^\varepsilon(x,t) h_{x_j}^\varepsilon)_{x_i}$. By using the upper bound for $Z^\varepsilon(x,\xi,t,\tau)$ given in [2], we can choose positive constants k_1 and k_2 that depend only on n and Λ in (4) (and are independent of ε) such that

$$h^\varepsilon(x,t) \leq e^{Mt} \int_{\mathbb{R}^n} k_1 G(k_2(x-\xi), t)(v_0(\xi) - u_0(\xi))\, d\xi,$$

where $G(z,s)$ is the usual heat kernel on $\mathbb{R}^n$. Consequently, since $v_0(\xi) - u_0(\xi)$ is bounded as $0 \leq v_0(\xi) - u_0(\xi) \leq m + \alpha + \eta + 1$ and since $v_0(\xi) - u_0(\xi) = 0$ if $|x| \leq L$, by choosing L large enough, we have

$$w^\varepsilon(x,T) \leq h^\varepsilon(x,T) \leq \frac{\delta}{2}, \quad |x| \leq R.$$

We again note that the constant L can be chosen independent of ε. By combining this with (61), we obtain $u^\varepsilon(x,T) \geq 1 - \delta$ for $|x| \leq R$. This completes the proof. □

We are now ready to prove Lemma 3.2. The estimate (51) in Lemma 3.2 is proved easily from the comparison principle. The estimates (52) and (53) are derived from Lemmas 3.3 and 3.4, respectively.

Proof of Lemma 3.2. Step 1: We first prove the estimate (51). Consider the functions $U_\pm(t)$ that satisfy $U_\pm' = f(U_\pm)$ with the initial conditions $U_\pm(0) = \pm\max\{\|u_0\|_{L^\infty(\mathbb{R}^n)}, 1\}$. Since f is of bistable type as in (2), we have $U_+(t) \to 1$ and $U_-(t) \to 0$ as $t \to \infty$. Thus, since the comparison principle given by Proposition 2.6 implies $U_-(t) \leq u(x,t) \leq U_+(t)$ on $\mathbb{R}^n \times [0,\infty)$, there exists a positive constant T_1 such that

$$-\delta \leq u(x,t) \leq 1 + \delta, \quad x \in \mathbb{R}^n, t \geq T_1.$$

Step 2: By the assumption (50), we can choose constants $\alpha_* \in (0,\alpha)$ and $K' > 0$ such that

$$u_0(x) \leq \alpha_*, \quad |x| \geq K'. \tag{63}$$

Let $u^+(x,t;\nu)$ be the super-solution given in Lemma 3.3 with $M = \max\{\sup_{x\in\mathbb{R}^n} u_0(x), 2\}$. Namely, we consider the super-solution of the form:

$$u^+(x,t;\nu) = M\Phi(\nu\cdot x - \sigma t - K) + \alpha_* e^{-\beta t}. \tag{64}$$

From (63), we can choose the constant K in (64) large enough to satisfy $u_0(x) \le u^+(x,0;\nu)$ in $x\in\mathbb{R}^n$ for all $\nu \in S^{n-1}$. Then the comparison principle given in Proposition 2.6 implies

$$u(x,t) \le \min_{\nu\in S^{n-1}} u^+(x,t;\nu), \quad x\in\mathbb{R}^n,\ t\ge 0.$$

Consequently, since we have $\lim_{z\to\infty}\Phi(z) = 0$, we can choose positive constants K_2 and T_2 such that

$$u(x,t) \le \delta, \quad |x| \ge K_2 + \sigma t,\ t \ge T_2. \tag{65}$$

Step 3: Let $\{u^{\varepsilon_i}\}$ be the approximating sequence for the weak solution u of (1) given in (ii) of Proposition 2.5. Then Lemma 3.4 implies that there exist constants $L > 0$ and $T \ge \max\{T_1, T_2\}$ that are both independent of ε_i such that

$$u^{\varepsilon_i}(x,T) \ge 1-\delta, \quad |x| \le R.$$

By passing to the limit as $i\to\infty$, we get $u(x,T) \ge 1-\delta$ for $|x| \le R$. This establishes the estimates (53). The estimate (52) follows from (65) since

$$u(x,T) \le \delta, \quad |x| \ge K_2 + \sigma T.$$

This completes the proof of Lemma 3.2. □

3.2. Super-solutions and sub-solutions

In this subsection, we construct a fine set of super-solutions and sub-solutions whose level sets are the expanding Wulff shapes. For this purpose, we make some preparations. First, choose constants $\mu > 0$ and $\delta_0 \in (0, \frac{1}{4})$ satisfying

$$-f'(s) \ge \mu, \quad s \in [-2\delta_0, 2\delta_0] \cup [1-2\delta_0, 1+2\delta_0]. \tag{66}$$

Since f is of bistable type, these constants are both well-defined. We also choose a positive constant σ large enough to satisfy

$$\sigma \ge \max\left\{\frac{3}{\mu}, (\delta_0)^{-\frac{2}{3}}\right\}, \tag{67}$$

$$\frac{1}{\sqrt{\sigma}}\left(c + \frac{2}{\lambda\sqrt{C_\Phi}} + 2\right) \le \frac{\mu}{4}, \tag{68}$$

$$\frac{\sqrt{\sigma}}{2}\cdot \max_{\Phi\in[\delta_0, 1-\delta_0]}(-\Phi') - \|f'\|_{L^\infty(0,1)} - \frac{3}{2} \ge \frac{\mu}{2}, \tag{69}$$

where λ and C_Φ are the positive constants defined in (15)–(16). We note that $\sigma > 1$ follows from (67) because of $\delta_0 < 1/4$. We next introduce the cut-off anisotropic signed distance function $\tilde{d_\gamma}(x; W_R(t))$ by

$$\begin{aligned}\tilde{d_\gamma}(x; W_R(t)) &= h(d_\gamma(x; W_R(t)), t)\\ &= h(\gamma^*(x) - \rho(t;R), t),\end{aligned} \tag{70}$$

where, setting $\eta(t) = 2\lambda^{-1}\log\left(\sqrt{C_\Phi}(\sigma + t)\right) + 1$, we define $h(s,t)$ as a smooth odd function that satisfies (see Fig. 2)

$$h(s,t) = s, \qquad |s| \le \eta(t),$$
$$h(s,t) = \eta(t) + 1, \qquad |s| \ge \eta(t) + 2,$$
$$0 \le h_s(s,t) \le 1, \qquad s\in\mathbb{R}, t \ge 0,$$
$$|h_{ss}(s,t)| \le 1, \qquad s\in\mathbb{R}, t\ge 0,$$
$$|h_t(s,t)| \le \eta'(t) = \frac{2}{\lambda\sqrt{C_\Phi}(\sigma + t)}, \qquad s\in\mathbb{R}, t\ge 0.$$

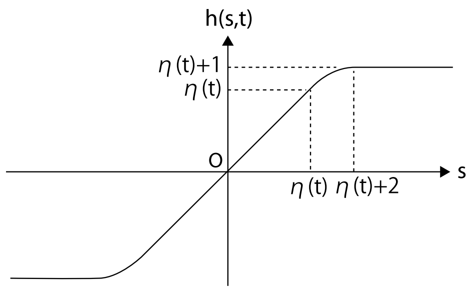

Fig. 2. Profile of the odd function $h(s,t)$ in (70).

Before constructing super-solutions and sub-solutions, we give two auxiliary lemmas. The first one gives basic estimates for $\rho(t;R)$ given in (8). The bounds (71) for $\rho(t;R)$ suffice for our analysis, however $\rho(t;R) = R + ct - ((n-1)/c)\log t + o(\log t)$ holds as is mentioned in the introduction.

Lemma 3.5 *(Estimate for $\rho(t;R)$). Let $\rho(t;R)$ be the solution of (8) with $R > (n-1)/c$. Then the following hold:*

(i) *One has*

$$R + ct - \frac{n-1}{c}\log\left(1 + \frac{ct}{R - \frac{n-1}{c}}\right) < \rho(t;R) < R + ct, \qquad t \geq 0. \tag{71}$$

(ii) *For any positive constants R_1, R_2 with $R_1 > R_2 > (n-1)/c$, one has*

$$R_1 - R_2 \leq \rho(t;R_1) - \rho(t;R_2) < \frac{R_2(R_1 - R_2)}{R_2 - \frac{n-1}{c}}, \quad t \geq 0. \tag{72}$$

Proof. Since $\rho'(t;R) < c$ holds obviously, the second inequality of (71) follows immediately. To show the first inequality of (71), we integrate (8) and obtain

$$\left[\rho(s;R) + \frac{n-1}{c}\log\left(\rho(s;R) - \frac{n-1}{c}\right)\right]_0^t = ct.$$

This implies

$$\begin{aligned}\rho(t;R) &= R + ct - \frac{n-1}{c}\log\left(\frac{\rho(t;R) - \frac{n-1}{c}}{R - \frac{n-1}{c}}\right)\\ &> R + ct - \frac{n-1}{c}\log\left(1 + \frac{ct}{R - \frac{n-1}{c}}\right),\end{aligned}$$

where the last inequality comes from $\rho(t;R) < R + ct$. Thus, we obtain (71).

Next we show (72). Since $\rho(t;R_2) \to \infty$ as $t \to \infty$ because of $R_2 > (n-1)/c$, there exists a positive constant t_0 such that $\rho(t_0;R_2) = R_1$. Thus we have

$$\rho(t;R_1) - \rho(t;R_2) = \rho(t+t_0;R_2) - \rho(t;R_2) = \int_t^{t+t_0} \rho'(s;R_2)\,ds.$$

Since $\rho'(t;R_2) = c - (n-1)/\rho(t;R_2)$ is monotone increasing in t, the first inequality of (72) follows from

$$\int_t^{t+t_0} \rho'(s;R_2)\,ds \geq \int_0^{t_0} \rho'(s;R_2)\,ds = R_1 - R_2.$$

On the other hand, since $\rho'(t; R_2) \geq c - (n-1)/R_2$, we have $t_0 \leq (R_1 - R_2)/(c - (n-1)/R_2)$. Thus the second inequality of (72) follows from

$$\int_t^{t+t_0} \rho'(s; R_2)\, ds < ct_0 \leq \frac{R_2(R_1 - R_2)}{R_2 - \frac{n-1}{c}}.$$

This completes the proof of Lemma 3.5. □

We give another auxiliary lemma. As is mentioned in Remark 2.3, we have $\operatorname{div} a_p(\nabla d_\gamma) = (n-1)/\gamma^*(x)$. Since it blows up at $x = 0$, to avoid this, we consider $\operatorname{div} a_p(\nabla \tilde{d}_\gamma)$ instead and obtain the estimate for $\partial_t \tilde{d}_\gamma - \operatorname{div} a_p(\nabla \tilde{d}_\gamma) + c$ as follows.

Lemma 3.6. *Let $\tilde{d}_\gamma(x; W_R(t))$ be the cut-off anisotropic signed distance function defined by (70). Then, for any constant $C \in (0, 1]$, there exists a positive constant $R_* > (n-1)/c$ such that, if $R \geq R_*$, one has*

$$|\partial_t \tilde{d}_\gamma - \operatorname{div} a_p(\nabla \tilde{d}_\gamma) + c| \leq \begin{cases} C(\sigma + t)^{-\frac{3}{2}}, & |d_\gamma(x; W_R(t))| \leq \eta(t), \\ c + \frac{2}{\lambda\sqrt{C_\Phi}} + 2, & |d_\gamma(x; W_R(t))| \geq \eta(t), \end{cases}$$

where λ and C_Φ are the positive constants defined in (15)–(16) and σ is the positive constant defined in (67)–(69).

Proof. By the definition of $\tilde{d}_\gamma$, we have

$$\partial_t \tilde{d}_\gamma(x; W_R(t)) = -\rho'(t; R) h_s + h_t = \left(\frac{n-1}{\rho(t; R)} - c \right) h_s + h_t.$$

Since $a(p)$ is homogeneous of degree two and since $\operatorname{div} a_p(\nabla \gamma^*) = (n-1)/\gamma^*$ as is mentioned in Remark 2.3, we have

$$\begin{aligned} \operatorname{div} a_p(\nabla \tilde{d}_\gamma) &= \operatorname{div} (h_s a_p(\nabla \gamma^*)) \\ &= \operatorname{div} \left(a_p(\nabla \gamma^*)\right) h_s + (a_p(\nabla \gamma^*) \cdot \nabla \gamma^*) h_{ss} \\ &= \frac{n-1}{\gamma^*(x)} h_s + 2a(\nabla \gamma^*) h_{ss}. \end{aligned}$$

Thus, since (33) in Remark 2.1 gives $2a(\nabla \gamma^*) = (\gamma(\nabla \gamma^*))^2 = 1$, we have

$$\partial_t \tilde{d}_\gamma - \operatorname{div} a_p(\nabla \tilde{d}_\gamma) + c = \left(\frac{n-1}{\rho(t; R)} - \frac{n-1}{\gamma^*(x)} \right) h_s + c(1 - h_s) + h_t - h_{ss}.$$

Let I_1 and I_2 be the first term and the remaining terms of the right-hand side of the above. If $|d_\gamma(x; W_R(t))| \geq \eta(t) + 2$, we have $h_s \equiv 0$ and thus $I_1 \equiv 0$ holds. On the other hand, if $|d_\gamma(x; W_R(t))| \leq \eta(t) + 2$, namely, if $|\gamma^*(x) - \rho(t; R)| \leq \eta(t) + 2$, we have

$$|I_1| \leq \left| \frac{(n-1)(\gamma^*(x) - \rho(t; R))}{\rho(t; R)\gamma^*(x)} \right| \leq \frac{(n-1)(\eta(t) + 2)}{\rho(t; R)|\rho(t; R) - \eta(t) - 2|},$$

if $\rho(t; R)$ is sufficiently large. Thus, by using the lower bound for $\rho(t; R)$ given by (71) in Lemma 3.5, we find that there exists a constant $R_* > (n-1)/c$ such that, if $R \geq R_*$, one has $|I_1| \leq C(\sigma + t)^{-\frac{3}{2}}$.

Finally, I_2 is estimated easily as

$$|I_2| \leq \begin{cases} 0, & |d_\gamma(x; W_R(t))| \leq \eta(t), \\ c + \frac{2}{\lambda\sqrt{C_\Phi}(\sigma + t)} + 1, & |d_\gamma(x; W_R(t))| \geq \eta(t). \end{cases}$$

Since $\sigma > 1$ and $C \leq 1$ by the definitions, by combining the above estimates for I_1 and I_2, we obtain the desired estimates. □

The next lemma gives a fine set of sub-solutions and super-solutions whose level sets are roughly the expanding Wulff shapes. An important point of this lemma is that the Hausdorff distance between the corresponding level sets of the super-solution and the sub-solution remains uniformly bounded up to $t = +\infty$, since the function $p(t)$ is bounded as $-1 \le p(t) \le 0$.

Lemma 3.7 *(Super-solutions and sub-solutions). Let $\tilde{d}_\gamma(x; W_R(t))$ be the cut-off anisotropic signed distance function defined in (70), and let σ be the positive constant defined in (67)–(69). Then there exists a positive constant $R_* > (n-1)/c$ such that, if $R \ge R_*$, the functions*

$$u^\pm(x,t;R) = \Phi(\tilde{d}_\gamma(x; W_R(t)) \pm p(t)) \pm q(t),$$

with

$$p(t) = \sigma^{\frac{1}{2}}(\sigma+t)^{-\frac{1}{2}} - 1, \quad q(t) = (\sigma+t)^{-\frac{3}{2}},$$

satisfy $\pm\mathcal{L}[u^\pm] \ge 0$ in the classical sense, where $\mathcal{L}[w] := w_t - \operatorname{div} a_p(\nabla w) - f(w)$.

Proof. We only prove $\mathcal{L}[u^+] \ge 0$, since $\mathcal{L}[u^-] \le 0$ can be proved in the same way. By direct computations, we have

$$\begin{aligned}
u_t^+ &= \left(\partial_t \tilde{d}_\gamma + p'\right)\Phi' + q',\\
\operatorname{div} a_p(\nabla u^+) &= \operatorname{div}\left(\Phi' a_p(\nabla \tilde{d}_\gamma)\right)\\
&= \operatorname{div}(a_p(\nabla \tilde{d}_\gamma))\Phi' + \left(a_p(\nabla \tilde{d}_\gamma)\cdot \nabla \tilde{d}_\gamma\right)\Phi'',\\
&= \operatorname{div}(a_p(\nabla \tilde{d}_\gamma))\Phi' + 2a(\nabla \tilde{d}_\gamma)\Phi'',\\
f(u^+) &= f(\Phi) + \int_0^1 f'(\Phi+\theta q)\,d\theta \cdot q,
\end{aligned}$$

where, to compute $\operatorname{div} a_p(\nabla u^+)$, we used the homogeneity of $a(p)$ and $\gamma(p)$; see Remark 2.1. By using the relation $\Phi'' + c\Phi' + f(\Phi) = 0$, we have $\mathcal{L}[u^+] = I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \left(\partial_t \tilde{d}_\gamma - \operatorname{div} a_p(\nabla \tilde{d}_\gamma) + c\right)\Phi',\\
I_2 &= \left(1 - 2a(\nabla \tilde{d}_\gamma)\right)\Phi'',\\
I_3 &= p'\Phi' - \int_0^1 f'(\Phi+\theta q)\,d\theta \cdot q + q'.
\end{aligned}$$

We first estimate I_1. From Lemma 3.6, there exists a positive constant $R_* > (n-1)/c$ such that, for any $R \ge R_*$,

$$|I_1| \le \left|\partial_t \tilde{d}_\gamma - \operatorname{div} a_p(\nabla \tilde{d}_\gamma) + c\right| \cdot \|\Phi'\|_{L^\infty(\mathbb{R})} \le \frac{\mu}{4}(\sigma+t)^{-\frac{3}{2}},$$

holds if $|d_\gamma(x; W_R(t))| \le \eta(t)$. On the other hand, if $|d_\gamma(x; W_R(t))| \ge \eta(t)$, we have

$$\begin{aligned}
|I_1| &\le \left(c + \frac{2}{\lambda\sqrt{C_\Phi}} + 2\right)|\Phi'(\tilde{d}_\gamma(x; W_R(t)) + p(t))|\\
&\le \left(c + \frac{2}{\lambda\sqrt{C_\Phi}} + 2\right) C_\Phi \exp(-\lambda(\eta(t) - 1))\\
&= \left(c + \frac{2}{\lambda\sqrt{C_\Phi}} + 2\right)(\sigma+t)^{-2},
\end{aligned}$$

where the first inequality comes from Lemma 3.6 and the second inequality comes from the estimates for $|\Phi'|$ given in (15)–(16) and $-1 \le p(t) \le 0$. Thus (68) gives $|I_1| \le \frac{\mu}{4}(\sigma+t)^{-\frac{3}{2}}$ if $|d_\gamma(x; W_R(t))| \ge \eta(t)$.

Next, we estimate I_2. From (33) in Remark 2.1, we have

$$2a(\nabla \tilde{d_\gamma}) = 2a(\nabla\gamma^*)(h_s)^2 = (\gamma(\nabla\gamma^*))^2(h_s)^2 = (h_s)^2.$$

If $|d_\gamma(x; W_R(t))| \le \eta(t)$, we have $h_s \equiv 1$ and hence $I_2 \equiv 0$. If $|d_\gamma(x; W_R(t))| \ge \eta(t)$, since $|h_s| \le 1$ by the definition, we have

$$\begin{aligned}|I_2| &\le \left(1-(h_s)^2\right)|\Phi''(\tilde{d_\gamma}(x,t)+p(t))| \\ &\le C_\Phi \exp(-\lambda(\eta(t)-1)) \\ &\le (\sigma+t)^{-2}.\end{aligned}$$

Since (68) gives $1/\sqrt{\sigma} \le \mu/4$, we obtain $|I_2| \le \frac{\mu}{4}(\sigma+t)^{-\frac{3}{2}}$.

Finally, we estimate I_3 and show $\mathcal{L}[u^+] \ge 0$. By computing p' and q', we have

$$I_3 = \left(-\frac{\sqrt{\sigma}}{2}\Phi' - \int_0^1 f'(\Phi+\theta q)\,d\theta - \frac{3}{2}(\sigma+t)^{-1}\right)\cdot(\sigma+t)^{-\frac{3}{2}}.$$

When $\Phi \in (0,\delta_0] \cup [1-\delta_0, 1)$ for the constant δ_0 defined in (66), since $\Phi' < 0$, the inequality (67) gives

$$\begin{aligned}I_3 &\ge \left(-\int_0^1 f'(\Phi+\theta q)\,d\theta - \frac{3}{2}(\sigma+t)^{-1}\right)\cdot(\sigma+t)^{-\frac{3}{2}} \\ &\ge \frac{\mu}{2}(\sigma+t)^{-\frac{3}{2}}.\end{aligned}$$

On the other hand, when $\Phi \in [\delta_0, 1-\delta_0]$, the inequality (69) gives

$$\begin{aligned}I_3 &\ge \left(\frac{\sqrt{\sigma}}{2}\cdot \max_{\Phi\in[\delta_0,1-\delta_0]}(-\Phi') - \|f'\|_{L^\infty(0,1)} - \frac{3}{2}(\sigma+t)^{-1}\right)\cdot(\sigma+t)^{-\frac{3}{2}} \\ &\ge \frac{\mu}{2}(\sigma+t)^{-\frac{3}{2}}.\end{aligned}$$

By combining the above estimates for I_1, I_2 and I_3, we obtain $\mathcal{L}[u^+] \ge 0$. □

3.3. Proof of Proposition 3.1

We here prove Proposition 3.1. To complete the proof, it suffices to show that the roughly generated front shown in Lemma 3.2 is captured between the sub-solution and the super-solution given in Lemma 3.7 for all large times.

Proof of Proposition 3.1. Step 1: Let $p(t)$ and $q(t)$ be the functions defined in Lemma 3.7 and let σ be the constant defined in (67)–(69). We will first prove

$$|\Phi(\tilde{d_\gamma} \pm p(t)) - \Phi(d_\gamma \pm p(t))| \le (\sigma+t)^{-2}, \tag{73}$$

where, for simplicity, d_γ and $\tilde{d_\gamma}$ denote $d_\gamma(x; W_R(t))$ and $\tilde{d_\gamma}(x; W_R(t))$ defined in (10) and (70). In the case of $|d_\gamma| \le \eta(t)$, we have $\tilde{d_\gamma} \equiv d_\gamma$ by the definition, and thus (73) holds.

In the case of $d_\gamma \ge \eta(t)$, since $\tilde{d_\gamma} \le d_\gamma$ in this case and since $\Phi(z)$ is monotone decreasing in $z \in \mathbb{R}$, we have

$$\Phi(\tilde{d_\gamma} \pm p(t)) - \Phi(d_\gamma \pm p(t)) \ge 0. \tag{74}$$

On the other hand, since $d_\gamma \ge \eta(t)$ implies $\tilde{d_\gamma} \ge \eta(t)$, we have

$$\begin{aligned}\Phi(\tilde{d_\gamma} \pm p(t)) &\le \Phi(\eta(t) \pm p(t)) \\ &\le C_\Phi \exp(-\lambda(\eta(t)-1)) \\ &= (\sigma+t)^{-2},\end{aligned}$$

where the second inequality follows from (15) and $-1 \le p(t) \le 0$. Since $\Phi(d_\gamma \pm p(t)) \ge 0$ holds obviously, this implies

$$\Phi(\tilde{d_\gamma} \pm p(t)) - \Phi(d_\gamma \pm p(t)) \le (\sigma + t)^{-2}.$$

This and (74) imply (73). The case of $d_\gamma \le -\eta(t)$ is proved in a similar way.

Step 2: We set $\delta = \sigma^{-\frac{3}{2}}$ for simplicity. Since $\Phi(-\infty) = 1$ and $\Phi(+\infty) = 0$, we can choose a positive constant M such that

$$\Phi(-M) \ge 1 - \frac{\delta}{2}, \quad \Phi(M) \le \frac{\delta}{2}.$$

Let R be the constant defined as in Lemma 3.7, and define $M_* = \min_{\nu\in S^{n-1}} \gamma^*(\nu)$ and $M^* = \max_{\nu\in S^{n-1}} \gamma^*(\nu)$. Then Lemma 3.2 implies that there exist positive constants L, T, and K such that, if $u_0(x) \ge \alpha + \eta$ for $|x| \ge L$, it holds that

$$-\frac{\delta}{2} \le u(x,T) \le 1 + \frac{\delta}{2}, \quad x \in \mathbb{R}^n, \tag{75}$$

$$u(x,T) \le \delta, \quad |x| \ge K, \tag{76}$$

$$u(x,T) \ge 1-\delta, \quad |x| \le \frac{R+M}{M_*}. \tag{77}$$

Then, since $M_*|x| \le \gamma^*(x)$, it holds from (77) that

$$u(x,T) \ge 1-\delta, \quad \gamma^*(x) \le R + M. \tag{78}$$

On the other hand, since $M^*|x| \ge \gamma^*(x)$, by choosing a large constant K', (76) gives

$$u(x,T) \le \delta, \quad \gamma^*(x) \ge R + K'. \tag{79}$$

Consequently, the inequalities (75), (78), and (79) imply

$$\Phi(\gamma^*(x) - R) - \delta \le u(x,T) \le \Phi(\gamma^*(x) - R - K' - M) + \delta, \quad x \in \mathbb{R}^n.$$

Then, by using the sub-solution given in Lemma 3.7, the lower bound is obtained as

$$\begin{aligned} u(x,T+t) &\ge \Phi\left(\tilde{d_\gamma}(x; W_R(t)) - p(t)\right) - q(t) \\ &\ge \Phi\left(d_\gamma(x; W_R(t)) - p(t)\right) - q(t) - (\sigma+t)^{-2} \\ &\ge \Phi\left(d_\gamma(x; W_R(t)) + 1\right) - 2(1+t)^{-3/2}, \end{aligned}$$

where the first inequality is obtained by applying the comparison principle and the second inequality follows from (73). Similarly, by using the super-solution given in Lemma 3.7, the upper bound is obtained as

$$\begin{aligned} u(x,T+t) &\le \Phi\left(\tilde{d_\gamma}(x; W_{R+K'+M}(t)) + p(t)\right) + q(t) \\ &\le \Phi\left(d_\gamma(x; W_{R+K'+M}(t)) + p(t)\right) + q(t) + (\sigma+t)^{-2} \\ &\le \Phi\left(d_\gamma(x; W_R(t)) - \frac{R(K'+M)}{R - \frac{n-1}{c}} - 1\right) + 2(1+t)^{-3/2}, \end{aligned}$$

where the last inequality follows from the second inequality of (72) in Lemma 3.5. The proof of Proposition 3.1 is complete. □

4. Fine formation of the front

In this section, we analyze the fine formation of the front and prove Theorems 1.1 and 1.4. We first give the local convergence result (Lemma 4.1) in Subsection 4.1. It says that, for each direction $\nu \in S^{n-1}$, the solution converges to the planar waves in the sense of $BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R})$ for a time sequence $0 < t_1 < t_2 < \cdots \to \infty$. In Subsection 4.2, we prove the strict monotonicity of the solution around the front and analyze the α-level set of the solution. Subsections 4.3 and 4.4 are devoted to the proof of Theorems 1.1 and 1.4, respectively.

4.1. Local convergence to the planar wave

The aim of this subsection is to prove Lemma 4.1 below. The key of its proof is Lemma 4.2, which characterizes weak entire solutions of the equation (1a).

Lemma 4.1 *(Local convergence to planar waves). Let the assumptions of Proposition 3.1 hold and let R be the constant defined in Proposition 3.1. Suppose further that there exists a sequence $\{(x_i, t_i)\} \subset \mathbb{R}^n \times (0, \infty)$ such that $0 < t_1 < t_2 < \cdots \to \infty$ and that*

$$\frac{x_i}{|x_i|} \to {}^{\exists}\nu \in S^{n-1}, \quad d_\gamma(x_i; W_R(t_i)) \to {}^{\exists}M \in \mathbb{R},$$

as $i \to \infty$. Then there exist a subsequence $\{(x_{k_i}, t_{k_i})\}$ of $\{(x_i, t_i)\}$ and a constant $\mu \in \mathbb{R}$ such that

$$\lim_{i\to\infty} u(x_{k_i} + x, t_{k_i} + t) = \Phi(\nabla\gamma^*(\nu) \cdot x - ct + \mu) \quad \text{in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}).$$

Proof. Step 1: We will first show that, for any fixed $x \in \mathbb{R}^n$ and any fixed $t \in \mathbb{R}$, it holds that

$$\lim_{i\to\infty} d_\gamma(x_i + x; W_R(t_i + t)) = \nabla\gamma^*(\nu) \cdot x + M - ct. \tag{80}$$

Fix $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ arbitrarily. In what follows, we estimate the terms

$$\begin{aligned} d_\gamma(x_i + x; W_R(t_i + t)) = {} & \gamma^*(x_i + x) - \gamma^*(x_i) \\ & + \gamma^*(x_i) - \rho(t_i; R) \\ & + \rho(t_i; R) - \rho(t_i + t; R). \end{aligned}$$

Let I_i, J_i, and K_i be the first two, the next two, and the last two terms of the right-hand side of the above. We first estimate I_i. For each i, there exists a constant $\theta_i \in [0, 1]$ such that

$$I_i = \nabla\gamma^*(x_i + \theta_i x) \cdot x = \nabla\gamma^* \left(\frac{x_i + \theta_i x}{|x_i + \theta_i x|} \right) \cdot x,$$

where the last equality comes from the homogeneity of $\nabla\gamma^*$. By the assumptions of the lemma, we have $\lim_{i\to\infty} |x_i| = \infty$. Thus, for any fixed $x \in \mathbb{R}^n$, we have

$$\lim_{i\to\infty} \frac{x_i + \theta_i x}{|x_i + \theta_i x|} = \lim_{i\to\infty} \frac{x_i}{|x_i|} = \nu.$$

Consequently, since $\nabla\gamma^*$ is continuous in $\mathbb{R}^n \setminus \{0\}$, we have $\lim_{i\to\infty} I_i = \nabla\gamma^*(\nu) \cdot x$. Secondly, J_i is estimated simply as $\lim_{i\to\infty} J_i = M$ by the assumption of the lemma. Finally, for any fixed $t \in \mathbb{R}$, we estimate K_i as

$$K_i = -\int_{t_i}^{t_i+t} \rho'(s; R) ds = -\int_{t_i}^{t_i+t} \left(c - \frac{n-1}{\rho(s; R)} \right) ds \to -ct \quad \text{as } i \to \infty.$$

By combining these estimates for I_i, J_i, and K_i, we obtain (80).

Step 2: From Lemma 2.8, by choosing a subsequence $\{(x_{k_i}, t_{k_i})\} \subset \{(x_i, t_i)\}$, we have a weak entire solution in the sense of Lemma 2.8 such that

$$u(x_{k_i} + x, t_{k_i} + t) \to w(x, t) \quad \text{in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}) \text{ as } i \to \infty.$$

Letting T, R, and K be the constants given in Proposition 3.1, the upper bound for $u(x, t)$ given in Proposition 3.1 implies

$$\begin{aligned} w(x, t) &= \lim_{i\to\infty} u(x_{k_i} + x, t_{k_i} + t) \\ &\le \lim_{i\to\infty} \Phi\left(d_\gamma(x_{k_i} + x; W_R(t_{k_i} + t - T)) - K\right) + 2(1 + t_{k_i} + t - T)^{-\frac{3}{2}} \\ &= \Phi\left(\nabla\gamma^*(\nu) \cdot x + M - c(t - T) - K\right), \end{aligned}$$

for any fixed $x \in \mathbb{R}^n$ and any fixed $t \in \mathbb{R}$, where we used (80) to obtain the last equality. Similarly, we obtain $w(x,t) \geq \Phi(\nabla\gamma^*(\nu)\cdot x + M - c(t-T) + K)$ by using the lower bound for $u(x,t)$ given in Proposition 3.1. Thus, Lemma 4.2 given below implies that there exists a constant μ such that $w(x,t) \equiv \Phi(\nabla\gamma^*(\nu)\cdot x - ct + \mu)$. This completes the proof. □

Lemma 4.2 *(Liouville type theorem). Let $u(x,t)$ be a weak entire solution in the sense of Lemma 2.8. Suppose that there exist $\nu \in S^{n-1}$ and $K > 0$ such that*

$$\Phi(\nabla\gamma^*(\nu)\cdot x - ct + K) \leq u(x,t) \leq \Phi(\nabla\gamma^*(\nu)\cdot x - ct - K), \tag{81}$$

holds for any $x \in \mathbb{R}^n$ and any $t \in \mathbb{R}$. Then there exists a constant $\mu \in [-K, K]$ such that

$$u(x,t) \equiv \Phi(\nabla\gamma^*(\nu)\cdot x - ct + \mu).$$

Lemma 4.2 implies that any weak entire solution sandwiched between two planar traveling waves is itself a planar wave. This result generalizes, in some sense, Theorem 3.1 of [7] to anisotropic equations. The proof goes along almost the same lines as that of [7], but for the convenience of the reader, we give it in Appendix C.

4.2. Strict monotonicity and level sets of the solution

In this subsection, we first prove the strict monotonicity of the solution $u(x,t)$ around the front at the large time (Lemma 4.3). Then we prove that the α-level set of the solution is a smooth hypersurface that is star-shaped with respect to the origin (Lemma 4.4). A similar argument is used in [25] to analyze the large time behavior of disturbed planar fronts in the isotropic Allen–Cahn equation.

Lemma 4.3 *(Strict monotonicity around the front). Let the assumptions of Proposition 3.1 hold and let R be the constant defined in Proposition 3.1. Then, for any $C > 0$, there exists a positive constant T such that*

$$\inf_{|d_\gamma(x;W_R(t))|\leq C}\left(-\frac{\partial u}{\partial \nu}\right) > 0, \quad t \geq T,$$

where $\partial/\partial\nu$ means the differential along $x/|x|$.

Proof. Assume that the conclusion does not hold. Then there exists a sequence $\{(x_i,t_i)\} \subset \mathbb{R}^n \times [0,\infty)$ such that $0 < t_1 < t_2 < \cdots \to \infty$, that $|d_\gamma(x_i; W_R(t_i))| \leq C$, and that

$$\limsup_{i\to\infty} \frac{\partial u(x_i,t_i)}{\partial \nu_i} \geq 0, \tag{82}$$

where $\nu_i = x_i/|x_i|$. By choosing a subsequence, which is denoted by $\{(x_i,t_i)\}$ again, we have

$$\nu_i \to {}^\exists\nu \in S^{n-1}, \quad d_\gamma(x_i; W_R(t_i)) \to {}^\exists M \in [-C, C],$$

as $i \to \infty$. Then Lemma 4.1 implies that, by choosing a subsequence again, which is also denoted by $\{(x_i,t_i)\}$, we have

$$\lim_{i\to\infty} u(x_i + x, t_i + t) \equiv \Phi(\nabla\gamma^*(\nu)\cdot x - ct + \mu) \quad \text{in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}),$$

for a constant $\mu \in \mathbb{R}$. This implies $\lim_{i\to\infty} \nabla u(x_i,t_i) = \Phi'(\mu)\nabla\gamma^*(\nu)$ and thus

$$\lim_{i\to\infty} \frac{\partial u(x_i,t_i)}{\partial \nu_i} = \lim_{i\to\infty} \nu_i \cdot \nabla u(x_i,t_i) = \nu\cdot(\Phi'(\mu)\nabla\gamma^*(\nu)) = \Phi'(\mu)\gamma^*(\nu).$$

This contradicts (82), because $\Phi'(z) < 0$ holds for all $z \in \mathbb{R}$. The proof of Lemma 4.3 is complete. □

Lemma 4.4 below is a simple corollary of Proposition 3.1 and Lemma 4.3. We note that a part of Theorem 1.1 follows immediately from Lemma 4.4.

Lemma 4.4 *(α-level set of the solution). Let the assumptions of Proposition 3.1 hold and let R be the constant defined in Proposition 3.1. Then there exist a positive constant T and a smooth bounded function $l : S^{n-1} \times [T,\infty) \to \mathbb{R}$ such that*

$$u(x,t) = \alpha \quad \text{if and only if} \quad d_\gamma(x; W_R(t)) = l\left(\frac{x}{|x|}, t\right), \qquad t \geq T. \tag{83}$$

Proof. Define $\delta = \min\{\alpha, 1-\alpha\}/2$. From Proposition 3.1, there exist positive constants C and T such that

$$\{x \in \mathbb{R}^n \mid |u(x,t) - \alpha| \leq \delta\} \subset \{x \in \mathbb{R}^n \mid |d_\gamma(x; W_R(t)| \leq C\},$$

holds for any $t \geq T$. Moreover, by choosing T larger if necessary, Lemma 4.3 gives

$$\inf_{|u-\alpha|\leq\delta}\left(-\frac{\partial u}{\partial \nu}\right) \geq \inf_{|d_\gamma(x;W_R(t))|\leq C}\left(-\frac{\partial u}{\partial \nu}\right) > 0, \quad t \geq T.$$

Thus, there exists a bounded function $l(\nu,t) : S^{n-1} \times [T,\infty) \to [-C, C]$ that satisfies (83). Here, l is smooth by the implicit function theorem, since $u(x,t)$ is smooth for $t > 0$ when $\nabla u \neq 0$. The proof of Lemma 4.4 is complete. □

By virtue of Lemma 4.4, we can refine Lemma 4.1 as follows, where the constant μ in Lemma 4.1 is replaced by the specific constant K. Moreover, the convergence to a planar wave takes place for $\{(x_i,t_i)\}$ itself, not for a subsequence.

Lemma 4.5 *(Local convergence to planar waves). Let the assumptions of Lemma 4.4 hold and let R, T, and $l : S^{n-1} \times [T,\infty) \to \mathbb{R}$ be the constants, and the smooth bounded function defined in Lemma 4.4. Suppose further that there exists a sequence $\{(x_i,t_i)\} \subset \mathbb{R}^n \times (0,\infty)$ such that $0 < t_1 < t_2 < \cdots \to \infty$ and that*

$$\frac{x_i}{|x_i|} \to {}^\exists\nu \in S^{n-1}, \quad d_\gamma(x_i; W_R(t_i)) - l\left(\frac{x_i}{|x_i|}, t_i\right) \to {}^\exists K \in \mathbb{R},$$

as $i \to \infty$. Then one has

$$\lim_{i\to\infty} u(x_i + x, t_i + t) = \Phi(\nabla\gamma^*(\nu)\cdot x - ct + K) \quad \text{in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}). \tag{84}$$

Proof. Lemma 4.1 implies that, by choosing a subsequence $\{(x_{k_i}, t_{k_i})\}$ of $\{(x_i,t_i)\}$, we have

$$\lim_{i\to\infty} u(x_{k_i} + x, t_{k_i} + t) \equiv \Phi(\nabla\gamma^*(\nu)\cdot x - ct + \mu) \quad \text{in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}),$$

for a constant $\mu \in \mathbb{R}$. For each x_{k_i}, we define the point $y_{k_i} \in \mathbb{R}^n$ by

$$y_{k_i} = \left(\rho(t_{k_i}; R) + l\left(\frac{x_{k_i}}{|x_{k_i}|}, t_{k_i}\right)\right)\frac{x_{k_i}}{\gamma^*(x_{k_i})}.$$

Then $u(y_{k_i}, t_{k_i}) = \alpha$ holds, since we have $\gamma^*(y_{k_i}) = \rho(t_{k_i}; R) + l(x_{k_i}/|x_{k_i}|, t_{k_i})$ and thus $d_\gamma(y_i; W_R(t_i)) = l(x_{k_i}/|x_{k_i}|, t_{k_i})$. Then we have

$$\begin{aligned}
\alpha &= \lim_{i\to\infty} u(y_{k_i}, t_{k_i}) \\
&= \lim_{i\to\infty} u\left(\left(\rho(t_{k_i}; R) + l\left(\frac{x_{k_i}}{|x_{k_i}|}, t_{k_i}\right)\right)\frac{x_{k_i}}{\gamma^*(x_{k_i})}, t_{k_i}\right) \\
&= \lim_{i\to\infty} u\left(x_{k_i} + \left(-d_\gamma(x_i; W_R(t_i)) + l\left(\frac{x_{k_i}}{|x_{k_i}|}, t_{k_i}\right)\right)\frac{x_{k_i}}{\gamma^*(x_{k_i})}, t_{k_i}\right) \\
&= \Phi\left(\nabla\gamma^*(\nu)\cdot\frac{-K\nu}{\gamma^*(\nu)} + \mu\right) \\
&= \Phi(-K + \mu).
\end{aligned}$$

This implies $\mu = K$ because $\Phi(0) = \alpha$ by the definition. Thus, the subsequence $\{(x_{k_i}, t_{k_i})\}$ satisfies (84). Moreover, since the constant K does not depend on the choice of subsequences of $\{(x_i,t_i)\}$, the given sequence $\{(x_i,t_i)\}$ itself satisfies (84). The proof is complete. □

4.3. Proof of Theorem 1.1

In this subsection, we complete the proof of Theorem 1.1. Before that, we prepare an auxiliary lemma that shows the convergences of the solution $u(x,t)$ and its gradient around the front.

Lemma 4.6 *(Convergence around the front). Let the assumptions of Lemma 4.4 hold and let R, T, and $l : S^{n-1} \times [T,\infty) \to \mathbb{R}$ be the constants, and the smooth bounded function defined in Lemma 4.4. Then, for any $C > 0$, one has*

$$\lim_{t\to\infty} \sup_{|d_\gamma(x;W_R(t))|\le C} \left| u(x,t) - \Phi\left(d_\gamma(x;W_R(t)) - l\left(\frac{x}{|x|},t\right)\right)\right| = 0, \tag{85}$$

$$\lim_{t\to\infty} \sup_{|d_\gamma(x;W_R(t))|\le C} \left| \nabla u(x,t) - \Phi'\left(d_\gamma(x;W_R(t)) - l\left(\frac{x}{|x|},t\right)\right)\nabla\gamma^*(x)\right| = 0. \tag{86}$$

Proof. We only prove (86), since (85) can be proved in a similar and easier way. Assume that (86) does not hold. Then there exist a positive constant δ and a sequence $\{(x_i,t_i)\}$ such that $|d_\gamma(x_i;W_R(t_i))| \le C$, that $0 < t_1 < t_2 < \cdots \to \infty$, and that

$$\left| \nabla u(x_i,t_i) - \Phi'\left(d_\gamma(x_i;W_R(t_i)) - l\left(\frac{x_i}{|x_i|},t_i\right)\right)\nabla\gamma^*(x_i)\right| \ge \delta, \tag{87}$$

for all $i = 1, 2, \cdots$. By choosing a subsequence, which is denoted by $\{(x_i,t_i)\}$ again, we have

$$\frac{x_i}{|x_i|} \to {}^\exists\nu \in S^{n-1}, \quad d_\gamma(x_i;W_R(t_i)) - l\left(\frac{x_i}{|x_i|},t_i\right) \to {}^\exists K \in \mathbb{R},$$

as $i \to \infty$. Then Lemma 4.5 implies that

$$\lim_{i\to\infty} u(x_i + x, t_i + t) = \Phi(\nabla\gamma^*(\nu)\cdot x - ct + K) \quad \text{in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}). \tag{88}$$

This implies

$$\lim_{i\to\infty} \nabla u(x_i,t_i) = \nabla_x \left(\Phi(\nabla\gamma^*(\nu)\cdot x - ct + K)\right)|_{(x,t)=(0,0)} = \Phi'(K)\nabla\gamma^*(\nu).$$

However, this contradicts (87), since we have

$$\lim_{i\to\infty} \Phi'\left(d_\gamma(x_i;W_R(t_i)) - l\left(\frac{x_i}{|x_i|},t_i\right)\right)\nabla\gamma^*(x_i) = \Phi'(K)\nabla\gamma^*(\nu).$$

Thus we obtain (86). The proof of Lemma 4.6 is complete. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Lemma 4.4 implies that, under the assumptions of Theorem 1.1, there exist positive constants R, T, and a bounded smooth function $l : S^{n-1} \times [T,\infty) \to \mathbb{R}$ such that (21) holds.

We next prove the convergence of $u(x,t)$. Fix $\varepsilon > 0$ arbitrarily. Since $l(x/|x|,t)$ is bounded, Proposition 3.1 implies that there exist positive constants C and t_* such that

$$\sup_{|d_\gamma(x;W_R(t))|\ge C, x\ne 0} \left| u(x,t) - \Phi\left(d_\gamma(x;W_R(t)) - l\left(\frac{x}{|x|},t\right)\right)\right| \le \varepsilon, \quad t \ge t_*.$$

Combining this with (85) of Lemma 4.6, we have

$$\sup_{x\in\mathbb{R}^n\setminus\{0\}} \left| u(x,t) - \Phi\left(d_\gamma(x;W_R(t)) - l\left(\frac{x}{|x|},t\right)\right)\right| \le \varepsilon, \quad t \ge t_*,$$

by choosing t_* larger if necessary. Thus we obtain (22). Moreover, $\lim_{t\to\infty} u(0,t) = 1$ follows immediately from Proposition 3.1.

We finally prove the convergence of $\nabla u(x,t)$. Fix $\varepsilon > 0$ arbitrarily. Then there exist positive constants C and t_* such that

$$\sup_{|d_\gamma(x;W_R(t))|\geq C} |\nabla u(x,t)| < \frac{\varepsilon}{2}, \tag{89}$$

$$\sup_{|d_\gamma(x;W_R(t))|\geq C,\, x\neq 0} \left| \Phi'\left(d_\gamma(x;\, W_R(t)) - l\left(\frac{x}{|x|}, t\right)\right) \nabla\gamma^*(x) \right| < \frac{\varepsilon}{2}, \tag{90}$$

for any $t \geq t_*$. Indeed, (89) holds since Proposition 2.5 gives $\|\nabla u\|_{C^{\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))} < +\infty$ and since the upper and the lower bounds given in Proposition 3.1 give

$$\lim_{C\to\infty}\lim_{t\to\infty}\operatorname*{osc}_{d_\gamma(x;W_R(t))\leq -C} u(x,t) = 0, \quad \lim_{C\to\infty}\lim_{t\to\infty}\operatorname*{osc}_{d_\gamma(x;W_R(t))\geq C} u(x,t) = 0,$$

where $\operatorname{osc}_A = \sup_A - \inf_A$. Moreover, (90) holds since $\nabla\gamma^*(x)$ is bounded on $\mathbb{R}^n \setminus \{0\}$ because it is positively homogeneous of degree zero and since $|\Phi'(z)|$ decays exponentially as $|z| \to \infty$ as in (15)–(16). By combining (89) with (90), we have

$$\sup_{|d_\gamma(x;W_R(t))|\geq C,\, x\neq 0} \left| \nabla u(x,t) - \Phi'\left(d_\gamma(x;\, W_R(t)) - l\left(\frac{x}{|x|}, t\right)\right) \nabla\gamma^*(x) \right| \leq \varepsilon,$$

for any $t \geq t_*$. Combining this with (86) of Lemma 4.6, we obtain (23). On the other hand, (89) gives $\lim_{t\to\infty} \nabla u(0,t) = 0$. The proof of Theorem 1.1 is complete. □

4.4. *Proof of Theorem 1.4*

To prove Theorem 1.4, we prepare two auxiliary results. The first one implies that the weak solution $u(x,t)$ of (1) in the sense of Definition 2.4 satisfies the equation (1a) in the classical sense around the front at the large time. The second one describes the large time behavior of u_t around the front. These results follow from the fact that $\nabla u \neq 0$ (and hence $a_{p_i p_j}(\nabla u)$ are sufficiently smooth) around the front at the large time.

Proposition 4.7 *(Regularity of solutions around the front). Let the assumptions of Theorem 1.1 hold and let R, T, and $l : S^{n-1} \times [T,\infty) \to \mathbb{R}$ be the constants, and the smooth bounded function defined in Theorem 1.1. Then, for any $C > 0$, there exists a constant $T_* \in [T,\infty)$ such that $u \in C^{2+\theta,1+\theta/2}(Q_C)$ and that*

$$\|u\|_{C^{2+\theta,1+\theta/2}(Q_C)} < \infty,$$

holds, where $Q_C = \left\{(x,t) \in \mathbb{R}^n \times [T_*,\infty) \mid |d_\gamma(x;\, W_R(t))| \leq C\right\}$.

Proof. Since $\Phi'(z) < 0$ holds for $z \in \mathbb{R}$ and since $\nabla\gamma^*(x) \neq 0$ holds if $x \neq 0$, it follows from (23) in Theorem 1.1 that there exists a constant $T_* \in [T,\infty)$ such that $\nabla u(x,t) \neq 0$ holds on

$$\widetilde{Q}_C := \left\{(x,t) \in \mathbb{R}^n \times [T_* - 1,\infty) \mid |d_\gamma(x;\, W_R(t))| \leq C + 1\right\}.$$

Then, since $a_{p_i p_j}(p)$ is continuous when $p \neq 0$ and since $\|\nabla u\|_{C^{\theta,\theta/2}(\mathbb{R}^n\times[0,\infty))} < \infty$ holds from Proposition 2.5, we find that $a_{p_i p_j}(\nabla u) \in C^{\theta,\theta/2}(\widetilde{Q}_C)$ and that

$$\|a_{p_i p_j}(\nabla u)\|_{C^{\theta,\theta/2}(\widetilde{Q}_C)} < \infty.$$

Consequently, since the equation (1a) is rewritten as $u_t = \sum_{i,j=1}^n a_{p_i p_j}(\nabla u) + f(u)$, the interior Schauder estimate gives the desired result. □

By combining Proposition 4.7 with Lemma 4.5, we obtain the convergence of u_t around the front. Note that Lemma 4.8 below implies that u_t is uniformly bounded and uniformly positive around $\Gamma(t)$ for all sufficiently large t. This fact is essential to prove Theorem 1.4.

Lemma 4.8 *(Convergence of u_t around the front). Let the assumptions of Theorem 1.1 hold and let R, T, and $l : S^{n-1} \times [T,\infty) \to \mathbb{R}$ be the constants, and the smooth bounded function defined in Theorem 1.1. Then, for any $C > 0$, one has*

$$\lim_{t\to\infty} \sup_{|d_\gamma(x;W_R(t))|\le C} \left| u_t(x,t) + c\Phi'\left(d_\gamma(x;W_R(t)) - l\left(\frac{x}{|x|},t\right)\right)\right| = 0.$$

Proof. Assume that the conclusion of the lemma does not hold. Then there exist a positive constant δ and a sequence $\{(x_i,t_i)\}$ such that $|d_\gamma(x_i;W_R(t_i))| \le C$, that $0 < t_1 < t_2 < \cdots \to \infty$, and that

$$\left| u_t(x_i,t_i) + c\Phi'\left(d_\gamma(x_i;W_R(t_i)) - l\left(\frac{x_i}{|x_i|},t_i\right)\right)\right| \ge \delta, \tag{91}$$

for all $i = 1, 2, \cdots$. By choosing a subsequence, which is denoted by $\{(x_i,t_i)\}$ again, we have

$$\frac{x_i}{|x_i|} \to {}^\exists\nu \in S^{n-1}, \quad d_\gamma(x_i;W_R(t_i)) - l\left(\frac{x_i}{|x_i|},t_i\right) \to {}^\exists K \in \mathbb{R},$$

as $i \to \infty$. Define $u^i(x,t) = u(x_i + x, t_i + t)$. Then Lemma 4.5 implies that

$$\lim_{i\to\infty} u^i(x,t) = \Phi(\nabla\gamma^*(\nu)\cdot x - ct + K) \quad \text{in } BC^{1,0}_{loc}(\mathbb{R}^n \times \mathbb{R}). \tag{92}$$

Let $C = \|l\|_{L^\infty(S^{n-1}\times[T,\infty))} + K + 1$. From Proposition 4.7, we can choose a positive constant $T_* \in [T,\infty)$ such that $u \in C^{2+\theta,1+\theta/2}(Q_C)$ and that

$$\|u\|_{C^{2+\theta,1+\theta/2}(Q_C)} < \infty,$$

holds, where $Q_C = \left\{(x,t) \in \mathbb{R}^n \times [T_*,\infty) \mid |d_\gamma(x;W_R(t))| \le C\right\}$. On the other hand, since

$$\limsup_{i\to\infty} |d_\gamma(x_i;W_R(t_i))| \le \|l\|_{L^\infty(S^{n-1}\times[T,\infty))} + K,$$

we can choose a positive constant r such that

$$B_r(x_i) \times [t_i - r, t_i + r] \subset Q_C \ \text{ for all sufficiently large } i,$$

where $B_r(x)$ is the closed ball with the center $x \in \mathbb{R}^n$ and the radius r. Then we have

$$\begin{aligned}\|u^i\|_{C^{2+\theta,1+\theta/2}(B_r(0)\times[-r,r])} &= \|u\|_{C^{2+\theta,1+\theta/2}(B_r(x_i)\times[t_i-r,t_i+r])}\\ &\le \|u\|_{C^{2+\theta,1+\theta/2}(Q_C)},\end{aligned}$$

for all sufficiently large i. Combining this with (92), we obtain

$$\lim_{i\to\infty} u^i(x,t) = \Phi(\nabla\gamma^*(\nu)\cdot x - ct + K) \quad \text{in } BC^{2,1}(B_r(0)\times[-r,r]).$$

Consequently, we have

$$\lim_{i\to\infty} u_t(x_i,t_i) = \lim_{i\to\infty} u^i_t(0,0) = -c\Phi'(K).$$

However, this contradicts (91), since we have

$$\lim_{i\to\infty} c\Phi'\left(d_\gamma(x_i;W_R(t_i)) - l\left(\frac{x_i}{|x_i|},t_i\right)\right) = c\Phi'(K).$$

The proof of the lemma is complete. □

We can also show the convergences of $u_{x_ix_j}$ around the front. We omit the proof of Lemma 4.9, since it is done by the same argument as Lemma 4.8.

Lemma 4.9 *(Convergences of $u_{x_i x_j}$ around the front). Let the assumptions of Theorem 1.1 hold and let R, T, and $l : S^{n-1} \times [T, \infty) \to \mathbb{R}$ be the constants, and the smooth bounded function defined in Theorem 1.1. Then, for any $C > 0$, one has*

$$\lim_{t\to\infty} \sup_{|d_\gamma(x;W_R(t))|\le C} \left| u_{x_i x_j}(x,t) - \gamma^*_{x_i}(x)\gamma^*_{x_j}(x)\Phi''\left(d_\gamma(x;W_R(t)) - l\left(\frac{x}{|x|},t\right)\right)\right| = 0.$$

Proof of Theorem 1.4. We first prove the statement (i) of Theorem 1.4. Set $\delta = \min\{\alpha, 1-\alpha\}/2$. From Proposition 3.1, there exist positive constants C and T' such that

$$\{x \in \mathbb{R}^n \mid |u(x,t) - \alpha| \le \delta\} \subset \{x \in \mathbb{R}^n \mid |d_\gamma(x;W_R(t)| \le C\},$$

holds for any $t \ge T'$. Since $\Phi' < 0$, Lemma 4.8 implies that, by choosing T' larger if necessary,

$$\inf_{|u(x,t)-\alpha|\le\delta} u_t(x,t) \ge \inf_{|d_\gamma(x;W_R(t))|\le C} u_t(x,t) > 0, \quad t \ge T'.$$

This implies that the region $\{x \in \mathbb{R}^n \mid u(x,t) \ge \alpha\}$ and hence the α-level set $\Gamma(t)$ of the solution $u(x,t)$ are monotonically expanding for $t \ge T'$. Thus the statement (i) of Theorem 1.4 is proved.

We next show the statements (iii). Since $x \in \Gamma(t)$ implies $d_\gamma(x;W_R(t)) - l(x/|x|,t) = 0$, Lemma 4.8 gives

$$\lim_{t\to\infty} u_t(x,t) = -c\Phi'(0) \quad \text{uniformly in } x \in \Gamma(t). \tag{93}$$

Similarly, (23) in Theorem 1.1 gives

$$\lim_{t\to\infty} \nabla u(x,t) = \Phi'(0)\nabla\gamma^*(x) \quad \text{uniformly in } x \in \Gamma(t). \tag{94}$$

For $\nu \in S^{n-1}$, let $x_\nu(t)$ be the intersection point between $\Gamma(t)$ and the half-line $\{\xi\nu \mid \xi > 0\}$, namely, we define

$$x_\nu(t) = \frac{1}{\gamma^*(\nu)}\left(\rho(t;R) + l(\nu,t)\right)\nu. \tag{95}$$

Since $x_\nu(t) \in \Gamma(t)$, we have $u(x_\nu(t),t) = \alpha$. By differentiating this with respect to t, we have

$$x'_\nu(t)\cdot\nabla u(x_\nu(t),t) + u_t(x_\nu(t),t) = 0.$$

By computing $x'_\nu(t)$ from (95), this implies

$$\frac{1}{\gamma^*(\nu)}\left(c - \frac{n-1}{\rho(t;R)} + l_t(\nu,t)\right)\nu\cdot\nabla u(x_\nu(t),t) + u_t(x_\nu(t),t) = 0,$$

since ρ satisfies (8). By passing to the limit as $t \to \infty$ and by using (93)–(94), we have

$$\lim_{t\to\infty}\left(\frac{1}{\gamma^*(\nu)}(c + l_t(\nu,t))\,\nu\cdot\nabla\gamma^*(x_\nu(t)) - c\right) = 0 \quad \text{uniformly in } \nu \in S^{n-1}.$$

Since $\nu\cdot\nabla\gamma^*(x_\nu(t)) = \nu\cdot\nabla\gamma^*(\nu) = \gamma^*(\nu)$, this implies $\lim_{t\to\infty} l_t(\nu,t) = 0$ uniformly in $\nu \in S^{n-1}$, namely, the statement (iii) is proved.

Finally we prove the statements (iv) and (ii). Since $u(x,t)$ is strictly monotone decreasing along ν at large time as in Lemma 4.3, the Euclidean outward normal vector $n(\nu,t)$ and the Euclidean normal velocity $V(\nu,t)$ at $x_\nu(t)$ are given by

$$n(\nu,t) = -\frac{\nabla u(x_\nu(t),t)}{|\nabla u(x_\nu(t),t)|}, \quad V(\nu,t) = \frac{u_t(x_\nu(t),t)}{|\nabla u(x_\nu(t),t)|}.$$

Thus the anisotropic normal velocity $V_\gamma(\nu,t)$ at $x_\nu(t)$ defined in Definition 2.2 is given by

$$V_\gamma(\nu,t) = \frac{V(\nu,t)}{\gamma(n(\nu,t))} = \frac{u_t(x_\nu(t),t)}{\gamma(-\nabla u(x_\nu(t),t))}.$$

Since $x_\nu(t) \in \Gamma(t)$, by using (93)–(94), we have

$$\lim_{t\to\infty} V_\gamma(\nu,t) = \frac{-c\Phi'(0)}{\gamma(-\Phi'(0)\nabla\gamma^*(\nu))} = \frac{c}{\gamma(\nabla\gamma^*(\nu))} = c \quad \text{uniformly in } \nu \in S^{n-1},$$

where the last equality comes from (33) in Remark 2.1. On the other hand, the anisotropic mean curvature $\kappa_\gamma(\nu,t)$ at $x_\nu(t)$ defined in Definition 2.2 is given by

$$\begin{aligned}\kappa_\gamma(\nu,t) &= -\mathrm{div}\nabla\gamma(n(\nu,t))\\ &= -\mathrm{div}\nabla\gamma(-\nabla u(x_\nu(t),t))\\ &= \sum_{i,j=1}^{n} \gamma_{x_i x_j}(-\nabla u(x_\nu(t),t)) u_{x_i x_j}(x_\nu(t),t),\end{aligned}$$

where, to derive the second equality, we used the fact that $\nabla\gamma$ is positively homogeneous of degree zero. Lemma 4.9 gives

$$\lim_{t\to\infty} u_{x_i x_j}(x,t) = \Phi''(0)\gamma^*_{x_i}(x)\gamma^*_{x_j}(x) \quad \text{uniformly in } x \in \Gamma(t).$$

Since $x_\nu(t) \in \Gamma(t)$, by using this and (94), we have

$$\lim_{t\to\infty} \kappa_\gamma(\nu,t) = -\Phi'(0)\Phi''(0) \sum_{i,j=1}^{n} \gamma_{x_i x_j}(\nabla\gamma^*(\nu))\gamma^*_{x_i}(\nu)\gamma^*_{x_j}(\nu) \quad \text{uniformly in } \nu \in S^{n-1}.$$

Since γ_{x_j} is positively homogeneous of degree zero, we have $\sum_{i=1}^{n} \gamma_{x_i x_j}(\nabla\gamma^*(\nu))\gamma^*_{x_i}(\nu) = 0$ for every $1 \le j \le n$. Thus, we obtain $\lim_{t\to\infty}\kappa_\gamma(\nu,t) = 0$ uniformly in $\nu \in S^{n-1}$. The statement (vi) of Theorem 1.4 is proved. The statement (ii) follows immediately from (iii) or (vi). The proof of Theorem 1.4 is complete. □

Conflict of interest statement

No conflict of interest.

Appendix A. Supplementary remarks on anisotropy

We here give supplementary remarks on the notion of anisotropy. We first recall further properties of the dual $\gamma^*(x)$ of $\gamma(p)$, and explain the relation between Euclidean normal vectors and anisotropic normal vectors. The latter will be important in understanding the properties of planar waves of the problem (1).

A.1. Dual and convex conjugate

There is a close relation between the dual $\gamma^*(x)$ of $\gamma(p)$ defined in (6) and the convex conjugate $a^*(x)$ of $a(p)$, namely, the Fenchel convex conjugate $a^*(x)$ defined by

$$a^*(x) = \sup_{p\in\mathbb{R}^n} (x\cdot p - a(p)). \tag{96}$$

From its definition, it is clear that $a^*(x)$ is convex and homogeneous of degree two. Furthermore the following holds:

Lemma A.1. *For the dual $\gamma^*(x)$ defined by (6) and the convex conjugate $a^*(x)$ defined by (96), one has $\gamma^*(x) = \sqrt{2a^*(x)}$.*

The above lemma implies, among other things, that the Wulff shape is the conjugate convex set of the Frank diagram (in the sense of Fenchel) up to dilation by a factor of two. To prove Lemma A.1, we provide two preliminary lemmas.

Lemma A.2. *The map $a_p : \mathbb{R}^n \to \mathbb{R}^n$ is bijective.*

Proof. Since a is strictly convex and homogeneous of degree two and satisfies (4), there exists a positive constant C such that

$$(a_p(p_1) - a_p(p_2)) \cdot (p_1 - p_2) \geq C|p_1 - p_2|^2, \quad p_1, p_2 \in \mathbb{R}^n,$$

see Lemma 4.3 in [1] for instance. Thus, $a_p : \mathbb{R}^n \to \mathbb{R}^n$ is injective, that is, $p_1 \neq p_2$ implies $a_p(p_1) \neq a_p(p_2)$. To show surjectivity, we use a homotopy argument. Choose $x \in \mathbb{R}^n$ arbitrarily and consider the equation

$$x = (1-t)p + ta_p(p) \ \text{ with } \ t \in [0, 1]. \tag{97}$$

If p satisfies (97), then we have

$$\begin{aligned} x \cdot p &= (1-t)p \cdot p + ta_p(p) \cdot p \\ &= (1-t)|p|^2 + 2ta(p) \\ &\geq ((1-t) + 2\Lambda^{-1}t)|p|^2, \end{aligned}$$

hence $|x| \geq \min\{1, 2\Lambda^{-1}\}|p|$, where Λ is the constant defined in (4). This implies that, for each $x \in \mathbb{R}^n$ and $0 \leq t \leq 1$, the equation (97) has no solution on the sphere $|p| = R$, if $R > |x|/\min\{1, 2\Lambda^{-1}\}$. On the other hand, if $t = 0$, the map $(1-t)p + ta_p(p)$ is an identity map. Therefore, the mapping degree of $(1-t)p + ta_p(p)$ on the ball $|p| < R$ with respect to the value x is 1, since $R > |x|$. It follows that the mapping degree of $(1-t)p + ta_p(p)$ with respect to the value x is equal to 1 for each $0 \leq t \leq 1$. Hence (97) with $t = 1$ has a solution in the ball $|p| < R$. This proves that $a_p(p) : \mathbb{R}^n \to \mathbb{R}^n$ is surjective. □

Lemma A.3. *For the convex conjugate $a^*(x)$ of $a(p)$ defined in (96), one has*

$$a^*(x) = a((a_p)^{-1}(x)), \quad x \in \mathbb{R}^n. \tag{98}$$

Proof. Since $a(p)$ is strictly convex and homogeneous of degree two, $x \cdot p - a(p)$ takes the maximum when $x = a_p(p)$. Since a_p is bijective from Lemma A.2, by substituting $p = p_x := (a_p)^{-1}(x)$, we have

$$a^*(x) = \sup_{p \in \mathbb{R}^n} (x \cdot p - a(p)) = x \cdot p_x - a(p_x). \tag{99}$$

By noting $x \cdot p_x = a_p(p_x) \cdot p_x = 2a(p_x)$, we obtain $a^*(x) = a(p_x)$, which implies (98). □

Proof of Lemma A.1. Recall that $\gamma^*(x)$ is given by

$$\gamma^*(x) = \sup_{p \in \mathbb{R}^n} \frac{x \cdot p}{\gamma(p)} = \sup_{p \in \mathbb{R}^n} \frac{x \cdot p}{\sqrt{2a(p)}}.$$

Differentiating $x \cdot p/\sqrt{2a(p)}$ by p, one easily finds that the maximum of this quantity is attained if and only if $a_p(p) = Cx$ for some (arbitrary) constant $C > 0$, or, equivalently, $p = C(a_p)^{-1}(x)$. Thus, by substituting $p = p_x := (a_p)^{-1}(x)$, we obtain

$$\gamma^*(x) = \frac{x \cdot p_x}{\gamma(p_x)} = \frac{a_p(p_x) \cdot p_x}{\sqrt{2a(p_x)}} = \sqrt{2a(p_x)} = \sqrt{2a^*(x)},$$

where the last equality follows from (98) of Lemma A.3. This completes the proof of Lemma A.1. □

Next we investigate further properties of the convex conjugate $a^*(x)$, in order to prove (33) and (34) in Section 2.

Lemma A.4. *For the convex conjugate $a^*(x)$ of $a(p)$ defined in (96), one has*

$$a^*_x(x) = (a_p)^{-1}(x) \ \text{for } x \in \mathbb{R}^n, \quad a_p(p) = (a^*_x)^{-1}(p) \ \text{for } p \in \mathbb{R}^n. \tag{100}$$

Proof. Letting $p_x := (a_p)^{-1}(x)$, we differentiate the expression (99), namely,

$$a^*(x) = x \cdot p_x - a(p_x) = x \cdot (a_p)^{-1}(x) - a((a_p)^{-1}(x)),$$

and obtain

$$a^*_x(x) = (a_p)^{-1}(x) + \partial_x((a_p)^{-1}(x))x - \partial_x((a_p)^{-1}(x))a_p((a_p)^{-1}(x)),$$

for $x \in \mathbb{R}^n \setminus \{0\}$, where $\partial_x((a_p)^{-1}(x))$ denotes the matrix whose (i, j)-component is the x_i-derivative of the jth-component of $(a_p)^{-1}(x)$. This implies the first equality of (100) for $x \in \mathbb{R}^n \setminus \{0\}$, since the third term of the right-hand side equals $\partial_x((a_p)^{-1}(x))x$. When $x = 0$, we have $a^*_x(x) = (a_p)^{-1}(x) = 0$, since it is easily found that $a^*_x(x)$ and $(a_p)^{-1}(x)$ are both homogeneous of degree one. Thus the first equality of (100) holds for all $x \in \mathbb{R}^n$. The second equality of (100) follows immediately from the bijectivity of $a_p : \mathbb{R}^n \to \mathbb{R}^n$. □

Lemma A.5. *$a^*_{xx}(x)$ is positive definite and satisfies*

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a^*_{x_i x_j}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^n, \tag{101}$$

for $x \in \mathbb{R}^n \setminus \{0\}$.

Proof. From the first equality of (100), we have $x = a_p(a^*_x(x))$, namely, $x_i = a_{p_i}(a^*_x(x))$ for $1 \le i \le n$. By differentiating this with respect to x_j, we have

$$\delta_{ij} = \sum_{k=1}^{n} a_{p_i p_k}(a^*_x(x))a^*_{x_k x_j}(x),$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$. This means that the Hessian matrix of a^* at x is the inverse of the Hessian matrix of a at $a^*_x(x)$. We here note that $x \ne 0$ if and only if $a^*_x(x) \ne 0$ since we have $a^*_x(x) = (a_p)^{-1}(x)$ from Lemma A.4 and since a_p is bijective from Lemma A.2 and satisfies $a_p(0) = 0$ by the homogeneity of degree two. Consequently, since a_{pp} is positive definite and satisfies (4) for all $p \ne 0$ by the assumptions, a^*_{xx} is positive definite and satisfies (101) for all $x \ne 0$. □

Remark A.6. The equalities (100) in Lemma A.4 give

$$x = a_p(a^*_x(x)) \ \text{ for } x \in \mathbb{R}^n, \quad p = a^*_x(a_p(p)) \ \text{ for } p \in \mathbb{R}^n.$$

On the other hand, the identities $\gamma(p) = \sqrt{2a(p)}$ and $\gamma^*(x) = \sqrt{2a^*(x)}$ imply $a_p(p) = \gamma(p)\nabla\gamma(p)$ and $a^*_x(x) = \gamma^*(x)\nabla\gamma^*(x)$. Combining these, we obtain

$$x = \gamma^*(x)\gamma(\nabla\gamma^*(x))\nabla\gamma(\nabla\gamma^*(x)), \tag{102}$$

$$p = \gamma(p)\gamma^*(\nabla\gamma(p))\nabla\gamma^*(\nabla\gamma(p)). \tag{103}$$

Now we take an inner product of (103) with the vector $\nabla\gamma(p)$, and apply the identities (32), to obtain

$$\gamma(p) = \gamma(p)\left(\gamma^*(\nabla\gamma(p))\right)^2.$$

This implies $\gamma^*(\nabla\gamma(p)) = 1$. Similarly we obtain $\gamma(\nabla\gamma^*(x)) = 1$, which proves (33). Substituting (33) into (102) and (103) establishes the identities (34).

A.2. Euclidean and anisotropic normal vectors

We here explain the relation between the Euclidean normal vectors and the anisotropic normal vectors on any given surface.

Let us start with studying the two notions of normal vectors on the surface of the Wulff shape, ∂W_1. By noting that (33) in Remark 2.1 implies

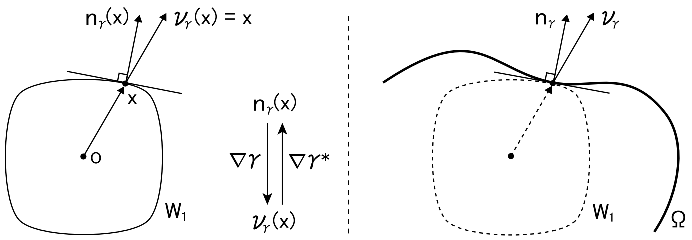

Fig. 3. (Left) Wulff shape W_1 and its Gauss map $\nabla\gamma^* : \nu_\gamma = x \mapsto n_\gamma$ and the inverse Gauss map $\nabla\gamma : n_\gamma \mapsto \nu_\gamma = x$. Note that the vector $\nu_\gamma(x)(= x)$ is normal to ∂W_1 at x in the anisotropic metric γ^*, since ∂W_1 is the unit sphere in this metric. (Right) Euclidean normal vector n_γ and the anisotropic normal vector ν_γ to the boundary of a general domain Ω.

$$\nabla\gamma(p) \in \partial W_1 = \{x \in \mathbb{R}^n \mid \gamma^*(x) = 1\},$$
$$\nabla\gamma^*(x) \in \partial F_1 = \{p \in \mathbb{R}^n \mid \gamma(p) = 1\},$$

if $p \neq 0$ and $x \neq 0$, we consider the restrictions

$$\nabla\gamma|_{\partial F_1} : \partial F_1 \to \partial W_1, \quad \nabla\gamma^*|_{\partial W_1} : \partial W_1 \to \partial F_1.$$

Then we find that

$$(\nabla\gamma|_{\partial F_1})^{-1} = \nabla\gamma^*|_{\partial W_1}, \quad (\nabla\gamma^*|_{\partial W_1})^{-1} = \nabla\gamma|_{\partial F_1}. \tag{104}$$

Indeed, since $\nabla\gamma$ and $\nabla\gamma^*$ are both positively homogeneous of degree zero, (34) in Remark 2.1 gives

$$\nabla\gamma^*\left(\nabla\gamma\left(\frac{p}{\gamma(p)}\right)\right) = \frac{p}{\gamma(p)}, \quad \nabla\gamma\left(\nabla\gamma^*\left(\frac{x}{\gamma^*(x)}\right)\right) = \frac{x}{\gamma^*(x)},$$

for any $p, x \in \mathbb{R}^n \setminus \{0\}$. This implies (104), since $p/\gamma(p) \in \partial F_1$ and $x/\gamma^*(x) \in \partial W_1$. From (104), we find that the Euclidean normal vector and the anisotropic normal vector are connected by the bijections $\nabla\gamma|_{\partial F_1}$ and $\nabla\gamma^*|_{\partial W_1}$.

Remark A.7 *(Conversion of normal vectors).* The anisotropic signed distance function for ∂W_1 is given by $d_\gamma(x; W_1) = \gamma^*(x) - 1$. At each point $x \in \partial W_1$, we consider the Euclidean outward (not necessarily unit) normal vector $n_\gamma(x)$ and the anisotropic outward normal vector $\nu_\gamma(x)$ for ∂W_1 given by

$$n_\gamma(x) = \nabla d_\gamma(x; W_1), \quad \nu_\gamma(x) = \nabla\gamma(n_\gamma(x)).$$

Here, $\gamma(n_\gamma(x)) = 1$ follows from (35) and $\gamma^*(\nu_\gamma(x)) = 1$ follows from the first formula of (33). Thus, we have $n_\gamma(x) \in \partial F_1$ and $\nu_\gamma(x) \in \partial W_1$. Moreover, we have

$$\nu_\gamma(x) = \nabla\gamma(\nabla d_\gamma(x; W_1)) = \nabla\gamma(\nabla\gamma^*(x)) = \frac{x}{\gamma^*(x)} = x, \quad x \in \partial W_1,$$

where the second equality from the last comes from the second formula of (34). Consequently, (104) implies that, at each point $x \in \partial W_1$,

$$\nabla\gamma \ \text{ maps } \ n_\gamma(x) \in \partial F_1 \ \text{ to } \ \nu_\gamma(x) = x \in \partial W_1,$$
$$\nabla\gamma^* \ \text{ maps } \ \nu_\gamma(x) = x \in \partial W_1 \ \text{ to } \ n_\gamma(x) \in \partial F_1.$$

In other words, $\nabla\gamma$ and $\nabla\gamma^*$ can be regarded as the inverse Gauss map and the Gauss map for ∂W_1, respectively. See Fig. 3 (left).

Now that the relation between the two notions of normal vectors on ∂W_1 has become clear, one can naturally extend this observation to understand the relation of the two notions of normal vectors on the boundary of general smooth domain Ω. More precisely, given any boundary point y on $\partial\Omega$, we consider a translation of W_1 that is tangent to $\partial\Omega$ at y from inside (see Fig. 3 (right)). The point of tangency on the shifted Wulff shape determines a position x on ∂W_1, and consequently the two outward normal vectors $n_\gamma(x)$ and $\nu_\gamma(x)$ as before. This defines the two outward normal vectors at $y \in \partial\Omega$.

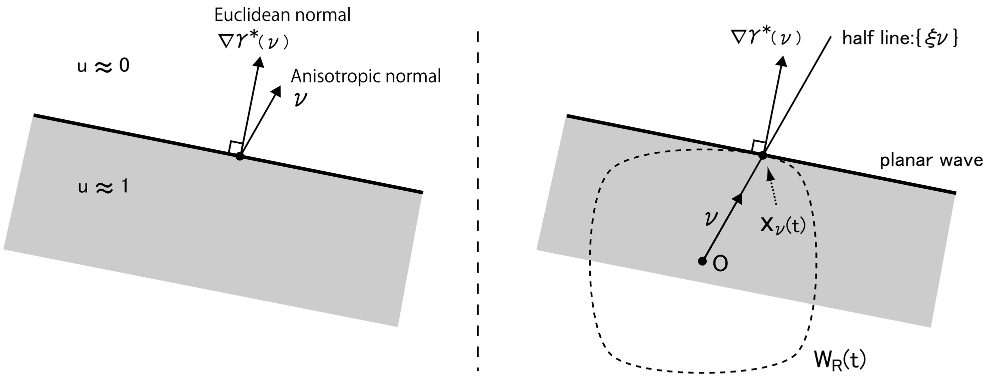

Fig. 4. The planar wave whose Euclidean normal is parallel to $\nabla\gamma^*(\nu)$.

A.3. *Wulff shape and planar waves*

For each $\nu \in S^{n-1}$, the equation (1a) has a planar wave solution of the form

$$u(x,t) = \Phi(\nabla\gamma^*(\nu)\cdot x - ct). \tag{105}$$

Indeed, this satisfies the equation (1a), since we have $u_t = -c\Phi'$ and $\operatorname{div} a_p(\nabla u) = \Phi''$. Note that, since $a_p(p) = \gamma(p)\nabla\gamma(p)$ by the definition and since $\nabla u = \Phi'\nabla\gamma^*(\nu)$, we have

$$\begin{aligned}
\operatorname{div} a_p(\nabla u) &= \operatorname{div}\left[\gamma(\Phi'\nabla\gamma^*(\nu))\nabla\gamma(\Phi'\nabla\gamma^*(\nu))\right] \\
&= \operatorname{div}\left[\Phi'\gamma(\nabla\gamma^*(\nu))\nabla\gamma(\nabla\gamma^*(\nu))\right] \\
&= \operatorname{div}\left[\Phi'\frac{\nu}{\gamma^*(\nu)}\right] \\
&= \Phi''\nabla\gamma^*(\nu)\cdot\frac{\nu}{\gamma^*(\nu)} \\
&= \Phi'',
\end{aligned}$$

where the second equality comes from the homogeneity of γ, the third one comes from the second formulas of (33)–(34), and the last one comes from the homogeneity of γ^*.

Clearly, the Euclidean normal of the planar wave (105) is parallel to $\nabla\gamma^*(\nu)$ and it propagates in this direction with the speed $c/|\nabla\gamma^*(\nu)|$ in the usual Euclidean distance. On the other hand, the anisotropic normal vector of this planar wave is parallel to ν, along which the planar wave moves with the speed $c/\gamma^*(\nu)$ in the Euclidean distance and c in the anisotropic distance.

Let $x_\nu(t)$ be the intersection point between the half line $\{\xi\nu \mid \xi > 0\}$ and the expanding Wulff shape $W_R(t)$, namely, $x_\nu(t) = \rho(t;R)\nu/\gamma^*(\nu)$. Then the α-level set of a translation of the planar wave (105) is tangential to $W_R(t)$ at $x_\nu(t)$, because the Euclidean normal of $\partial W_R(t)$ at $x_\nu(t)$ is $\nabla\gamma^*(\nu)$. The moving speed of $x_\nu(t)$ along ν in the anisotropic metric is not c but $\rho'(t;R)$, however $\lim_{t\to\infty}\rho'(t;R) = c$ holds obviously. Namely, the asymptotic shape and the speed of $\partial W_R(t)$ around $x_\nu(t)$ at the large time are described by those of the level set of the planar wave (105). See Fig. 4.

Finally, we remark that (105) can be rewritten as follows: for each $n \in S^{n-1}$, the equation (1a) has a planar wave solution of the form

$$u(x,t) = \Phi\left(\frac{n}{\gamma(n)}\cdot x - ct\right), \tag{106}$$

that propagates in the direction $n \in S^{n-1}$ with the speed $c\gamma(n)$ in the Euclidean distance. Indeed, since $\nabla\gamma^*|_{\partial W_1} : \partial W_1 \to \partial F_1$ is a bijection, for any Euclidean unit vector n we can find $\nu \in S^{n-1}$ such that $n = \nabla\gamma^*(\nu)/|\nabla\gamma^*(\nu)|$. Substituting this into (105) and using the homogeneity of γ along with (33), we obtain

$$\frac{n}{\gamma(n)} = \frac{\nabla\gamma^*(\nu)}{\gamma(\nabla\gamma^*(\nu))} = \nabla\gamma^*(\nu).$$

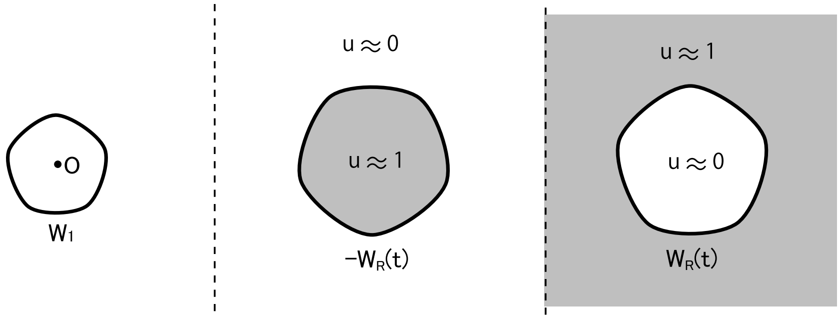

Fig. 5. Spreading fronts in the case where $a(p)$ is *positively* homogeneous. (left) the Wulff shape W_1; (center) the case of $\Phi' < 0$; (right) the case of $\Phi' > 0$.

Appendix B. Positive homogeneity

By the homogeneity of $a(p)$, we have $a(-p) = a(p)$. Thus, the Frank diagram and the Wulff shape are both symmetric with respect to the origin. On the other hand, if we only assume that $a(p)$ is *positively homogeneous* of degree two, namely, if $a(p)$ satisfies

$$a(\lambda p) = \lambda^2 a(p), \quad \lambda > 0,$$

$a(p)$ is not necessarily symmetric. Nonetheless, all the main results of the present paper remain valid with only a minor modification. More precisely, assume that $a(p)$ is positively homogeneous of degree two, then the conclusions of Theorems 1.1 and 1.4 and Corollary 1.2 remain true if we make *either one* of the following modifications:

(a) $\partial W_R(t)$ and $d_\gamma(x; W_R(t))$ are replaced by $\partial(-W_R(t))$ and $d_\gamma(x; -W_R(t))$, respectively, where $-W_R(t) := \{x \in \mathbb{R}^n \mid -x \in W_R(t)\}$;
(b) Condition (3) is replaced by $\int_0^1 f(s)ds < 0$, Φ satisfies $\Phi(-\infty) = 0$ and $\Phi(+\infty) = 1$ instead of (14b), and the conditions (17)–(19) are replaced by

$$\sup_{x\in\mathbb{R}^n} u_0(x) \le -m, \quad \max_{|x|\le L} u_0(x) \le \alpha - \eta, \quad \liminf_{|x|\to\infty} u_0(x) > \alpha.$$

Statement (a) implies that the position of the spreading front roughly coincides with the boundary of the *symmetric* image (with respect to the origin) of the expanding Wulff shape. On the other hand, statement (b) implies that Theorem 1.1 holds for "reverse" fronts, where the value of u is smaller behind the expanding front, and the front is facing the inward direction (see Fig. 5). The reason why we need modification (a) or (b) is the following. In the original setting of Theorem 1.1, in which the front is facing outward, the normal velocity and ∇u have opposite signs. Therefore, one needs to consider either the Wulff shape associated with the function $a(-p)$ as in (a) above, or consider propagation of a reverse front as in (b) above, in which the normal velocity and ∇u have roughly the same sign. We note that the behavior of reverse fronts can be analyzed by simply setting $v = 1 - u$ and rewriting the equation (1a) as

$$v_t = \operatorname{div}\left(-a_p(-\nabla v)\right) - f(1-v) = \operatorname{div}\tilde{a}_p(\nabla v) + \tilde{f}(v),$$

where $\tilde{a}(p) = a(-p)$ and $\tilde{f}(s) = -f(1-s)$ satisfying $\int_0^1 \tilde{f}(s)ds < 0$.

Appendix C. Proof of Lemma 4.2

The aim of this section is to prove Lemma 4.2, which is a Liouville type theorem for entire solutions of the Allen–Cahn equation. As we mentioned, this result is an anisotropic extension of Theorem 3.1 in [7]. We first state the strong maximum principle for weak entire solutions of the anisotropic Allen–Cahn equation.

Lemma C.1 *(Strong maximum principle). Let Ω be a (not necessarily bounded) sub-domain of $\mathbb{R}^n$. Let $u^-(x,t)$ and $u^+(x,t)$ be a weak entire sub-solution and a weak entire super-solution of (1a), namely, the functions that belong to $BC^{1,0}(\mathbb{R}^n \times \mathbb{R})$ and satisfy*

$$\pm \int_{\mathbb{R}} \int_{\mathbb{R}^n} -u^{\pm}\varphi_t + a_p(\nabla u^{\pm}) \cdot \nabla \varphi - f(u^{\pm})\varphi \, dx\, dt \geq 0,$$

for any non-negative $\varphi \in C_0^{\infty}(\Omega \times \mathbb{R})$. If there exists a point $(x_*, t_*) \in \Omega \times \mathbb{R}$ such that $u^-(x_*, t_*) = u^+(x_*, t_*)$ and that

$$u^-(x,t) \leq u^+(x,t), \quad (x,t) \in \Omega \times (-\infty, t_*],$$

then $u^- \equiv u^+$ holds on $\Omega \times (-\infty, t_*]$.

Proof. By setting $w = e^{-Mt}(u^- - u^+)$, we have

$$w_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x,t) w_{x_j}\right) - (B(x,t) - M)\, w \leq 0,$$

in the weak sense in $\Omega \times \mathbb{R}$, where A_{ij} and B are defined by

$$A_{ij}(x,t) = \int_0^1 a_{p_i p_j}(\nabla u^+ + \theta \nabla(u^- - u^+))d\theta, \quad B(x,t) = \int_0^1 f'(u^+ + \theta(u^- - u^+))d\theta.$$

We note that $A_{ij} \in L^{\infty}(\mathbb{R}^n \times \mathbb{R})$ and is positively definite. Since $u^{\pm}$ are both bounded by the assumption, we can choose M large enough to satisfy $B(x,t) - M \leq 0$ and apply the strong maximum principle for weak solutions given by Theorem 6.25 in [24]. Consequently, if $w(x_*, t_*) = 0$ holds, we have $w \equiv 0$ on $\Omega \times (-\infty, t_*]$. This completes the proof. □

Lemma C.2 below implies the uniqueness (up to shift) of traveling wave solution of the Allen–Cahn equation on $\mathbb{R}$ in the sense of weak solutions. Note that the function $u(x)$ in Lemma C.2 is not $BC^2(\mathbb{R})$ but $BC^1(\mathbb{R})$.

Lemma C.2 *(Uniqueness of traveling wave). Suppose that the constant c and the function $u \in BC^1(\mathbb{R})$ satisfy $u(-\infty) = 1$, $u(+\infty) = 0$, and*

$$u'' + cu' + f(u) = 0, \quad x \in \mathbb{R},$$

in the weak sense. Then there exists a constant $\mu \in \mathbb{R}$ such that

$$u(x) = \Phi(x + \mu), \quad x \in \mathbb{R}.$$

Lemma C.2 can be proved in essentially the same way as the proof of Theorem 2.1 in [9]. The only difference is to replace the usual strong maximum principle for the classical solutions by that for the weak solutions given in Lemma C.1. So we omit the proof.

We now prove Lemma 4.2. The argument is only a slight modification of the proof of Theorem 3.1 in [7].

Proof of Lemma 4.2. Step 1: Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$ and choose $Q \in SO(n)$ satisfying $Qe_1 = \nabla\gamma^*(\nu)/|\nabla\gamma^*(\nu)|$. We define the function $v(z,t)$ by

$$v(z,t) = u(x,t), \quad \text{where } x = Q\left(z + \frac{ct}{|\nabla\gamma^*(\nu)|} e_1\right).$$

In what follows, we use the notation $z = (z_1, z') \in \mathbb{R}^n$, where $z' = (z_2, \cdots, z_n) \in \mathbb{R}^{n-1}$. Then $v(z,t)$ is a weak entire solution of

$$v_t = \operatorname{div}\left(Q^{-1} a_p(Q\nabla v)\right) + \frac{c}{|\nabla\gamma^*(\nu)|} v_{z_1} + f(v), \quad z \in \mathbb{R}^n, t \in \mathbb{R}, \tag{107}$$

in the sense of Lemma 2.8. By noting that

$$\nabla\gamma^*(\nu) \cdot Qe_1 = |\nabla\gamma^*(\nu)|, \quad \nabla\gamma^*(\nu) \cdot Q(z - z_1 e_1) = 0,$$

we have $\nabla\gamma^*(\nu)\cdot x = |\nabla\gamma^*(\nu)|z_1 + ct$. Thus, the inequality (81) is reduced to

$$\Phi(|\nabla\gamma^*(\nu)|z_1 + K) \le v(z,t) \le \Phi(|\nabla\gamma^*(\nu)|z_1 - K), \quad z \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{108}$$

Step 2: Fix $\xi \in \mathbb{R}^{n-1}$ and $\tau \in \mathbb{R}$ arbitrarily. We define the function $v^\sigma(z,t)$ by

$$v^\sigma(z,t) = v(z_1 - \sigma, z' + \xi, t + \tau).$$

From (108) and the monotonicity of Φ, we have $v \le v^\sigma$ on $\mathbb{R}^n \times \mathbb{R}$ if σ is large enough. For the same reason, we have $v \ge v^\sigma$ on $\mathbb{R}^n \times \mathbb{R}$ if σ is sufficiently negative. Thus the constant σ_* below is well-defined:

$$\sigma_* = \inf\left\{\sigma \in \mathbb{R} \mid v(z,t) \le v^{\sigma'}(z,t) \text{ on } \mathbb{R}^n \times \mathbb{R} \text{ for all } \sigma' \ge \sigma\right\}.$$

Then $v \le v^{\sigma_*}$ on $\mathbb{R}^n \times \mathbb{R}$ from the continuity of v. Our goal is to show that $\sigma_* = 0$, from which the conclusion of the lemma easily follows. This will be done in the next two steps.

Step 3: Since f is of the bistable type as specified in (2), we can choose a positive constant δ_0 that satisfies

$$f'(s) \le 0, \quad s \in [0, 2\delta_0] \cup [1-\delta_0, 1+\delta_0]. \tag{109}$$

Since $\Phi(-\infty) = 1$ and $\Phi(+\infty) = 0$, the upper and the lower bounds for v in (108) imply that there exists a positive constant M such that

$$v \le v^{\sigma_*} \le \delta_0 \ \text{ if } \ z_1 \ge M - 1, \tag{110}$$

$$v^{\sigma_*} \ge v \ge 1 - \delta_0 \ \text{ if } \ z_1 \le -(M-2). \tag{111}$$

In what follows, setting $D_M = [-M, M] \times \mathbb{R}^{n-1} \times \mathbb{R}$, we will show

$$\inf_{(z_1,z',t)\in D_M} (v^{\sigma_*}(z_1,z',t) - v(z_1,z',t)) = 0. \tag{112}$$

Assume that (112) does not hold. Then we can choose a constant $\eta_0 \in (0,1]$ such that

$$v \le v^{\sigma_* - \eta} \ \text{ on } D_M, \tag{113}$$

for any $\eta \in [0, \eta_0]$. Fix $\eta \in [0, \eta_0]$ arbitrarily. Then, since $\eta \le 1$, (110) and (111) give

$$v,\ v^{\sigma_*-\eta} \in (0, \delta_0] \ \text{ if } \ z_1 \ge M - 1, \tag{114}$$

$$v,\ v^{\sigma_*-\eta} \in [1-\delta_0, 1) \ \text{ if } \ z_1 \le -(M-1). \tag{115}$$

By combining these inequalities with (113), we have $v \le v^{\sigma_*-\eta} + \delta_0$ on $\mathbb{R}^n \times \mathbb{R}$. Thus, we can define the constant $\delta_* \in [0, \delta_0]$ by

$$\delta_* = \inf\left\{\delta \in \mathbb{R} \mid v(z,t) \le v^{\sigma_*-\eta}(z,t) + \delta \text{ on } \mathbb{R}^n \times \mathbb{R}\right\}.$$

Assume $\delta_* > 0$. Then there exist a sequence $\{(z_{1,i}, z_i', t_i)\}$ and a constant $z_{1,\infty}$ such that

$$\lim_{i\to\infty} \left(v^{\sigma_*-\eta}(z_{1,i}, z_i', t_i) + \delta_* - v(z_{1,i}, z_i', t_i)\right) = 0, \tag{116}$$

$$\lim_{i\to\infty} z_{1,i} = z_{1,\infty}, \tag{117}$$

$$|z_{1,\infty}| \ge M, \tag{118}$$

where (117) follows from the boundedness of $\{z_{1,i}\}$. Indeed, by virtue of (108), we have $\lim_{|z_1|\to\infty}(v^{\sigma_*-\eta} + \delta_* - v) = \delta_*$ uniformly in $z' \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$. This implies that the set $\{z_{1,i}\} \subset \mathbb{R}$ is bounded. Moreover, (118) follows from (113).

From Lemma 2.8, by choosing a subsequence, which is denoted by $\{(z_{1,i}, z_i', t_i)\}$ again, we obtain a weak entire solution $w(z_1, z', t)$ of (107) such that $v(z_1, z' + z_i', t + t_i) \to w(z_1, z', t)$ as $i \to \infty$ in $BC^{1,0}_{loc}(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R})$. Then, we have

$$\begin{aligned} w(z_1, z', t) &= \lim_{i\to\infty} v(z_1, z' + z_i', t + t_i) \\ &\le \lim_{i\to\infty} v^{\sigma_* - \eta}(z_1, z' + z_i', t + t_i) + \delta_* \\ &= w^*(z_1, z', t) + \delta_*, \end{aligned}$$

where $w^*(z_1, z', t) := w(z_1 - \sigma_* + \eta, z' + \xi, t + \tau)$. On the other hand, since (114) and (115) imply

$$\begin{aligned} &0 < w^* < w^* + \delta_* \le 2\delta_0 \ \text{ if } \ z_1 \ge M - 1, \\ &1 - \delta_0 \le w^* < w^* + \delta_* < 1 + \delta_0 \ \text{ if } \ z_1 \le -(M - 1), \end{aligned}$$

the inequality (109) gives

$$\begin{aligned} \mathcal{L}[w^* + \delta_*] &:= w_t^* - \operatorname{div}_z \left(Q^{-1} a_p(Q \nabla_z w^*) \right) - \frac{c}{|\nabla\gamma^*(\nu)|} w_{z_1}^* - f(w^* + \delta_*) \\ &= f(w^*) - f(w^* + \delta_*) \ \ge \ 0, \end{aligned}$$

in the weak sense of Lemma C.1 if $|z_1| \ge M - 1$. We now apply the strong maximum principle in the region $|z_1| > M - 1$ to derive a contradiction. From (116)–(118), we have

$$w^*(z_{1,\infty}, 0, 0) + \delta_* = w(z_{1,\infty}, 0, 0), \tag{119}$$

where $z_{1,\infty}$ is as defined in (117) (hence it satisfies $|z_{1,\infty}| \ge M > M - 1$). On the other hand, from (113), we have $w^*|_{z_1 = M-1} \ge w|_{z_1 = M-1}$ and $w^*|_{z_1 = -M+1} \ge w|_{z_1 = -M+1}$; hence

$$w^* + \delta_*|_{z_1 = M-1} > w|_{z_1 = M-1}, \quad w^* + \delta_*|_{z_1 = -M+1} > w|_{z_1 = -M+1}. \tag{120}$$

In the case where $z_{1,\infty} \ge M$ (resp. $z_{1,\infty} \le -M$), the statements (119)–(120) contradict the strong maximum principle (Lemma C.1) in the region $z_1 > M - 1$ (resp. $z_1 < -M + 1$). Thus, we obtain $\delta_* = 0$, but it contradicts the minimality of σ_*. This contradiction establishes (112).

Step 4: From (112), there exist a sequence $\{(z_{1,i}, z_i', t_i)\}$ and a constant $z_{1,\infty}$ such that

$$\begin{aligned} &\lim_{i\to\infty} \left(v^{\sigma_*}(z_{1,i}, z_i', t_i) - v(z_{1,i}, z_i', t_i) \right) = 0, \\ &\lim_{i\to\infty} z_{1,i} = z_{1,\infty}, \\ &|z_{1,\infty}| \le M. \end{aligned}$$

From Lemma 2.8, by choosing a subsequence, which is denoted by $\{(z_{1,i}, z_i', t_i)\}$ again, we obtain a weak entire solution $w(z_1, z', t)$ such that $v(z_1, z' + z_i', t + t_i) \to w(z_1, z', t)$ as $i \to \infty$ in $BC^{1,0}_{loc}(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R})$. Then, we have

$$\begin{aligned} w(z_1, z', t) &= \lim_{i\to\infty} v(z_1, z' + z_i', t + t_i) \\ &\le \lim_{i\to\infty} v^{\sigma_*}(z_1, z' + z_i', t + t_i) \\ &= w^*(z_1, z', t), \end{aligned}$$

where $w^*(z_1, z', t) := w(z_1 - \sigma^*, z' + \xi, t + \tau)$. Moreover, we have

$$w(z_{1,\infty}, 0, 0) = \lim_{i\to\infty} v(z_{1,i}, z_i', t_i) = \lim_{i\to\infty} v^{\sigma_*}(z_{1,i}, z_i', t_i) = w^*(z_{1,\infty}, 0, 0).$$

Thus, the strong maximum principle given in Lemma C.1 implies $w \equiv w^*$ for $t \le 0$. Then $w \equiv w^*$ holds for $t \in \mathbb{R}$ by virtue of the uniqueness of the solution, which follows from the comparison principle given in Proposition 2.6 since Lemma 2.8 implies $w \in C^{1+\theta,\theta/2}(\mathbb{R}^n \times \mathbb{R})$. Consequently, the equality

$$w(z_1, z', t) = w^*(z_1, z', t) = w(z_1 - \sigma^*, z' + \xi, t + \tau) \ \text{ on } \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R},$$

gives

$$w(0, 0, 0) = w(-k\sigma^*, k\xi, k\tau) \ \text{ for all } k \in \mathbb{Z}.$$

This is a contradiction when $\sigma^* \neq 0$, because

$$\lim_{k\to-\infty} w(-k\sigma^*, k\xi, k\tau) = 1, \quad \lim_{k\to+\infty} w(-k\sigma^*, k\xi, k\tau) = -1, \quad \text{if } \sigma_* > 0,$$

while

$$\lim_{k\to-\infty} w(-k\sigma^*, k\xi, k\tau) = -1, \quad \lim_{k\to+\infty} w(-k\sigma^*, k\xi, k\tau) = 1, \quad \text{if } \sigma_* < 0.$$

Thus we obtain $\sigma^* = 0$, namely, we have

$$v(z_1, z', t) \leq v(z_1, z' + \xi, t + \tau), \quad \text{on } \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}.$$

Since $\xi \in \mathbb{R}^{n-1}$ and $\tau \in \mathbb{R}$ are both arbitrary, v is independent of both z' and t, namely, $v = v(z_1)$. Thus, from (107), it satisfies

$$\frac{1}{|\nabla\gamma^*(\nu)|^2} v_{z_1 z_1} + \frac{c}{|\nabla\gamma^*(\nu)|} v_{z_1} + f(v) = 0, \quad z_1 \in \mathbb{R},$$

in the weak sense. Consequently, Lemma C.2 implies that there exists a constant μ such that

$$v(z_1) = \Phi(|\nabla\gamma^*(\nu)| z_1 + \mu), \quad z_1 \in \mathbb{R}.$$

This completes the proof of Lemma 4.2. □

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