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Convergence to self-similarity for the Boltzmann equation for strongly inelastic Maxwell molecules

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Abstract

We prove propagation of regularity, uniformly in time, for the scaled solutions of the inelastic Maxwell model for any value of the coefficient of restitution. The result follows from the uniform in time control of the tails of the Fourier transform of the solution, normalized in order to have constant energy. By standard arguments this implies the convergence of the scaled solution towards the stationary state in Sobolev and L^1 norms in the case of regular initial data as well as the convergence of the original solution to the corresponding self-similar cooling state. In the case of weak inelasticity, similar results have been established by Carlen, Carrillo and Carvalho (2009) in [11] via a precise control of the growth of the Fisher information.

Keywords: Granular gases; Kinetic models; Boltzmann equation

1. Introduction

This paper concerns the regularity properties of solutions to the spatially homogeneous Boltzmann equation for Maxwellian molecules in $\mathbb{R}^3$ with inelastic collisions, introduced in [6]. This equation describes the relaxation towards equilibrium of the distribution function of particles interacting through inelastic binary collisions. Kinetic theory of granular gases becames popular in the last ten years, and various mathematical problems, arising from dissipation, have been considered so far [22]. Let $f(v,\tau)$ be the probability density for the velocity v of a particle chosen randomly from the collection at time τ. Let $\varphi(v)$ be any bounded and continuous function on $\mathbb{R}^3$. For a dilute gas, with a mean free path of size ε, the equation under investigation is given, in weak form, by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle f,\varphi\rangle = \frac{1}{\varepsilon}\big\langle Q(f,f),\varphi\big\rangle \tag{1}$$

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where

$$\langle f, \varphi\rangle = \int_{\mathbb{R}^3} \varphi(v) f(v,\tau)\, \mathrm{d}v,$$

and

$$\big\langle Q(f,f), \varphi\big\rangle = \frac{1}{4\pi} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f(v,\tau) f(w,\tau)\big[\varphi(v') - \varphi(v)\big]\, \mathrm{d}\sigma\, \mathrm{d}v\, \mathrm{d}w. \tag{2}$$

In (2) σ is a unit vector in S^2, $\mathrm{d}\sigma$ is the uniform measure on S^2 with total mass 4π and the post-collision velocities are given by the collision mechanism written as

$$\begin{aligned} v' &= \frac{1}{2}(v+w) + \frac{1-e}{4} q + \frac{1+e}{4}|q|\sigma, \\ w' &= \frac{1}{2}(v+w) - \frac{1-e}{4} q - \frac{1+e}{4}|q|\sigma \end{aligned} \tag{3}$$

with $q = v - w$. The positive parameter e, with $0 \leqslant e < 1$ is the restitution coefficient. From (3) it follows that the post-collision relative velocity is non-increasing, with

$$|v' - w'|^2 = |q'|^2 = |q|^2 - \frac{1-e^2}{2}\big(|q|^2 - |q| q\cdot\sigma\big) \leqslant |q|^2 = |v-w|^2.$$

Note that the dissipation increases with e decreasing. Thus the case $e = 0$ corresponds to the strongest dissipation. Since $e < 1$, the collisions are inelastic, energy is dissipated in each collision, and the collisions are not reversible. This makes a crucial difference with the elastic theory in which there is a complete time reversal symmetry between the pre- and post-collision velocities. As pointed out in [11], it is mainly for this reason that the Boltzmann equation is usually written in the weak form (1), and not because of any difficulty in constructing strong solutions. The post-collision velocities (3) represent one of the possible parameterizations of the inelastic collision mechanism. However, as exhaustively discussed in [11] other possible pairs of pre-collision velocities (v_*, w_*) that result in the pair of post-collision velocities (v, w) can be constructed [6,19]. It is remarkable that these parameterizations give equivalent collision terms only when the restitution coefficient satisfies $e > 0$ [11]. Consequently, any result which is valid for any value of e, including the case $e = 0$, cannot be translated, in this limit case, to other collision rules.

Since the total momentum $v + w$ is conserved in each individual collision $(v, w) \to (v', w')$, the first moment of $f(v,\tau)$ (i.e. the mean velocity) is conserved. In particular, choosing as initial datum a probability density f_0 satisfying

$$\int_{\mathbb{R}^3} f_0(v)\, \mathrm{d}v = 1, \qquad \int_{\mathbb{R}^3} v_i f_0(v)\, \mathrm{d}v = 0, \quad i = 1,2,3, \qquad \int_{\mathbb{R}^3} |v|^2 f_0(v)\, \mathrm{d}v = T_0 < \infty, \tag{4}$$

it follows that while both mass and momentum are preserved in time, the second moment (i.e. the temperature or energy) decays according to the law

$$T(\tau) = \int_{\mathbb{R}^3} |v|^2 f(v,\tau)\, \mathrm{d}v = T_0 \exp\left\{-\frac{1-e^2}{4\varepsilon}\tau\right\}. \tag{5}$$

This implies that $f(v,\tau)\, \mathrm{d}v$ converges towards a point mass at $v = 0$ as τ tends to infinity. The precise way in which the density collapses into a mass concentrated in zero has been investigated in various previous works [5,8,2,7,9]. It has been shown that, after a certain relaxation time, each solution converges towards a self-similar solution, known as the *homogeneous cooling state*. The argument to show that this happens, and the respective rate of convergence, is based on an argument which is commonly used in nonlinear diffusion equations [15]. If one defines a temperature invariant scaling $h(v,\tau)$ of $f(v,\tau)$ as

$$h(v,\tau) = \left(\frac{T(\tau)}{3}\right)^{3/2} f\left(\left(\frac{T(\tau)}{3}\right)^{1/2} v, \tau\right), \tag{6}$$

so that $\int_{\mathbb{R}^3} |v|^2 h(v,\tau)\, \mathrm{d}v = 3$ for all $\tau \geqslant 0$, then the scaled density tends to a universal equilibrium state h_∞, and $\int_{\mathbb{R}^3} |v|^2 h_\infty(v)\, \mathrm{d}v = 3$.

The equation for $h(v,\tau)$ now reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle h,\varphi\rangle = \frac{1}{\varepsilon}\big\langle Q(h,h),\varphi\big\rangle + \frac{1-e^2}{8\varepsilon}\langle h, v\cdot\nabla\varphi\rangle. \tag{7}$$

The speed at which the second moment converges to zero depends both on the mean free path ε and on the coefficient of restitution e. On the other hand, the dependence on the mean free path can be absorbed once and for all by scaling the time. By setting

$$E = \frac{8}{1-e^2}, \tag{8}$$

and scaling the time as

$$t = \frac{1-e^2}{8\varepsilon}\tau, \tag{9}$$

we obtain for

$$g(v,t) = h\left(v, \frac{8\varepsilon}{1-e^2}t\right) \tag{10}$$

the equivalent equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle g,\varphi\rangle = E\big\langle Q(g,g),\varphi\big\rangle + \langle g, v\cdot\nabla\varphi\rangle. \tag{11}$$

In the rest of the paper, we will study Eq. (11). We remark that any result on the asymptotic convergence of $g(v,t)$ towards $g_\infty = h_\infty$ can be easily translated into a result on the asymptotic convergence of $h(v,\tau)$. It is worth remarking that thanks to (6) and (10), the initial data g_0 of a scaled solution $g(t)$ is related to the initial data f_0 of the original solution $f(\tau)$ by

$$g_0(v) = \left(\frac{T_0}{3}\right)^{3/2} f_0\left(\left(\frac{T_0}{3}\right)^{1/2} v\right)$$

and so $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$ independently of the initial temperature T_0 of f_0.

Both the large-time behavior and the regularity of the solution of Eq. (11) have been recently studied in [11]. In particular, propagation of regularity was found by controlling the growth of the frequencies of the Fourier transform of the solution by means of a precise control of the growth of the Fisher information

$$I(f)(t) = 4\int_{\mathbb{R}^3} \big|\nabla\sqrt{f(v,t)}\big|^2\,\mathrm{d}v.$$

Convergence in L^1 follows therefore by coupling the uniform bound on the regularity of the solution with the time decay of the Fourier metric [2,16]

$$d_2(f,g)(t) = \sup_{\xi\neq 0}\frac{|\hat f(\xi,t) - \hat g(\xi,t)|}{|\xi|^2}. \tag{12}$$

This strategy is popular in the field of nonlinear diffusion equations (cf. [15,3] and the references therein), where, after scaling to obtain confinement, convergence in relative entropy is used to deduce rates of convergence in more standard norms, like L^1, or in C^k by interpolation. The results in [3], which are referred to fast diffusion equations, are linked to the present problem for one additional aspect, namely the fat tails of the asymptotic profile, which possesses only moments of low order. A related problem for a linear Fokker–Planck equation with coefficient of diffusion growing with the distance from the origin is treated in [10]. There, convergence in Fourier weak norm coupled with regularity has been used to prove strong convergence to equilibrium in the physical space. Last, we recall that other weak norms can be used to control the convergence of kinetic equations towards equilibrium, among others the well-known Wasserstein metric [21]. This metric has been successfully used to control convergence towards equilibrium in Fokker–Planck models of granular media [13,14].

Going back to the results of [11], their validity requires the assumption of weak inelasticity, which corresponds to fix the coefficient of restitution e sufficiently close to 1.

A different technique which allows to control the growth of the frequencies of the Fourier transform of the solution has been recently applied in [18] to a one-dimensional dissipative model introduced by Ben-Avraham and coworkers [1]. The results in [18] are independent of the degree of dissipation. By adapting this technique to the three-dimensional case, we will end up with a result which is independent of the coefficient of restitution e. The starting point of this analysis is the results of the paper [12], where the proof of uniform propagation of regularity makes use of the following property for the solution $f(t)$ of the elastic Boltzmann equation

$$\sup_{|\xi|\geqslant R} \left|\hat{f}(\xi,t)\right| \to 0, \quad R\to +\infty \tag{13}$$

uniformly in time. This property is proved by exploiting the pointwise convergence of the Fourier transform of the solution to the Maxwellian equilibrium $\hat{M}(\xi) = e^{-|\xi|^2/2}$ and the decreasing property (with respect to $|\xi|^2$) of the Maxwellian itself. In the dissipative case, condition (13) cannot be satisfied by $f(t)$ but still holds for the solution to the scaled Boltzmann equation $g(t)$, provided the Fourier transform of the initial data satisfies

$$\left(1+\kappa|\xi|\right)^{\mu}\left|\hat{g}_0(\xi)\right| \leqslant K, \quad \xi\in\mathbb{R}^3 \tag{14}$$

for some positive constants κ, K and μ. Indeed, we prove that the solution $g(t)$ satisfies an analogous estimate

$$\left(1+\kappa|\xi|\right)^{\mu}\left|\hat{g}(\xi,t)\right| \leqslant K, \quad \xi\in\mathbb{R}^3 \tag{15}$$

uniformly in time, for possibly different κ, K and μ. Condition (15) is difficult to prove directly from the equation satisfied by the scaled density $g(t)$, due to the presence of the drift term in Eq. (11). The key point of our approach is to consider a semi-implicit discretization of Eq. (11), where the drift term is absorbed in an integral term which is much easier to handle. The same trick has been successfully exploited in the one-dimensional case in [18] where it has been also pointed out that an equivalent approach could have been through the Duhamel formula for the solution $g(t)$ (see e.g. [16]).

Our main results are summarized into the following statements.

Theorem 1. *Assume* $e\in[0,1)$. *Let* $g(t)$ *be the weak solution of Eq.* (11)*, corresponding to the initial probability density* g_0 *with zero mean velocity,* $\int_{\mathbb{R}^3}|v|^2 g_0(v)\,\mathrm{d}v = 3$ *and such that* $\int_{\mathbb{R}^3}|v|^4 g_0(v)\,\mathrm{d}v < +\infty$. *If in addition*

$$\left|\hat{g}_0(\xi)\right| \leqslant \frac{1}{(1+\beta|\xi|)^{\nu}}, \quad |\xi|>R, \tag{16}$$

for some $R>0$, $\nu>0$ *and* $\beta>0$, *then there exist* $\rho>0$, $k>0$, $\nu'>0$ *such that* $g(t)$ *satisfies*

$$\left|\hat{g}(\xi,t)\right| \leqslant \begin{cases} \dfrac{1}{1+k|\xi|^2}, & |\xi|\leqslant\rho,\ t\geqslant 0,\\[2mm] \dfrac{1}{(1+\beta|\xi|)^{\nu'}}, & |\xi|>\rho,\ t\geqslant 0. \end{cases} \tag{17}$$

Theorem 2. *Assume* $e\in[0,1)$. *Let* $g(t)$ *be the weak solution of Eq.* (11)*, corresponding to the initial probability density* g_0 *with zero mean velocity,* $\int_{\mathbb{R}^3}|v|^2 g_0(v)\,\mathrm{d}v = 3$ *and satisfying* $\int_{\mathbb{R}^3}|v|^4 g_0(v)\,\mathrm{d}v < +\infty$. *Let us suppose moreover* $g_0\in\dot{H}^{\eta}(\mathbb{R}^3)$ *for some* $\eta>0$ *and* $\sqrt{g_0}\in\dot{H}^{\nu}(\mathbb{R}^3)$ *for some* $\nu>0$. *Then* $g(t)$ *converges strongly in* L^1 *with an exponential rate towards the stationary solution* g_∞*, i.e., there exist positive constants* C *and* γ *explicitly computable such that*

$$\left\|g(t)-g_\infty\right\|_{L^1(\mathbb{R}^3)} \leqslant Ce^{-\gamma t}, \quad t\geqslant 0.$$

Thanks to the scaling invariance of the L^1-norm, Theorem 2 allows to deduce also the strong convergence of the original non-scaled solution $f(\tau)$ to the self-similar state

$$\left\|f(\tau)-f_\infty(\tau)\right\|_{L^1(\mathbb{R}^3)} \leqslant Ce^{-\gamma\frac{1-e^2}{8\varepsilon}\tau}, \quad \tau\geqslant 0,$$

where

$$f_\infty(v,\tau)=\left(\frac{3}{T(\tau)}\right)^{3/2} g_\infty\left(\left(\frac{3}{T(\tau)}\right)^{1/2} v\right)$$

for $T(\tau)$ as in (5) and this independently of the initial temperature T_0 of f_0.

2. Preliminary results

Following Bobylev [4] it is convenient to rewrite Eq. (1) for Maxwell molecules in the dissipative case in the Fourier variables:

$$\frac{\partial}{\partial\tau}\hat f(\xi,\tau)=\frac{1}{4\pi\varepsilon}\int\limits_{\sigma\in S^2}\left(\hat f\left(\xi^+,\tau\right)\hat f\left(\xi^-,\tau\right)-\hat f(\xi,\tau)\hat f(\xi,0)\right)\mathrm{d}\sigma, \tag{18}$$

where

$$\begin{aligned}\xi^+&=\frac{3-e}{4}\xi+\frac{1+e}{4}|\xi|\sigma,\\ \xi^-&=\frac{1+e}{4}\left(\xi-|\xi|\sigma\right)=\xi-\xi^+.\end{aligned} \tag{19}$$

The existence and uniqueness of a solution of (18) for any initial data f_0 satisfying (4) can be established through the application of Wild sums [16].

Theorem 3 *(Theorem of existence and uniqueness [4,16]). We consider $f_0\geqslant 0$ satisfying the normalization conditions* (4) *and the following Cauchy problem*:

$$\begin{cases}\dfrac{\partial}{\partial\tau}\hat f(\xi,\tau)=\dfrac{1}{4\pi\varepsilon}\displaystyle\int\limits_{\sigma\in S^2}\left(\hat f\left(\xi^+,\tau\right)\hat f\left(\xi^-,\tau\right)-\hat f(\xi,\tau)\hat f(\xi,0)\right)\mathrm{d}\sigma, & \tau>0,\\ \hat f(\xi,0)=\hat f_0(\xi).\end{cases} \tag{20}$$

Then, there exists a unique nonnegative solution $f\in C^1([0,+\infty),L^1(\mathbb{R}^3))$ to Eq. (20). *This solution preserves mass and momentum, while the energy decays at the exponential rate given by* (5).

Let $g(v,t)$ be defined by (10). One can easily show that $g(v,t)$ preserves the temperature, and moreover

$$\int\limits_{\mathbb{R}^3}|v|^2 g(v,t)\,\mathrm{d}v=3,\quad t\geqslant 0.$$

Moreover if the initial data g_0 has a diagonal pressure tensor which is unitary

$$\int\limits_{\mathbb{R}^3} v_i v_j g_0(v)\,\mathrm{d}v=\delta_{i,j},\quad i,j=1,2,3, \tag{21}$$

then all the second moments are also preserved. If on the contrary the pressure tensor is not unitary, then its non-isotropic part ($\int_{\mathbb{R}^3} v_i v_j g(v,t)\,\mathrm{d}v$ for $i\neq j$) vanishes if it is initially vanishing (cf. [2]), whereas the isotropic part ($\int_{\mathbb{R}^3} v_i^2 g(v,t)\,\mathrm{d}v$) is in general not preserved. Nevertheless, the condition $\int_{\mathbb{R}^3}|v|^2 g_0(v)\,\mathrm{d}v=3$ implies that the matrix $[\int_{\mathbb{R}^3} v_i v_j g_0(v)\,\mathrm{d}v]_{i,j}$ is real and symmetric. Consequently there exists a suitable orthonormal system in $\mathbb{R}^3$ in which it is diagonal and therefore it remains diagonal for all $t>0$. Owing to this property, throughout this paper we will assume, without any additional assumption, that g_0 has diagonal pressure tensor, although not unitary.

It is well known that in Maxwell models the time evolution of moments can be evaluated exactly. In particular, this can be done for the diagonal terms $\int_{\mathbb{R}^3} v_i^2 g(v,t)\,\mathrm{d}v$. We have

Proposition 4. *Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of Eq.* (11)*, corresponding to the initial probability density g_0 with zero mean velocity, diagonal pressure tensor and satisfying $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$. Then, we have*

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2 g(v,t)\,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-\frac{1+e}{1-e}t} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-\frac{1+e}{1-e}t}, \quad t \geqslant 0, \tag{22}$$

where $C_1 = \int_{\mathbb{R}^3} v_2^2 g_0(v)\,\mathrm{d}v - 1$ and $C_2 = \int_{\mathbb{R}^3} v_3^2 g_0(v)\,\mathrm{d}v - 1$.

Proof. Recalling Eq. (11), the expression (3) of the post-collisional variables, the conservations of the mass and the vanishing both of the momentum and of the non-isotropic terms $\int_{\mathbb{R}^3} v_i v_j g(v,t)\,\mathrm{d}v$ along the solution, we obtain the following linear differential system:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \int_{\mathbb{R}^3} v_1^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2 g(v,t)\,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & -\frac{2}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & -\frac{2}{3}\frac{1+e}{1-e} \end{bmatrix} \begin{bmatrix} \int_{\mathbb{R}^3} v_1^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2 g(v,t)\,\mathrm{d}v \end{bmatrix}.$$

Since the matrix of coefficients has 0 as simple eigenvalue and $-\frac{1+e}{1-e}$ as double eigenvalue, with eigenspaces $\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$ and $\left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$ respectively, the general solution of the system is

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_2^2 g(v,t)\,\mathrm{d}v \\ \int_{\mathbb{R}^3} v_3^2 g(v,t)\,\mathrm{d}v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-\frac{1+e}{1-e}t} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-\frac{1+e}{1-e}t}, \quad t \geqslant 0,$$

for C_1, C_2 real constants to be determined by the initial conditions. □

In Fourier variables, the function $g(t)$ satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{g}(\xi,t) - (\xi\cdot\nabla_\xi)\hat{g}(\xi,t) = \dfrac{E}{4\pi} \displaystyle\int_{\sigma\in S^2} \big(\hat{g}(\xi^+,t)\hat{g}(\xi^-,t) - \hat{g}(\xi,t)\hat{g}(\xi,0)\big)\,\mathrm{d}\sigma, \\ \hat{g}(\xi,0) = \hat{g}_0(\xi). \end{cases} \tag{23}$$

We will denote

$$Q_+(\hat{g},\hat{g})(\xi,t) = \frac{1}{4\pi}\int_{\sigma\in S^2} \hat{g}(\xi^+,t)\hat{g}(\xi^-,t)\,\mathrm{d}\sigma.$$

Hence Eq. (23) can be written

$$\frac{\partial}{\partial t}\hat{g}(\xi,t) - (\xi\cdot\nabla_\xi)\hat{g}(\xi,t) = E\big(Q_+(\hat{g},\hat{g})(\xi,t) - \hat{g}(\xi,t)\big).$$

For $s > 0$ let $\mathcal{P}_s(\mathbb{R}^3)$ define the set of probability densities satisfying

$$\int_{\mathbb{R}^3} |v|^s f(v)\,\mathrm{d}v < +\infty. \tag{24}$$

Consider on $\mathcal{P}_s(\mathbb{R}^3)$ the distance

$$d_s(f,g) = \sup_{\xi\neq 0} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^s}.$$

It is not difficult to show through a Taylor expansion that $d_s(f,g)$ is finite for any pair of probability densities f and g with equal moments $\int_{\mathbb{R}^3} v^\beta f(v)\,\mathrm{d}v$ for any multi-index $\beta \in \mathbb{N}^3$ of length smaller than or equal to $[s]$ (if $s \in \mathbb{N}$, it is enough to suppose equal moments up to the $(s-1)$-th order). For $\alpha \in (0,1]$ and initial data $g_1, g_2 \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ which share the same moments up to the second order and have unitary pressure tensor as in (21), it is possible to estimate the $d_{2+\alpha}$ distance between the two solutions as follows.

Theorem 5 *(Strict contraction of $d_{2+\alpha}$ [2]). Assume $e \in [0,1)$. For $\alpha \in (0,1]$ there exists an explicit constant $C(\alpha,e) > 0$, $C(\alpha,e) \searrow 0$ as $\alpha \to 0$, such that for any $g_1(t)$ and $g_2(t)$ solutions of* (11) *corresponding to initial values g_1^0, g_2^0 in $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with unit mass, zero mean velocity and unitary pressure tensor, then*

$$d_{2+\alpha}\big(g_1(t), g_2(t)\big) \leqslant d_{2+\alpha}\big(g_1^0, g_2^0\big)e^{-C(\alpha,e)t}, \quad t \geqslant 0.$$

In our framework, the constant $C(\alpha,e)$ has the following expression:

$$C(\alpha,e) = E\big(1 - A(\alpha,e)\big) - (2+\alpha) \tag{25}$$

where

$$A(\alpha,e) = \frac{1}{4\pi}\int\limits_{S^2} \frac{|\xi^+|^{2+\alpha} + |\xi^-|^{2+\alpha}}{|\xi|^{2+\alpha}}\,\mathrm{d}\sigma. \tag{26}$$

A detailed analysis of $C(\alpha,e)$ can be found in [2].

It is worth noticing that we cannot deduce from the previous theorem the uniform boundedness in time neither of $\int_{\mathbb{R}^3} |v|^{2+\alpha} g_1(v,t)\,\mathrm{d}v$ nor of $\int_{\mathbb{R}^3} |v|^{2+\alpha} g_2(v,t)\,\mathrm{d}v$. Nevertheless, if the initial data g_0 belongs to $\mathcal{P}_4(\mathbb{R}^3)$, it is possible to prove that the solution keeps on satisfying the same property uniformly in time.

Theorem 6 *(Uniform control of 4th moment [9]). If g_0 is a Borel probability density in $\mathcal{P}_4(\mathbb{R}^3)$ then the solution $g(t)$ to* (11) *with initial data g_0 belongs to $\mathcal{P}_4(\mathbb{R}^3)$ for all $t \geqslant 0$ and*

$$\sup_{t\geqslant 0} \int\limits_{\mathbb{R}^3} |v|^4 g(v,t)\,\mathrm{d}v < \infty.$$

Thanks to the uniform boundedness of the fourth moment, it is possible to prove, via a fixed point argument, the existence of a universal stationary state in a suitable subspace of $\mathcal{P}_2(\mathbb{R}^3)$. Moreover, the strict contraction of the $d_{2+\alpha}$ metric allows also to prove the stability of this stationary state in this subspace.

Theorem 7 *(Stationary states [5,2,16]). Let $e \in [0,1)$ be fixed. Eq.* (11) *has a unique stationary state g_∞ which is a probability density in*

$$\mathcal{H} = \Bigg\{ g \geqslant 0,\ \int\limits_{\mathbb{R}^3} g(v)\,\mathrm{d}v = 1,\ \int\limits_{\mathbb{R}^3} v_i\, g(v)\,\mathrm{d}v = 0,\ \int\limits_{\mathbb{R}^3} v_i v_j g(v)\,\mathrm{d}v = \delta_{i,j},\ i,j = 1,2,3,$$
$$\int\limits_{\mathbb{R}^3} |v|^{2+\alpha} g(v)\,\mathrm{d}v < +\infty,\ \textit{for } \alpha \in (0,1] \Bigg\}.$$

This stationary state is a radial function and belongs to $\mathcal{P}_4(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} |v|^4 g_\infty(v)\,\mathrm{d}v \leqslant M_4$, where M_4 depends only on $e \in [0,1)$. Moreover, for $\alpha \in (0,1]$, given any $g(t)$ solution of (11) *issued from an initial datum $g_0 \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with unit mass, zero mean velocity and unitary pressure tensor, then*

$$d_{2+\alpha}\big(g(t), g_\infty\big) \leqslant d_{2+\alpha}(g_0, g_\infty)e^{-C(\alpha,e)t}, \quad t \geqslant 0,$$

where $C(\alpha,e)$ is the constant (25).

The hypothesis of unitary pressure tensor can in fact be removed. Under the only assumption that the initial datum belongs to $\mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$ (and same moments up to the first order as the stationary state), the d_2 distance between the solution and the stationary state is proved to be exponentially decreasing.

Theorem 8 *(Stability in d_2 without unitary tensor pressure [2,16]). Let $e \in [0,1)$ be fixed. For any $\alpha \in (0,1]$ and any initial datum $g_0 \in \mathcal{P}_{2+\alpha}(\mathbb{R}^3)$ with unit mass, zero mean velocity and $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$, there exist positive constants K_1 and K_2 depending on e, α, g_0 such that given $g(t)$ the solution of (23) issued from g_0 and g_∞ the stationary state, then*

$$d_2\big(g(t), g_\infty\big) \leqslant K_1 e^{-K_2 t}, \quad t \geqslant 0.$$

As a consequence, the uniqueness of the stationary state holds in a larger space where the second moments are not prescribed.

Corollary 9 *(Uniqueness of stationary states [5,16]). The uniqueness of the stationary state g_∞ found in Theorem 7 holds true in*

$$\tilde{\mathcal{H}} = \Bigg\{ g \geqslant 0, \int_{\mathbb{R}^3} g(v)\,\mathrm{d}v = 1, \int_{\mathbb{R}^3} v_i g(v)\,\mathrm{d}v = 0,\ i = 1,2,3, \int_{\mathbb{R}^3} |v|^2 g(v)\,\mathrm{d}v = 3, \int_{\mathbb{R}^3} |v|^{2+\alpha} g(v)\,\mathrm{d}v < +\infty, \textit{ for } \alpha \in (0,1] \Bigg\}.$$

As far as the smoothness of the stationary state is concerned, it is known that $g_\infty \in H^s(\mathbb{R}^3)$, for all $s \geqslant 0$. This is a consequence of the following result.

Theorem 10 *(Smoothness of the stationary state [5], Theorem 5.3). For $e \in [0,1)$, the stationary state g_∞ satisfies the bounds*

$$e^{-\frac{|\xi|^2}{2}} \leqslant \big|\hat{g}_\infty(\xi)\big| \leqslant \big(1 + |\xi|\big) e^{-|\xi|}, \quad \xi \in \mathbb{R}^3.$$

3. The iteration process

The goal of this section is to build up a sequence of functions $\{g^N(\xi,t)\}$ which approximates uniformly the solution $\hat{g}(\xi,t)$. In order to do this, for any fixed $T > 0$ we consider first a semi-implicit discretization in time of Eq. (23) by partitioning the interval $[0,T]$ into N subintervals and we define thus the approximate solution at any time $t = j\frac{T}{N}$ for $j = 0,\dots,N$. Second, we define $g^N(\xi,t)$ on the whole interval $[0,T]$ by interpolation and last we show the convergence of the approximation to the solution. Other details and the missing proofs of this section can be found in [18].

We begin by introducing a semi-implicit discretization in time of Eq. (23) as follows. Let $T > 0$ and $\Delta t = \frac{T}{N}$ for $N \in \mathbb{N}$, $N > T$. Let $\hat{\phi}_j^N(\xi)$, $j = 0,\dots,N$, be the sequence:

$$\begin{cases} \hat{\phi}_0^N(\xi) = \hat{g}_0(\xi), \\ \dfrac{\hat{\phi}_{j+1}^N(\xi) - \hat{\phi}_j^N(\xi)}{\Delta t} = \xi \cdot \nabla_\xi \hat{\phi}_{j+1}^N + E\big(Q_+\big(\hat{\phi}_j^N, \hat{\phi}_j^N\big)(\xi) - \hat{\phi}_j^N(\xi)\big), \quad j = 0,\dots,N-1. \end{cases} \tag{27}$$

Proposition 11. *Assume $e \in [0,1)$. If g_0 is a probability density with zero mean velocity satisfying $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$, then there exists a unique sequence of bounded functions $\hat{\phi}_j^N$ for $j = 1,\dots,N$, satisfying (27). This sequence is defined as follows*

$$\begin{cases} \hat{\phi}_0^N(\xi) = \hat{g}_0(\xi), \\ \hat{\phi}_{j+1}^N(\xi) = \dfrac{1}{\Delta t} \displaystyle\int_1^{+\infty} \big(E\Delta t\, Q_+\big(\hat{\phi}_j^N, \hat{\phi}_j^N\big)(\eta\xi) + (1 - E\Delta t)\hat{\phi}_j^N(\eta\xi)\big) \frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}, \quad j = 0,\dots,N-1. \end{cases} \tag{28}$$

Proof. Let us begin by proving that $\hat{\phi}_1^N$ is well defined. We proceed in the same way as in Proposition 7 in [18]. For $\xi \neq 0$ we multiply Eq. (27) by $(-\frac{1}{\Delta t})|\xi|^{-\frac{1}{\Delta t}-1}$ to obtain

$$\left(-\frac{1}{\Delta t}\right)|\xi|^{-\frac{1}{\Delta t}-1}\hat{\phi}_1^N(\xi) + |\xi|^{-\frac{1}{\Delta t}}\frac{\xi}{|\xi|}\cdot\nabla_\xi\hat{\phi}_1^N(\xi)$$
$$= \left(-\frac{1}{\Delta t}\right)|\xi|^{-\frac{1}{\Delta t}-1}\big(E\Delta t Q_+(\hat{g}_0,\hat{g}_0)(\xi) + (1-E\Delta t)\hat{g}_0(\xi)\big),$$

or, what is the same

$$\frac{\partial}{\partial|\xi|}\big(|\xi|^{-\frac{1}{\Delta t}}\hat{\phi}_1^N\big)(\xi) = \left(-\frac{1}{\Delta t}\right)|\xi|^{-\frac{1}{\Delta t}-1}\big(E\Delta t Q_+(\hat{g}_0,\hat{g}_0)(\xi) + (1-E\Delta t)\hat{g}_0(\xi)\big).$$

In spherical coordinates, $\xi = (|\xi|\cos\theta\sin\psi, |\xi|\sin\theta\sin\psi, |\xi|\cos\psi)$, where $\theta \in [0, 2\pi)$ and $\psi \in [0, \pi]$. Since $\hat{g}_0(\xi)$ is bounded, integrating over $[|\xi|, +\infty)$ we get

$$|\xi|^{-\frac{1}{\Delta t}}\hat{\phi}_1^N(\xi) = \frac{1}{\Delta t}\int_{|\xi|}^{+\infty}\big(E\Delta t Q_+(\hat{g}_0,\hat{g}_0)(s,\theta,\psi) + (1-E\Delta t)\hat{g}_0(s,\theta,\psi)\big)s^{-\frac{1}{\Delta t}-1}\,\mathrm{d}s.$$

Since

$$\big(\eta|\xi|\cos\theta\sin\psi, \eta|\xi|\sin\theta\sin\psi, \eta|\xi|\cos\psi\big) = \eta\xi,$$

through the change of variables $\eta = \frac{s}{|\xi|}$ we finally obtain

$$\hat{\phi}_1^N(\xi) = \frac{1}{\Delta t}\int_1^{+\infty}\big(E\Delta t Q_+(\hat{g}_0,\hat{g}_0)(\eta\xi) + (1-\Delta t)\hat{g}_0(\eta\xi)\big)\frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}. \tag{29}$$

Since the initial density g_0 has unit mass, zero mean velocity and bounded temperature, then $\hat{g}_0$ belongs to $\mathcal{C}^1(\mathbb{R}^3)$ and there exists $C > 0$ such that

$$\hat{g}_0(0) = 1, \qquad \big|\partial_k\hat{g}_0(\xi)\big| \leqslant C, \qquad \partial_k\hat{g}_0(0) = 0, \quad k = 1, 2, 3. \tag{30}$$

Therefore the function $\hat{\phi}_1^N$ defined by continuity in $\xi = 0$ as $\hat{\phi}_1^N(0) = 1$ is the unique bounded and $\mathcal{C}^1(\mathbb{R}^3)$ solution of (27).

By an iteration argument the same conclusion holds for any $\hat{\phi}_j^N$. Hence, for $j = 0, \ldots, N-1$ it holds

$$\hat{\phi}_{j+1}^N(\xi) = \frac{1}{\Delta t}\int_1^{+\infty}\big(E\Delta t Q_+\big(\hat{\phi}_j^N,\hat{\phi}_j^N\big)(\eta\xi) + (1-E\Delta t)\hat{\phi}_j^N(\eta\xi)\big)\frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}. \qquad \square$$

Remark 12. We remark that Fubini's theorem implies that any function $\hat{\phi}_{j+1}^N(\xi)$ is the Fourier transform of $\phi_{j+1}^N(v)$, where for $j = 0, \ldots, N-1$ and $v \in \mathbb{R}^3$

$$\begin{cases} \phi_0^N(v) = g_0(v), \\ \phi_{j+1}^N(v) = \dfrac{1}{\Delta t}\displaystyle\int_1^{+\infty}\left(E\Delta t\frac{1}{\eta^3}Q_+\big(\phi_j^N,\phi_j^N\big)\left(\frac{v}{\eta}\right) + (1-E\Delta t)\frac{1}{\eta^3}\phi_j^N\left(\frac{v}{\eta}\right)\right)\frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}}. \end{cases} \tag{31}$$

In the following proposition we gather the results on the moments of the approximation ϕ_j^N which will be used in the proof of Theorem 1. For the sake of clarity, the proof of the proposition will be postponed to Appendix A.

Proposition 13. *Assume $e \in [0, 1)$ and let g_0 be a probability density in $\mathcal{P}_4(\mathbb{R}^3)$ with zero mean velocity, satisfying*

$$\int_{\mathbb{R}^3} v_i v_k g_0(v)\,\mathrm{d}v = 0, \quad i \neq k, \qquad \int_{\mathbb{R}^3}|v|^2 g_0(v)\,\mathrm{d}v = 3. \tag{32}$$

Let ϕ_j^N be the approximation defined in (31). *Then, there exists $C>0$ such that for N large enough, for $j=0,\dots,N$ it holds*

$$\phi_j^N(v)\geqslant 0,\qquad \int_{\mathbb{R}^3}\phi_j^N(v)\,\mathrm{d}v=1,\qquad \int_{\mathbb{R}^3}v_k\phi_j^N(v)\,\mathrm{d}v=0,\quad k=1,2,3, \tag{33}$$

$$\int_{\mathbb{R}^3}v_iv_k\phi_j^N(v)\,\mathrm{d}v=0,\quad i\neq k, \tag{34}$$

$$\begin{bmatrix}\int_{\mathbb{R}^3}v_1^2\phi_j^N(v)\,\mathrm{d}v\\ \int_{\mathbb{R}^3}v_2^2\phi_j^N(v)\,\mathrm{d}v\\ \int_{\mathbb{R}^3}v_3^2\phi_j^N(v)\,\mathrm{d}v\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}+C_1\begin{bmatrix}-1\\1\\0\end{bmatrix}\left(\frac{1-\Delta t\frac{3-e}{1-e}}{1-2\Delta t}\right)^j+C_2\begin{bmatrix}-1\\0\\1\end{bmatrix}\left(\frac{1-\Delta t\frac{3-e}{1-e}}{1-2\Delta t}\right)^j, \tag{35}$$

$$\int_{\mathbb{R}^3}|v|^4\phi_j^N(v)\,\mathrm{d}v\leqslant C, \tag{36}$$

where $C_1=\int_{\mathbb{R}^3}v_2^2g_0(v)\,\mathrm{d}v-1$ and $C_2=\int_{\mathbb{R}^3}v_3^2g_0(v)\,\mathrm{d}v-1$.

We define the approximate solution $g^N(\xi,t)$ at any time $t=j\frac{T}{N}$ as $g^N(\xi,j\frac{T}{N})=\hat\phi_j^N(\xi)$ for $j=0,\dots,N$. We extend afterwards the definition on the whole interval by interpolation. More precisely, let us define

$$g^N(\xi,t)=\begin{cases}\hat g_0(\xi), & t=0,\\ \alpha(t)\hat\phi^N_{K_N-1}(\xi)+\big(1-\alpha(t)\big)\hat\phi^N_{K_N}(\xi), & 0<t\leqslant T,\end{cases}$$

where for $0<t\leqslant T$ and $K_N\in\{1,\dots,N\}$ we have $(K_N-1)\frac{T}{N}<t\leqslant K_N\frac{T}{N}$. Thus there is a function $0\leqslant\alpha(t)<1$ such that $t=\alpha(t)(K_N-1)\frac{T}{N}+(1-\alpha(t))K_N\frac{T}{N}$. Any $g^N(\xi,t)$ is continuous on $\mathbb{R}^3\times[0,T]$ and for any $t\in[0,T]$ it belongs to $\mathcal{C}^2(\mathbb{R}^3)$.

The result of convergence is therefore as follows.

Proposition 14. *There is a subsequence $\{g^{N_l}(\xi,t)\}_l$ of $\{g^N(\xi,t)\}_N$ which converges uniformly on any compact set of $\mathbb{R}^3\times[0,T]$ to the solution $\hat g(\xi,t)$.*

4. Control of tails of the Fourier transform of the solution

In this section we prove Theorem 1. Thanks to the uniform convergence of a subsequence of the approximate solutions $g^N(\xi,t)$ to the solution $\hat g(\xi,t)$ and to the definition of $g^N(\xi,t)$, it is enough to prove bounds (17) for any $\hat\phi_j^N(\xi)$ uniformly for $N\in\mathbb{N}$ large enough and $j=0,\dots,N$. We remark that the control of low frequencies follows directly from properties (4) and from the boundedness of the fourth moment of the initial data.

Lemma 15. *Assume $e\in[0,1)$. Let $g(t)$ be the weak solution of Eq.* (11)*, corresponding to the initial probability density $g_0\in\mathcal{P}_4(\mathbb{R}^3)$ with zero mean velocity and $\int_{\mathbb{R}^3}|v|^2g_0(v)\,\mathrm{d}v=3$. Let $\hat\phi_j^N$ be the approximation defined in* (28)*. There exist $k>0$ and $\rho>0$ such that for any fixed $T>0$ and any $N\in\mathbb{N}$ large enough we get*

$$\big|\hat g(\xi,t)\big|\leqslant\frac{1}{1+k|\xi|^2},\quad |\xi|\leqslant\rho,\ t\geqslant 0, \tag{37}$$

$$\big|\hat\phi_j^N(\xi)\big|\leqslant\frac{1}{1+k|\xi|^2},\quad |\xi|\leqslant\rho,\ j=0,\dots,N. \tag{38}$$

Proof. Both estimates will be achieved through a MacLaurin expansion in which all terms will be bounded uniformly thanks to Proposition 4, Theorem 6 and Proposition 13. In what follows, constants may vary from one line to another.

Let us begin by proving (37). From the hypotheses on g_0 and Proposition 4, the MacLaurin expansion in Fourier variables reads

$$\hat{g}(\xi,t) = 1 - \frac{1}{2}\sum_{k=1}^{3}\left(\int_{\mathbb{R}^3} v_k^2 g(v,t)\,\mathrm{d}v\right)\xi_k^2 + \int_0^1 D^3\hat{g}(s\xi,t)(\xi,\xi,\xi)\,\mathrm{d}s, \quad \xi\in\mathbb{R}^3,\ t\geqslant 0.$$

For all $i,j,k = 1,2,3$ and all $t\geqslant 0$ the conservation of the mass and the uniform boundedness of the fourth moment (Theorem 6) implies

$$\int_{\mathbb{R}^3} |v_i v_j v_k|\, g(v,t)\,\mathrm{d}v \leqslant C < +\infty.$$

On the other hand, for all $i,j,k=1,2,3$ and all $t\geqslant 0$

$$\left|\partial^3_{ijk}\hat{g}(\xi,t)\right| \leqslant C\int_{\mathbb{R}^3}|v_i v_j v_k|\, g(v,t)\,\mathrm{d}v.$$

Hence we have the uniform upper bound

$$\left|\int_0^1 D^3\hat{g}(s\xi,t)(\xi,\xi,\xi)\,\mathrm{d}s\right| \leqslant C|\xi|^3, \quad \xi\in\mathbb{R}^3,\ t\geqslant 0.$$

The time evolution of the second moments of $g(t)$ obtained in Proposition 4 implies a strictly positive uniform lower bound on $\int_{\mathbb{R}^3} v_k^2\, g(v,t)\,\mathrm{d}v$ for $k=1,2,3$ and $t\geqslant 0$. Therefore, for any $\xi\in\mathbb{R}^3$ and $t\geqslant 0$ we have the estimate

$$\left|\hat{g}(\xi,t)\right| \leqslant 1 - C|\xi|^2 + D|\xi|^3, \tag{39}$$

with C and D suitably chosen. Inequality (39) implies that there exist $\rho>0$ and $k>0$ such that

$$\left|\hat{g}(\xi,t)\right| \leqslant \frac{1}{1+k\xi^2}, \quad |\xi|\leqslant\rho,\ t\geqslant 0.$$

To prove (38), thanks to the uniform estimates collected in Proposition 13, we can proceed exactly in the same way. □

We are now in position to prove Theorem 1.

Theorem 1. *Assume $e\in[0,1)$. Let $g(t)$ be the weak solution of Eq.* (11)*, corresponding to the initial probability density $g_0\in\mathcal{P}_4(\mathbb{R}^3)$ with zero mean velocity and $\int_{\mathbb{R}^3}|v|^2 g_0(v)\,\mathrm{d}v = 3$. If in addition*

$$\left|\hat{g}_0(\xi)\right| \leqslant \frac{1}{(1+\beta|\xi|)^{\nu}}, \quad |\xi|>R, \tag{40}$$

for some $R>0$, $\nu>0$ and $\beta>0$, then there exist $\rho>0$, $k>0$, $\nu'>0$ such that $g(t)$ satisfies

$$\left|\hat{g}(\xi,t)\right| \leqslant \begin{cases} \dfrac{1}{1+k|\xi|^2}, & |\xi|\leqslant\rho,\ t\geqslant 0,\\[2mm] \dfrac{1}{(1+\beta|\xi|)^{\nu'}}, & |\xi|>\rho,\ t\geqslant 0. \end{cases} \tag{41}$$

Proof. The bound on the low frequencies $|\xi|\leqslant\rho$ has been established in Lemma 15. Moreover, in consequence of Proposition 3.3 in [17], we can suppose that condition (40) holds for any $|\xi|>\rho$ (with a possibly smaller exponent ν^*). We will now prove that, for any $N\in\mathbb{N}$ and $j=0,\dots,N$,

$$\left|\hat{\phi}_j^N(\xi)\right| \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}, \quad |\xi|>\rho,$$

for a positive constant ν' small enough. By induction we have only to check the bound on

$$\hat{\phi}_1^N(\xi) = \frac{1}{\Delta t}\int_{\eta=1}^{+\infty}\big(E\Delta t\, Q_+(\hat{g}_0,\hat{g}_0)(\eta\xi) + (1-E\Delta t)\hat{g}_0(\eta\xi)\big)\frac{d\eta}{\eta^{\frac{1}{\Delta t}+1}}.$$

Let $|\xi| > \rho$ and $\eta > 1$. We start to bound the term

$$Q_+(\hat{g}_0,\hat{g}_0)(\eta\xi) = \frac{1}{4\pi}\int_{S^2}\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\,d\sigma,$$

where $\tilde{\xi} = \eta\xi$. Since $|\eta| > 1$, we have $|\tilde{\xi}| = \eta|\xi| > |\xi| > \rho$. We recall that for $\sigma \in S^2$

$$\tilde{\xi}^+ = \frac{3-e}{4}\tilde{\xi} + \frac{1+e}{4}|\tilde{\xi}|\sigma,$$
$$\tilde{\xi}^- = \frac{1+e}{4}\big(\tilde{\xi} - |\tilde{\xi}|\sigma\big),$$

so that $|\tilde{\xi}^+ + \tilde{\xi}^-| = |\tilde{\xi}|$. Therefore

$$\big|\tilde{\xi}^+\big| + \big|\tilde{\xi}^-\big| \geqslant |\tilde{\xi}|, \tag{42}$$

and so either $|\tilde{\xi}^+| \geqslant \frac{|\tilde{\xi}|}{2}$ or $|\tilde{\xi}^-| \geqslant \frac{|\tilde{\xi}|}{2}$. Moreover, for any $\sigma \in S^2$

$$\big|\tilde{\xi}^+\big|^2 \geqslant |\tilde{\xi}|^2\left(\frac{1-e}{2}\right)^2 \geqslant |\xi|^2\left(\frac{1-e}{2}\right)^2. \tag{43}$$

The surface S^2 can be split into six non-overlapping domains

$$S^2 = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5,$$

where

$$A_0 = \big\{\sigma \in \mathcal{S}^2:\ \big|\tilde{\xi}^+\big| > \rho,\ \big|\tilde{\xi}^-\big| > \rho\big\},$$
$$A_1 = \big\{\sigma \in \mathcal{S}^2:\ \big|\tilde{\xi}^+\big| \leqslant \rho,\ \big|\tilde{\xi}^-\big| > \rho\big\},$$
$$A_2 = \left\{\sigma \in \mathcal{S}^2:\ \big|\tilde{\xi}^-\big| \leqslant \rho,\ \big|\tilde{\xi}^-\big| \leqslant \frac{|\tilde{\xi}|}{2},\ \big|\tilde{\xi}^+\big| > \frac{|\tilde{\xi}|}{2}\right\},$$
$$A_3 = \left\{\sigma \in \mathcal{S}^2:\ \big|\tilde{\xi}^-\big| \leqslant \rho,\ \big|\tilde{\xi}^-\big| > \frac{|\tilde{\xi}|}{2},\ \big|\tilde{\xi}^+\big| > \frac{|\tilde{\xi}|}{2},\ \big|\tilde{\xi}^+\big| \leqslant \rho\right\},$$
$$A_4 = \left\{\sigma \in \mathcal{S}^2:\ \big|\tilde{\xi}^-\big| \leqslant \rho,\ \big|\tilde{\xi}^-\big| > \frac{|\tilde{\xi}|}{2},\ \big|\tilde{\xi}^+\big| > \frac{|\tilde{\xi}|}{2},\ \big|\tilde{\xi}^+\big| > \rho\right\},$$
$$A_5 = \left\{\sigma \in \mathcal{S}^2:\ \big|\tilde{\xi}^-\big| \leqslant \rho,\ \big|\tilde{\xi}^+\big| \leqslant \frac{|\tilde{\xi}|}{2}\right\}.$$

We use the aforementioned decomposition to estimate

$$\big|Q_+(\hat{g}_0,\hat{g}_0)(\eta\xi)\big| \leqslant \frac{1}{4\pi}\sum_{i=0}^{5}\int_{A_i}\big|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\big|\,d\sigma.$$

If $\sigma \in A_0$, inequality (42) gives

$$\big|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\big| \leqslant \frac{1}{(1+\beta|\tilde{\xi}^+|)^{\nu'}}\frac{1}{(1+\beta|\tilde{\xi}^-|)^{\nu'}} \leqslant \frac{1}{(1+\beta(|\tilde{\xi}^+|+|\tilde{\xi}^-|))^{\nu'}} \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Hence

$$\int_{A_0}\big|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\big|\,d\sigma \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}|A_0|, \tag{44}$$

where $|A_0|$ denotes the Lebesgue measure of A_0. Let us choose now $\sigma \in A_1$. Thanks both to inequality (43) and to the lower bound $|\tilde{\xi}^-| \geqslant \frac{|\tilde{\xi}|}{2}$ (which holds true since $|\tilde{\xi}^-| \geqslant |\tilde{\xi}^+|$), we obtain

$$\begin{aligned}\left|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\right| &\leqslant \frac{1}{1+k|\tilde{\xi}^+|^2}\frac{1}{(1+\beta|\tilde{\xi}^-|)^{\nu'}} \leqslant \frac{1}{1+k(\frac{1-e}{2})^2|\xi|^2}\frac{1}{(1+\frac{\beta}{2}|\tilde{\xi}|)^{\nu'}}\\ &\leqslant \frac{1}{1+k(\frac{1-e}{2})^2|\xi|^2}\frac{1}{(1+\frac{\beta}{2}|\xi|)^{\nu'}}.\end{aligned}$$

On the other hand, if $\nu' > 0$ is small enough

$$\frac{1}{1+k(\frac{1-e}{2})^2|\xi|^2}\frac{1}{(1+\frac{\beta}{2}|\xi|)^{\nu'}} \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Indeed, since $(1+x)/(1+\frac{1}{2}x) \leqslant 2$ for any $x \geqslant 0$

$$\left(\frac{1+\beta|\xi|}{1+\frac{\beta}{2}|\xi|}\right)^{\nu'}\frac{1}{1+k(\frac{1-e}{2})^2|\xi|^2} \leqslant 2^{\nu'}\frac{1}{1+k(\frac{1-e}{2})^2\rho^2}$$

where the last term is smaller than 1 for $\nu' \leqslant \log_2(1+k(\frac{1-e}{2})^2\rho^2)$. Finally

$$\int_{A_1}\left|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\right|\mathrm{d}\sigma \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}|A_1|. \tag{45}$$

Let us now fix $\sigma \in A_2$. We get

$$\begin{aligned}\int_{A_2}\left|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\right|\mathrm{d}\sigma &\leqslant \frac{1}{(1+\beta|\tilde{\xi}^+|)^{\nu'}}\int_{A_2\cap\{|\tilde{\xi}^+|>\rho\}}\left|\hat{g}_0(\tilde{\xi}^-)\right|\mathrm{d}\sigma\\ &\quad+\frac{1}{1+k|\tilde{\xi}^+|^2}\int_{A_2\cap\{|\tilde{\xi}^+|\leqslant\rho\}}\left|\hat{g}_0(\tilde{\xi}^-)\right|\mathrm{d}\sigma.\end{aligned} \tag{46}$$

Thanks to (43), $|\tilde{\xi}^+|^2 > \rho^2(\frac{1-e}{2})^2$, so that for ν' small enough we obtain

$$\frac{1}{1+k|\tilde{\xi}^+|^2} \leqslant \frac{1}{(1+\beta|\tilde{\xi}^+|)^{\nu'}}.$$

Using the previous bound into (46) we conclude

$$\int_{A_2}\left|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\right|\mathrm{d}\sigma \leqslant \frac{1}{(1+\beta|\tilde{\xi}^+|)^{\nu'}}\int_{A_2}\left|\hat{g}_0(\tilde{\xi}^-)\right|\mathrm{d}\sigma \leqslant \frac{1}{(1+\frac{\beta}{2}|\xi|)^{\nu'}}\int_{A_2}\frac{1}{1+k|\tilde{\xi}^-|^2}\,\mathrm{d}\sigma.$$

To evaluate the integral over A_2, we introduce a spherical reference frame centered at the collision center between the two particles with z-axis defined by ξ (or $\tilde{\xi}$, which is just a multiple). We will denote $\vartheta \in [0,\pi]$ the longitude and $\varphi \in [0, 2\pi)$ the latitude of the point on the sphere corresponding to the vector σ. In this reference frame

$$\begin{aligned}\left|\tilde{\xi}^+\right|^2 &= \frac{10+2e^2-4e}{16}|\tilde{\xi}|^2 + \frac{(3-e)(1+e)}{8}|\tilde{\xi}|^2\cos\vartheta,\\ \left|\tilde{\xi}^-\right|^2 &= 2\left(\frac{1+e}{4}\right)^2|\tilde{\xi}|^2(1-\cos\vartheta).\end{aligned}$$

Consequently

$$A_2 = \left\{\varphi \in [0,2\pi),\ \vartheta \in [0,\pi]:\ (1-\cos\vartheta) \leqslant \min\left(\frac{\rho^2}{2(\frac{1+e}{4})^2|\tilde{\xi}|^2}, \frac{6}{(3-e)(1+e)}, \frac{1}{8(\frac{1+e}{4})^2}\right)\right\}.$$

Since in A_2 $|\tilde{\xi}^+| > \frac{|\tilde{\xi}|}{2}$ and $6/[(3-e)(1+e)] \geqslant 1/[8(\frac{1+e}{4})^2]$ for all $e > 0$,

$$A_2 = \left\{\varphi \in [0, 2\pi),\ \vartheta \in [0, \pi]:\ (1-\cos\vartheta) \leqslant \min\left(\frac{\rho^2}{2(\frac{1+e}{4})^2|\tilde{\xi}|^2}, \frac{1}{8(\frac{1+e}{4})^2}\right)\right\}.$$

Let us suppose first $\tilde{\xi}$ such that

$$|\tilde{\xi}|^2 \geqslant 4\rho^2.$$

In this case, we have

$$\int_{A_2} \frac{1}{1+k|\tilde{\xi}^-|^2}\,\mathrm{d}\sigma = \int_{\varphi\in[0,2\pi)} \int_{(1-\cos\vartheta)\leqslant \frac{\rho^2}{2(\frac{1+e}{4})^2|\tilde{\xi}|^2}} \frac{1}{1+2k(\frac{1+e}{4})^2|\tilde{\xi}|^2(1-\cos\vartheta)} \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

$$= \frac{2\pi}{2(\frac{1+e}{4})^2|\tilde{\xi}|^2} \int_0^{\rho^2} \frac{\mathrm{d}x}{1+kx} = \frac{2\pi\rho^2}{2(\frac{1+e}{4})^2|\tilde{\xi}|^2} \frac{\log(1+k\rho^2)}{k\rho^2} = \frac{\log(1+k\rho^2)}{k\rho^2}|A_2|.$$

If on the contrary

$$|\tilde{\xi}|^2 < 4\rho^2$$

then

$$\int_{A_2} \frac{1}{1+k|\tilde{\xi}^-|^2}\,\mathrm{d}\sigma = \int_{\varphi\in[0,2\pi)} \int_{(1-\cos\vartheta)\leqslant \frac{1}{8(\frac{1+e}{4})^2}} \frac{1}{1+2k(\frac{1+e}{4})^2|\tilde{\xi}|^2(1-\cos\vartheta)} \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

$$= \frac{2\pi}{2(\frac{1+e}{4})^2|\tilde{\xi}|^2} \int_0^{\frac{|\tilde{\xi}|^2}{4}} \frac{\mathrm{d}x}{1+kx} = \frac{2\pi}{2(\frac{1+e}{4})^2|\tilde{\xi}|^2} \frac{\log(1+k\frac{|\tilde{\xi}|^2}{4})}{k}$$

$$= \frac{2\pi}{8(\frac{1+e}{4})^2} \frac{\log(1+k\frac{|\tilde{\xi}|^2}{4})}{k\frac{|\tilde{\xi}|^2}{4}} = \frac{\log(1+k\frac{|\tilde{\xi}|^2}{4})}{k\frac{|\tilde{\xi}|^2}{4}}|A_2|.$$

In both cases we get

$$\int_{A_2} \frac{1}{1+k|\tilde{\xi}^-|^2}\,\mathrm{d}\sigma \leqslant \sup_{t\geqslant \frac{\rho^2}{4}} \frac{\log(1+kt)}{kt}|A_2|$$

so that

$$\int_{A_2} \left|\hat{g}_0(\tilde{\xi}^+)\hat{g}_0(\tilde{\xi}^-)\right|\mathrm{d}\sigma \leqslant \frac{1}{(1+\frac{\beta}{2}|\xi|)^{\nu'}} \sup_{t\geqslant \frac{\rho^2}{4}} \frac{\log(1+kt)}{kt}|A_2|.$$

By choosing $\nu' > 0$ small enough

$$\frac{1}{(1+\frac{\beta}{2}|\xi|)^{\nu'}} \sup_{t\geqslant \frac{\rho^2}{4}} \frac{\log(1+kt)}{kt} \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Indeed,

$$\left(\frac{1+\beta|\xi|}{1+\frac{\beta}{2}|\xi|}\right)^{\nu'} \sup_{t\geqslant \frac{\rho^2}{4}} \frac{\log(1+kt)}{kt} \leqslant 2^{\nu'} \sup_{t\geqslant \frac{\rho^2}{4}} \frac{\log(1+kt)}{kt} \leqslant 1,$$

as soon as

$$\nu' \leqslant \log_2\left(\inf_{t\geqslant \frac{\rho^2}{4}} \frac{kt}{\log(1+kt)}\right).$$

Finally

$$\int_{A_2} \big|\hat g_0(\tilde\xi^+)\hat g_0(\tilde\xi^-)\big|\,\mathrm{d}\sigma \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}|A_2|. \tag{47}$$

Let us fix now $\sigma \in A_3$. For $\nu' > 0$ small enough it holds

$$\big|\hat g_0(\tilde\xi^+)\hat g_0(\tilde\xi^-)\big| \leqslant \frac{1}{1+k|\tilde\xi^+|^2}\,\frac{1}{1+k|\tilde\xi^-|^2} \leqslant \frac{1}{1+k(|\tilde\xi^+|^2+|\tilde\xi^-|^2)} \leqslant \frac{1}{1+k\frac{|\xi|^2}{2}} \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Thus

$$\int_{A_3} \big|\hat g_0(\tilde\xi^+)\hat g_0(\tilde\xi^-)\big|\,\mathrm{d}\sigma \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}|A_3|. \tag{48}$$

For $\sigma \in A_4$ and $\nu' > 0$ small enough

$$\big|\hat g_0(\tilde\xi^+)\hat g_0(\tilde\xi^-)\big| \leqslant \frac{1}{(1+\beta|\tilde\xi^+|)^{\nu'}}\,\frac{1}{1+k|\tilde\xi^-|^2} \leqslant \frac{1}{(1+\frac{\beta}{2}|\xi|)^{\nu'}}\,\frac{1}{1+k\frac{|\xi|^2}{4}} \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Therefore

$$\int_{A_4} \big|\hat g_0(\tilde\xi^+)\hat g_0(\tilde\xi^-)\big|\,\mathrm{d}\sigma \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}|A_4|. \tag{49}$$

Last, let us consider $\sigma \in A_5$. It is worth noticing that

$$A_5 \subset \left\{ \big|\tilde\xi^-\big| > \frac{|\tilde\xi|}{2},\ \big|\tilde\xi^+\big| \leqslant \rho \right\}$$

so that, for $\nu' > 0$ small enough

$$\big|\hat g_0(\tilde\xi^+)\hat g_0(\tilde\xi^-)\big| \leqslant \frac{1}{1+k|\tilde\xi^+|^2}\,\frac{1}{1+k|\tilde\xi^-|^2} \leqslant \frac{1}{1+k\frac{|\xi|^2}{4}} \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

This implies

$$\int_{A_5} \big|\hat g_0(\tilde\xi^+)\hat g_0(\tilde\xi^-)\big|\,\mathrm{d}\sigma \leqslant \frac{1}{(1+\beta|\xi|)^{\nu'}}|A_5|. \tag{50}$$

Grouping inequalities (44), (45), (47), (48), (49) and (50) we conclude that it exists a suitably small positive constant ν' depending on ρ and k fixed once and for all in the first part of this proof such that, for $|\xi| > \rho$ and $\eta > 1$

$$\big|Q_+(\hat g_0, \hat g_0)(\eta\xi)\big| \leqslant \frac{1}{4\pi}\big|S^2\big|\frac{1}{(1+\beta|\xi|)^{\nu'}} = \frac{1}{(1+\beta|\xi|)^{\nu'}}.$$

Coming back to $\hat\phi_1^N(\xi)$ for N large enough and $|\xi| > \rho$ we obtain

$$\big|\hat\phi_1^N(\xi)\big| \leqslant \frac{1}{\Delta t}\int_{\eta=1}^{+\infty}\left(E\,\Delta t\,\frac{1}{(1+\beta|\xi|)^{\nu'}} + (1-E\,\Delta t)\frac{1}{(1+\beta|\xi|)^{\nu'}}\right)\frac{\mathrm{d}\eta}{\eta^{\frac{1}{\Delta t}+1}} = \frac{1}{(1+\beta|\xi|)^{\nu'}}. \qquad \square$$

Remark 16. Note that the bounds in (41) imply

$$\big|\hat g(\xi,t)\big| \leqslant \frac{C}{(1+\kappa|\xi|)^{\mu}}, \quad \xi \in \mathbb{R}^3,\ t \geqslant 0.$$

5. Propagation of regularity and strong convergence

The aim of this section is to prove Theorem 2 about the strong L^1-convergence of the solution $g(t)$ of Eq. (11) to the stationary state g_∞.

Theorem 2. *Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of Eq.* (11)*, corresponding to the initial probability density $g_0 \in \mathcal{P}_4(\mathbb{R}^3)$ with zero mean velocity and $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$. Let us suppose moreover $g_0 \in \dot{H}^\eta(\mathbb{R}^3)$ for some $\eta > 0$ and $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ for some $\nu > 0$. Then $g(t)$ converges strongly in L^1 with an exponential rate towards the stationary solution g_∞, i.e., there exist positive constants C and γ explicitly computable such that*

$$\left\|g(t) - g_\infty\right\|_{L^1(\mathbb{R}^3)} \leqslant Ce^{-\gamma t}, \quad t \geqslant 0.$$

Let us begin by the following lemma (cf. [20]), which makes a link between the condition $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ and the decay of the Fourier transform. For the proof we refer to [18].

Lemma 17. *Let g_0 be a probability density such that $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ for some $\nu > 0$. Then there exist positive constants C and β such that $\hat{g}_0$ satisfies*

$$\left|\hat{g}_0(\xi)\right| \leqslant \frac{C}{(1+\beta|\xi|)^\nu}, \quad \xi \in \mathbb{R}^3.$$

In order to prove Theorem 2, we need to pass from the weak convergence in the Fourier distance d_2 of the solution $g(t)$ to the stationary state g_∞ (Theorem 7) to the more natural L^1-convergence in the physical space. To this extent, we make use of the propagation of regularity established in Theorem 1. The additional ingredient is the uniform boundedness of the solution in Sobolev norms. The proof of this statement follows along the lines of the analogous proof in [11], Lemma 3.4 and Theorem 3.6. In [11], however, the decay of the Fourier transform of the solution (41) was proved only for small inelasticity $e \simeq 1$, making use of a precise control of the growth of the Fisher information (Theorem 1.2).

Theorem 18. *Assume $e \in [0,1)$. Let $g(t)$ be the weak solution of Eq.* (11)*, corresponding to the initial probability density $g_0 \in \mathcal{P}_4(\mathbb{R}^3)$ with zero mean velocity and $\int_{\mathbb{R}^3} |v|^2 g_0(v)\,\mathrm{d}v = 3$. Moreover, let $g_0 \in \dot{H}^\eta(\mathbb{R}^3)$ for some $\eta > 0$ and $\sqrt{g_0} \in \dot{H}^\nu(\mathbb{R}^3)$ for some $\nu > 0$. Then $g(t)$ is uniformly bounded in $\dot{H}^\eta(\mathbb{R}^3)$.*

Proof of Theorem 2. The proof of Theorem 2 is completed using the following interpolation bounds:

- there exists a positive constant C such that
$$\|h\|_{L^1} \leqslant C\left\||v|^2 h\right\|_{L^1}^{\frac{3}{7}} \|h\|_{L^2}^{\frac{4}{7}};$$
- for any $s \geqslant 0$ and any $\eta > 0$, there exists a positive constant C such that
$$\|h\|_{\dot{H}^s} \leqslant C\left(\sup_{\xi \neq 0} \frac{|\hat{h}(\xi)|}{|\xi|^2}\right)^{\frac{\eta}{2s+7+\eta}} \|h\|_{\dot{H}^{s+\eta}}^{1-\frac{\eta}{2s+7+\eta}}.$$

To prove the last bound, consider that, for any given constant $R > 0$

$$\begin{aligned}\|h\|_{\dot{H}^s}^2 &= \int_{\mathbb{R}^3} \left|\hat{h}(\xi)\right|^2 |\xi|^{2s}\,\mathrm{d}\xi = \int_{|\xi|\leqslant R} \frac{|\hat{h}(\xi)|^2}{|\xi|^4} |\xi|^{2s+4}\,\mathrm{d}\xi + \int_{|\xi|>R} \frac{1}{|\xi|^{2\eta}} \left|\hat{h}(\xi)\right|^2 |\xi|^{2s+2\eta}\,\mathrm{d}\xi \\ &\leqslant C\left(\sup_\xi \frac{|\hat{h}(\xi)|}{|\xi|^2}\right)^2 R^{2s+7} + \frac{1}{R^{2\eta}} \|h\|_{\dot{H}^{s+\eta}}^2.\end{aligned}$$

Optimizing over R gives the result.

Let $h = g(t) - g_\infty$. Keeping in mind the regularity of the stationary state (Theorem 10), for $s = 0$ we get

$$\|g(t) - g_\infty\|_{L^1} \leqslant C\big(\||v|^2 g(t)\|_{L^1} + \||v|^2 g_\infty\|_{L^1}\big)^{\frac{3}{7}} \times \Big(d_2\big(g(t), g_\infty\big)^{\frac{\eta}{7+\eta}} \big(\|g(t)\|_{\dot H^\eta} + \|g_\infty\|_{\dot H^\eta}\big)^{1-\frac{\eta}{7+\eta}}\Big)^{\frac{4}{7}}. \qquad \square$$

Appendix A

Proof of Proposition 13. The estimates on the first order moments (33) are easily obtained by a recursive procedure based on expression (31). A similar procedure applies to show that the non-isotropic second moments (34) vanish.

Let us now consider the diagonal second moments (35). Using the same procedure of Proposition 22 we arrive at the following first order linear difference system

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2 \phi_{j+1}^N \, dv \\ \int_{\mathbb{R}^3} v_2^2 \phi_{j+1}^N \, dv \\ \int_{\mathbb{R}^3} v_3^2 \phi_{j+1}^N \, dv \end{bmatrix} = \frac{1}{1-2\Delta t}(\mathrm{Id} + \Delta t A) \begin{bmatrix} \int_{\mathbb{R}^3} v_1^2 \phi_j^N \, dv \\ \int_{\mathbb{R}^3} v_2^2 \phi_j^N \, dv \\ \int_{\mathbb{R}^3} v_3^2 \phi_j^N \, dv \end{bmatrix}$$

where

$$A = \begin{bmatrix} -\frac{4}{3}\frac{2-e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & -\frac{4}{3}\frac{2-e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} \\ \frac{1}{3}\frac{1+e}{1-e} & \frac{1}{3}\frac{1+e}{1-e} & -\frac{4}{3}\frac{2-e}{1-e} \end{bmatrix}.$$

The matrix A has -2 as simple eigenvalue, with eigenspace $\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$ and $-\frac{3-e}{1-e}$ as double eigenvalue with eigenspace $\left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$. Therefore the matrix $\frac{1}{1-2\Delta t}(\mathrm{Id} + \Delta t A)$ has 1 as simple eigenvalue and $\frac{1}{1-2\Delta t}(1 - \Delta t \frac{3-e}{1-e})$ as double eigenvalue, with the same eigenspaces. Since for N large enough (depending on $e \in [0,1)$) we have $0 < \frac{1}{1-2\Delta t}(1 - \Delta t \frac{3-e}{1-e}) < 1$, we obtain

$$\begin{bmatrix} \int_{\mathbb{R}^3} v_1^2 \phi_j^N(v) \, dv \\ \int_{\mathbb{R}^3} v_2^2 \phi_j^N(v) \, dv \\ \int_{\mathbb{R}^3} v_3^2 \phi_j^N(v) \, dv \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \left(\frac{1 - \Delta t \frac{3-e}{1-e}}{1 - 2\Delta t} \right)^j + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \left(\frac{1 - \Delta t \frac{3-e}{1-e}}{1 - 2\Delta t} \right)^j,$$

with C_1, C_2 real constants determined by the initial conditions. By virtue of (35), for all N and for all $j = 0, \dots, N$

$$\int_{\mathbb{R}^3} |v|^2 \phi_j^N(v) \, dv = 3.$$

Finally, let us take into account the fourth moment (36). Making use again of (31) we find

$$\begin{aligned} &\int_{\mathbb{R}^3} |v|^4 \phi_{j+1}^N(v) \, dv \\ &\quad = \frac{1}{\Delta t} \int_1^{+\infty} \left(E\Delta t \int_{\mathbb{R}^3} \frac{|v|^4}{\eta^3} Q_+\big(\phi_j^N, \phi_j^N\big)\left(\frac{v}{\eta}\right) dv + (1 - E\Delta t) \int_{\mathbb{R}^3} \frac{|v|^4}{\eta^3} \phi_j^N\left(\frac{v}{\eta}\right) dv \right) \frac{d\eta}{\eta^{\frac{1}{\Delta t}+1}} \\ &\quad = \frac{1}{\Delta t} \frac{1}{\frac{1}{\Delta t} - 4} \left(E\Delta t \int_{\mathbb{R}^3} |v|^4 Q_+\big(\phi_j^N, \phi_j^N\big)(v) \, dv + (1 - E\Delta t) \int_{\mathbb{R}^3} |v|^4 \phi_j^N(v) \, dv \right). \end{aligned}$$

We denote

$$m_{4,j} = \int_{\mathbb{R}^3} |v|^4 \phi_j^N(v) \, dv.$$

By Lemma 5 in [9], we know that there exist λ, μ_1 and μ_2 positive constants depending only on e such that

$$\int_{\mathbb{R}^3} |v|^4 Q\big(\phi_j^N, \phi_j^N\big)(v)\,\mathrm{d}v = -\lambda m_{4,j} + 9\mu_1 + \mu_2 \int_{\mathbb{R}^3\times\mathbb{R}^3} (v\cdot w)^2 \phi_j^N(v)\phi_j^N(w)\,\mathrm{d}v\,\mathrm{d}w$$
$$\leqslant -\lambda m_{4,j} + C$$

for a suitable positive constant C depending on g_0. Remembering that

$$\int_{\mathbb{R}^3} |v|^4 Q_+\big(\phi_j^N, \phi_j^N\big)(v)\,\mathrm{d}v = \int_{\mathbb{R}^3} |v|^4 Q\big(\phi_j^N, \phi_j^N\big)(v)\,\mathrm{d}v + \int_{\mathbb{R}^3} |v|^4 \phi_j^N(v)\,\mathrm{d}v,$$

we end up with the recursive relation

$$\begin{aligned} m_{4,j+1} &\leqslant \frac{1}{1-4\Delta t}\big(E\Delta t\big((1-\lambda)m_{4,j} + C\big) + (1-E\Delta t)m_{4,j}\big) \\ &= \frac{1}{1-4\Delta t}\big((1-\lambda E\Delta t)m_{4,j} + CE\Delta t\big). \end{aligned}$$

By a Taylor expansion we get for $\Delta t \to 0$:

$$\begin{aligned} m_{4,j+1} &\leqslant (1-\lambda E\Delta t)\big(1+4\Delta t + o(\Delta t)\big)m_{4,j} + CE\Delta t\big(1+4\Delta t + o(\Delta t)\big) \\ &= \big(1-(\lambda E - 4)\Delta t + o(\Delta t)\big)m_{4,j} + CE\Delta t + o(\Delta t). \end{aligned}$$

Since $\lambda > 4/E$ for any $e \in [0,1)$ [9], we get for N large enough

$$m_{4,j+1} \leqslant \left(1 - \frac{\lambda E - 4}{2}\Delta t\right)m_{4,j} + 2CE\Delta t.$$

Thus

$$\begin{aligned} m_{4,j+1} &\leqslant \left(1 - \frac{\lambda E - 4}{2}\Delta t\right)^{j+1} m_{4,0} + 2CE\Delta t \sum_{k=0}^{j}\left(1 - \frac{\lambda E - 4}{2}\Delta t\right)^k \\ &\leqslant m_{4,0} + \frac{2CE\Delta t}{\frac{\lambda E-4}{2}\Delta t} = m_{4,0} + \frac{4CE}{\lambda E - 4}. \end{aligned}$$

Hence, if $g_0 \in \mathcal{P}_4(\mathbb{R}^3)$, $m_{4,j}$ is uniformly bounded for all $j = 0, \dots, N$, provided N is large enough. □

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