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Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints

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Abstract

We consider the well-known following shape optimization problem:

$$\lambda_1(\Omega^*) = \min_{\substack{|\Omega|=a \\ \Omega\subset D}} \lambda_1(\Omega),$$

where λ_1 denotes the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary condition, and D is an open bounded set (a box). It is well-known that the solution of this problem is the ball of volume a if such a ball exists in the box D (Faber–Krahn's theorem).

In this paper, we prove regularity properties of the boundary of the optimal shapes Ω^* in any case and in any dimension. Full regularity is obtained in dimension 2.

Keywords: Shape optimization; Eigenvalues of the Laplace operator; Regularity of free boundaries

1. Introduction and main results

Let D be a bounded open subset of $\mathbb{R}^d$. For all open subset Ω of D, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Laplace operator in Ω, with homogeneous boundary conditions, and by u_Ω a normalized eigenfunction, that is

$$\begin{cases} -\Delta u_\Omega = \lambda_1(\Omega) u_\Omega & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u_\Omega^2 = 1. \end{cases}$$

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We are interested here in the regularity of the optimal shapes of the following shape optimization problem, where $a \in (0, |D|)$ ($|D|$ denotes the Lebesgue measure of D):

$$\begin{cases} \Omega^* \text{ open}, \quad \Omega^* \subset D, \quad |\Omega^*| = a, \\ \lambda_1(\Omega^*) = \min\{\lambda_1(\Omega); \Omega^* \text{ open}, \ \Omega \subset D, \ |\Omega| = a\}. \end{cases} \tag{1}$$

By a well-known theorem of Faber and Krahn, if there is a ball $B \subset D$ with $|B| = a$, then this ball is an optimal shape and it is unique, up to translations (and up to sets of zero capacity).

Here we address the question of existence of a **regular** optimal set in all cases.

Existence of a *quasi-open* optimal set Ω^* may be deduced from a general existence result by G. Buttazzo and G. Dal Maso (see [5]) for an extended version of (1), where the variable sets Ω are not necessarily open. An optimal shape Ω^* may not be more than a quasi-open set if D is not connected (we reproduce in the appendix the example mentioned in [4]). On the other hand, it is proved in [4] or [12] that such an open optimal set Ω^* always exists for (1) and, if moreover D is connected, then all optimal shapes Ω^* are open. More precisely, it is proved in [4] that, for any D, u_{Ω^*} is locally Lipschitz continuous in D. If moreover D is connected, then Ω^* coincides with the support of u_{Ω^*} (and is therefore open). Let us summarize this as follows (see also [13]):

Proposition 1.1. *Assume D is open and bounded. The problem* (1) *has a solution Ω^*, and u_{Ω^*} is non-negative and locally Lipschitz continuous in D. If D is connected, $\Omega^* = \{x \in D, u_{\Omega^*} > 0\}$.*

Moreover, we have

$$\Delta u_{\Omega^*} + \lambda_1(\Omega^*) u_{\Omega^*} \geqslant 0 \quad \text{in } D, \tag{2}$$

which means that $\Delta u_{\Omega^*} + \lambda_1(\Omega^*) u_{\Omega^*}$ is a positive Radon measure.

Here, we are interested in the regularity of $\partial\Omega^*$ itself, and we prove the following theorem:

Theorem 1.2. *Assume D is open, bounded and connected. Then any solution of* (1) *satisfies*:

1. *Ω^* has locally finite perimeter in D and*
$$\mathcal{H}^{d-1}\big((\partial\Omega^* \setminus \partial^*\Omega^*) \cap D\big) = 0, \tag{3}$$
where $\mathcal{H}^{d-1}$ is the Hausdorff measure of dimension $d-1$, and $\partial^*\Omega^*$ is the reduced boundary (*in the sense of sets with finite perimeter, see* [9] *or* [10]).
2. *There exists $\Lambda > 0$ such that*
$$\Delta u_{\Omega^*} + \lambda_1(\Omega^*) u_{\Omega^*} = \sqrt{\Lambda}\,\mathcal{H}^{d-1}\lfloor\partial\Omega^*,$$
in the sense of distribution in D, where $\mathcal{H}^{d-1}\lfloor\partial\Omega^*$ is the restriction of the $(d-1)$*-Hausdorff measure to $\partial\Omega^*$.*
3. *$\partial^*\Omega^*$ is an analytic hypersurface in D.*
4. *If $d = 2$, then the whole boundary $\partial\Omega^* \cap D$ is analytic.*

We use the same strategy as in [3] (where the regularity is studied for another shape optimization problem). Theorem 1.2 essentially relies on the proof of the equivalence of (1) with a penalized version for the constraint $|\Omega| = a$, as stated in Theorem 1.5 below. Once we have this penalized version, we can use techniques and results from [1] (see also [11] and [3]).

Remark 1.3. According to the results in [1], the third point in Theorem 1.2 is a direct consequence of the second one which says that u_{Ω^*} is a "weak solution" in the sense of [1]. To obtain the full regularity of the boundary for $d = 2$, the fact that u_{Ω^*} is a weak solution is not sufficient, and more information has to be deduced from the variational problem. The approach is essentially the same as in Theorem 6.6 and Corollary 6.7 in [1]. The necessary adjustments are given at the end of this paper.

Remark 1.4. According to the result of [14,15,6,8], it is likely that full regularity of the boundary may be extended to higher dimension ($d \leqslant 6$?), and therefore that the estimate (3) can be improved.

But this needs quite more work and is under study.

By a classical variational principle, we know that, for all $\Omega \subset D$ open,

$$\lambda_1(\Omega) = \int_\Omega |\nabla u_\Omega|^2 = \min\left\{ \int_\Omega |\nabla u|^2, u \in H_0^1(\Omega), \int_\Omega u^2 = 1 \right\}. \tag{4}$$

Here, $\lambda_1(\Omega^*) \leqslant \lambda_1(\Omega)$ for all open set $\Omega \subset D$ with $|\Omega| = a$. Since $[\Omega \subset \widetilde{\Omega} \Rightarrow \lambda_1(\Omega) \geqslant \lambda_1(\widetilde{\Omega})]$, it follows that $\lambda_1(\Omega^*) \leqslant \lambda_1(\Omega)$ for all open set $\Omega \subset D$ with $|\Omega| \leqslant a$. Coupled with (4), this leads to the following variation property of Ω^* and u_{Ω^*} (see [4] for more details), where we denote $u = u_{\Omega^*}$, $\lambda_a = \lambda_1(\Omega^*)$, and $\Omega_v = \{x \in D; v(x) \neq 0\}$:

$$\lambda_a = \int_D |\nabla u|^2 = \min\left\{ \int_D |\nabla v|^2; v \in H_0^1(D), \int_D v^2 = 1, |\Omega_v| \leqslant a \right\}. \tag{5}$$

Let us rewrite this as follows. For $w \in H_0^1(D)$, we denote $J(w) = \int_D |\nabla w|^2 - \lambda_a \int_D w^2$. Then applying (5) with $v = w/(\int_D w^2)^{1/2}$, we obtain that u is a solution of the following optimization problem:

$$J(u) \leqslant J(w), \quad \text{for all } w \in H_0^1(D), \text{ with } |\Omega_w| \leqslant a. \tag{6}$$

One of the main ingredient in the proof of Theorem 1.2 is to improve the variational property (6) in two directions, as stated in Theorem 1.5 below. The approach is local.

Let B_R be a ball included in D and centered on $\partial\Omega_u \cap D$. We define

$$\mathcal{F} = \left\{ v \in H_0^1(D), \ u - v \in H_0^1(B_R) \right\}.$$

For $h > 0$, we denote by $\mu_-(h)$ the biggest $\mu_- \geqslant 0$ such that,

$$\forall v \in \mathcal{F} \text{ such that } a - h \leqslant |\Omega_v| \leqslant a, \quad J(u) + \mu_-|\Omega_u| \leqslant J(v) + \mu_-|\Omega_v|. \tag{7}$$

We also define $\mu_+(h)$ as the smallest $\mu_+ \geqslant 0$ such that,

$$\forall v \in \mathcal{F} \text{ such that } a \leqslant |\Omega_v| \leqslant a + h, \quad J(u) + \mu_+|\Omega_u| \leqslant J(v) + \mu_+|\Omega_v|. \tag{8}$$

The following theorem is a main step in the proof of Theorem 1.2:

Theorem 1.5. *Let u, B_R and $\mathcal{F}$ as above. Then for R small enough (depending only on u, a and D), there exists $\Lambda > 0$ and $h_0 > 0$ such that,*

$$\forall h \in (0, h_0), \quad 0 < \mu_-(h) \leqslant \Lambda \leqslant \mu_+(h) < +\infty,$$

and, moreover,

$$\lim_{h\to 0} \mu_+(h) = \lim_{h \to 0} \mu_-(h) = \Lambda. \tag{9}$$

Remark 1.6. We can compare the existence of $\mu_+(h)$ with Theorem 2.9 in [4]. This theorem shows that there exists μ_+ such that

$$\int_D |\nabla u|^2 \leqslant \int_D |\nabla v|^2 + \lambda_a \left[1 - \int_D v^2 \right]^+ + \mu_+\big(|\Omega_v| - a\big),$$

for $v \in H_0^1(D)$ and $|\Omega_v| \geqslant a$. The difference with [4] is that, in (8), we have the term $\lambda_a[1 - \int_D v^2]$ (not only the positive part), but we allowed only perturbations in B_R. We cannot expect to have something like (8) for perturbations in all D (because we may find v with $|\Omega_v| > a$ and $J(v) < 0$, so $\lim_{t\to+\infty} J(tv) = -\infty$).

In the next section, we will prove Theorem 1.5. In the third section, we will prove Theorem 1.2. In Appendix A, we discuss the case D non-connected.

2. Proof of Theorem 1.5

In the next lemma, we give an Euler–Lagrange equation for our problem. The proof follows the steps of the Euler–Lagrange equation in [7].

Lemma 2.1 *(Euler–Lagrange equation). Let u be a solution of* (6). *Then there exists $\Lambda \geqslant 0$ such that, for all $\Phi \in C_0^\infty(D, \mathbb{R}^d)$,*

$$\int_D 2(D\Phi\nabla u, \nabla u) - \int_D |\nabla u|^2 \nabla \cdot \Phi + \lambda_a \int_D u^2 \nabla \cdot \Phi = \Lambda \int_{\Omega_u} \nabla \cdot \Phi.$$

Proof. We start by a general remark that will be useful in the rest of the paper. If $v \in H_0^1(D)$ and if $\Phi \in C_0^\infty(D, \mathbb{R}^d)$, we define $v_t(x) = v(x + t\Phi(x))$; therefore, for t small enough, $v_t \in H_0^1(D)$. A simple calculus gives (when t goes to 0),

$$|\Omega_{v_t}| = |\Omega_v| - t \int_{\Omega_v} \nabla \cdot \Phi + \mathrm{o}(t),$$

$$J(v_t) = J(v) + t\left(\int_D 2(D\Phi\nabla v \cdot \nabla v) - \int_D |\nabla v|^2 \nabla \cdot \Phi + \lambda_a \int_D v^2 \nabla \cdot \Phi\right) + \mathrm{o}(t).$$

Now we apply this with $v = u$ and Φ such that $\int_{\Omega_u} \nabla \cdot \Phi > 0$. Such a Φ exists, otherwise we would get, using that D is connected, $\Omega_u = D$ or $\emptyset$ a.e. We have $|\Omega_{u_t}| < |\Omega_u|$ for $t > 0$ small enough and, by minimality,

$$\begin{aligned} J(u) &\leqslant J(u_t) \\ &= J(u) + t\left(\int_D 2(D\Phi\nabla u, \nabla u) - \int_D |\nabla u|^2 \nabla \cdot \Phi + \lambda_a \int_D u^2 \nabla \cdot \Phi\right) + \mathrm{o}(t), \end{aligned}$$

and so,

$$\int_D 2(D\Phi\nabla u, \nabla u) - \int_D |\nabla u|^2 \nabla \cdot \Phi + \lambda_a \int_D u^2 \nabla \cdot \Phi \geqslant 0. \tag{10}$$

Now, we take Φ with $\int_{\Omega_u} \nabla \cdot \Phi = 0$. Let Φ_1 be such that $\int_{\Omega_u} \nabla \cdot \Phi_1 = 1$. Writing (10) with $\Phi + \eta\Phi_1$ and letting η goes to 0, we get (10) with this Φ and, using $-\Phi$, we get (10) with an equality instead of the inequality. For a general Φ, we use this equality with $\Phi - \Phi_1(\int_{\Omega_u} \nabla \cdot \Phi)$ (we have $\int_{\Omega_u} \nabla \cdot (\Phi - \Phi_1(\int_{\Omega_u} \nabla \cdot \Phi)) = 0$), and we get the result with

$$\Lambda = \int_D 2(D\Phi_1\nabla u, \nabla u) - \int_D |\nabla u|^2 \nabla \cdot \Phi_1 + \lambda_a \int_D u^2 \nabla \cdot \Phi_1 \geqslant 0,$$

using (10). □

Remark 2.2. We will have to prove that, in fact, $\Lambda > 0$.

Let us remind our notations: let u be a solution of (6), and let B_R be a ball included in D and centered on $\partial\Omega_u \cap D$. We define

$$\mathcal{F} = \left\{v \in H_0^1(D),\ u - v \in H_0^1(B_R)\right\}.$$

Before proving Theorem 1.5, we give the following useful lemma:

Lemma 2.3. *Let u, B_R and $\mathcal{F}$ as above. Then there exists a constant C such that, for R small enough,*

$$\forall v \in \mathcal{F}, \quad J(v) \geqslant \frac{1}{2} \int_{B_R} |\nabla v|^2 - C.$$

Proof. We know that $\lambda_1(B_R) = \lambda_1(B_1)/(R^2)$ (we just use the change of variable $x \to x/R$). If R is small enough we have:

$$\lambda_1(B_R) \geqslant 1, \qquad \frac{4\lambda_a}{\lambda_1(B_R)} \leqslant 1/2. \tag{11}$$

Let $v \in \mathcal{F}$; so $u - v \in H_0^1(B_R)$, and using the variational formulation of $\lambda_1(B_R)$, we get

$$\|u - v\|^2_{L^2(B_R)} \leqslant \frac{\|\nabla(u - v)\|^2_{L^2(B_R)}}{\lambda_1(B_R)}.$$

We deduce that,

$$\begin{aligned}\|v\|^2_{L^2(B_R)} &\leqslant 2\frac{\|\nabla(u - v)\|^2_{L^2(B_R)}}{\lambda_1(B_R)} + 2\|u\|^2_{L^2(B_R)}\\ &\leqslant 4\frac{\|\nabla v\|^2_{L^2(B_R)}}{\lambda_1(B_R)} + \frac{C}{\lambda_a},\end{aligned}$$

(we use (11)) where C depends only on the L^2 norms of u and his gradient. Now we have

$$J(v) \geqslant \int_D |\nabla v|^2 - \lambda_a\left(4\frac{\|\nabla v\|^2_{L^2(B_R)}}{\lambda_1(B_R)} + \frac{C}{\lambda_a}\right),$$

and we get the result using (11). □

Remark 2.4. This lemma is interesting for two reasons. The first one is that J is bounded from below on $\mathcal{F}$. The second one is that, if $v_n \in \mathcal{F}$ is a sequence such that $J(v_n)$ is bounded, then $\|\nabla v_n\|_{L^2(B_R)}$ is also bounded. Since $v_n = u$ outside B_R we deduce that v_n is bounded in $H_0^1(D)$ (and so weakly converges up to a sub-sequence...).

Proof of Theorem 1.5. We divide our proof into four parts. Let $\Lambda \geqslant 0$ be as in Lemma 2.1.

First part: $\Lambda \leqslant \mu_+(h) < +\infty$.

We start the proof by showing that $\mu_+(h)$ is finite. Since B_R is centered on the boundary on $\partial\Omega_u$, we first show:

$$0 < |\Omega_u \cap B_R| < |B_R|.$$

The first inequality comes from the fact that Ω_u is open. The second one comes from the following lemma:

Lemma 2.5. *Let ω be an open subset of D, and let u be a solution of* (6). *If $|\Omega_u \cap \omega| = |\omega|$, then*

$$-\Delta u = \lambda_a u \quad \text{in } \omega,$$

and therefore $\omega \subset \Omega_u$.

Proof of Lemma 2.5. Since $u > 0$ a.e. on ω, we define $v \in H_0^1(D)$ by $v = u$ outside ω and $-\Delta v = \lambda_a u$ in ω. From the strict maximum principle, we get $v > 0$ on ω and $|\Omega_v| = |\Omega_u|$. By minimality ($J(u) \leqslant J(v)$) we have,

$$\int_\omega (\nabla u - \nabla v)\cdot(\nabla u - \nabla v + 2\nabla v) - \lambda_a\int_\omega (u - v)(u + v) \leqslant 0,$$

$$\int_\omega |\nabla u - \nabla v|^2 + \lambda_a\int_\omega (u - v)(2u - u - v) \leqslant 0,$$

(we use that $u - v \in H_0^1(\omega)$ and $-\Delta v = \lambda_a u$ in ω). We get that $u = v$ a.e. in ω and by continuity $u = v > 0$ everywhere in ω. □

If $|\Omega_u \cap B_R| = |B_R|$, applying this lemma to $\omega = B_R$, we would get $\Omega_u \cap B_R = B_R$, which is impossible since B_R is centered on $\partial\Omega_u$. If R is small enough we can also suppose,

$$0 < |\Omega_u \setminus B_R| < |D \setminus B_R|.$$

For the first inequality, we need that $|B_R| < a$, and for the second one we need $a < |D| - |B_R|$.

Let $h > 0$ be such that $h < |B_R| - |\Omega_u \cap B_R|$ (and so, if $v \in \mathcal{F}$ with $|\Omega_v| \leqslant a + h$, then $|\Omega_v \cap B_R| < |B_R|$). Let (μ_n) an increasing sequence to $+\infty$. There exists $v_n \in \mathcal{F}$ such that $|\Omega_{v_n}| \leqslant a + h$ and,

$$J(v_n) + \mu_n\big(|\Omega_{v_n}| - a\big)^+ = \min_{v\in\mathcal{F},\ |\Omega_v|\leqslant a+h}\big\{J(v) + \mu_n\big(|\Omega_v| - a\big)^+\big\}. \tag{12}$$

For this we use Remark 2.4, and so the functional $J(v) + \mu_n(|\Omega_v| - a)^+$ is bounded by below for $v \in \mathcal{F}$. Moreover, a minimizing sequence for this functional is bounded in $H_0^1(D)$ and so weakly converges in $H_0^1(D)$, strongly in $L^2(D)$ and almost everywhere (up to a sub-sequence) to some v_n. Using the lower semi-continuity of $v \to \int_D |\nabla v|^2$ for the weak convergence, the strong convergence in $L^2(D)$ and the lower semi-continuity of $v \to |\Omega_v|$ for the convergence almost everywhere we see that v_n is such that (12) is true.

If $|\Omega_{v_n}| \leqslant a$ then (8) is true with μ_n, so we will suppose to the contrary that $|\Omega_{v_n}| > a$ for all n.

Step 1. Euler–Lagrange equation for v_n. If we set $b_n = |\Omega_{v_n}|$, then v_n is also solution of

$$J(v_n) = \min_{v\in\mathcal{F},\ |\Omega_v|\leqslant b_n} J(v).$$

With the same proof as in Lemma 2.1, we can write an Euler–Lagrange equation for v_n in B_R. That is, there exists $\Lambda_n \geqslant 0$ such that, for $\Phi \in C_0^\infty(B_R, \mathbb{R}^d)$,

$$\int_D 2(D\Phi\nabla v_n \cdot \nabla v_n) - \int_D |\nabla v_n|^2\nabla\cdot\Phi + \lambda_a\int_D v_n^2\nabla\cdot\Phi = \Lambda_n\int_{\Omega_{v_n}} \nabla\cdot\Phi. \tag{13}$$

Step 2. $\Lambda_n \geqslant \mu_n$. There exists $\Phi \in C_0^\infty(B_R)$ such that $\int_{\Omega_{v_n}} \nabla\cdot\Phi = 1$. Let $v_n^t(x) = v_n(x + t\Phi(x))$. We have $v_n^t \in \mathcal{F}$ for $t \geqslant 0$ small enough, and using derivation results recalled in the proof of Lemma 2.1 and $|\Omega_{v_n}| > a$, we get

$$a < |\Omega_{v_n^t}| = |\Omega_{v_n}| - t + \mathrm{o}(t) \leqslant a + h,$$
$$J(v_n^t) = J(v_n) + t\Lambda_n + \mathrm{o}(t).$$

Now we use (12) with $v = v_n^t$ in order to get,

$$J(v_n) + \mu_n\big(|\Omega_{v_n}| - a\big) \leqslant J(v_n) + t\Lambda_n + \mathrm{o}(t) + \mu_n\big(|\Omega_{v_n}| - t - a\big),$$

and dividing by $t > 0$ and letting t goes to 0, we finally get $\Lambda_n \geqslant \mu_n$.

Step 3. v_n strongly converges to some v. Using (12) with $v = u$, we get

$$J(v_n) + \mu_n\big(|\Omega_{v_n}| - a\big) \leqslant J(u) \tag{14}$$

and so, using Remark 2.4, we can deduce that v_n weakly converge in H_0^1 (up to a sub-sequence) to some $v \in \mathcal{F}$ with $|\Omega_v| \leqslant a + h$. We also have the strong convergence in $L^2(D)$ and the convergence almost everywhere. Since J is bounded from below on $\mathcal{F}$, we see from (14) that $\mu_n(|\Omega_{v_n}| - a)$ is bounded and we get $\lim_{n\to\infty} |\Omega_{v_n}| = a$, and so $|\Omega_v| \leqslant a$. From $J(v_n) \leqslant J(u)$, we get $J(v) \leqslant \liminf J(v_n) \leqslant J(u)$ and so v is a solution of (6). Finally we can write, using (12), that $J(v_n) \leqslant J(v)$ and we get, using the strong convergence of v_n in L^2,

$$\limsup_{n\to\infty} \int_D |\nabla v_n|^2 \leqslant \int_D |\nabla v|^2.$$

We also have, with weak convergence in $H_0^1(D)$ that

$$\int_D |\nabla v|^2 \leqslant \liminf_{n\to\infty} \int_D |\nabla v_n|^2.$$

We deduce that $\lim_{n\to\infty}\|\nabla v_n\|_{L^2(D)} = \|\nabla v\|_{L^2(D)}$. With the weak-convergence, this gives the strong convergence of v_n to v in $H_0^1(D)$.

Step 4. $\boldsymbol{\lim \Lambda_n = \Lambda}$**.** We see that v is a solution of (6), so we can apply Lemma 2.1 to get that there exists a Λ_v such that

$$\forall \Phi \in C_0^\infty\big(D,\mathbb{R}^d\big),\quad \int_D 2(D\Phi\nabla v\cdot\nabla v) - \int_D |\nabla v|^2\nabla\cdot\Phi + \lambda_a\int_D v^2\nabla\cdot\Phi = \Lambda_v\int_{\Omega_v}\nabla\cdot\Phi.$$

We have $u = v$ outside B_R so, using this equation and the Euler–Lagrange equation for u we see that $\Lambda_v = \Lambda$. Now, we write the Euler–Lagrange for v_n and $\Phi \in C_0^\infty(D,\mathbb{R}^d)$ such that $\int_{\Omega_v}\nabla\cdot\Phi \neq 0$,

$$\int_D 2(D\Phi\nabla v_n\cdot\nabla v_n) - \int_D |\nabla v_n|^2\nabla\cdot\Phi + \lambda_a\int_D v_n^2\nabla\cdot\Phi = \Lambda_n\int_{\Omega_{v_n}}\nabla\cdot\Phi,$$

and, using the strong convergence of v_n to v, we get that

$$\begin{aligned}\lim_{n\to\infty}\Lambda_n &= \lim_{n\to\infty}\frac{\int_D 2(D\Phi\nabla v_n\cdot\nabla v_n) - \int_D|\nabla v_n|^2\nabla\cdot\Phi + \lambda_a\int_D v_n^2\nabla\cdot\Phi}{\int_{\Omega_{v_n}}\nabla\cdot\Phi}\\ &= \frac{\int_D 2(D\Phi\nabla v\cdot\nabla v) - \int_D|\nabla v|^2\nabla\cdot\Phi + \lambda_a\int_D v^2\nabla\cdot\Phi}{\int_{\Omega_v}\nabla\cdot\Phi}\\ &= \Lambda.\end{aligned}$$

Since $\lim\mu_n = +\infty$ we get the contradiction from Steps 2 and 4, and so $\mu_+(h)$ is finite.

To conclude this first part, we now have to see that $\Lambda \leqslant \mu_+(h)$. Let $\Phi \in C_0^\infty$ be such that $\int_{\Omega_u}\nabla\cdot\Phi = -1$, and let $u_t(x) = u(x + t\Phi(x))$. Using the calculus in the proof of Lemma 2.1 we have, for $t \geqslant 0$ small enough,

$$\begin{aligned}&a \leqslant |\Omega_{u_t}| = a + t + \mathrm{o}(t) \leqslant a + h,\\ &J(u_t) = J(u) - t\Lambda + \mathrm{o}(t).\end{aligned}$$

Now, using (8), we have

$$J(u) + \mu_+(h)a \leqslant J(u) - t\Lambda + \mu_+(h)(a+t) + \mathrm{o}(t),$$

and we get $\Lambda \leqslant \mu_+(h)$.

Second part: $\boldsymbol{\lim\mu_+(h) = \Lambda}$**.**

We first see that $\mu_+(h) > 0$ for $h > 0$. Indeed, if $\mu_+(h) = 0$ we write

$$\text{for every } \varphi \in C_0^\infty(B_R) \text{ with } \big|\{\varphi \neq 0\}\big| < h,\quad J(u) \leqslant J(u + t\varphi),$$

so

$$-\Delta u = \lambda_a u \quad \text{in } B_R,$$

which contradicts $0 < |\Omega_u \cap B_R| < |B_R|$.

Let $\varepsilon > 0$ and $h_n > 0$ a decreasing sequence tending to 0. Because $h \to \mu_+(h)$ is non-increasing, we just have to see that $\lim\mu_+(h_n) \leqslant \Lambda + \varepsilon$ for a sub-sequence of h_n. If $\Lambda > 0$, let $\varepsilon \in]0,\Lambda[$ and $0 < \alpha_n := \mu_+(h_n) - \varepsilon < \mu_+(h_n)$; if $\Lambda = 0$, let $0 < \alpha_n = \mu_+(h_n)/2 < \mu_+(h_n)$. There exists v_n such that

$$J(v_n) + \alpha_n\big(|\Omega_{v_n}| - a\big)^+ = \min_{v\in\mathcal{F},\ |\Omega_v|\leqslant a+h_n}\big\{J(v) + \alpha_n\big(|\Omega_v| - a\big)^+\big\}.$$

Since $\alpha_n < \mu_+(h_n)$ we see that $|\Omega_{v_n}| > a$ (otherwise we write $J(u) \leqslant J(v_n) + \alpha_n(|\Omega_{v_n}| - a)^+$). We now have 4 steps that are very similar to the 4 steps used in the previous part to show that $\mu_+(h_n)$ is finite.

Step 1. Euler–Lagrange equation for $\boldsymbol{v_n}$**.** If $v \in \mathcal{F}$ is such that $|\Omega_v| \leqslant |\Omega_{v_n}|$, we have $J(v_n) \leqslant J(v)$. Then, as in Lemma 2.1 we can write the Euler–Lagrange equation (13) for v_n in B_R for some Λ_n.

Step 2. $\Lambda_n \geqslant \alpha_n$. Since $|\Omega_{v_n}| > a$ the proof is the same as Step 2 in the first part, with α_n instead of μ_n.

Step 3. v_n strongly converge to some v. As in Step 3 above, we just write,

$$J(v_n) + \alpha_n\big(|\Omega_{v_n}| - a\big)^+ \leqslant J(u),$$

to get (up to a sub-sequence) that v_n weakly converges in $H_0^1(D)$, strongly in $L^2(D)$ and almost-everywhere to $v \in \mathcal{F}$. We have $a < |\Omega_{v_n}| \leqslant a + h_n$ and so $\lim_{n\to\infty} |\Omega_{v_n}| = a$. As in Step 3 above, we deduce that v is a solution of (6), and using

$$J(v_n) + \alpha_n\big(|\Omega_{v_n}| - a\big) \leqslant J(v),$$

we get the strong convergence in $H_0^1(D)$.

Step 4. $\lim \Lambda_n = \Lambda$. The proof is the same as in Step 4 of the first part of the proof. We write the Euler–Lagrange equation for v in D and use $u = v$ outside B_R. We get that $\lim \Lambda_n = \Lambda$ by letting n go to $+\infty$ in the Euler–Lagrange equation for v_n in B_R (using the strong convergence of v_n).

We can now conclude this second part: if $\Lambda > 0$, we have, for n large enough,

$$\mu_+(h_n) - \varepsilon = \alpha_n \leqslant \Lambda_n \leqslant \Lambda + \varepsilon,$$

and so $\mu_+(h_n) \leqslant \Lambda + 2\varepsilon$.

If $\Lambda = 0$ we have

$$\mu_+(h_n)/2 = \alpha_n \leqslant \Lambda_n \leqslant \varepsilon,$$

and so $0 \leqslant \mu_+(h_n) \leqslant 2\varepsilon$.

In both cases, we have $\Lambda \leqslant \mu_+(h_n) \leqslant \Lambda + 2\varepsilon$.

Third part: $\lim \mu_-(h) = \Lambda$.

Let h_n be a sequence decreasing to 0, and let $\varepsilon > 0$. Because $h \to \mu_-(h)$ is increasing, we just have to show that $\lim_{n\to\infty} \mu_-(h_n) \geqslant \Lambda - \varepsilon$ for a sub-sequence of h_n.

We first see that $\mu_-(h) \leqslant \Lambda$. Let $\Phi \in C_0^\infty(B_R, \mathbb{R}^d)$ be such that $\int_{B_R} \nabla \cdot \Phi = 1$ and let $u_t = u(x + t\Phi(x))$ for $t \geqslant 0$. We have (using the proof of Lemma 2.1),

$$a - h \leqslant |\Omega_{u_t}| = a - t + \mathrm{o}(t) \leqslant a,$$
$$J(u_t) = J(u) + t\Lambda + \mathrm{o}(t).$$

Now, using (7), we have

$$J(u) + \mu_-(h)a \leqslant J(u) + t\Lambda + \mu_-(h)(a - t) + \mathrm{o}(t),$$

and we get $\mu_-(h) \leqslant \Lambda$.

Let v_n be a solution of the following minimization problem,

$$J(v_n) + \big(\mu_-(h_n) + \varepsilon\big)\big(|\Omega_{v_n}| - (a - h_n)\big)^+ = \min_{w\in\mathcal{F},\ |\Omega_w|\leqslant a} \big\{J(w) + \big(\mu_-(h) + \varepsilon\big)\big(|\Omega_w| - (a - h_n)\big)^+\big\}. \qquad (15)$$

We will first see that,

$$a - h_n \leqslant |\Omega_{v_n}| < a.$$

If $|\Omega_{v_n}| = a$ we have,

$$J(u) + \big(\mu_-(h_n) + \varepsilon\big)|\Omega_u| \leqslant J(v_n) + \big(\mu_-(h_n) + \varepsilon\big)|\Omega_{v_n}| \leqslant J(w) + \big(\mu_-(h_n) + \varepsilon\big)|\Omega_w|,$$

for $w \in \mathcal{F}$ with $a - h_n \leqslant |\Omega_w| \leqslant a$ which contradicts the definition of $\mu_-(h_n)$.

Now, if $|\Omega_{v_n}| < a - h_n$, we have $J(v_n) \leqslant J(v_n + t\varphi)$ for every $\varphi \in C_0^\infty(B_R)$ with $|\{\varphi \neq 0\}| < a - h_n - |\Omega_{v_n}|$. And we get that $-\Delta v_n = \lambda_a v_n$ in B_R and so, we have $v_n \equiv 0$ on B_R or $v_n > 0$ on B_R, but this last case contradicts $|\Omega_{v_n}| < a$. If $v_n \equiv 0$ on B_R, because $v_n = u$ outside B_R, we get $u \in H_0^1(B_R)$, and using $J(u) \leqslant J(v_n)$,

$$\int_{B_R} |\nabla u|^2 - \lambda_a \int_{B_R} u^2 \leqslant 0.$$

We now deduce ($u \not\equiv 0$ on B_R) that $\lambda_a \geqslant \lambda_1(B_R)$, which is a contradiction, at least for R small enough.

We now study the sequence v_n in a very similar way than above.

Step 1. Euler–Lagrange equation for v_n. $J(v_n) \leqslant J(v)$ for $v \in \mathcal{F}$ with $|\Omega_v| \leqslant |\Omega_{v_n}|$, so we have an Euler–Lagrange equation (13) for v_n in B_R for some Λ_n.

Step 2. $\Lambda_n \leqslant (\mu_-(h_n) + \varepsilon)$. Since $|\Omega_{v_n}| < a$, we take $\Phi \in C_0^\infty(B_R, \mathbb{R}^d)$ with $\int_{B_R} \nabla \cdot \Phi = -1$ and $v_n^t(x) = v_n(x + t\Phi(x))$ for $t \geqslant 0$ small. We have $|\Omega_{v_n^t}| = |\Omega_{v_n}| + t + \mathrm{o}(t) \leqslant a$ and $J(v_n^t) = J(v_n) - \Lambda_n t + \mathrm{o}(t)$ and writing (15) with $w = v_n^t$ we get the result.

Step 3. v_n strongly converge to some v. As in Step 3 above we just write that

$$J(v_n) + \big(\mu_-(h_n) + \varepsilon\big)\big(|\Omega_{v_n}| - (a - h_n)\big) \leqslant J(u) + \big(\mu_-(h_n) + \varepsilon\big)h_n,$$

to get (up to a sub-sequence) that v_n weakly converge in $H_0^1(D)$, strongly in $L^2(D)$ and almost-everywhere to $v \in \mathcal{F}$. We have $a - h_n < |\Omega_{v_n}| \leqslant a$ and so $\lim_{n\to\infty} |\Omega_{v_n}| = a$. As in Step 3 above, we deduce that v is a solution of (6), and using

$$J(v_n) + \big(\mu_-(h_n) + \varepsilon\big)\big(|\Omega_{v_n}| - (a - h_n)\big) \leqslant J(v) + \big(\mu_-(h_n) + \varepsilon\big)\big(|\Omega_v| - (a - h_n)\big)^+,$$

we get the strong convergence in $H_0^1(D)$.

Step 4. $\lim \Lambda_n = \Lambda$. The proof is exactly the same as in Step 4 above in the study of the limit of $\mu_+(h_n)$.

Now we have, using steps 2 and 4, for n large enough,

$$\Lambda - \varepsilon \leqslant \Lambda_n \leqslant \mu_-(h_n) + \varepsilon \leqslant \Lambda + \varepsilon,$$

and so $\lim_{n\to\infty} \mu_-(h_n) = \Lambda$.

Fourth part: $\Lambda > 0$.

We would like to show that $\Lambda > 0$ (which implies $\mu_-(h) > 0$ for h small enough). We argue by contradiction and we suppose that $\Lambda = 0$. The proof is very close to the proof of Proposition 6.1 in [3]. We start with the following proposition:

Proposition 2.6. *Assume $\Lambda = 0$. Then, there exists η a decreasing function with $\lim_{r\to 0} \eta(r) = 0$ such that, if $x_0 \in B_{R/2}$ and $B(x_0, r) \subset B_{R/2}$ with $|\{u = 0\} \cap B(x_0, r)| > 0$, then*

$$\frac{1}{r} \fint_{\partial B(x_0,r)} u \leqslant \eta(r). \tag{16}$$

Proof of Proposition 2.6. Let x_0, r be as above, and we set $B_r = B(x_0, r)$. Let v be defined by,

$$\begin{cases} -\Delta v = \lambda_a u & \text{in } B_r, \\ v = u & \text{on } \partial B_r, \end{cases}$$

and $v = u$ outside B_r. We have $v > 0$ on B_r. We get, using (8),

$$\int_{B_r} \big(|\nabla u|^2 - |\nabla v|^2\big) - \lambda_a \int_{B_r} \big(u^2 - v^2\big) \leqslant \mu_+\big(\omega_d r^d\big)\big|\{u = 0\} \cap B_r\big|, \tag{17}$$

we also get (using $-\Delta v = \lambda_a u$ in B_r),

$$\int_{B_r} \left(|\nabla u|^2 - |\nabla v|^2\right) - \lambda_a \int_{B_r} u^2 - v^2 = \int_{B_r} \nabla(u-v)\cdot\nabla(u-v+2v) - \lambda_a \int_{B_r} u^2 - v^2$$
$$= \int_{B_r} \left|\nabla(u-v)\right|^2 + \lambda_a \int_{B_r} (u-v)^2. \tag{18}$$

Now, with the same computations as in [1,11] (with $\lambda_a u$ instead of f) we get,

$$\left|\{u=0\}\cap B_r\right| \left(\frac{1}{r} \fint_{\partial B_r} u\right)^2 \leqslant C \int_{B_r} \left|\nabla(u-v)\right|^2. \tag{19}$$

Now, using (17), (18) and (19) we get the result. $\square$

End of proof of Theorem 1.5. Now, the rest of the proof is the same as Proposition 6.2 in [3] with $\lambda_a u$ instead of $f\chi_{\Omega_u}$. The idea is that, from the estimate (16) of Proposition 2.6, ∇u tends to 0 at the boundary, and consequently the measure Δu does not charge the boundary $\partial\Omega_u$. It follows that $-\Delta u = \lambda_a u$ in B_R, which, by strict maximum principle, contradicts that u is zero on some part of B_R. $\square$

3. Proof of Theorem 1.2

Let Ω^* be a solution of (1). Then $u = u_{\Omega^*}$ is a solution of (6), and thus satisfies Proposition 1.1 and Theorem 1.5; moreover, $\Omega^* = \Omega_u$. Like in the previous section, we work in B, a small ball centered in $\partial\Omega_u$. Since the approach is local, we will show regularity for the part of $\partial\Omega_u$ included in B; but B can be centered on every point of $\partial\Omega_u \cap D$, so this is of course enough to lead to the announced results in Theorem 1.2.

Coupled with Remark 1.3, we conclude that it is sufficient to prove:

$$\left.\begin{array}{ll} \text{(a)} & \Omega^* \text{ has finite perimeter in } B \text{ and } \mathcal{H}^{d-1}\big((\partial\Omega^*\setminus\partial^*\Omega^*)\cap B\big)=0,\\ \text{(b)} & \Delta u_{\Omega^*} + \lambda_1(\Omega^*)u_{\Omega^*} = \sqrt{\Lambda}\mathcal{H}^{d-1}\lfloor\partial\Omega^* \text{ in } B,\\ \text{(c)} & \text{if } d=2,\ \partial\Omega^*\cap B = \partial^*\Omega^*\cap B. \end{array}\right\} \tag{20}$$

We use the same arguments as in [1] and [11], but we have to deal with the term in $\int u^2$ instead of $\int fu$ (in [11]). So we first start with the following technical lemma.

Lemma 3.1. *There exist $C_1, C_2, r_0 > 0$ such that, for $B(x_0,r)\subset B$ with $r\leqslant r_0$,*

$$\text{if } \frac{1}{r}\fint_{\partial B(x_0,r)} u \geqslant C_1 \quad \text{then} \quad u>0 \text{ on } B(x_0,r),$$
$$\text{if } \frac{1}{r}\fint_{\partial B(x_0,r)} u \leqslant C_2 \quad \text{then} \quad u\equiv 0 \text{ on } B(x_0,r/2). \tag{21}$$

Proof. The first point comes directly from the proof of Proposition 2.6. We take the same v and, using equation (19), we see that there exists C_1 such that if $\frac{1}{r}\fint_{\partial B(x_0,r)} u \geqslant C_1$, then $|\{u=0\}\cap B(x_0,r)| = 0$.

For the second part we argue as in Theorem 3.1 in [2]. We will denote B_r for $B(x_0,r)$. In this proof, C denotes (different) constants which depend only on a, d, D, u and B, but not on x_0 or r.

Let $\varepsilon > 0$ small and such that $\{u=\varepsilon\}$ is smooth (true for almost every ε), let $D_\varepsilon = (B_r\setminus\overline{B}_{r/2})\cap\{u>\varepsilon\}$ and v_ε be defined by

$$\begin{cases} -\Delta v_\varepsilon = \lambda_a u & \text{in } D_\varepsilon,\\ v_\varepsilon = u & \text{in } D\setminus B_r,\\ v_\varepsilon = u & \text{in } B_r\cap\{u\leqslant\varepsilon\},\\ v_\varepsilon = \varepsilon & \text{in } \overline{B}_{r/2}\cap\{u>\varepsilon\}. \end{cases}$$

We see that $u - v_\varepsilon$ is harmonic in D_ε.

We now show that $(v_\varepsilon - u)_\varepsilon$ is bounded in $H^1(D)$, for small $\varepsilon > 0$. Let φ be in $C_0^\infty(B_r)$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $\overline{B}_{r/2}$. Let $\Psi = (1-\varphi)u + \varepsilon\varphi = u + \varphi(\varepsilon - u)$. We have:

$$\Psi - u = 0 = v_\varepsilon - u \text{ on } \partial B_r \cup \big(\partial D_\varepsilon \cap (B_r \setminus \overline{B}_{r/2})\big),$$

and

$$\Psi - u = \varepsilon - u = v_\varepsilon - u \geqslant -\|u\|_\infty \text{ on } \partial D_\varepsilon \cap \partial B_{r/2},$$

so using that $v_\varepsilon - u$ is harmonic, we get $-\|u\|_\infty \leqslant v_\varepsilon - u \leqslant 0$ on D_ε and,

$$\int_{D_\varepsilon} \big|\nabla(v_\varepsilon - u)\big|^2 \leqslant \int_{D_\varepsilon} \big|\nabla(\Psi - u)\big|^2.$$

Now, using that $\nabla\Psi = \nabla u(1-\varphi) - (\nabla\varphi)u + \varepsilon\nabla\varphi$ and the L^∞ bounds for u and ∇u, we see that $v_\varepsilon - u$ is bounded in $H^1(D)$.

Now, up to a subsequence, v_ε weakly converges in $H_0^1(D)$ to v such that:

$$\begin{cases} -\Delta v = \lambda_a u & \text{in } (B_r \setminus \overline{B}_{r/2}) \cap \Omega_u, \\ v = u & \text{in } D \setminus B_r, \\ v = 0 & \text{in } \overline{B}_{r/2} \cup \big(B_r \cap \{u = 0\}\big). \end{cases}$$

Using (7) with $h = |B_{r/2}|$, and $u = v$ in $D \setminus B_r$, we have:

$$\int_{B_r} |\nabla u|^2 - \lambda_a \int_{B_r} u^2 + \mu_-(h)|\Omega_u \cap B_r| \leqslant \int_{B_r} |\nabla v|^2 - \lambda_a \int_{B_r} v^2 + \mu_-(h)|\Omega_v \cap B_r|,$$

and so,

$$\begin{aligned}
\int_{B_{r/2}} |\nabla u|^2 \mu_-(h)|\Omega_u \cap B_{r/2}| &\leqslant \int_{B_r\setminus B_{r/2}} \nabla(v-u)\cdot\nabla(u - v + 2v) - \lambda_a \int_{B_r\setminus B_{r/2}} \big(v^2 - u^2\big) + \lambda_a \int_{B_{r/2}} u^2 \\
&\leqslant \liminf_{\varepsilon\to 0} 2\int_{D_\varepsilon} \nabla(v_\varepsilon - u)\cdot\nabla v_\varepsilon - \lambda_a \int_{D_\varepsilon} \big(v_\varepsilon^2 - u^2\big) + \lambda_a \int_{B_{r/2}} u^2 \\
&= \liminf_{\varepsilon\to 0} 2\int_{\partial B_{r/2}\cap\{u>\varepsilon\}} (\varepsilon - u)\frac{\partial v_\varepsilon}{\partial n} + 2\lambda_a \int_{D_\varepsilon} (v_\varepsilon - u)u - \lambda_a \int_{D_\varepsilon} \big(v_\varepsilon^2 - u^2\big) + \lambda_a \int_{B_{r/2}} u^2 \\
&= \liminf_{\varepsilon\to 0} 2\int_{\partial B_{r/2}\cap\{u>\varepsilon\}} (\varepsilon - u)\frac{\partial v_\varepsilon}{\partial n} + \lambda_a \int_{D_\varepsilon} \big(2uv_\varepsilon - u^2 - v_\varepsilon^2\big) + \lambda_a \int_{B_{r/2}} u^2 \\
&\leqslant \liminf_{\varepsilon\to 0} 2\int_{\partial B_{r/2}\cap\{u>\varepsilon\}} (\varepsilon - u)\nabla v_\varepsilon\cdot\vec{n} + \lambda_a \int_{B_{r/2}} u^2, \qquad (22)
\end{aligned}$$

where $\vec{n}$ is the outward normal of D_ε and so the inward normal of $B_{r/2}$. Let w_ε be such that,

$$\begin{cases} -\Delta w_\varepsilon = \lambda_a u & \text{on } B_r \setminus \overline{B}_{r/2}, \\ w_\varepsilon = u & \text{on } \partial B_r \cap \{u > \varepsilon\}, \\ w_\varepsilon = \varepsilon & \text{on } \big(\partial B_r \cap \{u \leqslant \varepsilon\}\big) \cup \partial B_{r/2}. \end{cases}$$

Because $w_\varepsilon \geqslant \varepsilon$ on $\partial(B_r \setminus B_{r/2})$ and super-harmonic in $B_r \setminus \overline{B}_{r/2}$, we get that $w_\varepsilon \geqslant \varepsilon$ in $B_r \setminus \overline{B}_{r/2}$. In particular $w_\varepsilon \geqslant v_\varepsilon = \varepsilon$ in $\partial D_\varepsilon \cap (B_r \setminus \overline{B}_{r/2})$. Moreover, we also have $w_\varepsilon \geqslant v_\varepsilon$ on $\partial D_\varepsilon \cap (\partial B_r \cup \partial B_{r/2})$, and since $w_\varepsilon - v_\varepsilon$ is harmonic in D_ε, we get $w_\varepsilon \geqslant v_\varepsilon$ in D_ε. Using $w_\varepsilon = v_\varepsilon = \varepsilon$ on $\partial B_{r/2} \cap \{u > \varepsilon\}$, we can now compare the gradients of w_ε and v_ε on this set,

$$0 \leqslant -\nabla v_\varepsilon \cdot \vec{n} \leqslant -\nabla w_\varepsilon \cdot \vec{n} \text{ on } \partial B_{r/2} \cap \{u > \varepsilon\}. \qquad (23)$$

Let now w_ε^0 be defined by $w_\varepsilon^0 = w_\varepsilon$ on $\partial(B_r \setminus \overline{B}_{r/2})$ and harmonic in $B_r \setminus B_{r/2}$. We use now the following estimate:

$$0 \leqslant -\nabla w_\varepsilon^0 \cdot \vec{n} \leqslant \frac{C}{r} \fint_{\partial B_r} (u-\varepsilon)^+ \leqslant C\gamma \text{ on } \partial B_{r/2}, \tag{24}$$

where $\gamma = \frac{1}{r}\fint_{\partial B_r} u$ (to get this estimate, we can first prove, using a comparison argument, that $|\nabla w_\varepsilon^0| \leqslant \frac{C}{r}\|w_\varepsilon^0 - \varepsilon\|_{\infty, B_{3r/4}\setminus B_{r/2}}$, and then conclude using again maximum principle and Poisson formula for functions that are harmonic in a ball). Let $w_\varepsilon^1 = w_\varepsilon - w_\varepsilon^0$, we have $w_\varepsilon^1 = 0$ on $\partial(B_r \setminus B_{r/2})$ and $-\Delta w_\varepsilon^1 = \lambda_a u$ in $B_r \setminus \overline{B}_{r/2}$ and so,

$$\left\|\nabla w_\varepsilon^1\right\|_{\infty, B_r\setminus B_{r/2}} \leqslant Cr\|u\|_\infty \leqslant Cr. \tag{25}$$

Now using (22), (23), (24) and (25) we get,

$$L := \int_{B_{r/2}} |\nabla u|^2 + \mu_-(h)|\Omega_u \cap B_{r/2}| \leqslant C(\gamma + r)\int_{\partial B_{r/2}} u + \lambda_a \int_{B_{r/2}} u^2. \tag{26}$$

Our goal is now to bound from above the right-hand of this inequality with $CL(\gamma + r)$: and so if γ and r are small enough we will get $L = 0$ and so $u \equiv 0$ in $B_{r/2}$.

We now give an estimate of $\|u\|_{\infty, B_{r/2}}$ in term of γ. Let $w = 0$ on ∂B_r and $-\Delta w = \lambda_a u$ in B_r. We have (using (2)) $\Delta(u - w) = \Delta u + \lambda_a u \geqslant 0$ in B_r and $u - w = u$ on ∂B_r so,

$$\|u - w\|_{\infty, B_{r/2}} \leqslant C \fint_{\partial B_r} u \leqslant C\gamma r.$$

We also have that

$$\|w\|_{\infty, B_r} \leqslant Cr^2\|u\|_{\infty, B_r} \leqslant Cr^2,$$

and finally,

$$\|u\|_{\infty, B_{r/2}} \leqslant C\bigl(\gamma r + r^2\bigr). \tag{27}$$

We now write (using (27)),

$$\begin{aligned}\int_{\partial B_{r/2}} u &\leqslant C\biggl(\int_{B_{r/2}} |\nabla u| + \frac{1}{r}\int_{B_{r/2}} u\biggr)\\ &\leqslant C\biggl(\frac{1}{2}\int_{B_{r/2}} |\nabla u|^2 + \frac{1}{2}|\Omega_u \cap B_{r/2}| + \frac{1}{r}|\Omega_u \cap B_{r/2}|\|u\|_{\infty, B_{r/2}}\biggr).\end{aligned}$$

Here we use Theorem 1.5 to see that there exists h_0 such that

$$\frac{\Lambda}{2} \leqslant \mu_-(h) \leqslant \Lambda, \quad 0 < h \leqslant h_0.$$

And so, we have

$$\begin{aligned}\int_{\partial B_{r/2}} u &\leqslant C\biggl(\int_{B_{r/2}} |\nabla u|^2 + \mu_-(h)|\Omega_u \cap B_{r/2}| + C|\Omega_u \cap B_{r/2}|(\gamma + r)\biggr)\\ &\leqslant CL(1 + \gamma + r),\end{aligned} \tag{28}$$

with C independent of r for every r small enough such that $h = |B_{r/2}| \leqslant h_0$. We also have (using (27))

$$\int_{B_{r/2}} u^2 \leqslant C|\Omega_u \cap B_{r/2}|\bigl(\gamma r + r^2\bigr) \leqslant CL\bigl(\gamma r + r^2\bigr). \tag{29}$$

We now get, from (26), (28) and (29), if $\gamma \leqslant 1$ and $r \leqslant 1$,

$$L \leqslant C(\gamma + r)L(1 + \gamma + r) + CL\bigl(\gamma r + r^2\bigr) \leqslant CL(\gamma + r),$$

and, if we suppose $r \leqslant \frac{1}{2C}$ we get,

$$L \leqslant CL\gamma + \frac{L}{2},$$

and so, if $\gamma < \frac{1}{2C}$ we get $L = 0$ and $u \equiv 0$ on $B_{r/2}$. □

With the help of this lemma, we are now able to successively prove the three properties (a), (b) and (c) of (20).

Proof of (a). The proof is now, using (21) in Lemma 3.1, the same as in [11] or in [1]. Here are the main steps: we first show that there exists C_1, C_2 and r_0 such that, for every $B(x_0, r) \subset B$ with $r \leqslant r_0$,

$$0 < C_1 \leqslant \frac{|B(x_0, r) \cap \Omega_u|}{|B(x_0, r)|} \leqslant C_2 < 1,$$

and

$$C_1 r^{d-1} \leqslant (\Delta u + \lambda_a u)\big(B(x_0, r)\big) \leqslant C_2 r^{d-1}.$$

The proof is the same as in [11] with $\lambda_a u$ instead of f. It gives directly (using the Geometrical measure theory, see Section 5.8 in [9]) the first point of Theorem 1.2. □

Proof of (b). For the second point, we see that $\Delta u + \lambda_a u$ is absolutely continuous with respect to $\mathcal{H}^{d-1} \lfloor \partial\Omega_u$ which is a Radon–Measure (using the first point), so we can use Radon's Theorem. To compute the Radon's derivative, we argue as in Theorem 2.13 in [11] or (4.7, 5.5) in [1]. The main difference is that here, we have to use (9) in Theorem 1.5 to show that, if u_0 denotes a blow-up limit of $u(x_0 + rx)/r$ (when r goes to 0), then u_0 is such that,

$$\int_{B(0,1)} |\nabla u_0|^2 + \Lambda\big|\{u_0 \neq 0\} \cap B(0,1)\big| \leqslant \int_{B(0,1)} |\nabla v|^2 + \Lambda\big|\{v \neq 0\} \cap B(0,1)\big|,$$

for every v such that $v = u_0$ outside $B(0, 1)$. To show this, in [1] or in [11] the authors use only perturbations in $B(x_0, r)$ with r goes to 0, so using (9), we get the same result. We can compute the Radon's derivative and get (in B)

$$\Delta u + \lambda_a u = \sqrt{\Lambda}\,\mathcal{H}^{d-1} \lfloor \partial\Omega_u.$$

Now, u is a weak-solution in the sense of [11] and [1] and we directly get the analytic regularity of $\partial^*\Omega_u$ (this regularity is shown for weak-solutions). □

Proof of (c). If $d = 2$, in order to have the regularity of the whole boundary, we have to show that Theorem 6.6 and Corollary 6.7 in [1] (which are for solutions and not weak-solutions) are still true for our problem. The corollary directly comes from the theorem. So we need to show that, if $d = 2$ and $x_0 \in \partial\Omega_u$, then

$$\lim_{r \to 0} \fint_{B(x_0,r)} \max\big\{\Lambda - |\nabla u|^2, 0\big\} = 0. \tag{30}$$

We argue as in Theorem 6.6 in [1]. Let $\zeta \in C_0^\infty(B)$ be non-negative and let $v = \max\{u - \varepsilon\zeta, 0\}$. Using (7) with this v and $h = |0 < u \leqslant \varepsilon\zeta| \leqslant |\{\zeta \neq 0\}|$ we get,

$$\begin{aligned}
\mu_-(h)|0 < u \leqslant \varepsilon\zeta| &\leqslant \int |\nabla v|^2 - \int |\nabla u|^2 + \lambda_a \int \big(u^2 - v^2\big) \\
&= \int \big|\nabla \min\{\varepsilon\zeta, u\}\big|^2 - 2\int \nabla u \cdot \nabla \min\{\varepsilon\zeta, u\} \\
&\quad + \lambda_a \int_{\{u<\varepsilon\zeta\}} u^2 - \lambda_a \int_{\{u\geqslant\varepsilon\zeta\}} (\varepsilon\zeta)^2 + 2\lambda_a \int_{\{u\geqslant\varepsilon\zeta\}} u\varepsilon\zeta.
\end{aligned}$$

Using $-\Delta u = \lambda_a u$ in Ω_u we get:

$$\int \nabla u \cdot \nabla \min\{\varepsilon\zeta, u\} = \lambda_a \int u \min\{\varepsilon\zeta, u\} = \lambda_a \int_{\{u<\varepsilon\zeta\}} u^2 + \lambda_a \int_{\{u\geqslant\varepsilon\zeta\}} u\varepsilon\zeta,$$

and so,

$$\mu_-(h)|0<u\leqslant\varepsilon\zeta|\leqslant\int\limits_{\{u<\varepsilon\zeta\}}|\nabla u|^2+\int\limits_{\{u\geqslant\varepsilon\zeta\}}\varepsilon^2|\nabla\zeta|^2-\lambda_a\int\limits_{\{u<\varepsilon\zeta\}}u^2-\lambda_a\int\limits_{\{u\geqslant\varepsilon\zeta\}}(\varepsilon\zeta)^2,$$

and so, we can deduce that,

$$\int\limits_{\{0<u<\varepsilon\zeta\}}\left(\Lambda-|\nabla u|^2\right)\leqslant\int\limits_{\{u\geqslant\varepsilon\zeta\}}\varepsilon^2|\nabla\zeta|^2+\left(\Lambda-\mu_-(h)\right)h.$$

The only difference now with [1] is the last term. Using Theorem 1.5, we see that $(\Lambda-\mu_-(h))h=o(h)$, so we can choose the same kind of ζ and ε as in [1] to get (30) (see Theorem 5.7 in [3] for more details). □

Appendix A

In this appendix, we discuss the hypothesis "D is connected". We begin with the following example, taken from [4].

Example A.1. (From [4].) We take $D=D_1\cup D_2$, where D_1,D_2 are disjoint disks in $\mathbb{R}^2$ of radius R_1,R_2 with $R_1>R_2$. If $a=\pi R_1^2+\varepsilon$, then the solution u of (5) coincides with the first eigenfunction of D_1 and is identically 0 on D_2, and thus $\Omega_u=D_1$ and $|\Omega_u|<a$.

In this case, we can choose an open subset ω of D_2 with $|\omega|=\varepsilon$. Then $\Omega^*:=D_1\cup\omega$ is a solution of (1). Since ω may be chosen as irregular as one wants, this proves that optimal domains are not regular in general.

However, we are able to prove the following proposition.

Proposition A.2 *(The non-connected case). If we suppose that D is not connected, the problem* (5) *still has a solution u which is locally Lipschitz continuous in D. If ω is any open connected component of D, we have three cases*:

1. *either $u>0$ on ω,*
2. *or $u=0$ on ω,*
3. *or $0<|\Omega_u\cap\omega|<|\omega|$, and $\partial\Omega_u$ has the same regularity as stated in Theorem* 1.2.

If $|\Omega_u|<a$, then only the first two cases can appear.

Remark A.3. It follows from Proposition A.2 that we obtain the same regularity as in the connected case. Indeed, in the first two cases, $\partial\Omega^*\cap\omega=\partial\Omega_u\cap\omega=\emptyset$.

Remark A.4. To summarize, in all cases, there exists a solution Ω^* to (1) which is regular in the sense of Theorem 1.2, but there may be some other non-regular optimal shape. And if D is connected, any optimal shape is regular.

Proof. The existence and the Lipschitz regularity are stated in Proposition 1.1.

If $u=0$ a.e. on ω, then we get $u=0$ on ω by continuity.

If $u>0$ a.e. on ω, by Lemma 2.5, $u>0$ everywhere in ω.

If $0<|\Omega_u\cap\omega|<|\omega|$, the restriction of u to ω is of course solution of (6) with ω instead of D and $|\omega\cap\Omega_u|$ instead of a. We then may apply Theorem 1.2.

Finally, if $|\Omega_u|<a$, we may write $J(u)\leqslant J(u+t\varphi)$ for all $t\in(-\varepsilon,\varepsilon)$ and for all $\varphi\in C_0^\infty(D)$ such that $|\Omega_\varphi|<a-|\Omega_u|$ and so:

$$0=\left.\frac{\mathrm{d}J(u+t\varphi)}{\mathrm{d}t}\right|_{t=0}=2\int\limits_D(\nabla u\cdot\nabla\varphi)-2\lambda_a\int\limits_D u\varphi.$$

That is $-\Delta u=\lambda_a u$ in D and the third case is not possible since by maximum principle $u>0$ or $u=0$ on each connected component of D. □

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