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The optimal shape of a dendrite sealed at both ends

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Abstract

In this paper, we are interested in the geometric structures which appear in nature. We consider the example of a nerve fiber and we suppose that shapes in nature arise in order to optimize some criterion. Then, we try to solve the problem consisting in searching the shape of a nerve fiber for a given criterion. The first considered criterion represents the attenuation in space of the electrical message troughout the fiber and seems to be relevant. Our second criterion represents the attenuation in time of the electrical message and doesn't provide a realistic shape. We prove that the associated optimization problem has no solution.

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1. Introduction

1.1. Motivation

The observation of the nature and of the "perfection" of most of the mechanisms of living beings drives us to search *a principle of optimality* which governs those mechanisms. If a mathematical model exists to describe a biological phenomenon or component of living beings, there is a temptation to quantify the optimality by finding a functional which can lead to an optimality principle.

The confrontation between the computed optimum and the real one leads us to validate or invalidate the model and/or the choice of the functional. This inverse modeling method consists in finding the mathematical model starting from observations and their consequences. If the optimal shape obtained thanks to the mathematical model is close to the real shape, we have reasons to believe that the full model (equation and functional) is good. If not, one has to reject it and find another one, or improve it.

This point of view is very close to the idea developed by B. Mauroy, M. Filoche, E.R. Weibel and B. Sapoval in [11] (see also [12]). In these articles, they studied the compatibility between physical optimization and physiological robustness in the design of the human bronchial tree.

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In this paper, we will consider the example of a dendrite sealed at both ends. A dendrite is a branched extension of a nerve cell that conducts impulses from adjacent cells inward toward the cell body. A single nerve may possess many dendrites. Electrical stimulation is transmitted onto dendrites by upstream neurons via synapses which are located at various points throughout the dendritic arbor.

The aim of the work presented in this paper is to find the "optimal" shape which permits the best conduction of an electrical message into the dendrite. For this, we will consider two criterions: the first criterion is a good measure of the attenuation of the electrical message in space whereas the second is a good measure of the attenuation of the electrical message in time. We are consequently led to solve the problem of the minimization of these criterions. A similar study has been led on the shape of a dendrite connected to the soma, which is the part of the cell containing the nucleus (see [8]). It has been showed that the cylinder with constant radius is the optimal shape for such a dendrite and for each criterion. The results, in the case of a dendrite sealed at both ends are not exactly the same. In particular, the cylinder of constant radius is not an optimum for the first criterion and the mathematical technics developed in this paper are fairly different.

1.2. Mathematical model and notations

Let us consider a dendrite sealed at both ends, with a cylindrical symmetry, of length ℓ and radius $a(x)$ at point x. Passive classical cable theory uses mathematical models to calculate the flow of electric current (and accompanying voltage) along passive neuronal fibers. The word "Passive" refers to the membrane resistance being voltage-independent.

We denote by $v(x,t)$, the difference from rest of the membrane potential at point x and time t. According to W. Rall, the propagation of an electrical impulse in a dendrite fiber follows a parabolic p.d.e. (cf. [3,13–15])

$$\begin{cases} \frac{1}{2R_a}\frac{\partial}{\partial x}\left(a^2\frac{\partial v}{\partial x}\right) = a\sqrt{1+a'^2}\left(C_m\frac{\partial v}{\partial t}+G_m v\right), & (x,t)\in(0,\ell)\times]0;+\infty[,\\ \frac{\pi a^2(0)}{R_a}\frac{\partial v}{\partial x}(0,t) = -i_0(t), & t>0,\\ \frac{\partial v}{\partial x}(\ell,t)=0, & t>0,\\ v(x,0)=0, & x\in[0,\ell], \end{cases} \tag{1}$$

where R_a denotes the axial resistance (kΩcm), C_m the membrane capacitance (μF/cm^2), and G_m the fiber membrane conductance (mS/cm^2). We assume that the fiber is initially at rest. We consider an electrical impulsion at the beginning of the fiber: $i_0(t)=\delta_{\{t=0\}}$ (Dirac measure at $t=0$). This modelizes an explosive release of charge between a nerve cell (neuron) and its surroundings, called action potential.

Let us notice that the solution of Eq. (1) can be decomposed in a spectral basis $(\phi_n^a)_{n\geqslant 0}$ as did S. Cox and J. Raol in [3]

$$v(x,t)=\sum_{n=0}^{+\infty}\psi_n(t)\phi_n^a(x),\quad \forall x\in[0,\ell],\ \forall t>0, \tag{2}$$

where ϕ_n^a is the nth eigenfunction associated to the eigenvalue μ_n, solution of the system

$$\begin{cases} -(a^2\phi_n^{a\prime})' = \mu_n(a)a\sqrt{1+a'^2}\phi_n^a, & x\in(0,\ell),\\ \phi_n^{a\prime}(0)=\phi_n^{a\prime}(\ell)=0. \end{cases} \tag{3}$$

Eigenvalues problems with Neumann boundary conditions are well-known. The eigenvalues $\mu_n(a)$ verify $0=\mu_0(a)<\mu_1(a)<\cdots<\mu_n(a)$. It is common to normalize the eigenfunctions with the weighted norm

$$\|\phi_n^a\|_a^2 := \int_0^\ell a(x)\sqrt{1+a'^2(x)}\phi_n^{a2}(x)\,\mathrm{d}x. \tag{4}$$

We can now use a classical method of separation of variables to solve (1). The expression of $v(x,t)$ is given in the following theorem:

Theorem 1.1. *Let i_0 be a Radon measure. Then, the solution of* (1) *is given by*

$$v(x,t)=\frac{1}{2\pi C_m}\sum_{n=0}^{+\infty}\phi_n^a(0)\phi_n^a(x)\big(i_0 * e^{-\widetilde{\mu}_n t}\big)(t), \tag{5}$$

where ϕ_n^a denotes the solution of (3) *normalized with $\|.\|_a$, $*$ is the convolution product of distributions and $\widetilde{\mu}_n = \frac{\mu_n+2R_aG_m}{2R_aC_m}$.*

For a proof of this theorem, one can refer to [3].

We will use the following notations throughout the paper:

$W^{1,\infty}(0,\ell)$ the set of Lipschitz continuous functions defined on the interval $[0,\ell]$.

$\|.\|_\infty$ norm defined on the space of bounded functions $L^\infty(0,\ell)$ by

$$\|f\|_\infty := \inf\big\{C\geqslant 0\colon\ \big|f(x)\big|\leqslant C,\ \text{a.e. } x\in[0,\ell]\big\}.$$

$\big\langle \frac{\mathrm{d}J}{\mathrm{d}\nu}(\nu_0),h\big\rangle$ Gâteaux-derivative of a functional J at point ν_0 in direction h defined by

$$\left\langle \frac{\mathrm{d}J}{\mathrm{d}\nu}(\nu_0),h\right\rangle := \lim_{t\searrow 0}\frac{J(\nu_0+t.h)-J(\nu_0)}{t}.$$

$\widehat{v}$ Laplace transform in time of a function $v(x,t)$ defined by

$$\widehat{v}(x,p) := \int_0^{+\infty} e^{-pt}v(x,t)\,\mathrm{d}t.$$

ϕ' derivative of function ϕ with respect to the space variable x.

1.3. The optimization problems

We consider here that $i_0(t)=\delta_{\{t=0\}}$. Hence the expression of the solution v is given by the formula

$$v(x,t)=\frac{1}{2\pi C_m}\sum_{n=0}^{+\infty}\phi_n^a(0)\phi_n^a(x)e^{-\widetilde{\mu}_n t} \tag{6}$$

where $\widetilde{\mu}_n$ is given in Theorem 1.1. The main unknown of our problem will be the radial function $x\mapsto a(x)$ (the shape of the dendrite). Before introducing the optimization problems, let us define the class of functions in which we will search for a.

- Let us notice the presence of the term $a(x)\sqrt{1+a'^2(x)}$ in Eq. (1). So the minimal regularity desired for a is that the derivative a' of a exists almost everywhere. We consequently choose a in $W^{1,\infty}(0,\ell)$.
- The fiber must not collapse. That is why we assume $a(x)\geqslant a_0>0$, for all $x\in[0,\ell]$.
- We assume a constraint on the surface area of the fiber, which corresponds to the cost for Nature. Moreover, fractal-like objects are forbidden in our study. We thus impose:

$$\int_0^\ell a(x)\sqrt{1+a'^2(x)}\,\mathrm{d}x\leqslant S, \text{ where } S>0 \text{ is a given constant.}$$

We finally search a in the class $\mathcal{A}_{a_0,S}$ defined by

$$\mathcal{A}_{a_0,S} := \left\{a\in W^{1,\infty}(0,\ell)\ \middle|\ a(x)\geqslant a_0 \text{ and } \int_0^\ell a(x)\sqrt{1+a'^2(x)}\,\mathrm{d}x\leqslant S\right\}. \tag{7}$$

Remark 1.1. We choose $S > a_0\ell$ so that the class $\mathcal{A}_{a_0,S}$ be non-trivial.

We now introduce the two criterions we will study:

1st criterion: attenuation in space.

It is interesting to search the optimal shape which attenuates the least possible the average impulse in time between the beginning and the end of the fiber. Let us define the transfer function T by

$$T(a) := \frac{\int_0^{+\infty} v(0,t)\,\mathrm{d}t}{\int_0^{+\infty} v(\ell,t)\,\mathrm{d}t}, \tag{8}$$

where v denotes the solution of the p.d.e. (1). We will prove in Section 2.1 that the inequality $T(a) \leqslant 1$ holds in the class $\mathcal{A}_{a_0,S}$. To find the profile which produces the smallest attenuation, we are led to introduce the problem

$$\begin{cases} \min T \\ a \in \mathcal{A}_{a_0,S}. \end{cases} \tag{9}$$

2nd criterion: attenuation in time.

According to the decomposition (6) of the solution v of (1) in a spectral basis, we are led to minimize the exponential rate of decay in this equation. The asymptotic development of $v(x,t)$ at the second order when $t \to +\infty$ writes

$$v(x,t) \underset{t\to+\infty}{\sim} \frac{1}{2\pi C_m}\left(\phi_0^a\right)^2(0)e^{-\frac{G_m}{C_m}t} + \frac{1}{2\pi C_m}\phi_1^a(0)\phi_1^a(x)e^{-\widetilde{\mu}_1 t}. \tag{10}$$

Since we are interested in the shape which allows the best conduction of the electric impulse, it seems natural to look for a function a which minimizes the attenuation in time of the signal. The exponential rate of decay G_m/C_m of the first term of the previous development, is obviously independent on the shape $a(x)$ of the fiber. To answer this question, a first idea consists in solving the following problem:

$$\begin{cases} \max(\phi_0^a)^2(0), \\ a \in \mathcal{A}_{a_0,S}. \end{cases} \tag{11}$$

Nevertheless, let us notice that this eigenfunction associated to the eigenvalue $\mu_0 = 0$ is constant. Then, as a consequence of the normalization, one has:

$$\left[\phi_0^a(0)\right]^2 = \frac{1}{\int_0^\ell a(x)\sqrt{1+a'^2(x)}\,\mathrm{d}x} \leqslant \frac{1}{S}.$$

Thus, to solve problem (11), it is sufficient to exhibit an element a in $\mathcal{A}_{a_0,S}$ such that the inequality constraint $\int_0^\ell a(x)\sqrt{1+a'^2(x)}\,\mathrm{d}x \leqslant S$ is reached (e.g. the constant radius $a \equiv \frac{S}{\ell}$).

Nevertheless, the solution of the optimization problem (11) is not unique. One other choice of minimizer among many is given by

$$a(x) := \sqrt{\alpha^2 - \left(\sqrt{\alpha^2 - a_0^2} - \alpha x\right)^2}, \quad \text{with } \alpha := \frac{S}{\ell}.$$

Hence, because of the non-uniqueness, the previous radii are not satisfying answers of the biological associated problem. Then, to complete our answer, one can consider the second term of the asymptotic development (10) of the voltage $v(x,t)$. The exponential rate of decay of this term is $\widetilde{\mu}_1$ which is clearly a function of the shape of the fiber. To find the shape which furnishes the smallest attenuation in time of the signal, and taking into account the first term of the asymptotic development, we will introduce a new optimization problem

$$\begin{cases} \min \mu_1(a), \\ a \in \left\{a \in W^{1,\infty}(0,\ell) \mid a(x) \geqslant a_0 \text{ and } \int_0^\ell a(x)\sqrt{1+a'^2(x)}\,\mathrm{d}x = S\right\}. \end{cases} \tag{12}$$

Remark 1.2. Let us keep in mind that $\widetilde{\mu}_1 = \frac{\mu_1 + 2R_aG_m}{2R_aC_m}$. Then, the questions of minimizing μ_1 or $\widetilde{\mu}_1$ in $\mathcal{A}_{a_0,S}$ are equivalent.

Remark 1.3. In Section 3, we will prove in particular that problem (12) is equivalent to the following problem

$$\begin{cases} \min \mu_1(a), \\ a \in \mathcal{A}_{a_0,S}. \end{cases} \tag{13}$$

In other words, we will prove that a minimizing sequence for problem (13) has to achieve the inequality constraint.

In Sections 2 and 3, we solve the optimization problems (9) and (13). More precisely, we use classical methods of calculus of variations to prove the existence of a minimizer for the transfer function T in the class $\mathcal{A}_{a_0,S}$. On the contrary, we obtain a non-existence result for the minimization of the first eigenvalue $\mu_1(a)$ in the class $\mathcal{A}_{a_0,S}$ and we will say some words about the construction of the minimizing sequence. Nevertheless, we are able to find a relaxed formulation for this problem.

Remark 1.4. Problems linking the shape of a domain to the sequence of eigenvalues of Sturm–Liouville operators are a huge field of research. One can see [7] for a (non-exhaustive) review of such problems.

1.4. A change of variable

Let us now introduce a classical change of variable (used by S. Cox and R. Lipton in [2])

$$y = \int_0^x \frac{\mathrm{d}t}{a^2(t)}. \tag{14}$$

We denote by ℓ_1 the image of ℓ by this change of variable

$$\ell_1 = \int_0^\ell \frac{\mathrm{d}t}{a^2(t)}.$$

Let us notice that the interval $[0, \ell]$ becomes $[0, \ell_1]$.

We consider a new unknown ρ defined by $\rho(y) := a^3(x)\sqrt{1 + a'^2(x)}$. ρ will be used in this article as a new optimization variable, and since $a \in \mathcal{A}_{a_0,S}$, the function ρ must lie in the set

$$\mathcal{R}_{a_0,S,\ell_1} := \left\{ \rho \in L^\infty(0, \ell_1) \;\middle|\; a_0^3 \leqslant \rho(y) \text{ and } \int_0^{\ell_1} \rho(y)\,\mathrm{d}y \leqslant S \right\}.$$

In the different proofs, throughout the paper, we will also use the following subset of $L^\infty(0, \ell_1)$

$$\mathcal{R}^M_{a_0,S,\ell_1} := \left\{ \rho \in L^\infty(0, \ell_1) \;\middle|\; a_0^3 \leqslant \rho(y) \leqslant M \text{ and } \int_0^{\ell_1} \rho(y)\,\mathrm{d}y \leqslant S \right\},$$

for some $M > a_0^3$ and $\ell_1 \leqslant S a_0^{-3}$ (otherwise, this class would be empty). Noticing that, since $a(x) \geqslant a_0$, for all $x \in [0, \ell]$, $\ell_1 = \int_0^\ell \frac{\mathrm{d}t}{a^2(t)} \leqslant \ell / a_0^2$, and since $S \geqslant a_0 \ell$, $\ell / a_0^2 \leqslant S / a_0^3$. Then, we define $\mathcal{R}_{a_0,S}$ by

$$\mathcal{R}_{a_0,S} := \bigcup_{\ell_1 \in (0, \ell / a_0^2]} \mathcal{R}_{a_0,S,\ell_1}.$$

The use of this change of variable drives us to reformulate the optimization problems (9) and (13) with respect to the new variable ρ. These new problems are more convenient to employ standard technics of calculus of variation than the previous. Nevertheless, let us emphasize that the change of variable $x \mapsto y$ depends strongly on the optimization variable a. As a consequence, ℓ_1 depends on a. In the approach presented in the following sections, this difficulty is at first avoided, by considering that the quantity ℓ_1 is constant with respect to a and as a result, the new optimization problems are not equivalent with the initial problems. It is of course necessary to take then into account the fact that ℓ_1 is a function of a to solve the initial problems (9) and (13), as done in Sections 2.3 and 3.3.

2. Minimisation of the transfer function $T(a)$

2.1. Rewriting of the criterion $T(a)$ with the Laplace transform

Let us recall that the transfer function $T(a)$ models the space attenuation of the electrical message between the beginning and the end of the fiber. We are looking here for the solution of the problem (9), in other words the optimal shape which minimizes the transfer function T among the elements of $\mathcal{A}_{a_0,S}$. As did S.J. Cox and J.H. Raol in [3], we will use the Laplace transform in time of function v to rewrite criterion T. The method used to find a more suitable expression of the criterion is completely inspired by [8], and we refer to this paper for a proof of the following assertions.

It can be proved by standard semigroups arguments (e.g. see [4]), that the solution v of the p.d.e. (1) belongs to $L^2(0,T,H^1(0,\ell))$. It follows that the integrals $\int_0^{+\infty} v(0,t)\,\mathrm{d}t$ and $\int_0^{+\infty} v(\ell,t)\,\mathrm{d}t$ are well-defined for our choice of i_0 and we can consequently define the Laplace transform in time of v, denoted by $\widehat{v}$ (see the notations in Section 1.2).

Hence, it is possible to write

$$T(a)=\frac{\lim_{p\to 0}\widehat{v}(0,p)}{\lim_{p\to 0}\widehat{v}(\ell,p)},$$

where $\widehat{v}(.,p)$ is the solution of the following o.d.e.

$$\begin{cases} \frac{1}{2R_a}\frac{\partial}{\partial x}(a^2\frac{\partial\widehat{v}}{\partial x}) = a\sqrt{1+a'^2}(C_m p+G_m)\widehat{v}, & (x,p)\in(0,\ell)\times(0,+\infty),\\ \frac{\pi a^2(0)}{R_a}\frac{\partial\widehat{v}}{\partial x}(0,p)=-1, & p\in(0,+\infty),\\ \frac{\partial\widehat{v}}{\partial x}(\ell,p)=0, & p\in(0,+\infty). \end{cases}$$

Let us now use the change of variable introduced in Section 1.4. The function $\widehat{v}$ becomes w where $\widehat{v}(x,p)=w(y,p)$, for $x\in(0,\ell)$ and $y\in(0,\ell_1)$. w is obviously solution of the following o.d.e.

$$\begin{cases} \frac{1}{2R_a}\frac{\partial^2 w}{\partial y^2} = \rho(C_m p+G_m)w, & (y,p)\in(0,\ell_1)\times(0,+\infty),\\ \frac{\pi}{R_a}\frac{\partial w}{\partial y}(0,p)=-1, & p\in(0,+\infty),\\ \frac{\partial w}{\partial y}(\ell_1,p)=0, & p\in(0,+\infty). \end{cases}$$

We let p going to 0 and conclude that

$$T(a)=T_1(\rho):=\frac{w_0(0)}{w_0(\ell_1)},$$

where $w_0(y):=\lim_{p\to 0} w(y,p)$ for $y\in[0,\ell_1]$. It is easy to see that w_0 is solution of the o.d.e.

$$\begin{cases} \frac{\mathrm{d}^2 w_0}{\mathrm{d}y^2} = 2R_a G_m\rho w_0, & y\in(0,\ell_1),\\ \frac{\pi}{R_a}\frac{\mathrm{d}w_0}{\mathrm{d}y}(0)=-1,\\ \frac{\mathrm{d}w_0}{\mathrm{d}y}(\ell_1)=0. \end{cases} \tag{15}$$

Let us notice that we necessary have $w_0(0)\neq 0$, since $\frac{\mathrm{d}w_0}{\mathrm{d}y}(0)<0$. Otherwise, w_0 would be negative and concave, and this is in contradiction with the fact that $\frac{\mathrm{d}w_0}{\mathrm{d}y}(\ell_1)=0$. Then, it is possible to divide each member of (15) by $w_0(0)$. Denoting by $\widetilde{w}_0$ the function $w_0/w_0(0)$, it is easy to verify that $\widetilde{w}_0$ is solution of the following o.d.e.

$$\begin{cases} \frac{\mathrm{d}^2 \widetilde{w}_0}{\mathrm{d}y^2} = 2R_a G_m\rho \widetilde{w}_0, & y\in(0,\ell_1),\\ \widetilde{w}_0(0)=1,\\ \frac{\mathrm{d}\widetilde{w}_0}{\mathrm{d}y}(\ell_1)=0. \end{cases} \tag{16}$$

Conversely, let us notice that multiplying the solution of Eq. (16) by a well chosen constant gives the solution of Eq. (15).

Moreover, the criterion T_1 can be rewriting as

$$T_1(\rho) = \frac{1}{\widetilde{w}_0(\ell_1)}.$$

Let us notice that the well-possedness of this o.d.e. is clear, by Lax–Milgram theorem. Moreover, this gives also that $\widetilde{w}_0 \in H^2(0, \ell_1)$.

Finally, one can prove the assertion claimed in Section 1.3, that is $T(a) \geqslant 1$ for all $a \in \mathcal{A}_{a_0,S}$. It comes from the fact that, thanks to the change of variable, it is possible to associate to each element of $\mathcal{A}_{a_0,S}$ one element of $\mathcal{R}_{a_0,S,\ell_1}$ and from the fact that, thanks to the rewriting of the criterion T, one has $\widetilde{w}_0(\ell_1) \leqslant 1$. Let us keep in mind that the set $\mathcal{R}_{a_0,S,\ell_1}$ depends on the choice of the element a in $\mathcal{A}_{a_0,S}$.

Indeed, a direct consequence of Eq. (16) is that $\widetilde{w}_0 > 0$ on $[0, \ell_1]$. Else, $\widetilde{w}_0$ would change its sign and its convexity and the situation $\frac{\mathrm{d}\widetilde{w}_0}{\mathrm{d}y}(\ell_1) = 0$ would be impossible.

2.2. The main theorem

The new expression of the criterion $T(a)$ permits us to prove quite easily the existence and the uniqueness of a solution for the minimization problem (9).

Theorem 2.1. *Let a_0 and S be two (strictly) positive real numbers.*

Problem (9) *has a unique solution. Moreover, the minimizer of the transfer function T in the class $\mathcal{A}_{a_0,S}$ is the constant function $a \equiv a_0$.*

Let us remind that we are looking for the solution(s) of the problem (9), and that this problem rewrites

$$\begin{cases} \min T_1(\rho), \\ \rho(y) = a^3(x)\sqrt{1 + a'^2(x)}, \quad \forall x \in [0, \ell], \\ \text{where } a \in \mathcal{A}_{a_0,S} \text{ and } y = \int_0^\ell \frac{\mathrm{d}t}{a^2(t)}. \end{cases} \tag{17}$$

Then, we are driven to solve a new optimization problem:

$$\begin{cases} \min T_1(\rho), \\ \rho \in \mathcal{R}_{a_0,S,\ell_1}. \end{cases} \tag{18}$$

Nevertheless, we have to pay close attention to the fact that the map

$$\begin{aligned} X: \quad \mathcal{A}_{a_0,S} &\longrightarrow \mathcal{R}_{a_0,S}, \\ a &\longmapsto \rho \end{aligned}$$

is *a priori* not a one-to-one correspondence. Indeed, an element a of $\mathcal{A}_{a_0,S}$ can be associated by this map to an element $\rho \in \mathcal{R}_{a_0,S,\ell_1}$, with ℓ_1 depending on a, whereas the reverse property is not clear. Indeed, if an element $\rho \in \mathcal{R}_{a_0,S,\ell_1}$ has an antecedent $a \in \mathcal{A}_{a_0,S}$, then one has

$$\ell_1 = \int_0^\ell \frac{\mathrm{d}t}{a^2(t)}.$$

This equality can be seen as an overdetermined condition that has to be verified for an antecedent of the map $a \mapsto a^3\sqrt{1 + a'^2}$ to be an antecedent of X so much so that the onto character of X is not obvious.

The proof consists consequently in the following steps:

- Solve the new minimization problem (18).
- Verify that the solution of (18) belongs to the image of $\mathcal{A}_{a_0,S}$ by the map $a \longmapsto \rho$.

2.3. Proof of Theorem 2.1

The following lemma gives the answer of the first step:

Lemma 2.1. *The optimization problem*

$$\begin{cases} \min T_1(\rho), \\ \rho \in \mathcal{R}_{a_0,S,\ell_1} \end{cases} \tag{19}$$

has a unique solution $\rho^\star$ *defined by* $\rho^\star \equiv a_0^3$.

Proof. Let us consider two functions ρ_1 and ρ_2, elements of $\mathcal{R}_{a_0,S,\ell_1}$, such that $\rho_1 \geqslant \rho_2$ a.e. in $(0,\ell_1)$. Let us denote by $\widetilde{w}_0^1$ and $\widetilde{w}_0^2$ the respective solutions of the o.d.e. associated to problem (16), in other words, $\widetilde{w}_0^i$ is solution, for $i \in \{1,2\}$ of the following o.d.e.:

$$\begin{cases} \frac{d^2\widetilde{w}_0^i}{dy^2} = 2R_aG_m\rho_i\widetilde{w}_0^i, \quad y \in (0,\ell_1), \\ \widetilde{w}_0^i(0) = 1, \\ \frac{d\widetilde{w}_0^i}{dy}(\ell_1) = 0. \end{cases}$$

Then, since $\widetilde{w}_0^i > 0$ and $\rho_i > 0$ on $(0,\ell_1)$, one has:

$$\begin{cases} \frac{d^2(\widetilde{w}_0^2-\widetilde{w}_0^1)}{dy^2} \leqslant 2R_aG_m\rho_1(\widetilde{w}_0^2-\widetilde{w}_0^1), \quad y \in (0,\ell_1), \\ (\widetilde{w}_0^2-\widetilde{w}_0^1)(0) = 0, \\ \frac{d(\widetilde{w}_0^2-\widetilde{w}_0^1)}{dy}(\ell_1) = 0. \end{cases}$$

By comparison principle (see e.g. [6]), $\widetilde{w}_0^2 \geqslant \widetilde{w}_0^1$ on $[0,\ell_1]$.

Consequently, one can write

$$\forall \rho \in \mathcal{R}_{a_0,S,\ell_1}, \quad T_1(\rho) \geqslant T_1\big(a_0^3\big).$$

The conclusion of the lemma follows. □

Let us now conclude the proof of Theorem 2.1.

Lemma 2.1 proves that $\rho^\star$ is a global minimizer of T_1. Since for all $a \in \mathcal{A}_{a_0,S}$, $a \geqslant a_0$ and since ${a_0}^3\sqrt{1+{a_0'}^2} = {a_0}^3$, it follows that $a \equiv a_0$ is the unique antecedent in $\mathcal{A}_{a_0,S}$ of $\rho^\star$ by the change of variable (14). This ensures that $a \equiv a_0$ is the unique minimizer of T in the class $\mathcal{A}_{a_0,S}$.

3. Minimization of the eigenvalue $\mu_1(a)$

Let us recall that we are interested in the minimization of the first non-zero eigenvalue $\mu_1(a)$ of the problem (3) as stated in (13).

3.1. The main theorem

Theorem 3.1. *Let a_0 and S be two* (*strictly*) *positive real numbers.*

Problem (13) *has no solution.*

Remark 3.1. We are able to exhibit a minimizing sequence $(a_n)_{n\in\mathbf{N}}$ of elements in $\mathcal{A}_{a_0,S}$ for criterion $\mu_1(a)$ in the sense that $a_n \in \mathcal{A}_{a_0,S}$ for all $n \in \mathbf{N}$ and $\mu_1(a_n)$ converges to $\inf\{\mu_1(a),\ a \in \mathcal{A}_{a_0,S}\}$. In particular, in the proof of Theorem 3.1, we will show that the minimizing sequence $(a_n)_{n\in\mathbf{N}}$ has to verify the two following conditions:

1. $(a_n)_{n\in\mathbf{N}}$ converges uniformly to the constant function a_0 ;

2. Let us denote by $(b_n)_{n\in\mathbf{N}}$, the sequence of elements of $L^\infty(0,\ell)$ defined by: $b_n = a_n\sqrt{1+a_n'^2}$. Then, there exists $t\in[0,1]$ such that $(b_n)_{n\in\mathbf{N}}$ converges to $a_0+(S-a_0\ell)(t\delta_0+(1-t)\delta_\ell)$ in the sense of measures, where δ_0 and δ_ℓ denote the Dirac measures at $x=0$ and $x=\ell$.

The construction of such a minimizing sequence $(a_n)_{n\in\mathbf{N}}$ will be done in Section 3.4.

3.2. Variation of the Neumann-eigenvalue

Let $\rho\in\mathcal{R}_{a_0,S,\ell_1}$, we denote by $\mu(\rho)$ the first non-zero eigenvalue of the problem

$$\begin{cases} -w'' = \mu(\rho)\rho w, & y\in(0,\ell_1),\\ w'(0)=w'(\ell)=0. \end{cases} \tag{20}$$

To simplify the notations, $\dot\mu(\rho)$ will denote the Gâteaux derivative of $\rho\mapsto\mu(\rho)$ in an admissible given perturbation h, i.e. $\langle\frac{\mathrm{d}\mu}{\mathrm{d}\rho}(\rho),h\rangle$. We remind that, as a consequence of the properties of the Sturm–Liouville operators (see [5]), μ_1 is simple and it follows that μ_1 is differentiable with respect to ρ (see [10]). For a general survey on the calculus of shape derivative, one can refer to [9] and [16].

It is very classic to write the eigenvalue $\mu_1(\rho)$ as the solution of a min–max problem (see e.g. [7])

$$\mu_1(\rho)=\min_{\substack{V \text{ subspace of } H^1(0,\ell_1)\\ \text{of dim } 2}}\ \max_{v\in V} R(\rho,v),$$

where R is the Rayleigh quotient defined by

$$R(\rho,v):=\frac{\int_0^{\ell_1} v'^2(y)\,\mathrm{d}y}{\int_0^{\ell_1}\rho(y)v^2(y)\,\mathrm{d}y}.$$

Lemma 3.1. *Let ρ be an element of $\mathcal{R}_{a_0,S,\ell_1}$ and h, be an admissible perturbation. Then,*

$$\dot\mu(\rho)=-\mu(\rho)\int_0^{\ell_1} h(y)w^2(y)\,\mathrm{d}y,$$

where w denotes the normalized eigenfunction associated to $\mu(\rho)$, i.e. such that

$$\int_0^{\ell_1}\rho(y)w^2(y)\,\mathrm{d}y=1.$$

Proof. Using expression (20) and the first order optimality conditions for a min–max point, one can see that w verifies $\langle\frac{\mathrm{d}R}{\mathrm{d}v}(\rho,w),\delta v\rangle=0$, for some admissible perturbation δv. It follows that w is solution of (20).

Let us consider $\dot w$, the Gâteaux-derivative of w at ρ in direction h. $\dot w$ is solution of the following o.d.e.

$$\begin{cases} -\dot w'' = \dot\mu(\rho)\rho w+\mu(\rho)hw+\mu(\rho)\rho\dot w, & y\in(0,\ell_1),\\ \dot w'(0)=\dot w'(\ell)=0. \end{cases} \tag{21}$$

Multiplying Eq. (20) by $\dot w$ and integrating gives the relation

$$\int_0^{\ell_1}\dot w'(y)w'(y)\,\mathrm{d}y=\mu(\rho)\int_0^{\ell_1}\rho(y)w(y)\dot w(y)\,\mathrm{d}y \tag{22}$$

In the same way, multiplying Eq. (21) by w and integrating gives the relation

$$\int_0^{\ell_1}\dot w'(y)w'(y)\,\mathrm{d}y=\mu(\rho)\int_0^{\ell_1}\rho(y)w(y)\dot w(y)\,\mathrm{d}y$$
$$+\dot\mu(\rho)\int_0^{\ell_1}\rho(y)w^2(y)\,\mathrm{d}y+\mu(\rho)\int_0^{\ell_1}h(y)w^2(y)\,\mathrm{d}y. \tag{23}$$

The combination of (22) and (23) yields

$$\dot{\mu}(\rho) = -\frac{\mu(\rho)}{\int_0^{\ell_1} \rho(y)w^2(y)\,\mathrm{d}y} \int_0^{\ell_1} h(y)w^2(y)\,\mathrm{d}y. \qquad \square$$

3.3. Proof of Theorem 3.1

We will argue by contradiction. Let us suppose the existence of a minimizer $a^\star$ for problem (13). We consider $\rho^\star$, the image of $a^{\star 3}\sqrt{1+a^{\star\prime 2}}$ by the change of variable (14).

Let us denote by $\ell_1^\star$, the image of ℓ by this change of variable. Since $\rho^\star$ is clearly an element of $L^\infty(0,\ell_1^\star)\cap\mathcal{R}_{a_0,S,\ell_1^\star}$, there exists $M^\star > a_0^3$ such that $\rho^\star$ is an element of $\mathcal{R}^{M^\star}_{a_0,S,\ell_1^\star}$. Moreover, using an elementary property of the change of variable and the min–max formula, one can state that the eigenvalue $\mu_1(a^\star)$ verifies

$$\mu_1(a^\star) = \min_{\substack{V \text{ subspace of } H^1(0,\ell_1^\star)\\ \text{of dim } 2}} \max_{v\in V} R(\rho^\star, v). \tag{24}$$

Hence, we have $\mu_1(a^\star) = \mu(\rho^\star)$, where $\mu(\rho^\star)$ is defined as the first non-zero eigenvalue of the problem (20).

1st step: an auxiliary problem.
For $S > 0$, $\ell_1 \in (0, \ell/a_0^2]$ and $M > a_0^3$, let us consider the following problem

$$(\mathcal{P}_{M,\ell_1}) \begin{cases} \min \mu(\rho), \\ \rho \in \mathcal{R}^M_{a_0,S,\ell_1}. \end{cases} \tag{25}$$

The map $\rho \in \mathcal{R}_{a_0,S,\ell_1} \mapsto \mu(\rho)$ is continuous for the L^∞ weak-$*$ topology. To prove this, it suffices to adapt the proof in the Dirichlet case (see for example [7]). The set $\mathcal{R}^M_{a_0,S,\ell_1}$ is compact for this topology. This yields the existence of a minimizer for this problem. We denote by ρ_{M,ℓ_1}, a minimizer for $(\mathcal{P}_{M,\ell_1})$.

We prove now the following lemma, which gives an interesting precision on the profile of the solution of problem (25).

Lemma 3.2. *The solution ρ_{M,ℓ_1} of problem* (25) *is a bang–bang function. More precisely, there exists two real numbers ξ_1 and ξ_2 such that $\xi_1 \leqslant \xi_2$ and the function ρ_{M,ℓ_1} verifies*

$$\rho_{M,\ell_1}(y) = \begin{cases} M & \text{on } (0,\xi_1), \\ a_0^3 & \text{on } (\xi_1,\xi_2), \\ M & \text{on } (\xi_2,\ell_1) \end{cases} \quad \text{and} \quad \int_0^{\ell_1} \rho_{M,\ell_1}(y)\,\mathrm{d}y = S.$$

Moreover, the eigenfunction w associated to $\mu(\rho_{M,\ell_1})$ verifies $w^2(\xi_1) = w^2(\xi_2)$.

We prove now this lemma. For that purpose, let us introduce the Lagrangian of this problem, denoted by $\mathcal{L}$ and defined, for $(\rho,\lambda) \in \mathcal{R}^M_{a_0,S,\ell_1} \times \mathbf{R}_+$ by

$$\mathcal{L}(\rho,\lambda) := \mu(\rho) + \lambda\left(\int_0^{\ell_1} \rho(y)\,\mathrm{d}y - S\right).$$

The first order optimality conditions give the existence of a pair $(\rho_{M,\ell_1},\lambda) \in \mathcal{R}^M_{a_0,S,\ell_1} \times \mathbf{R}_+$ such that $\langle \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\rho}(\rho_{M,\ell_1},\lambda), h\rangle \geqslant 0$, for all admissible perturbation h, which can be written, by Lemma 3.1

$$\int_0^{\ell_1} h(y)\big(-\mu(\rho_{M,\ell_1})w^2(y) + \lambda\big)\,\mathrm{d}y \geqslant 0. \tag{26}$$

Let us introduce the sets

- $\mathcal{I}_0(\rho_{M,\ell_1})$: any element of the class of subsets of $[0,\ell_1]$ in which $\rho_{M,\ell_1}(y)=a_0^3$ a.e.;
- $\mathcal{I}_M(\rho_{M,\ell_1})$: any element of the class of subsets of $[0,\ell_1]$ in which $\rho_{M,\ell_1}(y)=M$ a.e.;
- $\mathcal{I}_\star(\rho_{M,\ell_1})$: any element of the class of subsets of $[0,\ell_1]$ in which $a_0^3<\rho_{M,\ell_1}(y)<M$ a.e.

We write

$$\mathcal{I}_\star(\rho_{M,\ell_1}):=\bigcup_{k=1}^{+\infty}\left\{y\in(0,\ell_1):\ a_0^3+\frac{1}{k}<\rho_{M,\ell_1}(y)<M-\frac{1}{k}\right\}=\bigcup_{k=1}^{+\infty}\mathcal{I}_{\star,k}(\rho_{M,\ell_1}).$$

We want to prove that $\mathcal{I}_{\star,k}(\rho_{M,\ell_1})$ has zero measure, for all integer $k\neq 0$. We argue by contradiction.

Let us assume that one of these sets $\mathcal{I}_{\star,k}(\rho_{M,\ell_1})$ is of positive measure. For any $y_0\in\mathcal{I}_{\star,k}(\rho_{M,\ell_1})$ and any measurable sequence of subsets $(G_{k,n})_{n\geqslant 0}\subset\mathcal{I}_{\star,k}(\rho_{M,\ell_1})$ containing y_0, perturbations $\rho_{M,\ell_1}+th$ and $\rho_{M,\ell_1}-th$ are admissible for t small enough. Let us choose $h=\chi_{G_{k,n}}$. Then

$$\begin{aligned}\left\langle\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\rho}(\rho_{M,\ell_1},\lambda),h\right\rangle &= \int_0^{\ell_1}h(y)\big(-\mu(\rho_{M,\ell_1})w^2(y)+\lambda_{M,\ell_1}\big)\,\mathrm{d}y=0\\ &\Longleftrightarrow \int_{G_{k,n}}\big(-\mu(\rho_{M,\ell_1})w^2(y)+\lambda_{M,\ell_1}\big)\,\mathrm{d}y=0.\end{aligned}$$

We can deduce that $\int_{G_{k,n}}(-\mu(\rho_{M,\ell_1})w^2(y)+\lambda)\,\mathrm{d}y=0$. We divide by $|G_{k,n}|$ and we let $G_{k,n}$ shrink to y_0 as $n\to+\infty$.

The Lebesgue density theorem shows that $w^2(y_0)=\frac{\mu(\rho_{M,\ell_1})}{\lambda}$, a.e. for $y_0\in\mathcal{I}_{\star,k}(\rho_{M,\ell_1})$. This is clearly a contradiction, since $\mu(\rho_{M,\ell_1})$ is a non-zero eigenvalue and this justifies that the associated eigenfunction cannot be constant on a set of non-zero measure.

This proves that $|\mathcal{I}_{\star,k}(\rho_{M,\ell_1})|=0$ and then $\mathcal{I}_\star(\rho_{M,\ell_1})$ has also zero measure, which implies that ρ_{M,ℓ_1} equals a_0^3 or M almost everywhere.

Moreover, standard arguments on the nodal domains (see [5] and [1]) show that w, the eigenfunction associated to $\mu(\rho_{M,\ell_1})$ has two nodal domains. We choose to normalize w by taking $w(0)\geqslant 0$ and hence $w(\ell_1)\leqslant 0$. On the set $\{w\geqslant 0\}$, w is concave and since $w'(0)=0$, $w'\leqslant 0$ on this set and w is decreasing. On the set $\{w\leqslant 0\}$, w is convex and since $w'(\ell_1)=0$, $w'\leqslant 0$ on this set and w is decreasing. It follows that w is monotone decreasing.

Since ρ_{M,ℓ_1} is *bang–bang* and by the optimality conditions, we know that

- $w^2(y_0)\leqslant\frac{\mu(\rho_{M,\ell_1})}{\lambda}$ on $\mathcal{I}_0(\rho_{M,\ell_1})$.
- $w^2(y_0)\geqslant\frac{\mu(\rho_{M,\ell_1})}{\lambda}$ on $\mathcal{I}_M(\rho_{M,\ell_1})$.

Moreover, let us notice that if ρ_1 and ρ_2 denote two functions of $\mathcal{R}^M_{a_0,S,\ell_1}$ such that $\rho_1\leqslant\rho_2$ almost everywhere, then, we clearly have $\mu(\rho_2)\leqslant\mu(\rho_1)$ by formulae (24).

Hence, it follows that $\int_0^{\ell_1}\rho_{M,\ell_1}(y)\,\mathrm{d}y=S$.

We deduce immediately from this the existence of two real numbers ξ_1 and ξ_2 such that function ρ_{M,ℓ_1} verifies

$$\rho_{M,\ell_1}(y)=\begin{cases}M & \text{on } (0,\xi_1),\\ a_0^3 & \text{on } (\xi_1,\xi_2),\\ M & \text{on } (\xi_2,\ell_1^\star)\end{cases}\quad\text{and}\quad\int_0^{\ell_1}\rho_{M,\ell_1}(y)\,\mathrm{d}y=S.$$

The fact that $w^2(\xi_1)=w^2(\xi_2)$ is an immediate consequence of the construction of the optimum. The graph below illustrates this construction.

Consequence of the 1st step on $\rho^\star$.

By definition, one has $\rho_{M^\star,\ell_1^\star}\in\operatorname{argmin}\{\mu_1(\rho),\ \rho\in\mathcal{R}^{M^\star}_{a_0,S,\ell_1^\star}\}$. Since $\rho^\star\in\mathcal{R}^{M^\star}_{a_0,S,\ell_1^\star}$, we obviously have $\mu_1(\rho_{M^\star,\ell_1^\star})\leqslant\mu_1(\rho^*)$. In the next step, we will (in particular) consider small perturbations of $\rho_{M^\star,\ell_1^\star}$ and prove that it is possible to exhibit an element ρ of $\mathcal{R}_{a_0,S,\ell_1^*}$ such that $\mu_1(\rho)<\mu_1(\rho^\star)$.

2nd step: Variations around of the optimum $\rho^\star$.

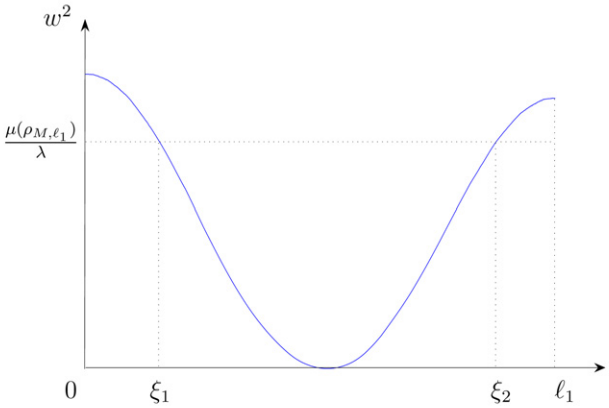

Fig. 1. Representation of the eigenfunction w^2 and construction of ξ_1 and ξ_2.

Let us denote by

- $\rho_\varepsilon^\star$, an element of $\mathcal{R}_{a_0,S,\ell_1^\star}^{M^\star+\varepsilon}$ verifying

$$\rho_\varepsilon^\star(y)=\begin{cases} M^\star+\varepsilon & \text{on } (0,\xi_1'),\\ a_0^3 & \text{on } (\xi_1',\xi_2'),\\ M^\star+\varepsilon & \text{on } (\xi_2',\ell_1^\star),\end{cases}$$

  with $0\leqslant\xi_1'\leqslant\xi_1<\xi_2\leqslant\xi_2'\leqslant\ell_1^\star$ and $\int_0^{\ell_1^\star}\rho_\varepsilon^\star(y)\,\mathrm{d}y=S$ (the representation of a possible function $\rho_\varepsilon^\star$ is done in Appendix A.1).
- ρ_{M,ℓ_1} a minimizer of $\{\mu(\rho),\ \rho\in\mathcal{R}_{a_0,S,\ell_1}^M\}$, as before. In other words, $\rho_{M,\ell_1}\in\operatorname{argmin}\{\mu(\rho),\ \rho\in\mathcal{R}_{a_0,S,\ell_1}^M\}$.

One has obviously $\mu(\rho_{M^\star+\varepsilon,\ell_1^\star})=\min\{\mu(\rho),\ \rho\in\mathcal{R}_{a_0,S,\ell_1^\star}^{M^\star+\varepsilon}\}$. Then, since $\mathcal{R}_{a_0,S,\ell_1^\star}^{M^\star}\subset\mathcal{R}_{a_0,S,\ell_1^\star}^{M^\star+\varepsilon}$, we have $\mu(\rho_{M^\star+\varepsilon,\ell_1^\star})\leqslant\mu(\rho^\star)$, and by virtue of Lemma A.1, $\mu(\rho_{M^\star+\varepsilon,\ell_1^\star})\leqslant\mu(\rho_\varepsilon^\star)<\mu(\rho^\star)$.

Let us recall the existence of a upper bound for ℓ_1: if $a\in\mathcal{A}_{a_0,S}$, then, necessary, $\ell_1=\int_0^\ell\frac{\mathrm{d}t}{a^2(t)}\leqslant\frac{\ell}{a_0^2}$. That is why we will now consider that ℓ_1 is an element of $(0,\ell/a_0^2)$.

Then, let us choose $M>M^\star+\varepsilon$ and $\ell_1<\frac{\ell}{a_0^2}$. By the conclusion of the first step, we know that ρ_{M,ℓ_1} is a *bang–bang* function which verifies $\int_0^{\ell_1}\rho(y)\,\mathrm{d}y=S$. Moreover, by virtue of Lemma A.2, we know that $\ell_1\in(0,\ell/a_0^2)\mapsto\mu(\rho_{M,\ell_1})$ is a decreasing function. Let M (resp. ℓ_1) going to $+\infty$ (resp. ℓ/a_0^2). The achievement of the upper constraint ($\int_0^{\ell_1}\rho_{M,\ell_1}(y)\,\mathrm{d}y=S$) and the *bang–bang* profile of ρ_{M,ℓ_1} prove the existence of a real $t\in[0,1]$ such that (ρ_{M,ℓ_1}) converges when $M\to+\infty$ and $\ell_1\to\ell/a_0^2$ up to a subsequence in the sense of measure to

$$\rho_\infty:=a_0^3+(S-a_0\ell)\big(t\delta_0+(1-t)\delta_{\ell/a_0^2}\big).$$

This leads us to define $\mu(\rho_\infty)$ as the solution of the following eigenvalue problem

$$\begin{cases} -v''=\mu a_0^3v+\mu(S-a_0\ell)(t\delta_0+(1-t)\delta_{\ell/a_0^2})v, & y\in(0,\ell/a_0^2),\\ v'(0)=v'(\ell/a_0^2)=0.\end{cases}\tag{27}$$

By taking the variational formulation of this problem and using the change of variable $y=x/a_0^2$, we can easily show that first eigenvalue of problem (27) is the first non-zero eigenvalue of

$$\begin{cases} -u''=\frac{\mu}{a_0}u, & x\in(0,\ell),\\ u'(0)-\mu(S-a_0\ell)tu(0)=0,\\ u'(\ell)+\mu(S-a_0\ell)(1-t)u(\ell)=0,\end{cases}\tag{28}$$

where $u(x)=v(y)$, for all $x\in[0,\ell]$ and $y\in[0,\frac{\ell}{a_0^2}]$. Moreover, $\mu(\rho_{M,\ell_1})$ converges to $\mu(\rho_\infty)$ when M goes to $+\infty$. It can be easily proved by standard arguments (adapting e.g. the proof of Appendix A in [8]), and by construction, $\mu(\rho_\infty)<\mu(\rho^\star)$.

Remark 3.2. The well-possedness of problems (27) and (28) is well known. For example, one can refer to [17].

3rd step: Conclusion.

Let us denote by $(a_n)_{n\in\mathbf{N}}$, a sequence of functions of $\mathcal{A}_{a_0,S}$ which verifies:

1. $a_n\sqrt{1+a_n'^2}\underset{n\to\infty}{\rightharpoonup} a_0+(S-a_0\ell)(t\delta_0+(1-t)\delta_\ell)$ in the sense of measure.
2. $a_n\xrightarrow[n\to+\infty]{L^\infty(0,\ell)} a_0$.
3. $\int_0^\ell a_n(x)\sqrt{1+a_n'^2(x)}\,\mathrm{d}x=S$.

The construction of such a sequence will be done in Section 3.4. Then, by the same classical argument as before, one can prove that the sequence $(\mu(a_n))_{n\in\mathbf{N}}$ converges to $\mu(\rho_\infty)$. However, we have seen in the previous step that $\mu(\rho_\infty)\leqslant\mu(\rho_\varepsilon^\star)<\mu(\rho^\star)$ and we have consequently found a better function than $\rho^\star$ for our criterion, which is absurd.

Direct consequence: The theorem is proved and the sequence $(a_n)_{n\in\mathbf{N}}$ constructed above is a minimizing sequence of $\mu_1(a)$.

3.4. An example of minimizing sequence

Let n be a non-zero integer and $(u_n)_{n\geqslant 0}$, the sequence of functions defined on the interval $[0,\ell]$ by

$$u_n(x)=\begin{cases}\sqrt{n^2-(-x+n)^2} & \text{on } \big[0,\frac{1}{2n^2}\big];\\ \sqrt{n^2-(x+n-\frac{1}{n^2})^2} & \text{on } \big[\frac{1}{2n^2},\frac{1}{n^2}\big];\\ u_n(x-\frac{i}{n^2}) & \text{on } \big[\frac{i}{n^2},\frac{i+1}{n^2}\big],\ \forall i\in\{1,\dots,n-1\};\\ 0 & \text{on } \big[\frac{1}{n},\ell\big].\end{cases}$$

Let $(a_n)_{n\geqslant 0}$ be the sequence defined by

$$\forall x\in[0,\ell],\quad a_n(x)=a_0+(S-a_0\ell)\big(tu_n(x)+(1-t)u_n(\ell-x)\big).$$

Then, it is easy to verify that

$$\begin{cases} a_n\xrightarrow[n\to+\infty]{\|\cdot\|_\infty} a_0;\\ a_n\sqrt{1+{a_n}'^2}=a_0+(S-a_0\ell)[tn\chi_{[0,\frac{\ell}{n}]}+(1-t)n\chi_{[\ell-\frac{\ell}{n},\ell]}];\\ a_n\sqrt{1+{a_n}'^2}\underset{n\to+\infty}{\rightharpoonup} a_0+(S-a_0\ell)(t\delta_0+(1-t)\delta_\ell);\\ \int_0^\ell a_n(x)\sqrt{1+{a_n}'^2(x)}\,\mathrm{d}x=S.\end{cases}$$

Fig. 2 represents the sequence $(u_n)_{n\in\mathbf{N}}$ used to build the sequence $(a_n)_{n\in\mathbf{N}}$ in every case.

3.5. Some remarks on the problem (13)

Remark 3.3. Relaxation of problem (13).

Since we have proved the non-existence of a solution for the problem of the minimization of $\mu_1(a)$ in the class $\mathcal{A}_{a_0,S}$, it seems natural to define a relaxed problem. Let us define the imbedding τ by

$$\begin{aligned}\tau:\quad W^{1,\infty}(0,\ell)&\hookrightarrow L^\infty(0,\ell)\times\mathcal{M}_b(0,\ell),\\ a&\longmapsto\big(a,a\sqrt{1+a'^2}\big),\end{aligned}$$

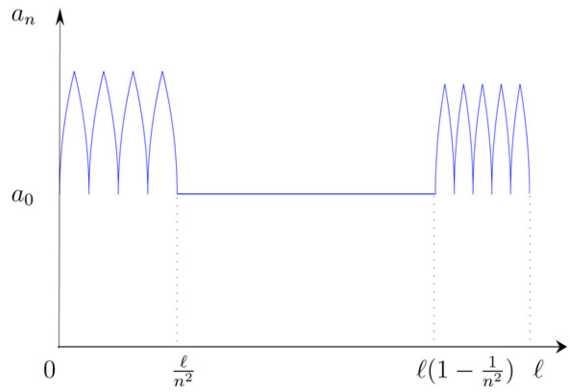

Fig. 2. Representation of a minimizing sequence.

where $\mathcal{M}_b(0,\ell)$ denotes the set of bounded Radon measures. Let us introduce $\widehat{\mathcal{A}}_{a_0,S}$, the completion of $\mathcal{A}_{a_0,S}$ for the topology induced by τ. Then, it is possible to define $\widehat{\mu}_1(a,b)$ as the second eigenvalue of the following problem

$$\begin{cases} -(a^2u')' = \widehat{\mu}(a,b)bu, & x \in (0,\ell), \\ u'(0) = u'(\ell) = 0. \end{cases}$$

Since b is a measure, this problem has to be understood with its variational formulation. Moreover, the existence of $\widehat{\mu}(a,b)$ is a direct consequence of the classical spectral decomposition theorem and one has

- $\widehat{\mu}_1$ is an extension of μ_1 in the class $\widehat{\mathcal{A}}_{a_0,S}$.
- $\inf\{\mu_1(a),\ a \in \mathcal{A}_{a_0,S}\} = \min\{\widehat{\mu}_1(a,b),\ (a,b) \in \widehat{\mathcal{A}}_{a_0,S}\}$.

Remark 3.4. Generalization of problem (13).

Let us introduce the generalized problem, consisting in minimizing $\mu_k(a)$, the kth non-zero eigenvalue of problem (3), with $k \geqslant 1$, among the elements of $\mathcal{A}_{a_0,S}$. The same result as before holds for this problem

Theorem 3.2. *Let S and a_0 be two (strictly) positive real numbers.*

The following problem:

$$\begin{cases} \min \mu_k(a), \\ a \in \mathcal{A}_{a_0,S} \end{cases} \tag{29}$$

has no solution. Moreover there exists $k+2$ elements of $[0,\ell]$ $\xi_0 = 0, \xi_1, \ldots, \xi_{k+1} = \ell$ and $k+2$ elements $t_0, \ldots, t_{k+1}$ of $[0,1]$ which verify $\sum_{i=0}^{k+1} t_i = 1$, such that any $(a_n)_{n\in\mathbf{N}}$ satisfying

$$\begin{cases} a_n \xrightarrow[n\to+\infty]{\|\cdot\|_\infty} a_0; \\ a_n\sqrt{1+{a_n'}^2} = a_0 + (S - a_0\ell)\sum_{i=0}^{k+1} t_i\delta_{\xi_i}; \\ \int_0^\ell a_n(x)\sqrt{1+{a_n'}^2(x)}\,\mathrm{d}x = S. \end{cases}$$

is a minimizing sequence of elements of $\mathcal{A}_{a_0,S}$ for the criterion $\mu_k(a)$.

The proof of this theorem is just an adaptation of the proof of Theorem 3.1. The principle is exactly the same as before. The main difference comes from the profile of w, the eigenfunction associated to $\mu_k(a)$. We have to notice that w has $k+1$ nodal domains (to prove this, one can refer to [1] or [5]), which implies that the solution of the following problem (after the change of variable (1.4))

$$\begin{cases} \min \mu_k(\rho), \\ \rho \in \mathcal{R}^M_{S,a_0,\ell_1}, \end{cases}$$

for some $\ell_1 > 0$ and $M > a_0^3$, is a *bang–bang* function, with $k+1$ discontinuities. This profile explains the construction of the new minimizing sequence.

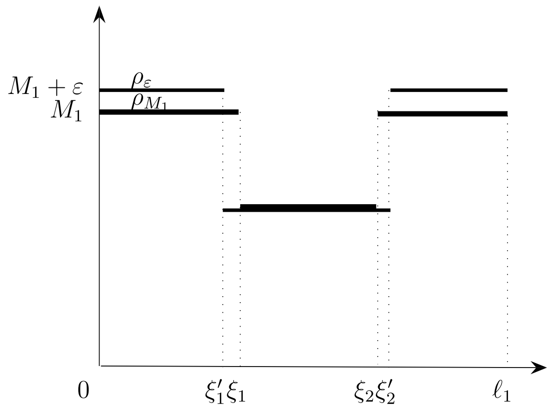

Fig. 3. Representation of functions ρ_{M_1} and ρ_ε.

Appendix A. Monotonicity of $\mu(\rho)$ with respect to some parameters

A.1. Monotonicity of $\mu(\rho)$ with respect to M

Lemma A.1. *Let M_1 and M_2 be two real numbers such that $M_2 > M_1 > a_0^3$. Let ℓ_1 and S be two (strictly) positive numbers. Then*

$$\min\{\mu(\rho),\ \rho \in \mathcal{R}^{M_2}_{a_0,S,\ell_1}\} < \min\{\mu(\rho),\ \rho \in \mathcal{R}^{M_1}_{a_0,S,\ell_1}\}.$$

Proof. Since $\mathcal{R}^{M_2}_{a_0,S,\ell_1} \supset \mathcal{R}^{M_1}_{a_0,S,\ell_1}$, we clearly have

$$\min\{\mu(\rho),\ \rho \in \mathcal{R}^{M_2}_{a_0,S,\ell_1}\} \leqslant \min\{\mu(\rho),\ \rho \in \mathcal{R}^{M_1}_{a_0,S,\ell_1}\}.$$

Let us denote by ρ_{M_1}, the solution of the optimization problem $\min\{\mu(\rho),\ \rho \in \mathcal{R}^{M_1}_{a_0,S,\ell_1}\}$. We have already seen that $\int_0^{\ell_1} \rho_{M_1}(y)\,\mathrm{d}y = S$ and that

$$\rho_{M_1}(y) = \begin{cases} M_1 & \text{on } (0,\xi_1), \\ a_0^3 & \text{on } (\xi_1,\xi_2), \\ M_1 & \text{on } (\xi_2,\ell_1), \end{cases}$$

for some ξ_1 and ξ_2 such that $0 \leqslant \xi_1 < \xi_2 \leqslant \ell_1$. Let h be a perturbation of ρ_{M_1} in $\mathcal{R}_{a_0,S,\ell_1}$ such that the function ρ_ε defined for some $\varepsilon > 0$ such that $M_1 + \varepsilon < M_2$ by $\rho_\varepsilon := \rho_{M_1} + h$ verifies

$$\rho_\varepsilon(y) = \begin{cases} M_1+\varepsilon & \text{on } (0,\xi_1'), \\ a_0^3 & \text{on } (\xi_1',\xi_2'), \\ M_1+\varepsilon & \text{on } (\xi_2',\ell_1), \end{cases}$$

with $0 \leqslant \xi_1' \leqslant \xi_1 < \xi_2 \leqslant \xi_2' \leqslant \ell_1$ and $\int_0^{\ell_1} \rho_\varepsilon(y)\,\mathrm{d}y = S$. Such a choice of ξ_1' and ξ_2' is always possible. This is illustrated by Fig. 3.

We now prove that $\mu(\rho_\varepsilon) < \min\{\mu(\rho), \rho \in \mathcal{R}^{M_1}_{a_0,S,\ell_1}\}$ for some $\varepsilon > 0$ small enough.

By Lemma 3.1, one can write

$$\mu(\rho_\varepsilon) - \mu(\rho_{M_1}) = -\mu(\rho_{M_1}) \int_0^{\ell_1} w^2(y)h(y)\,\mathrm{d}y + \underset{\varepsilon\to 0}{o}(\varepsilon). \tag{A.1}$$

Let us recall that w denotes the eigenfunction associated to $\mu(\rho)$. Since $h = \rho_\varepsilon - \rho_{M_1}$, one has

$$\int_0^{\ell_1} w^2(y)h(y)\,\mathrm{d}y \underset{\varepsilon\to 0}{\sim} \varepsilon\left(\int_0^{\xi_1'} w^2(y)\,\mathrm{d}y + \int_{\xi_2'}^{\ell_1} w^2(y)\,\mathrm{d}y\right) + \left(a_0^3 - M_1\right)\left(\int_{\xi_1}^{\xi_1'} w^2(y)\,\mathrm{d}y + \int_{\xi_2}^{\xi_2'} w^2(y)\,\mathrm{d}y\right).$$

Using the facts that $\int_0^{\ell_1} \rho_{M_1}(y)\,\mathrm{d}y = \int_0^{\ell_1} \rho_\varepsilon(y)\,\mathrm{d}y = S$ and $w^2(\xi_1) = w^2(\xi_2)$ (which comes from the optimality conditions detailed in Section 3.3), an expansion at the first order yields

$$\int_{\xi_1}^{\xi_1'} w^2(y)\,\mathrm{d}y + \int_{\xi_2}^{\xi_2'} w^2(y)\,\mathrm{d}y \underset{\varepsilon\to 0}{\sim} (\xi_1 - \xi_1' + \xi_2' - \xi_2)w^2(\xi_1) \underset{\varepsilon\to 0}{\sim} \varepsilon\frac{S - a_0^3\ell_1}{(M_1 - a_0^3)^2} w^2(\xi_1).$$

And according to the profile of w^2 (see Fig. 1), one can deduce that

$$\int_0^{\ell_1} w^2(y)h(y)\,\mathrm{d}y \underset{\varepsilon\to 0}{\sim} \varepsilon\left(\int_0^{\xi_1} w^2(y)\,\mathrm{d}y + \int_{\xi_2}^{\ell_1} w^2(y)\,\mathrm{d}y - \frac{S - a_0^3\ell_1}{M_1 - a_0^3} w^2(\xi_1)\right) > \varepsilon\left(w^2(\xi_1)(\ell_1 - \xi_2 + \xi_1) - \frac{S - a_0^3\ell_1}{M_1 - a_0^3} w^2(\xi_1)\right) = 0.$$

The previous inequality associated with formula (A.1) give the desired result. □

A.2. Monotonicity of $\mu(\rho)$ with respect to ℓ_1

Lemma A.2. *Let a_0, S and $M > a_0^3$ be three real (strictly) positive numbers.*
The map $\ell_1 \in \mathbf{R}_+ \mapsto \min\{\mu(\rho), \rho \in \mathcal{R}^M_{a_0,S,\ell_1}\}$ is strictly decreasing.

Proof. Like in the proof of Lemma A.1, let us consider a function ρ_{ℓ_1} realizing the minimum of μ in the class $\mathcal{R}^M_{a_0,S,\ell_1}$. We consider $\rho_\varepsilon := \rho_{\ell_1} + h$, where h denotes the perturbation

$$h := -\left(M - a_0^3\right)[\chi_{[\xi_1',\xi_1]} + \chi_{[\xi_2,\xi_2']}] + M\chi_{[\ell_1,\ell_1+\varepsilon]},$$

ε, ξ_1' and ξ_2' are chosen such that the following equality holds

$$\int_0^{\ell_1} \rho_{\ell_1}(y)\,\mathrm{d}y = \int_0^{\ell_1+\varepsilon} \rho_\varepsilon(y)\,\mathrm{d}y = S.$$

We use the same notations as in the proof of Lemma A.1. Fig. 4 represents the profile of ρ_{ℓ_1} and ρ_ε.

According to Lemma 3.1, one has the following expansion

$$\mu(\rho_\varepsilon) - \mu(\rho_{\ell_1}) = \mu(\rho_{\ell_1})\left(M - a_0^3\right)\int_{[\xi_1',\xi_1]\cup[\xi_2,\xi_2']} w^2(y)\,\mathrm{d}y - \mu(\rho_{\ell_1})M\int_{[\ell_1,\ell_1+\varepsilon]} w^2(y)\,\mathrm{d}y + \underset{\varepsilon\to 0}{o}(\varepsilon).$$

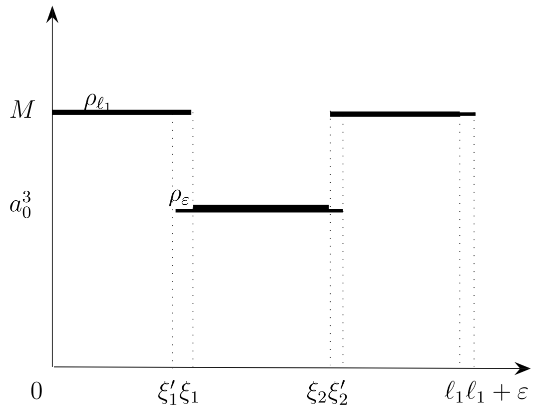

Fig. 4. Representation of functions ρ_{ℓ_1} and ρ_ε.

As before, by noticing that $\int_0^{\ell_1} \rho_{\ell_1}(y)\,\mathrm{d}y = \int_0^{\ell_1+\varepsilon} \rho_\varepsilon(y)\,\mathrm{d}y = S$, one can write

$$\begin{aligned}\mu(\rho_\varepsilon) - \mu(\rho_{\ell_1}) &\underset{\varepsilon\to 0}{\sim} -\mu(\rho_{\ell_1})\varepsilon\big(\big(M - a_0^3\big)w^2(\xi_1)\big(\xi_1' - \xi_1 - \xi_2' + \xi_2\big) + Mw^2(\ell_1)\big)\\ &\underset{\varepsilon\to 0}{\sim} -\mu(\rho_{\ell_1})\varepsilon M\big(w^2(\ell_1) - w^2(\xi_1)\big) < 0.\end{aligned}$$

The conclusion follows. □

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